7ème Congrès Pan Africain de Mathématiques
Yamoussoukro, Côte d'Ivoire du 3 au 8 août 2009

Nombres transcendants: résultats récents et problèmes ouverts

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CIMPA Centre International de Mathématiques Pures et Appliquées

http://www.math.jussieu.fr/∼miw/

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Lecture given on August 5, 2009

SMF – CIMPA – CDC SME

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Résumé

Après un bref aperçu historique *(le passé)*, nous présentons certaines des conjectures les plus importantes *(le futur)*, avant de faire le point sur les principales avancées récentes *(le présent)*.

Les résultats classiques sont ceux de Liouville, Hermite, Lindemann, Gel'fond et Schneider et plus récemment Baker, Chudnovsky, Nesterenko.

Les conjectures les plus importantes sont celles de Borel, Schanuel, Grothendieck, Rohrlich et Lang, André, Kontsevich et Zagier.

Les résultats récents dont nous parlerons sont dus à Rivoal, Adamczewski et Bugeaud, Roy.

Abstract

We start with a short historical introduction *(the past)*, then we state some of the most important conjectures *(the future)*, and we conclude with the state of the art on the main new results *(the present)*.

Classical results are due to Liouville, Hermite, Lindemann, Gel'fond and Schneider and more recently Baker, Chudnovsky, Nesterenko.

Among the most important conjectures are those of Borel, Schanuel, Grothendieck, Rohrlich and Lang, André, Kontsevich and Zagier.

The recent results we plan to state have been achieved by Rivoal, Adamczewski and Bugeaud, Roy.

Questions

- **Irrationality**. Given a real number x, decide whether it is *rational* : $x \in \mathbf{Q}$ or else *irrational* : $x \notin \mathbf{Q}$.

- **Transcendence**. Given a complex number x, decide whether or not it is a root of a non–zero polynomial with integer coefficients. In the first case x is *algebraic*, in the second case x is a *transcendental* number.

- **Algebraic independence**. Given complex numbers $x_1, \ldots, x_n$, decide whether or not there exists a non–zero polynomial in n variables with integer coefficients which vanishes at the point $(x_1, \ldots, x_n)$. In the first case $x_1, \ldots, x_n$ are *algebraically dependent*, in the second case $x_1, \ldots, x_n$ are *algebraically independent*.

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§ 1 : Historical survey

Irrationality :

H. Lambert 1767

Transcendence :

Liouville 1844

Hermite 1873

Lindemann 1882

Gel'fond and Schneider 1934

Baker 1968

Algebraic independence :

Chudnovsky 1976

Nesterenko 1996.

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Irrationality of π

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Lambert and Frederick II, King of Prussia

— Que savez vous, Lambert ?
— Tout, Sire.
— Et de qui le tenez–vous ?
— De moi-même !

Existence of transcendental numbers (1844)

J. Liouville (1809 - 1882)

gave the first examples of
transcendental numbers.
For instance

$$\sum_{n\geq 1} \frac{1}{10^{n!}} = 0.110\,001\,000\,000\,0\ldots$$

is a transcendental number.

Charles Hermite and Ferdinand Lindemann

Hermite (1873) :
Transcendence of e
$e = 2.718\,281\,828\,4\ldots$

Lindemann (1882) :
Transcendence of π
$\pi = 3.141\,592\,653\,5\ldots$

Hermite–Lindemann Theorem

For any non-zero complex number z, one at least of the two numbers z and e^z is transcendental.

Corollaries : Transcendence of $\log \alpha$ and of e^β for α and β non-zero algebraic complex numbers, provided $\log \alpha \neq 0$.

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A.O. Gel'fond and Th. Schneider

Solution of Hilbert's seventh problem (1934) : *Transcendence of* α^β *and of* $(\log \alpha_1)/(\log \alpha_2)$ *for algebraic* α, β, α_1 *and* α_2.

