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Recent developments of Ramanujan's work

Michel Waldschmidt

http://www.math.jussieu.fr/∼miw/

`http://mathworld.wolfram.com/RamanujanConstant.html`

Ramanujan Constant

The irrational constant

$$R = e^{\pi\sqrt{163}} = 262537412640768743.99999999999925\ldots$$

which is very close to an integer. Numbers such as the Ramanujan constant can be found using the theory of modular functions. In fact, the nine Heegner numbers (which include 163) share a deep number theoretic property related to some amazing properties of the j-function that leads to this sort of near-identity.

Although Ramanujan (1913-14) gave few rather spectacular examples of almost integers (such $e^{\pi\sqrt{58}}$), he did not actually mention particular near-identity give above. In fact, the first to observe this property of 163 was Hermite (1859). The name "Ramanujan's constant" seems to derive from an April Fool's joke played by Martin Gardner (Apr. 1975) on the readers of Scientific American. In his column, Gardner claimed that $e^{\pi\sqrt{163}}$ was exactly an integer, and that Ramanujan had conjectured this in his 1914 paper. Gardner admitted his hoax a few months later (Gardner, July 1975).

Gardner, M. *"Mathematical Games: Six Sensational Discoveries that Somehow or Another have Escaped Public Attention."* Sci. Amer. **232**, 127-131, Apr. 1975.

Gardner, M. *"Mathematical Games: On Tessellating the Plane with Convex Polygons"* Sci. Amer. **232**, 112-117, Jul. 1975.

Hermite, C. *"Sur la théorie des équations modulaires."* Comptes Rendus Acad. Sci. Paris **49**, 16-24, 110-118, and 141-144, 1859. Reprinted in Œuvres complètes, Tome II. Paris: Hermann, p. 61, 1912.

Ramanujan, S. *"Modular equations and approximations to π."* Quaterly J. Math. **45** (1914), 350–372.

$$e^{\pi\sqrt{22}} = 2\ 508\ 951.998\ 2\ldots$$

$$e^{\pi\sqrt{37}} = 199\ 148\ 647.999\ 978\ldots$$

$$e^{\pi\sqrt{58}} = 24\ 591\ 257\ 751.999\ 999\ 82\ldots$$

D. Shanks. *"Dihedral quartic approximations and series for π"*. J. Number Theory **14** (1982), 397–423.

Euler statement (1748, *Introductio in Analysin Infinitorum*):
For any rational $d>0$ *not a square,* $e^{\pi\sqrt{d}}$ *is not a rational integer, nor even a rational number.*

Hilbert's seventh problem (1900): *Show that* $e^{\pi\sqrt{d}}$ *is not an algebraic number.*

Gel'fond (1929). Interpolation series for the entire function $e^{\pi z}$ at the points of $\mathbf{Z}[i]$.

Gel'fond and Schneider (1934). *Transcendence of* a^b – *example:* $a=e^{i\pi}$, $b=-i\sqrt{d}$.

Back to

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743{,}999\,999\,999\,999\,2\ldots$$

Define $\tau = (1+\sqrt{-163})/2$; the imaginary quadratic field $k = \mathbf{Q}(\tau)$ has class number 1. Hence the modular function

$$J(q) = \frac{1}{q} + 744 + 196\,884q + 21\,493\,760q^2 + 864\,299\,970q^3 + \cdots$$

which is defined for $0 < |q| < 1$ satisfies

$$J(e^{2i\pi\tau}) \in \mathbf{Z}.$$

For

$$\tau = (1 + i\sqrt{163})/2$$

the number

$$q = e^{2i\pi\tau} = -e^{-\pi\sqrt{163}}$$

has

$$-\frac{1}{2}10^{-17} < q < 0$$

and

$$-10^{-12} < 196\,884q + 21\,493\,760q^2 + \cdots < 0.$$

Hence

$$-10^{-12} < e^{\pi\sqrt{163}} + J(q) - 744 < 0$$

with $J(q) \in \mathbf{Z}$.

