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Strategies of transcendence proofs

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Abstract

The purpose of this course is to explain the ideas behind transcendence proofs, avoiding technical details. We start with the Schneider-Lang Theorem. A reference for this part of the course is

http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/Strategy.pdf

The two solutions (in 1934) by Gel'fond and Schneider to Hilbert's seventh problem are linked by a duality related with the Fourier-Borel transform. This leads to the replacement of auxiliary functions with auxiliary functionals. The next step is to introduce the interpolation determinants of Michel Laurent. Finally we will introduce a conjecture by D. Roy which he proved to be equivalent to Schanuel's Conjecture.

- Hermite, Lindemann
- Gel'fond's method; Sketch of proof
- Schneider – Lang Theorem; Sketch of proof
- Meromorphic functions
- Schneider's Theorem on $j(\tau)$ (Gel'fond's method)
- Schneider's method
- Gel'fond vs Schneider. Duality
- Six Exponentials and Four Exponentials
- Baker's method

Hermite– Lindemann

Hermite 1873 transcendence of e

Lindemann 1882 transcendence of π

Theorem (Hermite– Lindemann).

Let α be a non–zero complex number. Then one at least of the two numbers α, e^{α} is transcendental.

For $\alpha = 1$, this is Hermite's result. For $\alpha = \mathrm{i}\pi$, this is Lindemann's result.

Also: the numbers $\log 2$, $\mathrm{e}^{\sqrt{2}}$ are transcendental.

Hermite– Lindemann: ideas of the proof

The function $f(z) = \mathrm{e}^z$ is entire, it satisfies the differential equation $f' = f$ and the functional equation $f(z_1 + z_2) = f(z_1)f(z_2)$;
in particular $f(mz) = f(z)^m$ for $m \in \mathbb{Z}$.

The two functions $f(z)$ and $g(z) = z$ are algebraically independent: the function f is transcendental over the field $\mathbb{C}(z)$ of rational functions.

If $\alpha \in \mathbb{C}$ is such that α and e^α are algebraic, then the two functions f and g take their values in the number field $K := \mathbb{Q}(\alpha, \mathrm{e}^\alpha)$ when $z = m\alpha$, with $m \in \mathbb{Z}$.

If $\alpha \neq 0$, then the set of $m\alpha$, with $m \in \mathbb{Z}$ is infinite.

Gel'fond's method 1

Assume $\alpha \neq 0$ and e^{α} are algebraic. We wish to get a contradiction with the algebraic independence of the two functions z, e^{z}, hence to prove the existence of a non zero polynomial $P \in \mathbb{Z}[X, Y]$ such that the function $F(z) := P(z, \mathrm{e}^{z})$ is the zero function.

If

$$P(X, Y) = \sum_{\tau=0}^{T_0-1} \sum_{t=0}^{T_1-1} a_{\tau,t} X^{\tau} Y^{t}$$

with $a_{\tau,t} \in \mathbb{Z}$, then

$$F(z) = P(z, \mathrm{e}^{z}) = \sum_{\tau=0}^{T_0-1} \sum_{t=0}^{T_1-1} a_{\tau,t} f_{\tau,t}(z) \text{ where } f_{\tau,t}(z) := z^{\tau} \mathrm{e}^{tz}.$$

Gel'fond's method 2

Notice that the numbers

$$f_{\tau,t}^{(\sigma)}(s\alpha) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\sigma} (z^{\tau}\mathrm{e}^{tz})(s\alpha)$$

($\tau \geqslant 0$, $t \geqslant 0$, $\sigma \geqslant 0$, $s \geqslant 0$) are all in the number field K.

We first request $P(X, Y)$ to satisfy finitely many conditions

$$F^{(\sigma)}(s\alpha) = 0 \quad \text{for} \quad 0 \leqslant \sigma < S_0,\ 0 \leqslant s < S_1.$$

The existence of such a $P \neq 0$ follows from linear algebra as soon as $T_0T_1 > \delta S_0S_1$ with $\delta = [K : \mathbb{Q}]$.

We will need an upper bound for the coefficients $a_{\tau,t}$ - such an estimate follows from Dirichlet's box principle and this estimate is sharp as soon as $T_0T_1 \geqslant 2\delta S_0S_1$. We refer to this argument as the Thue–Siegel Lemma.

Gel'fond's method 3

Recall that the goal is to prove $F = 0$. It suffices to prove

$$F^{(\sigma)}(s\alpha) = 0 \quad \text{for all} \quad \sigma \geqslant 0 \quad \text{and} \quad 0 \leqslant s < S_1.$$

Assume that this is true for $0 \leqslant \sigma < U$ with $U \geqslant S_0$. Let j satisfy $0 \leqslant j < S_1$. Consider the number $\gamma := F^{(U)}(j\alpha)$. From Cauchy's inequality we deduce

$$|\gamma| \leqslant U!|F|_r$$

for any $r \geqslant 1 + S_1|\alpha|$, where

$$|F|_r := \sup_{|z|=r} |F(z)|.$$

The idea is that, since F has many zeroes in the disc $|z| = r$, the number $|F|_r$ is small. More precisely we will use Schwarz's Lemma below.

