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CIMPA School on Functional Equations: Theory, Practice and Interaction.

Introduction to Transcendental Number Theory **6**

Early History of Transcendental Number Theory

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Algebraic vs transcendental numbers

An algebraic number is a complex number which is a root of a polynomial with rational coefficients. For instance $\sqrt{2}$, $i = \sqrt{-1}$, the Golden Ratio $(1+\sqrt{5})/2$, the roots of unity $\mathrm{e}^{2i\pi a/b}$, the roots of the polynomial $X^5 - 6X + 3$ are algebraic numbers. A **transcendental number** is a complex number which is not algebraic.

Abstract

The existence of transcendental numbers was proved in 1844 by J. Liouville who gave explicit ad-hoc examples. The transcendence of constants from analysis is harder ; the first result was achieved in 1873 by Ch. Hermite who proved the transcendence of e. In 1882, the proof by F. Lindemann of the transcendence of π gave the final (and negative) answer to the Greek problem of squaring the circle. The transcendence of $2^{\sqrt{2}}$ and e^{π}, which was included in Hilbert's seventh problem in 1900, was proved by Gel'fond and Schneider in 1934. During the last century, this theory has been extensively developed, and these developments gave rise to a number of deep applications. In spite of that, most questions are still open. In this talk we survey some of the early results of this theory.

Rational, algebraic irrational, transcendental

Goal : decide upon the arithmetic nature of "given" numbers : rational, algebraic irrational, transcendental.

Rational integers : $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$.

Rational numbers :

$$\mathbb{Q} = \{p/q \;\mid\; p \in \mathbb{Z}, q \in \mathbb{Z}, q > 0,\; \gcd(p,q) = 1\}.$$

Algebraic number : root of a polynomial with rational coefficients.

A **transcendental number** is complex number which is not algebraic.

Rational, algebraic irrational, transcendental

Goal : decide wether a “given” real number is rational, algebraic irrational or else transcendental.

- **Question** : what means ”given” ?

- Criteria for irrationality : development in a given basis (e.g. : decimal expansion, binary expansion), continued fraction.

- Analytic formulae, limits, sums, integrals, infinite products, any limiting process.

Algebraic irrational numbers

Examples of algebraic irrational numbers :

- $\sqrt{2}$, $i = \sqrt{-1}$, the Golden Ratio $(1+\sqrt{5})/2$,

- $\sqrt{d}$ for $d \in \mathbb{Z}$ not the square of an integer (hence not the square of a rational number),

- the roots of unity $\mathrm{e}^{2i\pi a/b}$, for $a/b \in \mathbb{Q}$,

- and, of course, any root of an irreducible polynomial with rational coefficients of degree > 1.

Rule and compass ; squaring the circle

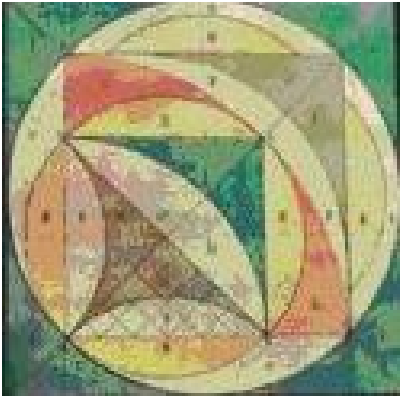
Construct a square with the same area as a given circle by using only a finite number of steps with compass and straightedge.

Any constructible length is an algebraic number, though not every algebraic number is constructible
(for example $\sqrt[3]{2}$ is not constructible).

Pierre Laurent Wantzel (1814 – 1848)
Recherches sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas. Journal de Mathématiques Pures et Appliquées **1** (2), (1837), 366–372.

Quadrature of the circle

Marie Jacob
La quadrature du cercle
Un problème
à la mesure des Lumières
Fayard (2006).



Resolution of equations by radicals

The roots of the polynomial $X^5 - 6X + 3$ are algebraic numbers, and are not expressible by radicals.

Evariste Galois
(1811 – 1832)

Gottfried Wilhelm Leibniz

Introduction of the concept of the transcendental in mathematics by Gottfried Wilhelm Leibniz in 1684 : *"Nova methodus pro maximis et minimis itemque tangentibus, qua nec fractas, nec irrationales quantitates moratur, ..."*

Breger, Herbert. *Leibniz' Einführung des Transzendenten*, 300 Jahre "Nova Methodus" von G. W. Leibniz (1684-1984), p. 119-32. Franz Steiner Verlag (1986).

Serfati, Michel. *Quadrature du cercle, fractions continues et autres contes*, Editions APMEP, Paris (1992).