S. Chowla:

$$J(q) = -262\,537\,412\,640\,768\,000 = -(640\,320)^3.$$

$$e^{\pi\sqrt{163}/3} = 640\,319.999\,1\ldots$$

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The equation

$$163y^2 = x^3 + 1728$$

has the solution

$$x = 640\,320, \qquad y = 40\,133\,016.$$

Euler:

$x^2 - x + 41$ *is a prime number for* $x = 1, 2, \ldots, 40.$

The discriminant is $1 - 4 \times 41 = -163$.

Ramanujan, S. *"On certain arithmetical functions."* Trans. Camb. Phil. Soc. **22** n°9 (1916), 159–184.

Eisenstein Series:

$$P = E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n,$$

$$Q = E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n,$$

$$R = E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n$$

where

$$\sigma_k(n) = \sum_{d|n} d^k.$$

Further let

$$\Delta = \frac{1}{1728}(Q^3 - R^2).$$

Special values

Values of P, Q, R and Δ at the points $e^{-2\pi}$ (lemniscate: $\tau = i$) and $-e^{-\pi\sqrt{3}}$ (anharmonic case: $\tau = \varrho$ cubic root of unity)

Let ω_1 be the smallest real period of an elliptic curve of equation $y^2 = 4x^3 - g_2 x - g_3$ whose modular invariant is j with $j(\tau) = J(e^{2i\pi\tau})$ and $\tau = \omega_2/\omega_1$ is the quotient of two fundamental periods.

In the lemniscate case $\tau = i$, $j = 1728$ and an equation of the curve is $y^2 = x^3 - x$, while in the anharmonic case $\tau = \varrho$, $j = 0$ and an equation of the curve is $y^2 = x^3 - 1$.

LEMNISCATE: $\tau = i$, $j = 1728$

$$\omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.6220575542\dots$$

$$P(e^{-2\pi}) = \frac{3}{\pi}$$

$$Q(e^{-2\pi}) = 3\left(\frac{\omega_1}{\pi}\right)^4$$

$$R(e^{-2\pi}) = 0$$

$$\Delta(e^{-2\pi}) = \frac{1}{2^6}\left(\frac{\omega_1}{\pi}\right)^{12}$$

ANHARMONIC: $\tau = \varrho$, $j = 0$

$$\omega_1 = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428650648\ldots$$

$$P(-e^{-\pi\sqrt{3}}) = \frac{2\sqrt{3}}{\pi}$$

$$Q(-e^{-\pi\sqrt{3}}) = 0$$

$$R(-e^{-\pi\sqrt{3}}) = \frac{27}{2}\left(\frac{\omega_1}{\pi}\right)^6$$

$$\Delta(-e^{-\pi\sqrt{3}}) = -\frac{27}{256}\left(\frac{\omega_1}{\pi}\right)^{12}$$

Theorem *(Nesterenko, 1996). For $0 < |q| < 1$, three at least of the four numbers*

$$q, \quad P(q), \quad Q(q), \quad R(q)$$

are algebraically independent.

Corollary. *The three numbers*

$$\pi, \quad e^{\pi}, \quad \Gamma(1/4)$$

are algebraically independent and the three numbers

$$\pi, \quad e^{\pi/\sqrt{3}}, \quad \Gamma(1/3)$$

are algebraically independent.

The Modular Function Δ

$$J(q) = \frac{Q}{\Delta} \quad \text{with} \quad \Delta = \frac{1}{1728}(Q^3 - R^2).$$

The function Δ has a product expansion

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24}$$

and a series expansion

$$\Delta = \sum_{n \geq 1} \tau(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - \cdots$$

with Ramanujan's tau function τ.

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and for

$$R(z) = 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n \qquad \text{(weight } 6\text{).}$$

The modular form $\Delta(z)$ is a *prototype.* In fact, three of Ramanujan's observations about the coefficients $\tau(n)$ provide a foundation for much of the modern theory of modular forms.