Transcendence of α^β and $\log \alpha_1 / \log \alpha_2$: examples

The following numbers are transcendental :

$$2^{\sqrt{2}} = 2.665\,144\,142\,6\ldots$$

$$\frac{\log 2}{\log 3} = 0.630\,929\,753\,5\ldots$$

$$e^{\pi} = 23.140\,692\,632\,7\ldots \qquad (e^{\pi} = (-1)^{-i})$$

$$e^{\pi\sqrt{163}} = 262\ 537\ 412\ 640\ 768\ 743.999\ 999\ 999\ 999\ 25\ldots$$

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Beta values : Th. Schneider 1948

Euler Gamma and Beta functions

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

$$\Gamma(z) = \int_0^\infty e^{-t}t^z \cdot \frac{dt}{t}$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Algebraic independence : A.O. Gel'fond 1948

The two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent.

More generally, if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an algebraic number of degree $d \geq 3$, then two at least of the numbers

$$\alpha^{\beta}, \; \alpha^{\beta^2}, \; \ldots, \alpha^{\beta^{d-1}}$$

are algebraically independent.

Alan Baker 1968

Transcendence of numbers like

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

or

$$e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_1^{\beta_1}$$

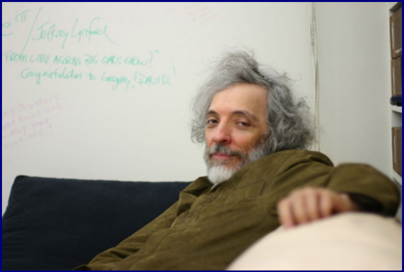
for algebraic α_i's and β_j's.

Example (Siegel) :

$$\int_0^1 \frac{dx}{1+x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right)$$

is transcendental.

Gregory V. Chudnovsky



G.V. Chudnovsky (1976)
Algebraic independence of the numbers π and $\Gamma(1/4)$.
Also : algebraic independence of the numbers π and $\Gamma(1/3)$.

Corollaries : *Transcendence of* $\Gamma(1/4) = 3.625\,609\,908\,2\ldots$ *and* $\Gamma(1/3) = 2.678\,938\,534\,7\ldots$.

Yuri V. Nesterenko

Yu.V.Nesterenko (1996)
Algebraic independence of
$\Gamma(1/4)$, π and e^π.
Also : Algebraic independence of
$\Gamma(1/3)$, π and $e^{\pi\sqrt{3}}$.

Corollary : *The numbers* $\pi = 3.1415926535\ldots$ *and* $e^\pi = 23.140\,692\,632\,7\ldots$ *are algebraically independent.*

§ 2 : Conjectures

Borel 1909, 1950

Schanuel 1964

Grothendieck 1960's

Rohrlich and Lang 1970's

André 1990's

Kontsevich and Zagier 2001.

Émile Borel (1871–1956)

- *Les probabilités dénombrables et leurs applications arithmétiques,*
  Palermo Rend. **27**, 247-271 (1909).
  Jahrbuch Database JFM 40.0283.01
  `http ://www.emis.de/MATH/JFM/JFM.html`

- *Sur les chiffres décimaux de $\sqrt{2}$ et divers problèmes de probabilités en chaînes,*
  C. R. Acad. Sci., Paris **230**, 591-593 (1950).
  Zbl 0035.08302

Émile Borel : 1950

Let $g \geq 2$ be an integer and x a real irrational algebraic number. *The expansion in basis* g *of* x *should satisfy some of the laws which are valid for almost all real numbers (for Lebesgue's measure).*

Decimal digits of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.41421356237309504880168872420969807856967187537694807317667973
7990732478462107038850387534327641572735013846230912297024924836
0558507372126441214970999358314132226659275055927557999505011527
8206057147010955997160597027453459686201472851741864088919860955
2329230484308714321450839762603627995251407989687253396546331808
8296406206152583523950547457502877599617298355752203375318570113
5437460340849884716038689997069900481503054402779031645424782306
8492936918621580578463111596668713013015618568987237235288509264
8612494977154218334204285686060146824720771435854874155657069677
6537202264854470158588016207584749226572260020855844665214583988
9394437092659180031138824646815708263010059485870400318648034219
4897278290641045072636881313739855256117322040245091227700226941
1275736272804957381089675040183698683684507257993647290607629969
4138047565482372899718032680247442062926912485905218100445984215
0591120249441341728531478105803603371077309182869314710171111683
9165817268894197587165821521282295184884 ...