Gel'fond's method 4

The last tool is due to Liouville, it is a lower bound for a non zero algebraic number.

The main such instance is the lower bound $|n| \geqslant 1$ for a nonzero rational number n.

In general, if γ is a nonzero algebraic number, multiplying by a denominator and taking the norm over $\mathbb{Q}$ yields a nonzero rational integer, hence of absolute value at least 1. This gives a lower bound for $|\gamma|$, which we call Liouville's estimate.

The initial example is the lower bound

$$\left|\alpha - \frac{p}{q}\right| \geqslant \frac{c(\alpha)}{q^d}$$

when α is an algebraic number of degree d and p/q a rational number $\neq \alpha$, which enabled Liouville to give the first examples of transcendental numbers in 1844.

Schwarz's Lemma

Let f be an analytic function in a disc $|z| \leqslant R$ of $\mathbb{C}$, with at least N zeroes in a disc $|z| \leqslant r$ with $r < R$, counting multiplicities. Then

$$|f|_r \leqslant \left(\frac{3r}{R}\right)^N |f|_R.$$

When $R > 3r$, this improves the maximum modulus bound $|f|_r \leqslant |f|_R$.

Proof.

Let $\zeta_1, \ldots, \zeta_N$ be zeroes of f in the disc $|z| \leqslant r$. Then the function

$$g(z) := f(z) \prod_{j=1}^{N} (X - \zeta_j)^{-1}$$

is analytic in the disc $|z| \leqslant R$, hence satisfies $|g|_r \leqslant |g|_R$. $\square$

Schneider–Lang (simplified)

An entire function f is said to be of finite order if there exists $\varrho > 0$ and $R_0 > 0$ such that $|f|_R \leqslant \mathrm{e}^{R^\varrho}$ for all $R > R_0$. In other terms

$$\limsup_{R\to\infty} \frac{\log\log|F|_R}{\log R} < \infty.$$

Theorem (Schneider–Lang).

Let f_1, f_2 be two entire functions of finite order which are algebraically independent over $\mathbb{Q}$. Let K be a number field. Assume that the derivatives f_1' and f_2' belong to $K[f_1, f_2]$. Then the set

$$\{w \in \mathbb{C} \mid f_1(w) \text{ and } f_2(w) \text{ belong to } K\}$$

is finite.

Hilbert's seventh problem

Theorem (Gel'fond –Schneider 1934).

If β is an irrational algebraic number, then for any $\ell \in \mathbb{C} \setminus \{0\}$ one at least of the two numbers e^{ℓ}, $\mathrm{e}^{\beta\ell}$ is transcendental.

Proof.

Assume $\alpha := \mathrm{e}^{\ell}$. Write α^{β} for $\mathrm{e}^{\beta\ell}$. Assume α, β and α^{β} are algebraic. Let K be the number field $\mathbb{Q}(\alpha, \beta, \alpha^{\beta})$

The two functions $f_1(z) = \mathrm{e}^{z}$, $f_2(z) = \mathrm{e}^{\beta z}$ have order 1, are algebraically independent ($\beta \notin \mathbb{Q}$) and satisfy the differential equations $f_1' = f_1$, $f_2' = \beta f_2$ with coefficients in K. These two functions take their values in K for $z = m\ell$, which is an infinite set since $\ell \neq 0$. □

Hilbert's Seventh Problem

Theorem (Gel'fond –Schneider).

Let α be a nonzero algebraic number, $\log \alpha$ a nonzero logarithm of α and β an irrational algebraic number. Then $\alpha^\beta := \mathrm{e}^{\beta \log \alpha}$ is transcendental.

With $\alpha = -1$, $\log \alpha = \mathrm{i}\pi$ and $\beta = \mathrm{i}$ this gives the transcendence of e^π. Proved by A.O. Gel'fond in 1929 using an interpolation method introduced by Pólya to prove that 2^z is the entire function of least growth taking values in $\mathbb{Z}$ for $z \in \mathbb{N}$.

Also if α_1 and α_2 are nonzero algebraic numbers and $\log \alpha_1$, $\log \alpha_2$ nonzero logarithms of α_1 and α_2 respectively, then $\log \alpha_1 / \log \alpha_2$ is either rational or transcendental.

If two logarithms of algebraic numbers are linearly independent over $\mathbb{Q}$, they are linearly independent over the field of algebraic numbers.