Irrationality

Given a basis $b \geq 2$, a real number x is rational if and only if its expansion in basis b is ultimately periodic.

$b = 2$: binary expansion.

$b = 10$: decimal expansion.

For instance the decimal number

$$0.123456789012345678901234567890\ldots$$

is rational :

$$= \frac{1\,234\,567\,890}{9\,999\,999\,999} = \frac{137\,174\,210}{1\,111\,111\,111}.$$

First decimal digits of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.41421356237309504880168872420969807856967187537694807317667973
799073247846210703885038753432764157273501384623091229702492483
605585073721264412149709993583141322266592750559275579995050115
278206057147010955997160597027453459686201472851741864088919860
955232923048430871432145083976260362799525140798968725339654633
180882964062061525835239505474575028775996172983557522033753185
701135437460340849884716038689997069900481503054402779031645424
782306849293691862158057846311159666871301301561856898723723528
850926486124949771542183342042856860601468247207714358548741556
570696776537202264854470158588016207584749226572260020855844665
214583988939443709265918003113882464681570826301005948587040031
864803421948972782906410450726368813137398552561173220402450912
277002269411275736272804957381089675040183698683684507257993647
290607629969413804756548237289971803268024744206292691248590521
810044598421505911202494413417285314781058036033710773091828693
1471017111168391658172688941975871658215212822951848847 . . .

First binary digits of $\sqrt{2}$ http://wims.unice.fr/wims/wims.cgi

1.011010100000100111100110011001111111001110111100110010010000
1000101100101111101100010011011001101110101010010101011110100
1111100011101011011110110000010111010100010010011101110101000 0
1001100111011010001011110101100100001011000001100110011100 1100
1000101010100101011111100100000110000010000111010101110001 0100
0101100001110101000101100011111111001101111111011100100000 11110
1101100111001000011110111010010101000010111100100001110011 1000
1111011010010100111100000000010010000111001101100011110111 11101
0001001110110100011010010001000000010111010000111010000101 0101
1110001111101001110010100110000010110011100011000000000100 01101
1110000110011011110111100101010110001101111001001000100010 1101
0001000001000101100010100100011000001010101111000111001000 10111
1011111000100111000110011110001101101010110101000101000111 0001
0111011011111101001110111001100101100101010011000110100001 1001
1000111110011110010000100110111110101001011110001001000001 1111
0000011011011100101100000101110111010101010010010100000010 00100
11001000000100000011001010010010101000000010011100101001010 . . .

Computation of decimals of $\sqrt{2}$

1 542 decimals computed by hand by Horace Uhler in 1951

14 000 decimals computed in 1967

1 000 000 decimals in 1971

$137 \cdot 10^9$ decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

- Motivation : computation of π.

Square root of 2 on the web

The first decimal digits of $\sqrt{2}$ are available on the internet

1, 4, 1, 4, 2, 1, 3, 5, 6, 2, 3, 7, 3, 0, 9, 5, 0, 4, 8, 8, 0, 1,

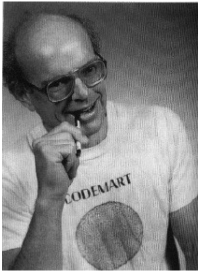
6, 8, 8, 7, 2, 4, 2, 0, 9, 6, 9, 8, 0, 7, 8, 5, 6, 9, 6, 7, 1, 8, ...

http://oeis.org/A002193

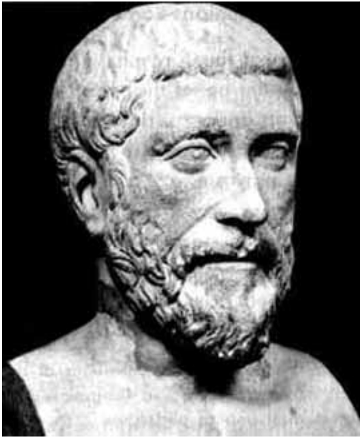
The On-Line Encyclopedia of Integer Sequences

http://oeis.org/

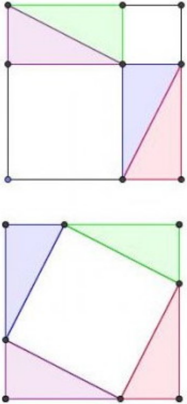
Neil J. A. Sloane



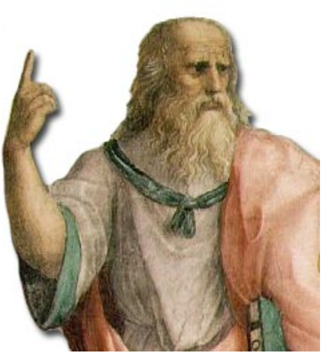
Pythagoras of Samos ∼ 569 BC – ∼ 475 BC



$$a^2 + b^2 = c^2 = (a+b)^2 - 2ab.$$



Irrationality in Greek antiquity



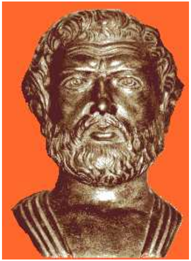
Plato, Republic :
incommensurable lines,
irrational diagonals.

