History of irrational and transcendental numbers

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Abstract

The transcendence proofs for constants of analysis are essentially all based on the seminal work by Ch. Hermite : his proof of the transcendence of the number e in 1873 is the prototype of the methods which have been subsequently developed. We first show how the founding paper by Hermite was influenced by earlier authors (Lambert, Euler, Fourier, Liouville), next we explain how his arguments have been expanded in several directions : Padé approximants, interpolation series, auxiliary functions.

Numbers : rational, irrational

Numbers = real or complex numbers $\mathbf{R}$, $\mathbf{C}$.

Natural integers : $\mathbf{N} = \{0, 1, 2, \ldots\}$.

Rational integers : $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Rational numbers :
a/b with a and b rational integers, $b > 0$.

Irreducible representation :
p/q with p and q in $\mathbf{Z}$, $q > 0$ and $\gcd(p, q) = 1$.

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Numbers : algebraic, transcendental

Algebraic number : a complex number which is root of a non-zero polynomial with rational coefficients.

Examples :

rational numbers : a/b, root of $bX - a$.
$\sqrt{2}$, root of $X^2 - 2$.
i, root of $X^2 + 1$.

The sum and the product of algebraic numbers are algebraic numbers. The set of complex algebraic numbers is a field, the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$.

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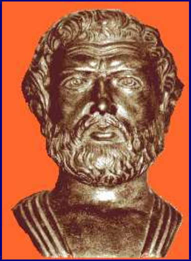
Irrationality of $\sqrt{2}$

Pythagoreas school



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Sulba Sutras, Vedic civilization in India, ∼800-500 BC.

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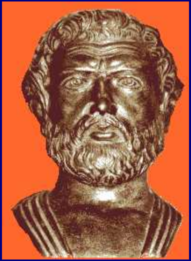

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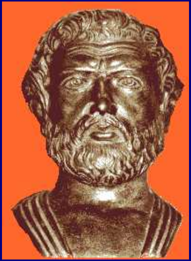

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Irrationality of $\sqrt{2}$: geometric proof

- Start with a rectangle have side length 1 and $1+\sqrt{2}$.
- Decompose it into two squares with sides 1 and a smaller rectangle of sides $1+\sqrt{2}-2=\sqrt{2}-1$ and 1.
- This second small rectangle has side lenghts in the proportion

$$\frac{1}{\sqrt{2}-1}=1+\sqrt{2},$$

  which is the same as for the large one.
- Hence the second small rectangle can be split into two squares and a third smaller rectangle, the sides of which are again in the same proportion.
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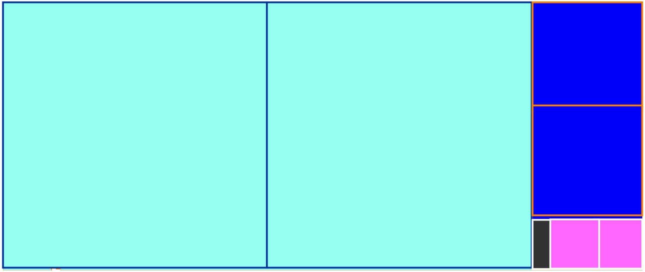
Rectangles with proportion $1+\sqrt{2}$



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If we start with a rectangle having integer side lengths, then this process stops after finitely may steps (the side lengths are positive decreasing integers).

Also for a rectangle with side lengths in a rational proportion, this process stops after finitely may steps (reduce to a common denominator and scale).

Hence $1+\sqrt{2}$ is an irrational number, and $\sqrt{2}$ also.

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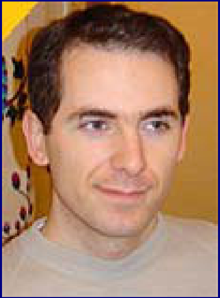

- Benoît Rittaud, Éditions *Le Pommier* (2006).

http://www.math.univ-paris13.fr/~rittaud/RacineDeDeux

Continued fraction

The number

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\ \ldots$$

satisfies

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1}\cdot$$

Hence

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{\sqrt{2}+1}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}$$

We write the continued fraction expansion of $\sqrt{2}$ using the shorter notation

$$\sqrt{2} = [1;\ 2,\ 2,\ 2,\ 2,\ 2,\ \ldots] = [1;\ \overline{2}].$$

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- H.W. Lenstra Jr, *Solving the Pell Equation*, Notices of the A.M.S. **49** (2) (2002) 182–192.

Irrationality criteria

A real number is rational if and only if its continued fraction expansion is finite.

A real number is rational if and only if its binary (or decimal, or in any basis $b \geq 2$) expansion is *ultimately periodic*.

Consequence : it should not be so difficult to decide whether a given number is rational or not.

To prove that certain numbers (occurring as constants in analysis) are irrational is most often an impossible challenge. However to construct irrational (even transcendental) numbers is easy.

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Euler–Mascheroni constant

Euler's Constant is

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$
$$= 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots$$

Is–it a rational number ?

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{1}{k}-\log\left(1+\frac{1}{k}\right)\right) = \int_1^{\infty}\left(\frac{1}{[x]}-\frac{1}{x}\right)dx$$
$$= -\int_0^1\int_0^1 \frac{(1-x)dxdy}{(1-xy)\log(xy)}.$$

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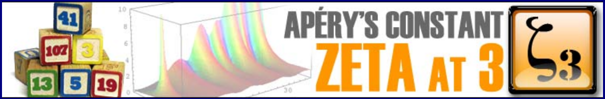

The number

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1.202\,056\,903\,159\,594\,285\,399\,738\,161\,511\ \ldots$$

is irrational (Apéry 1978).