Proof of Schneider–Lang Theorem - 1

Let K be a number field, $\delta := [K : \mathbb{Q}]$, $w_1, \ldots, w_m$ satisfy

$$f_i(w_j) \in K \text{ for } i = 1, 2 \text{ and } j = 1, \ldots, m,$$

ϱ_1, ϱ_2 and C satisfy

$$\log |f_i|_R \leqslant C R^{\varrho_i}$$

for $i = 1, 2$ and for all sufficiently large R.
We sketch the proof of the upper bound:

$$m \leqslant (\varrho_1 + \varrho_2)\delta.$$

Let S be a sufficiently large integer. We introduce the following parameters T_1 and T_2:

$$T_1 = S^{\frac{\varrho_2}{\varrho_1+\varrho_2}} \sqrt{\log S}, \quad T_2 = S^{\frac{\varrho_1}{\varrho_1+\varrho_2}} \sqrt{\log S}.$$

Proof of Schneider–Lang Theorem - 2

The first step is to prove the existence of a nonzero polynomial in $\mathbb{Z}[X_1, X_2]$, of degree $< T_1$ in X_1 and $< T_2$ in X_2, such that the function $F := P(f_1, f_2)$ satisfies the mS equations

$$F^{(\sigma)}(w_j) = 0$$

for $0 \leqslant \sigma < S$ and $1 \leqslant j \leqslant m$. Since

$$T_1 T_2 = S \log S,$$

the existence of P is a consequence of linear algebra as soon as $\log S > \delta m$. For sufficiently large S, the Thue–Siegel Lemma implies the existence of P with an upper bound for its height:

$$\log \mathrm{H}(P) = O(S).$$

Proof of Schneider–Lang Theorem - 3

Since the functions f_1 and f_2 are algebraically independent, the numbers $F^{(\sigma)}(w_j)$ for $1 \leqslant j \leqslant m$ and $\sigma \geqslant 0$ are not all zero. Let U be the least integer such that the function F satisfies $F^{(\sigma)}(w_j) = 0$ for $0 \leqslant \sigma < U$ and $1 \leqslant j \leqslant m$. From the first step we infer $U \geqslant S$. Since U is minimal, there exists j_0 which satisfies $1 \leqslant j_0 \leqslant m$ such that the number

$$\gamma := F^{(U)}(w_{j_0})$$

is not 0. Using Schwarz's Lemma with $R = U^{\frac{1}{\varrho_1+\varrho_2}}$, we obtain

$$\log|\gamma| \leqslant \left(1 - \frac{m}{\varrho_1+\varrho_2} + o(1)\right) U \log U.$$

Liouville's estimate for the nonzero element γ of K yields

$$\log|\gamma| \geqslant -(\delta - 1 + o(1))U \log U.$$

The upper bound for m follows.

Full version of Schneider–Lang Theorem

Theorem (Schneider–Lang).

Let $f_1, \ldots, f_m$ be meromorphic functions with $m \geqslant 2$. Assume f_1 and f_2 have finite order $\leqslant \varrho_1$ and ϱ_2 respectively, and are algebraically independent over $\mathbb{Q}$. Let K be a number field. Assume that for $1 \leqslant j \leqslant m$ the derivative f_j' of f_j belongs to $K[f_1, \ldots, f_m]$. Then the set

$$\{w \in \mathbb{C} \mid f_j(w) \in K \text{ for } j = 1, \ldots, m\}$$

has at most $(\varrho_1 + \varrho_2)[K : \mathbb{Q}]$ elements.

The main point is that this set is finite.

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Appendix 1: The transcendence of e and π.

Schneider 1949 Satz 1

Satz I. Es seien $f(z)$ bzw. $g(z)$ ganze Funktionen der Wachstumsordnungen μ_1 bzw. μ_2, d. h. es seien

$$\lim_{R\to\infty} \frac{\log\log \operatorname{Max}_{|z|\leqq R} |f(z)|}{\log R} = \mu_1, \quad \lim_{R\to\infty} \frac{\log\log \operatorname{Max}_{|z|\leqq R} |g(z)|}{\log R} = \mu_2.$$

$\zeta_1, \zeta_2, \ldots, \zeta_m, \ldots$ bezeichne eine unendliche Zahlenfolge. Mit $z_0^{\{m\}} = z_0$, $z_1^{\{m\}} = z_1, \ldots, z_k^{\{m\}} = z_k$ mögen die voneinander Verschiedenen unter den Zahlen $\zeta_1, \zeta_2, \ldots, \zeta_m$ benannt werden, und die Vielfachheit von $z_\varkappa$ in $\zeta_1, \zeta_2, \ldots, \zeta_m$ heiße $l_\varkappa^{\{m\}} + 1 = l_\varkappa + 1$. Dann ist $\sum_{\varkappa=0}^{k} (l_\varkappa + 1) = m$. Nun seien für jedes m die Funktionswerte $f(z_\varkappa)$, $g(z_\varkappa)$ $(\varkappa = 0, \ldots, k)$ und die Werte der Ableitungen von $f(z)$ und $g(z)$