Theodorus of Cyrene
(about 370 BC.)
irrationality of $\sqrt{3}, \ldots, \sqrt{17}$.

Theetetes : if an integer $n > 0$ is the square of a rational number, then it is the square of an integer.

Irrationality of $\sqrt{2}$

Pythagoreas school



Hippasus of Metapontum (around 500 BC).

Sulba Sutras, Vedic civilization in India, $\sim$800-500 BC.

Émile Borel : 1950

The sequence of decimal digits of $\sqrt{2}$ *should behave like a random sequence, each digit should be occurring with the same frequency* $1/10$*, each sequence of* 2 *digits occurring with the same frequency* $1/100$ *. . .*

Émile Borel (1871–1956)

- *Les probabilités dénombrables et leurs applications arithmétiques,*
  Palermo Rend. **27**, 247-271 (1909).
  Jahrbuch Database JFM 40.0283.01
  `http://www.emis.de/MATH/JFM/JFM.html`

- *Sur les chiffres décimaux de* $\sqrt{2}$ *et divers problèmes de probabilités en chaînes,*
  C. R. Acad. Sci., Paris **230**, 591-593 (1950).
  Zbl 0035.08302

Complexity of the b–ary expansion of an irrational algebraic real number

Let $b \geq 2$ be an integer.

• É. Borel (1909 and 1950) : *the b–ary expansion of an algebraic irrational number should satisfy some of the laws shared by almost all numbers (with respect to Lebesgue's measure).*

• **Remark** : no number satisfies **all** the laws which are shared by all numbers outside a set of measure zero, because the intersection of all these sets of full measure is empty!

$$\bigcap_{x \in \mathbb{R}} \mathbb{R} \setminus \{x\} = \emptyset.$$

• More precise statements by B. Adamczewski and Y. Bugeaud.

Conjecture of Émile Borel

Conjecture (É. Borel). *Let x be an irrational algebraic real number, $b \geq 3$ a positive integer and a an integer in the range $0 \leq a \leq b-1$. Then the digit a occurs at least once in the b–ary expansion of x.*

Corollary. *Each given sequence of digits should occur infinitely often in the b–ary expansion of any real irrational algebraic number.*

(consider powers of b).

- An irrational number with a *regular* expansion in some basis b should be transcendental.

The state of the art

There is no explicitly known example of a triple (b, a, x), where $b \geq 3$ is an integer, a is a digit in $\{0, \dots, b-1\}$ and x is an algebraic irrational number, for which one can claim that the digit a occurs infinitely often in the b–ary expansion of x.

A stronger conjecture, also due to Borel, is that algebraic irrational real numbers are *normal* : each sequence of n digits in basis b should occur with the frequency $1/b^n$, for all b and all n.

What is known on the decimal expansion of $\sqrt{2}$?

The sequence of digits (in any basis) of $\sqrt{2}$ is not ultimately periodic

Among the decimal digits

$$\{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\},$$

at least two of them occur infinitely often.

Almost nothing else is known.

Complexity of the expansion in basis b of a real irrational algebraic number

Theorem (B. Adamczewski, Y. Bugeaud 2005 ; conjecture of A. Cobham 1968).
If the sequence of digits of a real number x is produced by a finite automaton, then x is either rational or else transcendental.

Schmidt's Subspace Theorem

The main tool of the proof of the theorem of Adamczewski and Bugeaud is Schmidt's Subspace Theorem

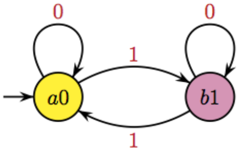
W.M. Schmidt

Finite automata

The Prouhet – Thue – Morse sequence : A010060 OEIS

$$(t_n)_{n\geq 0} = (01101001100101101001011001101001\dots)$$

Write the number n in binary. If the number of ones in this binary expansion is odd then $t_n = 1$, if even then $t_n = 0$.



Fixed point of the morphism $0 \mapsto 01$, $1 \mapsto 10$.

