TRANSCENDENTAL NUMBERS AND WEIERSTRASS SIGMA FUNCTIONS

MICHEL WALDSCHMIDT

1. Introduction

The theorems of Schneider [3] (Chap. II, §4) on the functions $\wp$ and ζ of Weierstrass can be interpreted in terms of algebraic points over an elliptic curve or over an extension of an elliptic curve by the additive group (these algebraic groups correspond to elliptic integrals of he first or the second kind). Then they follow from a theorem of Lang [2] (Chap. III, § 4 theorem 4). If we apply the same theorem of Lang to the extension of an elliptic curve by the multiplicative group (corresponding to elliptic integrals of the third kind), for which the description has been given by Serre (see [4], Appendice), we obtain a transcendence theorem on the values of the sigma functions [4, Chap. III].

We prove this theorem directly using the transcendence criterion of Schneider-Lang, without using algebraic groups.

2. Weierstrass Sigma functions

Let $\mathrm{L} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. The canonical Weierstrass product associated to L is the entire function

$$\sigma(z) = z \prod_{\omega \in L, \omega \neq 0} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega_2}\right).$$

With the usual notations, one checks

$$\sigma(z + \omega_i) = -\sigma(z) \exp\left(\eta_i \left(z + \frac{\omega_i}{2}\right)\right), \quad (i = 1, 2)$$

and

$$\sigma(mz) = (-1)^{m-1} \sigma(z)^{m^2} \psi_m\left(\wp(z), \wp'(z)\right), \quad \left(m \in \mathbb{Z}, m > 0\right),$$

where $\psi_m(X, Y)$ is a rational function of X and Y with coefficients in $\mathbb{Q}(g_2, g_3)$ (see for instance [1, p.205]).

One then deduces that, for non-negative integers p and q, $\omega \in \mathrm{L}$ and $\eta = \zeta(z + \omega) - \eta(z)$, the number

$$\sigma\left(\frac{p}{q}\omega\right) \exp\left(-\frac{p^2}{2q^2}\eta\omega\right)$$

is algebraic over the field $\mathbb{Q}(g_2, g_3)$. For example,

$$\sigma\left(\frac{\omega_1}{2}\right)^8 = -\frac{e^{\eta_1\omega_1}}{\psi_3\left(\wp\left(\frac{\omega_1}{2}\right)\right)}.$$

Date: June 2015.

This is a translation by K. Senthil Kumar and S. Kannappan of the text in French *Nombres transcendants et fonctions sigma de Weierstrass* (C. R. Math. Rep. Acad. Sci. Canada- Vol. 1 (1979) No.2, 111–114).

In the particular case $\mathrm{L} = \mathbb{Z} + \mathbb{Z}\mathrm{i}$, one checks that

$$\sigma\left(\frac{1}{2}\right) = 2^{5/4}\pi^{1/2}e^{\pi/8}\Gamma(1/4)^{-2}.$$

Finally, for $u \in \mathbb{C}, u \notin \mathrm{L}$, the number

$$\frac{\sigma(u+(\omega/2))}{\sigma(u)}\exp(-\eta(u/2+\omega/8))$$

is algebraic over the field $\mathbb{Q}(g_2, g_3, \wp(u))$.

3. A pseudo-periodic multiplicative function

Let $u_0 \in \mathbb{C}, u_0 \in \mathrm{L}$. The meromorphic function

$$\mathrm{F}(z) = \frac{\sigma(z+u_0)}{\sigma(u)\sigma(u_0)}e^{-\zeta(u_0)z},$$

satisfies, for $\omega \in \mathrm{L}$,

$$\mathrm{F}(z+\omega) = \mathrm{F}(z)\exp(\eta u_0 - \omega\zeta(u_0)).$$

Its logarithmic derivative is

$$\frac{\mathrm{F}'(z)}{\mathrm{F}(z)} = \frac{1}{2}\frac{\wp'(z)-\wp'(u_0)}{\wp(z)-\wp(u_0)}.$$

The function of three variables u_0, u_1, u_2

$$\frac{\mathrm{F}(u_1+u_2)}{\mathrm{F}(u_1).\mathrm{F}(u_2)} = \frac{\sigma(u_0+u_1+u_2)\sigma(u_0)\sigma(u_1)\sigma(u_2)}{\sigma(u_0+u_1)\sigma(u_1+u_2)\sigma(u_2+u_0)}$$

is a rational function of $\wp(u_0), \wp(u_1), \wp(u_2), \wp'(u_0), \wp'(u_1), \wp'(u_2)$ with coefficients in $\mathbb{Q}(g_2, g_3)$.