$$\frac{d^\lambda f(z)}{dz^\lambda}\bigg|_{z=z_\varkappa} = f^{(\lambda)}(z_\varkappa), \quad \frac{d^\lambda g(z)}{dz^\lambda}\bigg|_{z=z_\varkappa} = g^{(\lambda)}(z_\varkappa) \qquad \begin{pmatrix} \lambda = 1, \ldots, l_\varkappa; \\ \varkappa = 0, \ldots, k \end{pmatrix}$$

ganzrationale Zahlen. Es werde $\operatorname{Max}_{(\varkappa=0,\ldots,k)} l_\varkappa$ mit l bezeichnet und $l \leqq \frac{m}{\log m}$ vorausgesetzt. r sei das Minimum der Längen der Radien aller Kreise um den Koordinatenursprung, die die Stellen $z_0, \ldots, z_k$ der komplexen Ebene enthalten. Mit der Bezeichnung

$$\alpha = \varliminf_{m\to\infty} \frac{\log m}{\log r}$$

lautet dann die Behauptung:

Ist $\mu_1 + \mu_2 < \alpha$, so sind $f(z)$ und $g(z)$ voneinander algebraisch abhängig.

Schneider 1949 Satz III

Satz III. Gegeben seien n ganze oder meromorphe Funktionen $f_1(z), \ldots, f_n(z)$. Die sämtlichen Ausdrücke

$$f_\nu^{(\lambda)}(z_\varkappa) \qquad (\lambda = 0, \ldots, l_\varkappa;\ \varkappa = 0, \ldots, k;\ \nu = 1, \ldots, n)$$

seien algebraische Zahlen aus K. $H_\nu(z_\varkappa)$ seien solche ganzrationale positive Zahlen, daß für alle $\lambda = 0, \ldots, l_\varkappa$ die Produkte $H_\nu(z_\varkappa) \cdot f_\nu^{(\lambda)}(z_\varkappa)$ $\binom{\varkappa = 0, \ldots, k}{\nu = 1, \ldots, n}$ ganze algebraische Zahlen in K sind. Es sei $l \leqq \frac{m}{\log m}$ und

$$\lim_{m \to \infty} \frac{\log m}{\log r} = \alpha . \tag{1}$$

Die Funktionen $f_\nu(z)$ $(\nu = 1, \ldots, n)$ sollen, sofern sie ganz sind, die Wachstumsordnungen μ_ν besitzen, und sofern sie meromorph sind, gebe es für jedes ν eine ganze Funktion $G_\nu(z)$ der Wachstumsordnung μ_ν derart, daß $G_\nu(z) f(z)$ ganz ist und ebenfalls die Wachstumsordnung μ_ν hat. Es werde gefordert

$$\frac{\mu_1 + \cdots + \mu_n}{n - 1} < \alpha . \tag{2}$$

Daraus folgt $\alpha > 0$, und nun nenne $\frac{\mu_\nu}{\alpha} = \eta_\nu$ $(\nu = 1, \ldots, n)$. Mit diesen η_ν sollen die folgenden Ungleichungen gelten:

$$\overline{\lim} \frac{\log \log \operatorname{Max}_{\varkappa = 0, \ldots, k} (|G_\nu(z_\varkappa)|^{-1})}{\log m} \leqq \eta_\nu \qquad (\nu = 1, \ldots, n) \tag{3}$$

und

$$\overline{\lim_{m \to \infty}} \frac{\log \log \operatorname{Max}_{\substack{\varkappa = 0, \ldots, k \\ \lambda = 0, \ldots, l_\varkappa}} \left(\left| f_\nu^{(\lambda)}(z_\varkappa) \right|, H_\nu(z_\varkappa) \right)}{\log m} \leqq \eta_\nu \qquad (\nu = 1, \ldots, n) . \tag{4}$$

Dann, so wird behauptet, sind $f_1(z), \ldots, f_n(z)$ voneinander algebraisch abhängig.

Meromorphic functions

A meromorphic function in a domain D is a function which is analytic outside a discrete subset of D, the points of which are poles (no essential singularity).

It is a fact that a meromorphic function in $\mathbb{C}$ is a quotient of two entire function.

Definition. *A meromorphic function f has order $\leqslant \varrho$ if there exist two entire functions f_1 and f_2 of order $\leqslant \varrho$ such that $f = f_1/f_2$.*

Order of a meromorphic function

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Height of a rational fraction.

The height $H(P)$ of a polynomial $P \in \mathbb{C}[X]$ is the maximum of modulus of its coefficients.

"Definition". *A rational fraction $R \in \mathbb{C}(X)$ has height $\leqslant H$ if there exist two polynomials P_1 and P_2 of height $\leqslant H$ such that $R = P_1/P_2$.*

$$X^2 + 2X + 1 = \frac{(X-1)(X+1)^2}{X-1} = \frac{X^3 + X^2 - X - 1}{X-1}.$$

Lemma.