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Note: Only the third observation was proved by Ramanujan.

The first observation, on the multiplicativity of the τ function

$$\tau(n)\tau(m) = \tau(nm) \quad \text{if} \quad \gcd(n,m) = 1$$

and other relations which are related with the product expansion

$$\sum_{n\geq 1} \tau(n)n^{-s} = \prod_p \bigl(1 - \tau(p)p^{-s} + p^{11-2s}\bigr)^{-1},$$

was proved by Mordell, based on a theory initiated by Hurwitz which let to the theory of *Hecke operators*.

Hecke operators

$$T_n f(z) = \sum_{\substack{ad=n \\ b \mod n}} d^{n-1} f\left(\frac{az+b}{d}\right).$$

This yields the spectral interpretation of $\tau(n)$. The function Δ appears as a simultaneous eigenvector of a ring of operators:

$$T_n \Delta = \tau(n)\Delta.$$

The second observation

For all primes p *we have* $|\tau(p)| \leq 2p^{11/2}$

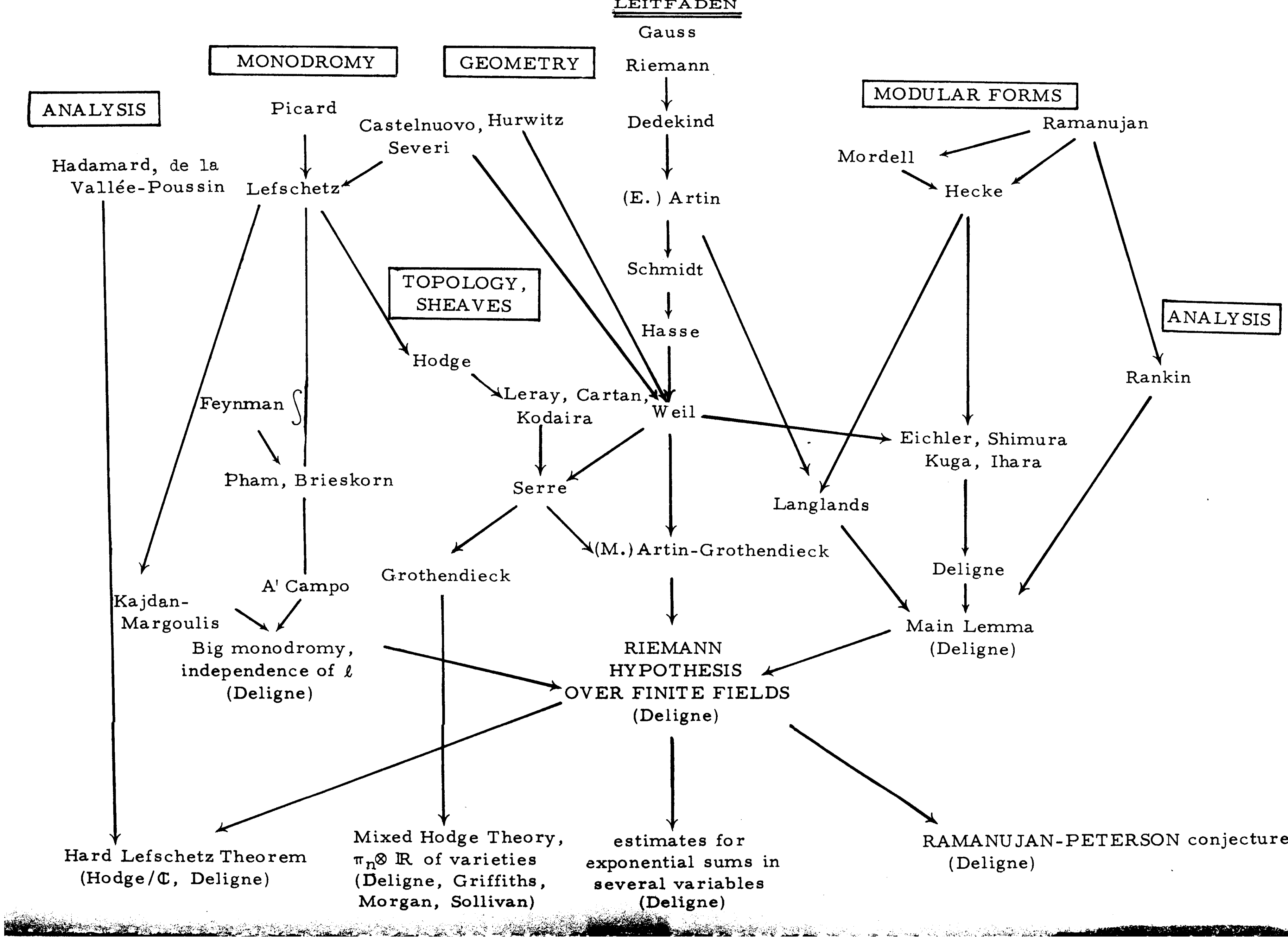
led to Deligne's proof of the Weil Conjectures. Deligne won the Fields Medal for his work, which has been called

