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Nombres transcendants: résultats récents et problèmes ouverts

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Transcendental Numbers: Recent Results and Open Problems

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Lecture given on August 5, 2009

SMF – CIMPA – CDC SME

SMF http ://smf.emath.fr/
(Société Mathématique de France)

CIMPA http ://www.cimpa-icpam.org
(Centre International de Mathématiques Pures et Appliquées)

CDC EMS http ://people.maths.ox.ac.uk/∼tsou/ems/
(Committee for Developing Countries, European Mathematical Society)

Résumé

Après un bref aperçu historique *(le passé)*, nous présentons certaines des conjectures les plus importantes *(le futur)*, avant de faire le point sur les principales avancées récentes *(le présent)*.

Les résultats classiques sont ceux de Liouville, Hermite, Lindemann, Gel'fond et Schneider et plus récemment Baker, Chudnovsky, Nesterenko.

Les conjectures les plus importantes sont celles de Borel, Schanuel, Grothendieck, Rohrlich et Lang, André, Kontsevich et Zagier.

Les résultats récents dont nous parlerons sont dus à Rivoal, Adamczewski et Bugeaud, Roy.

Abstract

We start with a short historical introduction *(the past)*, then we state some of the most important conjectures *(the future)*, and we conclude with the state of the art on the main new results *(the present)*.

Classical results are due to Liouville, Hermite, Lindemann, Gel'fond and Schneider and more recently Baker, Chudnovsky, Nesterenko.

Among the most important conjectures are those of Borel, Schanuel, Grothendieck, Rohrlich and Lang, André, Kontsevich and Zagier.

The recent results we plan to state have been achieved by Rivoal, Adamczewski and Bugeaud, Roy.

Questions

- **Irrationality**. Given a real number x, decide whether it is *rational* : $x \in \mathbf{Q}$ or else *irrational* : $x \notin \mathbf{Q}$.

- **Transcendence**. Given a complex number x, decide whether or not it is a root of a non–zero polynomial with integer coefficients. In the first case x is *algebraic*, in the second case x is a *transcendental* number.

- **Algebraic independence**. Given complex numbers $x_1, \dots, x_n$, decide whether or not there exists a non–zero polynomial in n variables with integer coefficients which vanishes at the point $(x_1, \dots, x_n)$. In the first case $x_1, \dots, x_n$ are *algebraically dependent*, in the second case $x_1, \dots, x_n$ are *algebraically independent*.

§ 1 : Historical survey

Irrationality :
H. Lambert 1767

Transcendence :
Liouville 1844
Hermite 1873
Lindemann 1882
Gel'fond and Schneider 1934
Baker 1968

Algebraic independence :
Chudnovsky 1976
Nesterenko 1996.

Irrationality of π

Johann Heinrich Lambert (1728 - 1777)
Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques,
Mémoires de l'Académie des Sciences de Berlin, **17** (1761), p. 265-322 ;
read in 1767 ; Math. Werke, t. II.

$\tan(v)$ is irrational for any rational value of $v \neq 0$
and $\tan(\pi/4) = 1$.

Lambert and Frederick II, King of Prussia

— Que savez vous, Lambert ?
— Tout, Sire.
— Et de qui le tenez–vous ?
— De moi-même !

Existence of transcendental numbers (1844)

J. Liouville (1809 - 1882)

gave the first examples of transcendental numbers.
For instance

$$\sum_{n\geq 1} \frac{1}{10^{n!}} = 0.110\,001\,000\,000\,0\ldots$$

is a transcendental number.

Charles Hermite and Ferdinand Lindemann

Hermite (1873) :
Transcendence of e
$e = 2.718\,281\,828\,4\ldots$

Lindemann (1882) :
Transcendence of π
$\pi = 3.141\,592\,653\,5\ldots$

Hermite–Lindemann Theorem

For any non-zero complex number z, one at least of the two numbers z and e^z is transcendental.

Corollaries : Transcendence of $\log \alpha$ and of e^β for α and β non-zero algebraic complex numbers, provided $\log \alpha \neq 0$.