If an entire function f is a quotient of two entire functions of order $\leqslant \varrho$, then f has order $\leqslant \varrho$.

Weierstrass elliptic functions

A Weierstrass elliptic function $\wp$, solution of the differential equation $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, has order $\leqslant 2$ (quotient of two sigma functions).

Elliptic analog of Hermite–Lindemann Theorem:

Corollary (Schneider).

Assume g_2 and g_3 are algebraic. Let α be a nonzero algebraic number. Then α is not a pole of $\wp$ and $\wp(\alpha)$ is transcendental.

Proof.

Let $f_1(z) = z$, $f_2(z) = \wp(z)$, $f_3(z) = \wp'(z)$. Then $f_3'(z) = \wp''(z) = 6\wp^2 - g_2/2$:

$$f_1' = 1, \quad f_2' = f_3, \quad f_3' = 6f_2^2 - g_2/2.$$

Corollaries (Schneider)

Assume g_2 and g_3 are algebraic.

1. *(Elliptic analog of Hermite–Lindemann)*
   Let α be a nonzero algebraic number. Then α is not a pole of $\wp$ and $\wp(\alpha)$ is transcendental.
2. *Elliptic analog of Gel'fond–Schneider)*
   Let α be a nonzero algebraic number and $\log \alpha$ a nonzero logarithm of α. Then $\log \alpha$ is not a pole of $\wp$ and $\wp(\log \alpha)$ is transcendental.
   $f_1(z) = \mathrm{e}^z$, $f_2(z) = \wp(z)$, $f_3(z) = \wp'(z)$.
3. Let $\wp^*$ be another Weierstrass elliptic function with g_2^* and g_3^* algebraic. Assume $\wp$ and $\wp^*$ are algebraically independent. Then for any $z \in \mathbb{C}$ not pole of $\wp$ nor of $\wp^*$ one at least of the two numbers $\wp(z)$, $\wp^*(z)$ is transcendental.

Consequences of Schneider's result on $\wp$ and $\wp^*$

Let $\wp$ be a Weierstras elliptic function with algebraic invariants g_2, g_3. Let ω_1, ω_2 be a pair of fundamental periods of $\wp$.

Assume that the quotient $\tau = \omega_2/\omega_1$ is algebraic.

Then the Weierstrass elliptic function $\wp^*(z) := \tau^2\wp(\tau z)$ has algebraic invariants g_2^*, g_3^* and the two functions $\wp(z)$, $\wp^*(z)$ take values in a number field for $z \in (\omega_1/2) + \mathbb{Z}\omega_1$.

Hence, by Schneider's result, these two functions are algebraically dependent. It follows that $\wp$ has complex multiplication and that τ is quadratic.

The modular function $j(\tau)$

Let j be the modular function, analytic in the upper half plane $\mathfrak{H} := \{\tau \mid \Im\mathrm{m}(\tau) > 0\}$, satisfying

$$j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

having a simple pole at infinity with residue 1.
For $\tau \in \mathfrak{H}$, τ and $j(\tau)$ are both algebraic if and only if τ is quadratic (τ is a special point*).*

Proof.

According to the theory of complex multiplication, if τ is quadratic then $j(\tau)$ is algebraic (Hilbert's class field).
Conversely, let $\Delta(\tau)$ be the discriminant of the elliptic curve of periods $\mathbb{Z} + \mathbb{Z}\tau$, let ω_1 be a 12–th root of $\Delta(\tau)$ and let $\omega_2 = \tau\omega_1$. If $j(\tau)$ is algebraic then the invariants g_2, g_3 of $\wp$ are algebraic. If τ is also algebraic, then Schneider's theorem implies that τ is imaginary quadratic. □

Schneider's method 1

Recall:

Theorem (Gel'fond –Schneider).

If β is an irrational algebraic number, then for any $\ell \in \mathbb{C} \setminus \{0\}$ one at least of the two numbers e^{ℓ}, $\mathrm{e}^{\beta\ell}$ is transcendental.

Let β be an irrational number and let $\ell \in \mathbb{C} \setminus \{0\}$. Assume β, e^{ℓ} and $\mathrm{e}^{\beta\ell}$ are algebraic. Let $\alpha := \mathrm{e}^{\ell}$, $\mathrm{e}^{\beta\ell} = \alpha^{\beta}$, $K = \mathbb{Q}(\alpha, \beta, \alpha^{\beta})$.

Gel'fond :

Uses the two functions $f_1(z) = \mathrm{e}^{z}$, $f_2(z) = \mathrm{e}^{\beta z}$ with derivatives evaluated at points in $\mathbb{Z}\ell$.

Schneider:

Uses the two functions z and $\mathrm{e}^{\ell z}$ which are algebraically independent and take algebraic values at points in $\mathbb{Z} + \mathbb{Z}\beta$.
No differential equation with algebraic coefficients.