"one of the crowning achievements of mathematics."

Katz, Nicholas M. –

An overview of Deligne's proof of the Riemann hypothesis for varieties over finite fields."

Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974), (275–305). Amer. Math. Soc., Providence, R.I., 1976.


LEITFADEN
Gauss
Riemann
Dedekind
(E.) Artin
Schmidt
Hasse
MONODROMY
GEOMETRY
ANALYSIS
MODULAR FORMS
Picard
Castelnuovo, Severi
Hurwitz
Hadamard, de la Vallée-Poussin
Lefschetz
Ramanujan
Mordell
Hecke
TOPOLOGY, SHEAVES
ANALYSIS
Hodge
Rankin
Feynman ∫
Leray, Cartan, Kodaira
Weil
Eichler, Shimura Kuga, Ihara
Pham, Brieskorn
Serre
Langlands
(M.) Artin-Grothendieck
A' Campo
Grothendieck
Deligne
Kajdan-Margoulis
Main Lemma (Deligne)
Big monodromy, independence of ℓ (Deligne)
RIEMANN HYPOTHESIS OVER FINITE FIELDS (Deligne)
Hard Lefschetz Theorem (Hodge/ℂ, Deligne)
Mixed Hodge Theory, $\pi_n \otimes \mathbb{R}$ of varieties (Deligne, Griffiths, Morgan, Sollivan)
estimates for exponential sums in several variables (Deligne)
RAMANUJAN-PETERSON conjecture (Deligne)

The third observation led to Deligne and Serre's theory of "modular Galois representations." This theory plays a central role, for example, in the proof of Fermat's Last Theorem by Wiles.

The Conjectures of Ramanujan and Weil.

Ramanujan Conjecture:

$$|\tau(p)| \leq 2p^{11/2} \qquad \textit{for all primes } p.$$

Recall: $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$ has weight 12.

Generalized Ramanujan Conjecture: *If*

$$f(z) = \sum_{n=1}^{\infty} a(n) q^n$$

has weight k*, then*

$$|a(p)| \le 2p^{(k-1)/2} \qquad \textit{for all primes } p.$$

Weil Conjectures

Consider a polynomial $f(x_1, x_2, \ldots, x_n)$ with rational integer coefficients, and a prime p.

The Weil Conjectures describe the number of solutions modulo p to the congruence

$$f(x_1, x_2, \ldots, x_n) \equiv 0 \pmod{p}.$$

This means that we work in the finite field $\mathbf{F}_p$ with p elements and consider the equation $f(x_1, x_2, \ldots, x_n) = 0$ in $\mathbf{F}_p$.

Example

An *elliptic curve* E is defined by an equation

$$E \quad : \quad y^2 = x^3 + ax^2 + bx + c,$$

where a, b, $c \in \mathbf{Z}$.

If p is a prime number, then let $N(p)$ be the number of points on E modulo p.

