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King Khalid University,
Wams school on Introductory topics
in Number Theory and Differential Geometry

Diophantine Equations

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$$u_0 = 1,\ u_1 = (1-\sqrt{5})/2, \qquad u_n = u_{n-1} + u_{n-2}$$

Question : compute u_{100}.

$$\frac{1-\sqrt{5}}{2} = -0.6180339887498948482045868343656\ldots$$

https://oeis.org/A001622

Excel file Column A : n Column B : u_n

$$u_0 = 1,\ u_1 = (1-\sqrt{5})/2, \qquad u_n = u_{n-1} + u_{n-2}$$

	A	B
1	0	1
2	1	`=(1-RACINE(5))/2`

	A	B
1	0	1
2	1	-0.618034

	A	B
1	0	1
2	1	-0.618034
3	=1+A2	=B1+B2

	A	B
1	0	1
2	1	-0.618034
3	2	0.38196601

Copy $A3$ $B3$ down

Excel file : u_1 to u_{39}

1	-0,61803399
2	0,381966011
3	-0,23606798
4	0,145898034
5	-0,09016994
6	0,05572809
7	-0,03444185
8	0,021286236
9	-0,01315562
10	0,008130619
11	-0,005025
12	0,00310562
13	-0,00191938
14	0,001186241
15	-0,00073314
16	0,000453104
17	-0,00028003
18	0,00017307
19	-0,00010696
20	6,6107E-05
21	-4,0856E-05
22	2,52506E-05
23	-1,5606E-05
24	9,64487E-06
25	-5,9609E-06
26	3,68401E-06
27	-2,2769E-06
28	1,40715E-06
29	-8,6971E-07
30	5,37445E-07
31	-3,3226E-07
32	2,05185E-07
33	-1,2708E-07
34	7,8109E-08
35	-4,8967E-08
36	2,91423E-08
37	-1,9824E-08
38	9,31784E-09
39	-1,0507E-08

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Exact value of u_n

Observations : The signs of u_n alternate, the absolute value is decreasing.

Set $\widetilde{\Phi} = (1 - \sqrt{5})/2$. Notice that $\widetilde{\Phi}$ is a root of $X^2 - X - 1$, the other root is $\Phi = (1 + \sqrt{5})/2$, the golden ratio.

From $\widetilde{\Phi}^n = \widetilde{\Phi}^{n-1} + \widetilde{\Phi}^{n-2}$ with $u_0 = 1$, $u_1 = \widetilde{\Phi}$, we deduce by induction $u_n = \widetilde{\Phi}^n$.

Exact value of u_{39}

Numerical values :

$$\widetilde{\Phi} = -0.618\,033\,988\,749\,895\ldots,$$

$$\log|\widetilde{\Phi}| = -0.481\,211\,825\,059\,603\,4\ldots$$

$$u_{39} = -\widetilde{\Phi}^{39} = -\mathrm{e}^{-18.767\,261\,177\,324{,}453\ldots} = -7.071\,019\ldots 10^{-9}.$$

PARI GP : https://pari.math.u-bordeaux.fr/ PARI GP

Comparing the excel values with the exact values

	excel value	exact value
30	5,37445E-07	5,3749E-07
31	-3,32261E-07	-3,32187E-07
32	2,05185E-07	2,05303E-07
33	-1,27076E-07	-1,26884E-07
34	7,8109E-08	7,84188E-08
35	-4,89667E-08	-4,84655E-08
36	2,91423E-08	2,99533E-08
37	-1,98244E-08	-1,85122E-08
38	9,31784E-09	1,14411E-08
39	-1,05066E-08	-7,07102E-09
40	-1,18878E-09	4,37013E-09
41	-1,16954E-08	-2,70089E-09

Exact value of u_{100}

The answer to initial question is

$$u_{100} = \widetilde{\Phi}^{100}$$

$$\widetilde{\Phi} = -0.618\,033\,988\,749\,895\ldots, \ \log|\widetilde{\Phi}| = -0.481\,211\,825\,059\,603\,4\ldots$$

$$\widetilde{\Phi}^{100} = \mathrm{e}^{-48.121\,182\,505\,960\,34\ldots} = 1.262\,513\,338\,064\ldots 10^{-21}.$$

Excel (continued)

$$u_{100} = -19\,241.901\,833\,167\ldots$$

38	9,31784E-09
39	-1,05066E-08
40	-1,18878E-09
41	-1,16954E-08
42	-1,28842E-08
43	-2,45796E-08
44	-3,74637E-08
45	-6,20433E-08
46	-9,9507E-08
47	-1,6155E-07
48	-2,61057E-07
49	-4,22608E-07
50	-6,83665E-07
51	-1,10627E-06
52	-1,78994E-06

85	-14,10695857
86	-22,82553845
87	-36,93249702
88	-59,75803546
89	-96,69053248
90	-156,4485679
91	-253,1391004
92	-409,5876684
93	-662,7267688
94	-1072,314437
95	-1735,041206
96	-2807,355643
97	-4542,396849
98	-7349,752492
99	-11892,14934
100	-19241,90183

The linear recurrence sequence $u_n = u_{n-1} + u_{n-2}$

From the two solutions Φ^n and $\widetilde{\Phi}^n$ one deduces that any solution is of the form $u_n = a\Phi^n + b\widetilde{\Phi}^n$.

Since $|\Phi| > 1$, the term Φ^n tends to ∞.

Since $|\widetilde{\Phi}| < 1$, the term $b\widetilde{\Phi}^n$ tends to 0.

If $a \neq 0$, then $|u_n|$ tends to infinity like $a\Phi^n$.

If $a = 0$, then $u_n = b\widetilde{\Phi}^n$ tends to 0.

If two consecutive terms are of the same sign, then all the next ones have the same sign and $|u_n|$ tends to infinity.