Binary digits of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.011010100000100111100110011001111111001110111100110010010000
1000101100101111101100010011011001101110101010010101011111010 0
1111100011101011011110110000010111010100010010011101110101 0000
1001100111011010001011110101100100000101100000011001100111001100
1000101010100101011111100100000011000001000011101010111000 10100
0101100001110101000101100011111111001101111110111001000001 1110
1101100111001000011110111010010101000010111100100001110011 1000
1111011010010100111100000000010010000111001101100011110111 11101
0001001110110100011010010001000000001011101000011101000010 10101
1110001111101001110010100110000010110011100011000000000100 01101
1110000110011011110111100101010110001101111001001000100010 1101
0001000010001011000101001000110000001010101111000111001000 10111
1011111000100111000110011110001101101010110101000101000111 0001
0111011011111101001110111001100101100101010011000110100001 1001
1000111110011110010000100110111111010100101111000100100000 11111
0000011011011100101100000010111011101010101001001010000001 000100
1100100000010000000110010100100101010000000100111001010010 10 ...

Borel's conjecture

If x is a real irrational algebraic number and $g \geq 2$ an integer, then, in the expansion of x in basis g, each digit $a \in \{0, \dots, g-1\}$ should occur with the frequency $1/g$, and each given sequence of k digits should occur with the frequency $1/g^k$.

In other terms, a real number with a regular expansion in some basis g should be either rational or else transcendental, but not algebraic irrational.

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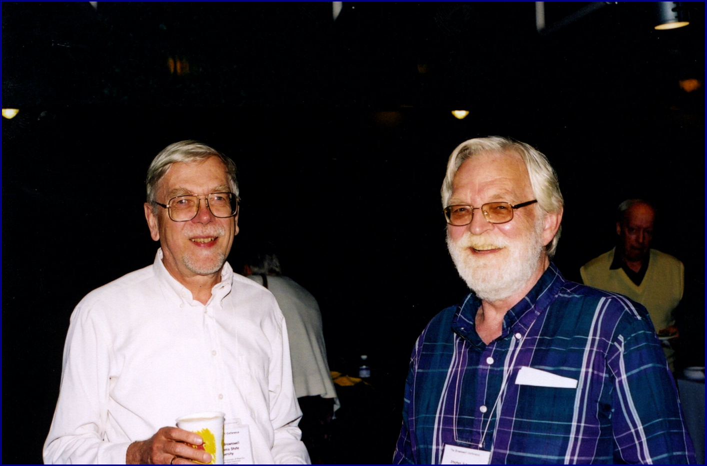
In other terms, a real number with a regular expansion in some basis g should be either rational or else transcendental, but not algebraic irrational.

State of the art

There is no known example of a triple (g, a, x), with $g \geq 3$ an integer, a a digit in $\{0, \ldots, g-1\}$ and x a real irrational algebraic number, for which one can claim that the digit a occurs infinitely often in the expansion in basis g of x.

Schanuel's Conjecture

Let $x_1, \ldots, x_n$ be $\mathbf{Q}$-linearly independent complex numbers. Then at least n of the $2n$ numbers $x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n}$ are algebraically independent.