A.O. Gel'fond and Th. Schneider

Solution of Hilbert's seventh problem (1934) : *Transcendence of* α^β *and of* $(\log\alpha_1)/(\log\alpha_2)$ *for algebraic* α, β, α_1 *and* α_2.

Transcendence of α^β and $\log\alpha_1/\log\alpha_2$: examples

The following numbers are transcendental :

$$2^{\sqrt{2}} = 2.665\,144\,142\,6\ldots$$

$$\frac{\log 2}{\log 3} = 0.630\,929\,753\,5\ldots$$

$$e^\pi = 23.140\,692\,632\,7\ldots \qquad (e^\pi = (-1)^{-i})$$

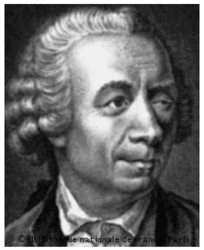
$$e^{\pi\sqrt{163}} = 262\ 537\ 412\ 640\ 768\ 743.999\ 999\ 999\ 999\ 25\ldots$$

Beta values : Th. Schneider 1948

Euler Gamma and Beta functions

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

$$\Gamma(z) = \int_0^\infty e^{-t}t^z \cdot \frac{dt}{t}$$



$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Algebraic independence : A.O. Gel'fond 1948

The two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent.

More generally, if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an algebraic number of degree $d \geq 3$, then two at least of the numbers

$$\alpha^\beta,\ \alpha^{\beta^2},\ \ldots,\alpha^{\beta^{d-1}}$$

are algebraically independent.

Alan Baker 1968

Transcendence of numbers like

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

or

$$e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_1^{\beta_1}$$

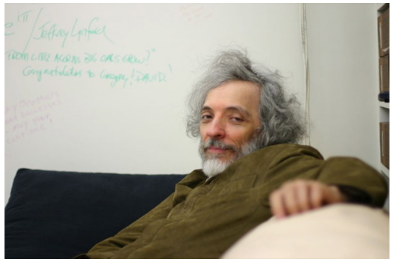
for algebraic α_i's and β_j's.

Example (Siegel) :

$$\int_0^1 \frac{dx}{1+x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right)$$

is transcendental.

Gregory V. Chudnovsky



G.V. Chudnovsky (1976)
Algebraic independence of the numbers π and $\Gamma(1/4)$.
Also : algebraic independence of the numbers π and $\Gamma(1/3)$.

Corollaries : *Transcendence of* $\Gamma(1/4) = 3.625\,609\,908\,2\ldots$ *and* $\Gamma(1/3) = 2.678\,938\,534\,7\ldots$.

Yuri V. Nesterenko

Yu.V.Nesterenko (1996)
Algebraic independence of $\Gamma(1/4)$, π and e^π.
Also : Algebraic independence of $\Gamma(1/3)$, π and $e^{\pi\sqrt{3}}$.

Corollary : *The numbers* $\pi = 3.1415926535\ldots$ *and* $e^\pi = 23.140\,692\,632\,7\ldots$ *are algebraically independent.*

§ 2 : Conjectures

Borel 1909, 1950

Schanuel 1964

Grothendieck 1960's

Rohrlich and Lang 1970's

André 1990's

Kontsevich and Zagier 2001.

Émile Borel (1871–1956)

- *Les probabilités dénombrables et leurs applications arithmétiques,*
  Palermo Rend. **27**, 247-271 (1909).
  Jahrbuch Database JFM 40.0283.01
  http ://www.emis.de/MATH/JFM/JFM.html

- *Sur les chiffres décimaux de* $\sqrt{2}$ *et divers problèmes de probabilités en chaînes,*
  C. R. Acad. Sci., Paris **230**, 591-593 (1950).
  Zbl 0035.08302

Émile Borel : 1950

Let $g \geq 2$ be an integer and x a real irrational algebraic number. *The expansion in basis* g *of* x *should satisfy some of the laws which are valid for almost all real numbers (for Lebesgue's measure).*