Schneider's method 2

We argue by contradiction: let again K be the number field $\mathbb{Q}(\alpha, \beta, \alpha^\beta)$. Notice that the two functions z and $\alpha^z = \mathrm{e}^{z \log \alpha}$ are algebraically independent since $\log \alpha \neq 0$. The auxiliary function G is now a linear combination of the functions

$$g_{\sigma,s}(z) = z^\sigma \alpha^{sz},$$

say

$$G(z) = \sum_{\sigma=0}^{S_0-1} \sum_{s=0}^{S_1-1} b_{\sigma,s} g_{\sigma,s}(z)$$

with $b_{\sigma,s} \in \mathbb{Z}$.
For $\sigma \geq 0$, $s \geq 0$, $\tau \geq 0$, $t \geq 0$, we have

$$g_{\sigma,s}(\tau + t\beta) \in K.$$

Schneider's method 3

The first step is the existence of a nonzero polynomial

$$P(X,Y) = \sum_{\sigma=0}^{S_0-1} \sum_{s=0}^{S_1-1} b_{\sigma,s} X^\sigma Y^s \in \mathbb{Z}[X,Y]$$

such that the auxiliary function $G(z) = P(z, \alpha^z)$ satisfies

$$G(\tau + t\beta) = 0$$

for $0 \leqslant \tau < T_0$, $0 \leqslant t < T_1$. The numbers $\tau + t\beta$ are pairwise distinct since β is irrational. The Thue–Siegel Lemma shows the existence of such a polynomial with an upper bound for the height as soon as we request S_0, S_1, T_0, T_1 to satisfy

$$S_0 S_1 \geqslant 2[K : \mathbb{Q}] T_0 T_1.$$

Schneider's method 4

By induction on $T'_0 + T'_1$, one proves that the equations $G(\tau + t\beta) = 0$ are satisfied for $0 \leqslant \tau < T'_0$, $0 \leqslant t < T'_1$.

For proving that a number of the form $\gamma = G(\tau + t\beta)$ is zero, one uses the analytic estimate given by Schwarz's Lemma to show that $|\gamma|$ is small, thanks to the inductive assumptions. One concludes the inductive process by means of Liouville's inequality.

Therefore the equations $G(\tau + t\beta) = 0$ are satisfied for all $\tau \geqslant 0$, and $t \geqslant 0$.

The last step is to deduce that $G = 0$. There are several ways of achieving this goal, the easiest is to remark that in the inductive argument in the second step, we show that $|G|_r$ is small for arbitrarily large values of r, hence $G = 0$ by the maximum modulus principle.

Duality Gel'fond – Schneider

The methods of Gel'fond and Schneider follow the same sketch of proof. There is something more that they share in common: the numbers which occur in Gel'fond 's proof and in Schneider's proof are the same.

Gel'fond 's method:

$$f_{\tau,t}(z) = (\mathrm{e}^z)^\tau (\mathrm{e}^{\beta z})^t = \mathrm{e}^{(\tau+t\beta)z}, \quad z = s\log\alpha.$$

Schneider's method:

$$g_{\sigma,s}(z) = z^\sigma \alpha^{sz}, \quad z = \tau + t\beta.$$

We have

$$f_{\tau,t}^{(\sigma)}(s\log\alpha) = (\tau+t\beta)^\sigma \alpha^{\tau s}(\alpha^\beta)^{st} = g_{\sigma,s}(\tau+t\beta).$$

Connection with Fourier–Borel duality.

Schneider's problems

Wir nennen die folgenden Probleme:

1. Es seien α, β, γ algebraische Zahlen und $\alpha \neq 0, 1$ sowie $\frac{\log \beta}{\log \alpha}$ und $\frac{\log \gamma}{\log \alpha}$ irrational. Es ist $\delta = \exp\left(\frac{\log \beta \log \gamma}{\log \alpha}\right)$ auf Transzendenz zu untersuchen.

2. Der Satz über die Transzendenz der Werte der Modulfunktion $J(\tau)$ soll durch direkte Untersuchung der transzendenten Modulfunktion und nicht durch Untersuchung von $\wp$-Funktionen gezeigt werden.

3. Es ist zu versuchen, Transzendenzresultate über elliptische Integrale dritter Gattung zu beweisen.

4. Die Transzendenzsätze über elliptische Integrale erster und zweiter Gattung sind in weitestmöglichem Umfang auf analoge Sätze über ABELsche Integrale zu verallgemeinern.

5. Es ist die Approximation bekannter transzendenter Zahlen durch solche rationale oder auch algebraische Zahlen zu untersuchen, die arithmetischen Bedingungen genügen.

6. Die Aussagen über das Maß der S-Zahlen (Satz 24) sind in Richtung auf die MAHLERsche Vermutung, wonach die Menge der S-Zahlen vom Typus $\vartheta > 1$ ein LEBESGUEsches Maß Null hat, zu verbessern.