$N(p)$ is the number of solutions $(x, y) \pmod{p}$ to the equation

$$y^2 \equiv x^3 + ax^2 + bx + c \pmod{p}.$$

Modularity (Andrew Wiles and others)

There is a weight two modular form

$$f(z) = \sum_{n=1}^{\infty} a(n) q^n$$

such that

$$a(p) = p - N(p) \qquad \textit{for all primes } p.$$

So the Generalized Ramanujan Conjecture tells us that

$$|N(p) - p| \leq 2\sqrt{p}.$$

Ramanujan's Conjecture is related to the problem of counting the number of solutions of equations modulo p.

Completing the program initiated by Grothendieck, Deligne first deduced Ramanujan's Conjecture from the Weil Conjectures (much before Wiles work on the modularity Conjecture), and shortly after proved Weil Conjectures (hence also Ramanujan's one).

The Weil Conjectures deal with more general polynomial equations

$$f(x_1, x_2, \ldots, x_n) = 0.$$

They give a precise description of the number of solutions to these equations modulo p.

(Also they are an analog of Riemann's Hypothesis for function fields).

In fact, Ramanujan's original Conjecture that

$$|\tau(p)| \leq 2p^{11/2}$$

is *equivalent* to a special case of the Weil Conjectures.

Moreover, methods developed to attack Ramanujan's Conjecture turned out to be instrumental in Deligne's proof.

Write

$$\frac{\tau(p)}{p^{11/2}} = \alpha(p) + \beta(p) \quad \text{with} \quad \alpha(p)\beta(p) = 1.$$

Ramanujan's Conjecture amounts to

$$|\alpha(p)| = |\beta(p)| = 1.$$

Deligne interprets $\alpha(p)$ and $\beta(p)$ as eigenvalues of Frobenius acting on certain cohomology groups.

Back to the third observation:

$$\tau(p) \equiv 1 + p^{11} \pmod{691}$$

Further similar congruences :

$$\tau(p) \equiv 1 + p^{11} \pmod{2^5} \qquad p \neq 2,$$

$$\tau(p) \equiv 1 + p \pmod{3} \qquad p \neq 3,$$

$$\tau(p) \equiv p^{30} + p^{-41} \pmod{5^3} \qquad p \neq 5,$$

$$\tau(p) \equiv p + p^4 \pmod{7} \qquad p \neq 7,$$

These all have the form

$$\tau(p) \equiv p^a + p^{11-a} \pmod{\ell^k}$$

for some prime ℓ.

Recall Fermat's little theorem: $p^{\ell-1} \equiv 1 \pmod{\ell}$.

Actually, it turns out that these are the only such congruences.

However, J.-P. Serre's interpretation of these *accidents* led to some of the most important discoveries of the last half-century.

Let us focus on

$$\tau(p) \equiv 1 + p^{11} \pmod{691}.$$

What is $1 + p^{11}$ *?*

First answer: It is the trace of the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{11} \end{pmatrix}.$$

We view everything modulo 691. So this matrix lies in the group

$$\mathrm{GL}_2(\mathbf{Z}/691\mathbf{Z}).$$

Serre's brilliant contribution was to conjecture *a profound relationship with Galois Theory.*

Recall that if K is a (nice) field containing $\mathbf{Q}$, then the *Galois Group*

$$\mathrm{Gal}(K/\mathbf{Q})$$

is the group of automorphisms of K.

If p is a (nice) prime, then $\mathrm{Gal}(K/\mathbf{Q})$ contains a distinguished automorphism

$$\mathrm{Frob}_p$$

called the *Frobenius* at p.

The automorphism Frob_p

acts like raising to the p-th power.

There is also a homomorphism

$$\chi \ : \ \mathrm{Gal}(K/\mathbf{Q}) \longrightarrow (\mathbf{Z}/691\mathbf{Z})^{\times}$$

such that

$$\chi(\mathrm{Frob}_p) = p$$

for all primes p.