Decimal digits of $\sqrt{2}$ http://wims.unice.fr/wims/wims.cgi

1.4142135623730950488016887242096980785696718753769480731766797
3
7990732478462107038850387534327641572735013846230912297024924
83
6055850737212644121497099935831413222665927505592755799950501
15
2782060571470109559971605970274534596862014728517418640889198
60
9552329230484308714321450839762603627995251407989687253396546
33
1808829640620615258352395054745750287759961729835575220337531
85
7011354374603408498847160386899970699004815030544027790316454
24
7823068492936918621580578463111596668713013015618568987237235
28
8509264861249497715421833420428568606014682472077143585487415
56
5706967765372022648544701585880162075847492265722600208558446
65
2145839889394437092659180031138824646815708263010059485870400
31
8648034219489727829064104507263688131373985525611732204024509
12
2770022694112757362728049573810896750401836986836845072579936
47
2906076299694138047565482372899718032680247442062926912485905
21
8100445984215059112024944134172853147810580360337107730918286
93
147101711116839165817268894197587165821521282295184884 7 ...

Binary digits of $\sqrt{2}$ http://wims.unice.fr/wims/wims.cgi

1.01101010000010011110011001100111111110011101111100110010010000
10001011001011111011000100110110011011101010100101010111110100
11111000111010110111101100000101110101000100100111011101010000
10011001110110100010111101011001000010110000011001100111001100
10001010101001010111111001000000110000010000111010101110001010 0
01011000011101010001011000111111110011011111101110010000011110
11011001110010000111101110100101010000101111001000011100111000
11110110100101001111000000000100100001110011011000111101111110 1
00010011101101000110100100010000000101110100001110100001010101
11100011111010011100101001100000101100111000110000000010001101
11100000110011011110111100101010110001101111001001000100010110 1
00010000010001011000101001000110000010101011110001110010001011 1
10111110001001110001100111100011011010101101010001010001110001
01110110111111010011101110011001011001010100110001101000011001
10001111100111100100001001101111101010010111100010010000011111
00000110110111001011000001011101110101010100100101000000100010 0
110010000001000000110010100100101010000000100111001010010 10 ...

Borel's conjecture

If x is a real irrational algebraic number and $g \geq 2$ an integer, then, in the expansion of x in basis g, each digit $a \in \{0, \ldots, g-1\}$ should occur with the frequency $1/g$, and each given sequence of k digits should occur with the frequency $1/g^k$.

In other terms, a real number with a regular expansion in some basis g should be either rational or else transcendental, but not algebraic irrational.

State of the art

There is no known example of a triple (g, a, x), with $g \geq 3$ an integer, a a digit in $\{0, \ldots, g-1\}$ and x a real irrational algebraic number, for which one can claim that the digit a occurs infinitely often in the expansion in basis g of x.

Schanuel's Conjecture

Let $x_1, \ldots, x_n$ be $\mathbf{Q}$-linearly independent complex numbers. Then at least n of the $2n$ numbers $x_1, \ldots, x_n$, $e^{x_1}, \ldots, e^{x_n}$ are algebraically independent.

Easy consequence of Schanuel's Conjecture

According to Schanuel's Conjecture, the following numbers are algebraically independent :

$$e + \pi,\ e\pi,\ \pi^e,\ e^e, e^{e^2}, \ldots,\ e^{e^e}, \ldots,\ \pi^\pi, \pi^{\pi^2}, \ldots,\ \pi^{\pi^\pi} \ldots,$$

$$\log \pi,\ \log(\log 2),\ \pi \log 2,\ (\log 2)(\log 3),\ 2^{\log 2},\ (\log 2)^{\log 3} \ldots$$

Proof : Use Schanuel's Conjecture several times.

Daniel Bertrand

Daniel Bertrand

Schanuel's conjecture for non-isoconstant elliptic curves over function fields.

Model theory with applications to algebra and analysis. Vol. 1, 41–62, London Math. Soc. Lecture Note Ser., **349**, Cambridge Univ. Press, Cambridge, 2008.

