**RUPP Masters in Mathematics Program: Number Theory**

**Problem Set March 2012**

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Do at least 5 of the following 7 items.

1. Find integers $x$, $y$ such that $34x + 19y = 7$.
2. Let $p$ and $q$ be distinct odd primes. Show that $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$.
3. Recall the definition of a Euclidean domain: An integral domain $R$ is a Euclidean domain if there is a map $\lambda : R \setminus \{0\} \longrightarrow \{0, 1, 2, \ldots\}$ that satisfies the following property: for any $a$, $b \in R$ with $b \neq 0$, there exist $q$, $r \in R$ such that $a = bq + r$, where $r = 0$ or $\lambda(r) < \lambda(b)$. Show that $\mathbb{Z}[i]$ is a Euclidean domain.
4. For each of the following, determine if Gaussian prime or not. If not, give its factorization as a product of Gaussian primes.
   (a) 53
   (b) 71
   (c) 187
5. If $d$ is a non-perfect square integer, $\mathbb{Q}(\sqrt{d})$ is defined to be the smallest field that contains both $\mathbb{Q}$ and $\sqrt{d}$. Prove that $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$.
6. The norm function on $K = \mathbb{Q}(\sqrt{d})$ is defined by $N(a + b\sqrt{d}) = a^2 - db^2$. Prove that $\alpha \in \mathcal{O}_K$ is a unit if and only if $N(\alpha) = \pm 1$.
7. Prove that the units in $\mathbb{Z}[\omega]$ are $\pm 1$, $\pm\omega$ and $\pm\omega^2$. ($\omega := e^{\frac{2\pi i}{3}}$)