So we can construct a homomorphism

$$\rho \ : \ \mathrm{Gal}(K/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\mathbf{Z}/691\mathbf{Z})$$

by defining

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & \chi^{11}(\sigma) \end{pmatrix}.$$

Note: It is clear that

$$\rho(\sigma_1\sigma_2) = \rho(\sigma_1)\rho(\sigma_2).$$

Notice that

$$\rho(\mathrm{Frob}_p) = \begin{pmatrix} 1 & 0 \\ 0 & \chi^{11}(\mathrm{Frob}_p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^{11} \end{pmatrix}.$$

Therefore,

$$\mathrm{Trace}(\rho(\mathrm{Frob}_p)) = 1 + p^{11} \equiv \tau(p) \pmod{691}.$$

This is Serre's answer to the question

$$\textit{What is } \ 1 + p^{11} \quad \textit{?}$$

For any prime power ℓ^k, Serre conjectured that there is a representation

$$\rho \ : \ \mathrm{Gal}(K/\mathbf{Q}) \to \mathrm{GL}_2\left(\mathbf{Z}/\ell^k\mathbf{Z}\right)$$

such that for all p we have

$$\mathrm{Trace}(\rho(\mathrm{Frob}_p)) \equiv \tau(p) \pmod{\ell^k}.$$

Note: Moreover,

$$\mathrm{Det}(\rho(\mathrm{Frob}_p)) \equiv p^{11} \pmod{\ell^k}.$$

Note:

All of the congruences

$$\tau(p) \equiv p^a + p^{11-a} \pmod{\ell^k}$$

can be explained analogously via representations of the form

$$\rho(\sigma) = \begin{pmatrix} \chi^a(\sigma) & 0 \\ 0 & \chi^{11-a}(\sigma) \end{pmatrix}.$$

The existence of these representations (for any modular form) was proved by Deligne.

Typically, these representations do not have such a simple form.

However, the simplicity of the *accidents* like Ramanujan's

$$\tau(p) \equiv 1 + p^{11} \pmod{691}$$

provided crucial insight into the general situation.

It is hard to overstate the importance of these representations. For example, the proof of Fermat's Last Theorem is written in this language.

Summary:

Ramanujan had an amazing talent for uncovering phenomena which would turn out to be of fundamental importance in Number Theory.

Hecke's Theory

Weil Conjectures

Fermat's Last Theorem

Also:

Goldbach's Conjecture

Waring's Problem

Asymptotics for the partition function.

$p(n) :=$ number of partitions of n.

Example: The partitions of 4:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Hence

$$p(4) = 5.$$

Recall the *asymptotic formula*:

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}.$$

This means that

$$p(n) = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} + \qquad \textit{something smaller} \quad .$$

Ramanujan believed (unrealistically?) that

$$\textit{something smaller}$$

could be replaced by an *exact* formula for $p(n)$.

Hardy & Ramanujan proved that this belief is true. Their work marks the birth of the

Circle method,

which has grown into one of the most powerful tools in analytic Number Theory.

We sketch a refinement of their work by Rademacher.

Recall the generating function

$$F(x) := \prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n) x^n, \quad |x| < 1.$$

Notice that

$$\frac{F(x)}{x^{n+1}} = \cdots + \frac{p(n-1)}{x^2} + \frac{p(n)}{x} + p(n+1) + \cdots$$

So by Cauchy's Residue Theorem

$$p(n) = \frac{1}{2\pi i} \int_C \frac{F(x)}{x^{n+1}} \, dx.$$

Main contributions come from poles of $F(x)$. These lie at *every* root of unity.

Circle Method: Estimate contribution from arcs of C near q-th root of unity by function

$$T_q(n).$$

Estimate is so good that we in fact have

$$p(n) = \sum_{q=1}^{\infty} T_q(n).$$

Since $p(n)$ is an integer, computing *enough* terms gives its exact value.