Known

For $n = 1$, Schanuel's Conjecture is the Hermite–Lindemann Theorem :
For any non-zero complex number x, one at least of the two numbers x and e^x is transcendental.

Not known

For $n = 2$, Schanuel's Conjecture is not yet known :

> *? If x_1, x_2 are $\mathbf{Q}$–linearly independent complex numbers, then among the 4 numbers x_1, x_2, e^{x_1}, e^{x_2}, at least 2 are algebraically independent.*

A few consequences :

With $x_1 = 1$, $x_2 = i\pi$: algebraic independence of e and π.

With $x_1 = 1$, $x_2 = e$: algebraic independence of e and e^e.

With $x_1 = \log 2$, $x_2 = (\log 2)^2$: algebraic independence of $\log 2$ and $2^{\log 2}$.

With $x_1 = \log 2$, $x_2 = \log 3$: algebraic independence of $\log 2$ and $\log 3$.

Not known

It is not known that *there exist two logarithms of algebraic numbers which are algebraically independent.*

Even the non–existence of non–trivial quadratic relations among logarithms of algebraic numbers is not yet established.

According to the *four exponentials Conjecture*, *any quadratic relation $(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3)$ is trivial : either $\log \alpha_1$ and $\log \alpha_2$ are linearly dependent, or else $\log \alpha_1$ and $\log \alpha_3$ are linearly dependent.*

Conjectures by A. Grothendieck and Y. André



Generalized Conjecture on Periods by Grothendieck : Dimension of the Mumford–Tate group of a smooth projective variety.

Generalization by Y. André to motives.

Case of 1–motives : Elliptico-Toric Conjecture of C. Bertolin.

Gamma and Beta values

$$\Gamma(z) = \int_0^\infty e^{-t}t^z \cdot \frac{dt}{t} = e^{-\gamma z}z^{-1}\prod_{n=1}^\infty \left(1+\frac{z}{n}\right)^{-1} e^{z/n}.$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

Standard relations among Gamma values

Translation :

$$\Gamma(a+1) = a\Gamma(a)$$

Reflexion :

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

Multiplication : for any positive integer n,

$$\prod_{k=0}^{n-1}\Gamma\left(a+\frac{k}{n}\right) = (2\pi)^{(n-1)/2}n^{-na+(1/2)}\Gamma(na).$$

Rohrlich's Conjecture

Conjecture (D. Rohrlich) *Any multiplicative relation*

$$\pi^{b/2}\prod_{a\in\mathbf{Q}}\Gamma(a)^{m_a} \in \overline{\mathbf{Q}}$$

with b and m_a in $\mathbf{Z}$ lies in the ideal generated by the standard relations.

Examples :

$$\Gamma\left(\frac{1}{14}\right)\Gamma\left(\frac{9}{14}\right)\Gamma\left(\frac{11}{14}\right) = 4\pi^{3/2}$$

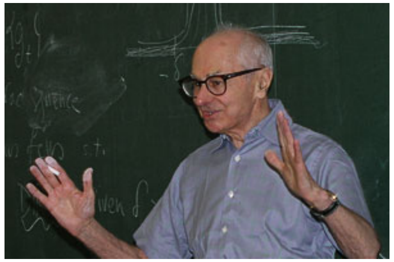
$$\prod_{\substack{1\le k\le n\\(k,n)=1}}\Gamma(k/n) = \begin{cases}(2\pi)^{\varphi(n)/2}/\sqrt{p} & \text{if } n=p^r \text{ is a prime power,}\\(2\pi)^{\varphi(n)/2} & \text{otherwise.}\end{cases}$$

Small Gamma Products with Simple Values

Albert Nijenhuis, *Small Gamma products with Simple Values*
http ://arxiv.org/abs/0907.1689, July 9, 2009.

Greg Martin, *A product of Gamma function values at fractions with the same denominator*
http ://arxiv.org/abs/0907.4384, July 24, 2009.