Easy consequence of Schanuel's Conjecture

According to Schanuel's Conjecture, the following numbers are algebraically independent :

$$e+\pi,\ e\pi,\ \pi^e,\ e^e, e^{e^2}, \dots,\ e^{e^e}, \dots,\ \pi^\pi, \pi^{\pi^2}, \dots,\ \pi^{\pi^\pi} \dots,$$

$$\log\pi,\ \log(\log 2),\ \pi\log 2,\ (\log 2)(\log 3),\ 2^{\log 2},\ (\log 2)^{\log 3} \dots$$

Proof : Use Schanuel's Conjecture several times.

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Daniel Bertrand

Daniel Bertrand

Schanuel's conjecture for non-isoconstant elliptic curves over function fields.

Model theory with applications to algebra and analysis. Vol. 1, 41–62, London Math. Soc. Lecture Note Ser., **349**, Cambridge Univ. Press, Cambridge, 2008.

Known

For $n = 1$, Schanuel's Conjecture is the Hermite–Lindemann Theorem :
For any non-zero complex number x, one at least of the two numbers x and e^x is transcendental.

Not known

For $n = 2$, Schanuel's Conjecture is not yet known :

? If x_1, x_2 are $\mathbf{Q}$–linearly independent complex numbers, then among the 4 numbers x_1, x_2, e^{x_1}, e^{x_2}, at least 2 are algebraically independent.

A few consequences :

With $x_1 = 1$, $x_2 = i\pi$: algebraic independence of e and π.

With $x_1 = 1$, $x_2 = e$: algebraic independence of e and e^e.

With $x_1 = \log 2$, $x_2 = (\log 2)^2$: algebraic independence of $\log 2$ and $2^{\log 2}$.

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Not known

It is not known that *there exist two logarithms of algebraic numbers which are algebraically independent.*

Even the non–existence of non–trivial quadratic relations among logarithms of algebraic numbers is not yet established.

According to the *four exponentials Conjecture*, *any quadratic relation* $(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3)$ *is trivial : either* $\log \alpha_1$ *and* $\log \alpha_2$ *are linearly dependent, or else* $\log \alpha_1$ *and* $\log \alpha_3$ *are linearly dependent.*

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Conjectures by A. Grothendieck and Y. André

Generalized Conjecture on Periods by Grothendieck : Dimension of the Mumford–Tate group of a smooth projective variety.

Generalization by Y. André to motives.

Case of 1–motives : Elliptico-Toric Conjecture of C. Bertolin.

Gamma and Beta values

$$\begin{aligned}\Gamma(z) &= \int_0^\infty e^{-t} t^z \cdot \frac{dt}{t} \\ &= e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}.\end{aligned}$$

$$\begin{aligned}B(a,b) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \\ &= \int_0^1 x^{a-1}(1-x)^{b-1} dx.\end{aligned}$$

Standard relations among Gamma values

Translation :

$$\Gamma(a+1) = a\Gamma(a)$$

Reflexion :

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

Multiplication : for any positive integer n,

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)} \Gamma(na).$$

Rohrlich's Conjecture

Conjecture (D. Rohrlich) *Any multiplicative relation*

$$\pi^{b/2} \prod_{a \in \mathbf{Q}} \Gamma(a)^{m_a} \in \overline{\mathbf{Q}}$$

with b and m_a in $\mathbf{Z}$ lies in the ideal generated by the standard relations.

Examples :

$$\Gamma\left(\frac{1}{14}\right)\Gamma\left(\frac{9}{14}\right)\Gamma\left(\frac{11}{14}\right) = 4\pi^{3/2}$$

$$\prod_{\substack{1 \le k \le n \\ (k,n)=1}} \Gamma(k/n) = \begin{cases} (2\pi)^{\varphi(n)/2}/\sqrt{p} & \text{if } n = p^r \text{ is a prime power,} \\ (2\pi)^{\varphi(n)/2} & \text{otherwise.} \end{cases}$$

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$$\Gamma\left(\frac{1}{14}\right)\Gamma\left(\frac{9}{14}\right)\Gamma\left(\frac{11}{14}\right) = 4\pi^{3/2}$$

$$\prod_{\substack{1 \le k \le n \\ (k,n)=1}} \Gamma(k/n) = \begin{cases} (2\pi)^{\varphi(n)/2}/\sqrt{p} & \text{if } n = p^r \text{ is a prime power,} \\ (2\pi)^{\varphi(n)/2} & \text{otherwise.} \end{cases}$$