E.g. the first 8 terms give

$$p(200) \approx 3,972,999,029,388.004.$$

Over last 80 years circle method has played major role in *additive number theory.*

Some examples:

Lagrange Theorem. *Every natural number is the sum of at most four squares.*

Waring's Problem. *Every (sufficiently large) natural number is the sum of at most $G(k)$ k-th powers.*

The quantity $G(k)$ can be estimated with circle method.

How does circle method apply? An example:

Let

$$f(x) := \sum_{n=0}^{\infty} x^{n^2}.$$

Then

$$f(x)^2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x^{n^2+m^2} = \sum_{N=0}^{\infty} R(N) x^N,$$

where

$R(N) :=$ the number of ways to write $N = n^2 + m^2$.

As above we have

$$R(N) = \frac{1}{2\pi i} \int_C \frac{f(x)^2}{x^{N+1}} \, dx.$$

Goldbach's Conjecture.

Every even number $n > 2$ is the sum of two primes.
Every odd number $n > 5$ is the sum of three primes.

Circle method proves that conjecture true for *almost* every even number, and for every *large* odd number.

The circle method has been used to predict entropy and area of black holes in *supergravity* theory.

Congruences for the partition function.

$p(n) :=$ number of partitions of n.

Ramanujan's congruences:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Ramanujan (1919): *I have proved a number of arithmetical properties of* $p(n)$ *... in particular that*

$$p(5n+4) \equiv 0 \pmod 5,$$

and

$$p(7n+5) \equiv 0 \pmod 7 \ldots.$$

I have since found another method which enables me to prove all of these properties and a variety of others, of which the most striking is

$$p(11n+6) \equiv 0 \pmod{11}.$$

There are corresponding properties in which the moduli are powers of 5, 7, *or* $11 \ldots.$ *It appears that there are no equally simple properties for any moduli involving primes other than these three.*

Theorem *(Ahlgren+Boylan, Invent. Math., 2003) Suppose that* ℓ *is a prime for which there exists a Ramanujan congruence*

$$p(\ell n + \alpha) \equiv 0 \pmod{\ell} \qquad \textit{for all } n,$$

with some $\alpha \in \mathbf{Z}$*. Then this congruence is one of Ramanujan's original three.*

Conjecture of Erdős *(1980)*. *If* ℓ *is prime, then there exists an* $n > 0$ *such that* $p(n) \equiv 0 \pmod{\ell}$.

A few examples of congruences

$$p(An+B) \equiv 0 \pmod{\ell^k}$$

with

$$\ell = 13, 17, 19, 23, 29, 31$$

were found in the 1960s.

- Simplest congruence modulo 13 (Atkin, O'Brien):

$$p(11^3 \cdot 13n + 237) \equiv 0 \pmod{13}.$$

Typical congruence modulo 13^2:

$$p(13^2 \cdot 97^3 \cdot 103^3 \cdot n - 6950975499605) \equiv 0 \pmod{13^2}.$$

Natural Questions:

Are these *extra* congruences accidents?

Are there congruences for primes $\ell > 31$?

Theorem *(Ono, 2000) If $\ell \geq 5$ is prime, then there are infinitely many congruences of the form*

$$p(An + B) \equiv 0 \pmod{\ell}.$$

Ono's result has been extended with

Theorem *(Ahlgren, 2000) If $\ell \geq 5$ is prime and m is a positive integer, then there are infinitely many congruences of the form*

$$p(An + B) \equiv 0 \pmod{\ell^m}.$$

These theorems show that congruences like Ramanujan's are *everywhere.*

However, the new congruences are

so far out there

that they are hard to actually see.

In recent joint work, Ono and Ahlgren have shown that such congruences are actually much more widespread than previously known.

Their latest result explains every known example of a partition function congruence.

Interestingly, to prove these results requires many results from the theory of *modular forms* whose roots can be traced back to

Ramanujan.

Ramanujan's impact on many different aspects of mathematics is amazing

Peter Sarnak, India-AMS Meeting Bangalore, Dec 19, 2003