Lang's Conjecture



Conjecture (S. Lang) *Any algebraic dependence relation among the numbers* $(2\pi)^{-1/2}\Gamma(a)$ *with* $a \in \mathbf{Q}$ *lies in the ideal generated by the standard relations.*
(Universal odd distribution).

Periods : Maxime Kontsevich and Don Zagier

Periods, *Mathematics unlimited—2001 and beyond*, Springer 2001, 771–808.

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients , over domains in $\mathbf{R}^n$ given by polynomial inequalities with rational coefficients.

The number π

Basic example of a *period* :

$$e^{z+2i\pi} = e^z$$

$$2i\pi = \int_{|z|=1} \frac{dz}{z}$$

$$\begin{aligned} \pi &= \int\int_{x^2+y^2\le 1} dxdy &&= 2\int_{-1}^{1} \sqrt{1-x^2}dx \\ &= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} &&= \int_{-\infty}^{\infty} \frac{dx}{1-x^2}. \end{aligned}$$

Elliptic integrals : arc length of a lemniscate

An explicit value for a pair of fundamental periods of the elliptic curve

$$y^2 = 4x^3 - 4x$$

follows from computations by Legendre using Gauss's lemniscate function

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}} = \frac{1}{2} B(1/4, 1/2) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}} = 2.622\,057\,554\,2\dots$$

and $\omega_2 = i\omega_1$. The lattice $\mathbf{Z}[i]$ has $g_2 = 4\omega_1^4$, thus

$$\sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} (m+ni)^{-4} = \frac{\Gamma(1/4)^8}{2^6 \cdot 3 \cdot 5 \cdot \pi^2} = 3.151\,212\,002\,153\,8\dots$$

Elliptic integrals : perimeter of an ellipse

The perimeter of an ellipse with radii a and b is the elliptic integral of second kind

$$2\int_{-b}^{b} \sqrt{1 + \frac{a^2x^2}{b^4 - b^2x^2}}\, dx.$$

For $a = b$, the perimeter is $2a\pi$.

Further examples of periods

$$\sqrt{2} = \int_{2x^2 \le 1} dx$$

and all algebraic numbers.

$$\log 2 = \int_{1<x<2} \frac{dx}{x}$$

and all logarithms of algebraic numbers.

$$\pi = \int_{x^2+y^2\le 1} dx dy,$$

$$\frac{\pi^2}{6} = \zeta(2) = \sum_{n\ge 1} \frac{1}{n^2} = \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2}.$$

$\zeta(2) = 1.644\,934\,066\,8\dots$ is a period

$$\begin{aligned}\int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} &= \int_0^1 \left(\int_0^{t_1} \frac{dt_2}{1-t_2} \right) \frac{dt_1}{t_1} \\ &= \int_0^1 \left(\int_0^{t_1} \sum_{n\ge 1} t_2^{n-1} dt_2 \right) \frac{dt_1}{t_1} \\ &= \sum_{n\ge 1} \frac{1}{n} \int_0^1 t_1^{n-1} dt_1 \\ &= \sum_{n\ge 1} \frac{1}{n^2} = \zeta(2).\end{aligned}$$

$\zeta(s)$ is a period

For any integer $s \geq 2$,

$$\zeta(s) = \int_{1>t_1>t_2\cdots>t_s>0} \frac{dt_1}{t_1} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s}.$$

Induction :

$$\int_{t_1>t_2\cdots>t_s>0} \frac{dt_2}{t_2} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s} = \sum_{n\geq 1} \frac{t_1^{n-1}}{n^{s-1}}.$$

Relations among periods

[1] Additivity
(in the integrand and in the domain of integration)

$$\int_a^b \bigl(f(x)+g(x)\bigr)dx = \int_a^b f(x)dx + \int_a^b g(x)dx,$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

[2] Change of variables :
if $y = f(x)$ is an invertible change of variables, then

$$\int_{f(a)}^{f(b)} F(y)dy = \int_a^b F\bigl(f(x)\bigr)f'(x)dx.$$

Relations among periods (continued)