Rohrlich's Conjecture

Conjecture (D. Rohrlich) *Any multiplicative relation*

$$\pi^{b/2} \prod_{a \in \mathbf{Q}} \Gamma(a)^{m_a} \in \overline{\mathbf{Q}}$$

with b and m_a in $\mathbf{Z}$ lies in the ideal generated by the standard relations.

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Small Gamma Products with Simple Values

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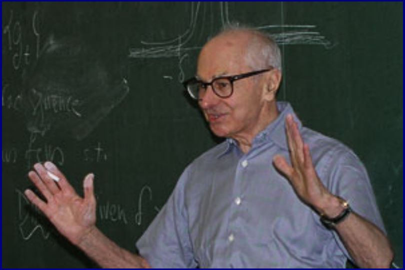
Lang's Conjecture



Conjecture (S. Lang) *Any algebraic dependence relation among the numbers* $(2\pi)^{-1/2}\Gamma(a)$ *with* $a \in \mathbf{Q}$ *lies in the ideal generated by the standard relations.*
(Universal odd distribution).

Periods : Maxime Kontsevich and Don Zagier

Periods, *Mathematics unlimited—2001 and beyond*, Springer 2001, 771–808.

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients , over domains in $\mathbf{R}^n$ given by polynomial inequalities with rational coefficients.

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The number π

Basic example of a *period* :

$$e^{z+2i\pi} = e^z$$

$$2i\pi = \int_{|z|=1} \frac{dz}{z}$$

$$\begin{aligned} \pi &= \iint_{x^2+y^2\leq 1} dx dy &&= 2\int_{-1}^{1} \sqrt{1-x^2} dx \\ &= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} &&= \int_{-\infty}^{\infty} \frac{dx}{1-x^2}. \end{aligned}$$

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Elliptic integrals : arc length of a lemniscate

An explicit value for a pair of fundamental periods of the elliptic curve

$$y^2 = 4x^3 - 4x$$

follows from computations by Legendre using Gauss's lemniscate function

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}} = \frac{1}{2} B(1/4, 1/2) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}} = 2.622\,057\,554\,2\ldots$$

and $\omega_2 = i\omega_1$. The lattice $\mathbf{Z}[i]$ has $g_2 = 4\omega_1^4$, thus

$$\sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} (m+ni)^{-4} = \frac{\Gamma(1/4)^8}{2^6 \cdot 3 \cdot 5 \cdot \pi^2} = 3.151\,212\,002\,153\,8\ldots$$

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Elliptic integrals : perimeter of an ellipse

The perimeter of an ellipse with radii a and b is the elliptic integral of second kind

$$2\int_{-b}^{b}\sqrt{1+\frac{a^2x^2}{b^4-b^2x^2}}dx.$$

For $a=b$, the perimeter is $2a\pi$.

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Further examples of periods

$$\sqrt{2} = \int_{2x^2 \leq 1} dx$$

and all algebraic numbers.

$$\log 2 = \int_{1<x<2} \frac{dx}{x}$$

and all logarithms of algebraic numbers.

$$\pi = \int_{x^2+y^2 \leq 1} dx dy,$$

$$\frac{\pi^2}{6} = \zeta(2) = \sum_{n \geq 1} \frac{1}{n^2} = \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2}.$$

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For any integer $s \geq 2$,

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Relations among periods

1. Additivity

   (in the integrand and in the domain of integration)

$$\int_a^b \big(f(x)+g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx,$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

2. Change of variables :

   if $y = f(x)$ is an invertible change of variables, then

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Relations among periods (continued)



3 Newton–Leibniz–Stokes Formula

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Conjecture of Kontsevich and Zagier

A widely-held belief, based on a judicious combination of experience, analogy, and wishful thinking, is the following

Conjecture (Kontsevich–Zagier). *If a period has two integral representations, then one can pass from one formula to another by using only rules* $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ *in which all functions and domains of integration are algebraic with algebraic coefficients.*

Conjecture of Kontsevich and Zagier (continued)

In other words, we do not expect any miraculous coïncidence of two integrals of algebraic functions which will not be possible to prove using three simple rules.