Start with 0 and successively append the Boolean complement of the sequence obtained thus far.

$$t_0 = 0, \qquad t_{2n} = t_n, \qquad t_{2n+1} = 1 - t_n$$

Sequence without cubes XXX.

Introductio in analysin infinitorum

Leonhard Euler (1737)

(1707 – 1783)

Introductio in analysin infinitorum

e is irrational.

$\mathrm{e} = [2, \overline{1, 2n, 1}^{n \geq 1}]$

Continued fraction of e :

$$\mathrm{e} = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\ddots}}}}}}$$

Joseph Fourier

Fourier (1815) : proof by means of the series expansion

$$\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{N!} + r_N$$

with $r_N > 0$ and $N! r_N \to 0$ as $N \to +\infty$.

Course of analysis at the École Polytechnique Paris, 1815.

Variant of Fourier's proof : e^{-1} is irrational

C.L. Siegel : Alternating series

For odd N,

$$1 - \frac{1}{1!} + \frac{1}{2!} - \cdots - \frac{1}{N!} < \mathrm{e}^{-1} < 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{(N+1)!}$$

$$\frac{a_N}{N!} < \mathrm{e}^{-1} < \frac{a_N}{N!} + \frac{1}{(N+1)!}, \quad a_N \in \mathbb{Z}$$

$$a_N < N!\mathrm{e}^{-1} < a_N + 1.$$

Hence $N!\mathrm{e}^{-1}$ is not an integer.

Irrationality of π

Āryabhaṭa, born 476 AD : $\pi \sim 3.1416$.

Nīlakaṇṭha Somayājī, born 1444 AD : *Why then has an approximate value been mentioned here leaving behind the actual value ? Because it (exact value) cannot be expressed.*

K. Ramasubramanian, *The Notion of Proof in Indian Science*, 13th World Sanskrit Conference, 2006.

Irrationality of π

Johann Heinrich Lambert (1728 – 1777)
Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques,
Mémoires de l'Académie des Sciences de Berlin, **17** (1761), p. 265-322 ;
lu en 1767 ; Math. Werke, t. II.

$\tan(v)$ is irrational when $v \neq 0$ is rational.
As a consequence, π is irrational, since $\tan(\pi/4) = 1$.

Lambert and Frederick II, King of Prussia

— Que savez vous, Lambert ?
— Tout, Sire.
— Et de qui le tenez–vous ?
— De moi-même !

Known and unknown transcendence results

Known :

$$\mathrm{e},\ \pi,\ \log 2,\ \mathrm{e}^{\sqrt{2}},\ \mathrm{e}^{\pi},\ 2^{\sqrt{2}},\ \Gamma(1/4).$$

Not known :

$$\mathrm{e}+\pi,\ \mathrm{e}\pi,\ \log \pi,\ \pi^{\mathrm{e}},\ \Gamma(1/5),\ \zeta(3),\ \text{Euler constant}$$

Why is e^{π} known to be transcendental while π^{e} is not known to be irrational ?
Answer : $\mathrm{e}^{\pi}=(-1)^{-i}$.

Transcendental numbers

- Liouville (1844)
- Hermite (1873)
- Lindemann (1882)
- Hilbert's Problem 7th (1900)
- Gel'fond–Schneider (1934)
- Baker (1968)

Existence of transcendental numbers (1844)

J. Liouville (1809 - 1882)

gave the first examples of
transcendental numbers.
For instance

$$\sum_{n\geq 1} \frac{1}{10^{n!}} = 0.110\,001\,000\,000\,0\ldots$$

is a transcendental number.

Charles Hermite and Ferdinand Lindemann

Hermite (1873) :
Transcendence of e
$\mathrm{e} = 2.718\,281\,828\,4\ldots$

Lindemann (1882) :
Transcendence of π
$\pi = 3.141\,592\,653\,5\ldots$

Hermite–Lindemann Theorem

For any non-zero complex number z, one at least of the two numbers z and e^z is transcendental.

Corollaries : Transcendence of $\log \alpha$ and of e^β for α and β non-zero algebraic complex numbers, provided $\log \alpha \neq 0$.

Transcendence of e, π, $\log 2$, $e^{\sqrt{2}}$.

Lindemann–Weierstraß Theorem (1885)

Let $\beta_1, \ldots, \beta_n$ *be algebraic numbers which are linearly independent over* $\mathbb{Q}$*. Then the numbers* $e^{\beta_1}, \ldots, e^{\beta_n}$ *are algebraically independent (over* $\mathbb{Q}$*, or over* $\overline{\mathbb{Q}}$*).*

Ferdinand von Lindemann
(1852 – 1939)

Karl Weierstrass
(1815 - 1897)

Equivalent forms of the Lindemann–Weierstraß Theorem

Let $\beta_1, \dots, \beta_n$ be algebraic. Then the numbers $e^{\beta_1}, \dots, e^{\beta_n}$ are algebraically independent (over $\mathbb{Q}$ of over $\overline{\mathbb{Q}}$) if and only if $\beta_1, \dots, \beta_n$ are linearly independent over $\mathbb{Q}$.