[3] Newton–Leibniz–Stokes Formula

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Conjecture of Kontsevich and Zagier

A widely-held belief, based on a
judicious combination
of experience, analogy,
and wishful thinking,
is the following

Conjecture (Kontsevich–Zagier). *If a period has two integral representations, then one can pass from one formula to another by using only rules* [1], [2], [3] *in which all functions and domains of integration are algebraic with algebraic coefficients.*

Conjecture of Kontsevich and Zagier (continued)

In other words, we do not expect any miraculous coïncidence of two integrals of algebraic functions which will not be possible to prove using three simple rules.

This conjecture, which is similar in spirit to the Hodge conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.

Further conjectures : irrationality of

Euler–Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) = 0,577\,215\,664\,9\ldots$$

Sloane's A001620

Catalan constant

$$G = \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)^2} = 1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\cdots = 0.915\,965\,594\,1\ldots$$

Sloane's A006752

http ://www.research.att.com/∼njas/sequences

§ 3 : Recent results

Rivoal, Fischler, Zudilin

Adamczewski and Bugeaud

Roy.

Riemann zeta function at odd positive integers

- Apéry (1978) : *The number*

$$\zeta(3) = \sum_{n\geq 1}\frac{1}{n^3} = 1.202\,056\,903\,1\ldots$$

is irrational.

- Rivoal (2000) + Ball, Zudilin... *Among the numbers* $\zeta(2k+1)$, *infinitely many are irrational.*

Tanguy Rivoal

Let $\epsilon > 0$. *For any odd positive integer* a, *the dimension of the* $\mathbf{Q}$*–space spanned by the numbers* $1,\ \zeta(3),\ \zeta(5),\ \cdots, \zeta(a)$ *is at least*

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

Wadim Zudilin

- *One at least of the four numbers* $\zeta(5),\quad \zeta(7),\quad \zeta(9),\quad \zeta(11)$ *is irrational.*
- *There is an odd integer* j *in the interval* $[5, 69]$ *such that the three numbers* $1,\ \zeta(3),\ \zeta(j)$ *are linearly independent over* $\mathbf{Q}$.

References

S. Fischler
Irrationalité de valeurs de zêta,
(d'après Apéry, Rivoal, ...),
Sém. Nicolas Bourbaki, 2002-2003,
N° 910 (Novembre 2002).
http://www.math.u-psud.fr/~fischler/publi.html

C. Krattenthaler et T. Rivoal, *Hypergéométrie et fonction zêta de Riemann,* Mem. Amer. Math. Soc. **186** (2007), 93 p.
http://www-fourier.ujf-grenoble.fr/~rivoal/articles.html

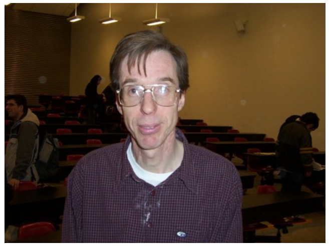
Recent developments

Stéphane Fischler and Wadim Zudilin, *A refinement of Nesterenko's linear independence criterion with applications to zeta values.* Preprint MPIM 2009-35.

Complexity of the expansion in basis g of a real irrational algebraic number

Theorem (B. Adamczewski, Y. Bugeaud 2005 ; conjecture of A. Cobham 1968).
If the sequence of digits of a real number x is produced by a finite automaton, then x is either rational or else transcendental.

Roy's approach to Schanuel's Conjecture (1999)



New conjecture equivalent to Schanuel's one, in the spirit of known transcendence criteria by Gel'fond (1949), Chudnovsky, Philippon, Nesterenko, Laurent. . .

D. Roy. *An arithmetic criterion for the values of the exponential function.* Acta Arith., **97** N° 2 (2001), 183–194.

Small value estimates for algebraic groups

D. Roy. *Small value estimates for the additive group.* Intern. J. Number Theory, to appear.

D. Roy. *Small value estimates for the multiplicative group.* Acta Arith. **135** (2008), 357–393.

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