This conjecture, which is similar in spirit to the Hodge conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.

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Further conjectures : irrationality of

Euler–Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) = 0,577\,215\,664\,9\ldots$$

Sloane's A001620

Catalan constant

$$G = \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)^2} = 1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\cdots = 0.915\,965\,594\,1\ldots$$

Sloane's A006752

http://www.research.att.com/∼njas/sequences

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§ 3 : Recent results

Rivoal, Fischler, Zudilin

Adamczewski and Bugeaud

Roy.

Riemann zeta function at odd positive integers

- Apéry (1978) : *The number*

$$\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3} = 1.202\,056\,903\,1\ldots$$

is irrational.

- Rivoal (2000) + Ball, Zudilin... *Among the numbers* $\zeta(2k+1)$, *infinitely many are irrational.*

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Tanguy Rivoal

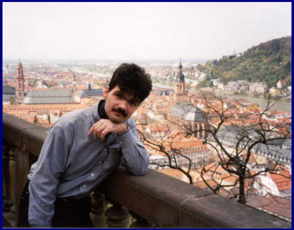
Let $\epsilon > 0$. For any odd positive integer a, the dimension of the $\mathbf{Q}$*–space spanned by the numbers* $1,\ \zeta(3),\ \zeta(5),\ \cdots, \zeta(a)$ *is at least*

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

Wadim Zudilin

- *One at least of the four numbers*
  $\zeta(5), \quad \zeta(7), \quad \zeta(9), \quad \zeta(11)$
  is irrational.

- *There is an odd integer* j *in the interval* $[5, 69]$ *such that the three numbers*
  $1, \zeta(3), \zeta(j)$
  are linearly independent over $\mathbb{Q}$.



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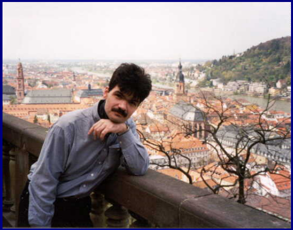

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http://www-fourier.ujf-grenoble.fr/~rivoal/articles.html

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Recent developments

Stéphane Fischler and Wadim Zudilin, *A refinement of Nesterenko's linear independence criterion with applications to zeta values.* Preprint MPIM 2009-35.

Complexity of the expansion in basis g of a real irrational algebraic number

Theorem (B. Adamczewski, Y. Bugeaud 2005 ; conjecture of A. Cobham 1968).
If the sequence of digits of a real number x is produced by a finite automaton, then x is either rational or else transcendental.

Roy's approach to Schanuel's Conjecture (1999)

New conjecture equivalent to Schanuel's one, in the spirit of known transcendence criteria by Gel'fond (1949), Chudnovsky, Philippon, Nesterenko, Laurent...

D. Roy. *An arithmetic criterion for the values of the exponential function.* Acta Arith., **97** N° 2 (2001), 183–194.

Small value estimates for algebraic groups

D. Roy. *Small value estimates for the additive group*. Intern. J. Number Theory, to appear.

D. Roy. *Small value estimates for the multiplicative group*. Acta Arith. **135** (2008), 357–393.

7th Pan African Congress of Mathematics

Yamoussoukro, Ivory Coast August 3 – 8, 2009

Transcendental Numbers: Recent Results and Open Problems

Michel Waldschmidt

Institut de Mathématiques de Jussieu & Paris VI

CIMPA International Center of Pure and Applied Mathematics

http://www.math.jussieu.fr/∼miw/

Lecture given on August 5, 2009