7. Es seien α und $\beta_1, \ldots, \beta_n$ algebraische Zahlen sowie $\alpha \neq 0, 1$ und $\beta_1, \ldots, \beta_n$ irrational und mit rationalen Koeffizienten linear unabhängig voneinander. Dann untersuche die transzendenten Zahlen $\alpha^{\beta_1}, \ldots, \alpha^{\beta_n}$ auf algebraische Unabhängigkeit bei algebraischen Koeffizienten.

8. Zeige, daß mindestens eine der Zahlen e^e und e^{e^2} transzendent sein muß.

Mit dieser kleinen Auswahl wollen wir uns bescheiden. Wir haben nicht gefragt nach der Irrationalität der EULERschen Konstanten,

nach der Transzendenz von Werten der RIEMANNschen ζ-Funktion, von weiteren Werten der Γ-Funktion, nach der algebraischen Unabhängigkeit von e und π, um nur einige von solchen häufig aufgeworfenen Problemen zu nennen, deren Lösung beim derzeitigen Stande der Theorie völlig aussichtslos erscheint, aber natürlich kann der nächste methodische Fortschritt auch hier ein völlig neues Bild erwirken.

Schneider's problems 1 and 2

1. Es seien α, β, γ algebraische Zahlen und $\alpha \neq 0, 1$ sowie $\frac{\log \beta}{\log \alpha}$ und $\frac{\log \gamma}{\log \alpha}$ irrational. Es ist $\delta = \exp\left(\frac{\log \beta \log \gamma}{\log \alpha}\right)$ auf Transzendenz zu untersuchen.

2. Der Satz über die Transzendenz der Werte der Modulfunktion $J(\tau)$ soll durch direkte Untersuchung der transzendenten Modulfunktion und nicht durch Untersuchung von $\wp$-Funktionen gezeigt werden.

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Die Grundlehren der Mathematischen Wissenschaften. Band **81**.
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Four Exponentials Conjecture

Conjecture.

Let x_1, x_2 be two $\mathbb{Q}$–linearly independent complex numbers and y_1, y_2 be two $\mathbb{Q}$–linearly independent complex numbers. Then one at least of the four numbers

$$\mathrm{e}^{x_1y_1},\ \mathrm{e}^{x_1y_2},\ \mathrm{e}^{x_2y_1},\ \mathrm{e}^{x_2y_2}$$

is transcendental.

Same as Schneider's first problem:

$$x_1 = \log\alpha,\ x_2 = \log\beta,\ y_1 = 1,\ y_2 = \frac{\log\gamma}{\log\alpha} = \frac{\log\delta}{\log\beta}$$

Open problem:
Assume 2^t and 3^t are integers. Prove that t is an integer.

Six Exponentials and Four Exponentials

Let $x_1, \ldots, x_d$ be $\mathbb{Q}$–linearly independent complex numbers and $y_1, \ldots, y_\ell$ be $\mathbb{Q}$–linearly independent complex numbers.

Theorem (Six Exponentials Theorem: Siegel – Schneider – Lang – Ramachandra).

If $\ell d > \ell + d$ then one at least of the ℓd numbers

$$\mathrm{e}^{x_i y_j} \quad (1 \leqslant i \leqslant d,\ 1 \leqslant j \leq \ell)$$

is transcendental. The relevant case is $\{\ell, d\} = \{2, 3\}$.

Compare with the Four Exponentials Conjecture:
If $\ell d \geqslant \ell + d$ then one at least of the ℓd numbers

$$\mathrm{e}^{x_i y_j} \quad (1 \leqslant i \leqslant d,\ 1 \leqslant j \leq \ell)$$

is transcendental. The relevant case is $\ell = d = 2$.

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Resonance

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Resonance, 27, 599–607 (2022)
doi 10.1007/s12045-022-1351-0

Proof of the Six Exponentials Theorem 1

Since $x_1, \ldots, x_d$ are $\mathbb{Q}$–linearly independent, the functions $\mathrm{e}^{x_1 z}, \ldots, \mathrm{e}^{x_d z}$ are algebraically independent.
Here there is no differential equation with algebraic coefficients.
Let K be the number field $\mathbb{Q}(\mathrm{e}^{x_i y_j} \ ; \ 1 \leqslant i \leqslant d, \ 1 \leqslant j \leq \ell)$.
The auxiliary function is an exponential polynomial with integer coefficients

$$F(z) = P(\mathrm{e}^{x_1 z}, \ldots, \mathrm{e}^{x_d z}).$$

The first step is to show the existence of a nonzero polynomial P of partial degrees $< T$ such that

$$F(s_1 y_1 + \cdots + s_\ell y_\ell) = 0$$

for $0 \leqslant s_j < S$, $(1 \leqslant j \leqslant \ell)$. Linear algebra show that such a polynomial exists as long as $T^d > [K : \mathbb{Q}] S^\ell$.