Let $\gamma_1, \dots, \gamma_n$ be algebraic numbers. Then the numbers $e^{\gamma_1}, \dots, e^{\gamma_n}$ are linearly independent (over $\mathbb{Q}$ or over $\overline{\mathbb{Q}}$) if and only if $\gamma_1, \dots, \gamma_n$ are pairwise distinct.

Exercise : Prove the equivalence.

Georg Cantor (1845 - 1918)

The set of algebraic numbers is countable, not the set of real (or complex) numbers.

Cantor (1874 and 1891).

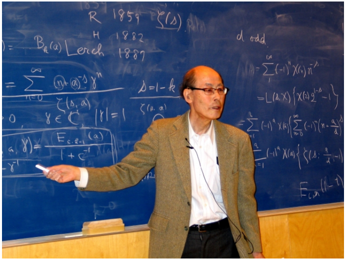
Henri Léon Lebesgue (1875 – 1941)

Almost all numbers for Lebesgue measure are transcendental numbers.

Most numbers are transcendental

Meta conjecture : *any number given by some kind of limit, which is not obviously rational (resp. algebraic), is irrational (resp. transcendental).*

Goro Shimura



Hilbert's Problems

August 8, 1900

Second International Congress of Mathematicians in Paris.

David Hilbert (1862 - 1943)

Twin primes,

Goldbach's Conjecture,

Riemann Hypothesis

Transcendence of e^{π} and $2^{\sqrt{2}}$

A.O. Gel'fond and Th. Schneider

Solution of Hilbert's seventh problem (1934) : *Transcendence of* α^β *and of* $(\log\alpha_1)/(\log\alpha_2)$ *for algebraic* α, β, α_1 *and* α_2.

Transcendence of α^β and $\log\alpha_1/\log\alpha_2$: examples

The following numbers are transcendental :

$$2^{\sqrt{2}} = 2.665\,144\,142\,6\ldots$$

$$\frac{\log 2}{\log 3} = 0.630\,929\,753\,5\ldots$$

$$\mathrm{e}^\pi = 23.140\,692\,632\,7\ldots \qquad (\mathrm{e}^\pi = (-1)^{-i})$$

$$\mathrm{e}^{\pi\sqrt{163}} = 262\ 537\ 412\ 640\ 768\ 743.999\ 999\ 999\ 999\ 25\ldots$$

$e^{\pi} = (-1)^{-i}$

Example : Transcendence of the number

$$e^{\pi\sqrt{163}} = 262\ 537\ 412\ 640\ 768\ 743.999\ 999\ 999\ 999\ 2\ldots.$$

Remark. For

$$\tau = \frac{1 + i\sqrt{163}}{2}, \quad q = e^{2i\pi\tau} = -e^{-\pi\sqrt{163}}$$

we have $j(\tau) = -640\ 320^3$ and

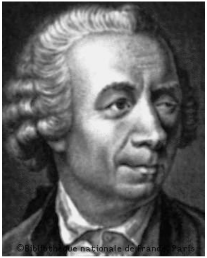
$$\left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}.$$

Beta values : Th. Schneider 1948

Euler Gamma and Beta functions

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^z \cdot \frac{dt}{t}$$



$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Algebraic independence : A.O. Gel'fond 1948

The two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent.

More generally, if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an algebraic number of degree $d \geq 3$, then two at least of the numbers

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \ldots,\alpha^{\beta^{d-1}}$$

are algebraically independent.

Alan Baker 1968

Transcendence of numbers like

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

or

$$e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$$

for algebraic α_i's and β_j's.

Example (Siegel) :

$$\int_0^1 \frac{dx}{1+x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right) = 0.835\,648\,848\,\ldots$$

is transcendental.

More results before 1968

- Thue Siegel Roth.
- Gel'fond and Schneider works.
- C.L. Siegel, *E* and *G* functions (1929, 1949).
- Mahler's method.

Reference

N.I. Fel'dman & A.B. Shidlovskiĭ.
The development and present state of the theory of transcendental numbers.
Russ. Math. Surv. **22**, No. 3, 1–79 (1967) ; translation from Uspehi Mat. Nauk **22** no. 3 (135), 3–81 (1967).

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