Let $\mathrm{B}_m(X) \in \mathbb{Q}(g_2, g_3)[X]$ be the polynomial defined, for non-negative integers m, by

$$\mathrm{B}_m(\wp(z)) = \Big(\psi_m\big(\wp(z), \wp'(z)\big)\Big)^2.$$

From the relation

$$\zeta(mz) = m\zeta(z) + \wp'(z)\frac{\mathrm{B}'_m(\wp(z))}{2m\mathrm{B}_m(\wp(z))},$$

one deduces that, if u_0 is a torsion point, there exists a number β_0, algebraic over the field $\mathbb{Q}(g_2, g_3, \wp(u_0))$, such that the function

$$\mathrm{F}(z)e^{\beta_0 z}$$

is algebraic over the field $\mathbb{Q}(g_2, g_3, \wp(u_0), \wp(z))$.

4. A transcendence theorem

We suppose that $g_2, g_3, \wp(u_0)$ are algebraic. We propose to apply the criterion of Schneider-Lang [2, Chap. III, §1 theorem 1] to the functions

$$\wp(z),\ \mathrm{F}(z)e^{\beta z},\ \wp'(z),\ \frac{1}{\wp(z)-\wp(u_0)},$$

with $\beta \in \overline{\mathbb{Q}}$. If the function $\mathrm{F}(z)e^{\beta z}$ is algebraic over $\overline{\mathbb{Q}}(\wp(z))$, one deduces from the pseudo-periodicity of F and from the relation of Legendre that u_0 is a torsion point. As the transcendence theorem corresponding to $\wp(z), e^z, \wp'(z)$, is known [3, Chap. II, theorem 18], we suppose that u_0 is not a torsion point.

Let $u \in \mathbb{C}$ with $u \notin \mathrm{L}$, $u + u_0 \notin \mathrm{L}$. To begin with, suppose $u - u_0 \notin \mathrm{L}$. As there are infinitely many $m \in \mathbb{Z}$ such that $mu \notin \mathrm{L}, mu - u_0 \notin \mathrm{L}, mu + u_0 \notin \mathrm{L}$, we obtain the transcendence of one at least of the two numbers

$$\wp(u), \quad \mathrm{F}(u)e^{\beta u}.$$

Choose u such that $2u - u_0 \in \mathrm{L}$ (hence $u \notin \mathrm{L}$ and $u \pm u_0 \notin \mathrm{L}$) and $\wp(u) \in \overline{\mathbb{Q}}$. After an easy computation, one deduces the transcendence of the number

$$\sigma(u_0)^2 \exp\bigl(\eta u_0 + (u_0 + \omega)(\beta - \zeta(u_0))\bigr),$$

and this shows that the assumption $u - u_0 \notin \mathrm{L}$ is superfluous. One deduces the following result:

Theorem 1. *Let $\wp$ be the Weierstrass function associated to the invariants g_2 and g_3, which are assumed to be algebraic, u, u_0 two algebraic points of $\wp$, and β an algebraic number. Suppose that u_0 is not a torsion point and that u and $u + u_0$ are not poles of $\wp$. Then, the number*

$$\frac{\sigma(u + u_0)}{\sigma(u)\sigma(u_0)} e^{(\beta - \zeta(u_0))u}$$

is transcendental.

5. Corollaries

The first corollary shows that the derivative of F takes transcendental values at the points where F is zero.

Corollary 1. *Let $\omega \in \mathrm{L}, \eta = \zeta(z + \omega) - \zeta(u_0)$. Then, the number*

$$\sigma(u_0)^2 \exp\bigl(\eta u_0 + (u_0 + \omega)(\beta - \zeta(u_0))\bigr)$$

is transcendental.

Therefore, assuming again that u_0 is an algebraic point which is not a torsion point, one deduces that the number

$$\sigma(u_0)e^{-\frac{1}{2}u_0\zeta(u_0)}$$

is transcendental.

Finally, when we choose for u in the theorem, a quotient of ω by a power of 2, one obtains the transcendence of the pseudo-periods of $F(z)e^{\beta z}$.

Corollary 2. *With the notations of the Corollary 1, for $\omega \neq 0$, the number*

$$\exp\bigl(\omega\zeta(u_0) - \eta u_0 + \beta\omega\bigr)$$

is transcendental.

References

[1] B. Fricke, *Die elliptichen Funktionen und ihre Anwendungen*, vol. II, Teubner, (Leipzig-Berlin), 1922.
[2] S. Lang, *Introduction to transcendental numbers*, Addison Wesley, (Reading, Mass.), 1966.
[3] Th. Schneider, *Einfürung in die transcendenten Zahlen*, Springer, Berlin, 1957. *Introduction aux nombres transcendants*, Gauthier-Villars (Paris), 1959.
[4] M. Waldschmidt, *Nombres transcendants et groupes algébriques*, Appendices par Daniel Bertrand et Jean-Pierre Serre. Astèrisque (Société Mathématique de France), 69-70 (1979 - Second edition 1987).

Received November 9, 1978
AMS MOS Subject Classification (1970): 10F35, 33A25