Proof of the Six Exponentials Theorem 2

Assume $T^d \geqslant 2[K:\mathbb{Q}]S^\ell$. A Lemma due to Thue and Siegel provides a solution with an upper bound for the height of P:

$$\mathrm{H}(P) \leqslant \mathrm{e}^{c_1 ST}.$$

The constant c_1 depends on the algebraic numbers $\mathrm{e}^{x_i y_j}$ but not on S nor T.
By induction on $\max\{s_j\}$ one proves that

$$F(s_1 y_1 + \cdots + s_\ell y_\ell) = 0$$

for all $(s_1, \ldots, s_\ell)$ with $s_j \geqslant 0$.
We explain the first step of the induction: assume $\max\{s_j\} = S$. From Schwarz's Lemma, using the fact that we already know S^ℓ zeroes of F, we deduce

$$|F|_{c_2 S} \leqslant \mathrm{e}^{-c_3 S^\ell}$$

Proof of the Six Exponentials Theorem 3

In particular

$$|F(s_1y_1 + \cdots + s_\ell y_\ell)| \leqslant \mathrm{e}^{-c_3 S^\ell}$$

for $0 \leqslant s_j \leqslant S$, $(1 \leqslant j \leq \ell)$. The left hand side is an algebraic number, Liouville's inequality yields that it is zero.

This is the first step of the inductive argument. Once we complete the induction, one deduces $F = 0$, hence the contradiction.

Assumption $\ell d > \ell + d$:

$$T^d \simeq S^\ell, \quad TS \simeq S^{(\ell+d)/d}, \quad TS < S^\ell.$$

Baker's Theorem

Let $\ell_1, \ldots, \ell_n$ be $\mathbb{Q}$–linearly independent complex numbers such that $\alpha_1 = \mathrm{e}^{\ell_1}, \ldots, \alpha_n = \mathrm{e}^{\ell_n}$ are algebraic. Then the numbers $1, \ell_1, \ldots, \ell_n$ are-linearly independent over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

Corollaries:

- Transcendence of numbers like $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$, $e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$.
- Siegel integral

$$\int_0^1 \frac{\mathrm{d}x}{1+x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right)$$

Siegel, Carl Ludwig
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Baker's method 1

Let $\ell_1, \ldots, \ell_n$ be $\mathbb{Q}$–linearly independent complex numbers such that $\alpha_i = \mathrm{e}^{\ell_i}$, $(i = 1, \ldots, n)$ are algebraic. Write $\ell_i = \log \alpha_i$. Let $\beta_0, \beta_1, \ldots, \beta_{n-1}$ be algebraic numbers such that

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n.$$

Consider the functions

$$z_0, \mathrm{e}^{z_1}, \ldots, \mathrm{e}^{z_{n-1}}, \mathrm{e}^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}}$$

and their derivatives at the points in

$$\mathbb{Z}(1, \log \alpha_1, \ldots, \log \alpha_{n-1}).$$

Baker's method 2

The first step is the existence of a nonzero polynomial $P \in \mathbb{Z}[X_0, \ldots, X_n]$ such that the auxiliary function of n variables

$$F(z_0, \ldots, z_{n-1}) = P(z_0, \mathrm{e}^{z_1}, \ldots, \mathrm{e}^{z_{n-1}}, \mathrm{e}^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}})$$

has a zero of high multiplicity, say T, at several points in

$$\mathbb{Z}\underline{u}, \text{ where } \underline{u} = (1, \log \alpha_1, \ldots, \log \alpha_{n-1}).$$

If F has a zero of multiplicity $\geqslant T$ at a point $s\underline{u}$, then for a derivative $D^{\underline{t}_1}$ of multiplicity $\leqslant T/2$ the function of a single variable

$$G(z) = D^{\underline{t}_1} F(z\underline{u})$$

has a zero of multiplicity $\geqslant T/2$ at s. Schwarz's Lemma in a single variable produces an upper bound for $|G|_r$. In the inductive process one decreases the order of derivation, and increases the number of points. To complete the proof (and reach a contradiction) one needs to use a zero estimate.

p–adic transcendence

The series

$$\sum_{n\geqslant 0} \frac{z^n}{n!}$$

which defines the exponential function has a finite radius of convergence in the p–adic case.

The transcendence proofs which require to take a large radius when applying the analytic estimate do not work in the p–adic case.

Two open problems:

- p–adic Lindemann–Weierstrass
- Algebraic independence of $2^{\sqrt[3]{2}}$, $2^{\sqrt[3]{4}}$


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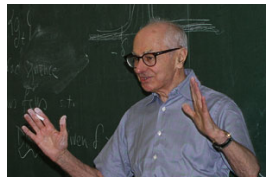

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