Is–it the same for

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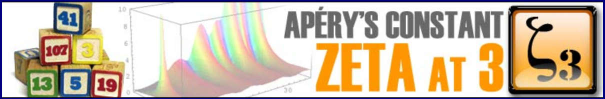

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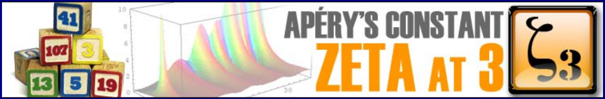

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Open problems (irrationality)

- Is the number
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- Is the number
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Is Catalan's constant

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an irrational number ?

This is the value at $s=2$ of the Dirichlet L–function $L(s, \chi_{-4})$ associated with the Kronecker character

$$\chi_{-4}(n) = \left(\frac{n}{4}\right),$$

which is the quotient of the Dedekind zeta function of $\mathbf{Q}(i)$ and the Riemann zeta function.

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Is the number

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$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} = \int_0^{\infty} e^{-t} t^z \cdot \frac{dt}{t}$$

Here is the set of rational values for z for which the answer is known (and, for these arguments, the Gamma value is a transcendental number) :

$$r \in \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}\right\} \pmod 1.$$

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Here is the set of rational values for z for which the answer is known (and, for these arguments, the Gamma value is a transcendental number) :

$$r \in \left\{ \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \right\} \pmod 1.$$

Known results

Irrationality of the number π :

Āryabhaṭa, b. 476 AD : $\pi \sim 3.1416$.

Nīlakaṇṭha Somayājī, b. 1444 AD : *Why then has an approximate value been mentioned here leaving behind the actual value ? Because it (exact value) cannot be expressed.*

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Continued fraction expansion of $\tan(x)$

$$\tan(x) = \frac{1}{i}\tanh(ix), \qquad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\cdot$$

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \cfrac{x^2}{9 - \cfrac{x^2}{\ddots}}}}}}\cdot$$

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$$e = \lim_{n\to\infty} (1+1/n)^n$$
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$$= [2; \overline{1,\ 2m,\ 1}]_{m \geq 1}.$$

e is neither rational (J-H. Lambert, 1766) nor quadratic irrational (J-L. Lagrange, 1770).

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Continued fraction expansion for $e^{1/a}$

Starting point : $y = \tanh(x/a)$ satisfies the differential equation $ay' + y^2 = 1$.

This leads Euler to

$$\begin{aligned} e^{1/a} &= [1\,;\, a-1,\ 1,\ 1,\ 3a-1,\ 1,\ 1,\ 5a-1, \dots] \\ &= [\overline{1,\ (2m+1)a-1,\ 1}]_{m\ge 0}. \end{aligned}$$

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A geometric proof that e is irrational and a new measure of its irrationality, Amer. Math. Monthly **113** (2006) 637-641.

Start with an interval I_1 with length 1. The interval I_n will be obtained by splitting the interval I_{n-1} into n intervals of the same length, so that the length of I_n will be $1/n!$.

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The origin of I_n will be

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\cdot$$

Hence we start from the interval $I_1 = [2, 3]$. For $n \geq 2$, we construct I_n inductively as follows : split I_{n-1} into n intervals of the same length, and call the second one I_n :

$$I_1 = \left[1 + \frac{1}{1!}\, , 1 + \frac{2}{1!}\right] = [2, 3],$$

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Irrationality of e, following J. Sondow

The origin of I_n is

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the length is $1/n!$, hence $I_n = [a_n/n!, (a_n+1)/n!]$.

The number e is the intersection point of all these intervals, hence it is inside each I_n, therefore it cannot be written $a/n!$ with a an integer.
Since

$$\frac{p}{q} = \frac{(q-1)!\,p}{q!},$$

we deduce that the number e is irrational.

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Irrationality measure for e, following J. Sondow

For any integer $n > 1$,

$$\frac{1}{(n+1)!} < \min_{m \in \mathbf{Z}} \left| e - \frac{m}{n!} \right| < \frac{1}{n!}.$$

Smarandache function : $S(q)$ is the least positive integer such that $S(q)!$ is a multiple of q :

$$S(1) = 1,\ S(2) = 2,\ S(3) = 3,\ S(4) = 4,\ S(5) = 5,\ S(6) = 3 \ldots$$

$S(p) = p$ for p prime. Also $S(n!) = n$.
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Joseph Fourier

Course of analysis at the École Polytechnique Paris, 1815.

Irrationality of e, following J. Fourier

$$e = \sum_{n=0}^{N} \frac{1}{n!} + \sum_{m \geq N+1} \frac{1}{m!}.$$

Multiply by $N!$ and set

$$B_N = N!, \qquad A_N = \sum_{n=0}^{N} \frac{N!}{n!}, \quad R_N = \sum_{m \geq N+1} \frac{N!}{m!},$$

so that $B_N e = A_N + R_N$. Then A_N and B_N are in $\mathbf{Z}$, $R_N > 0$ and

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In the formula

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Since e is irrational, the same is true for $e^{1/b}$ when b is a positive integer. That e^2 is irrational is a stronger statement.

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The number e is not quadratic

Recall (Euler, 1737) : $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ which is not a periodic expansion. J.L. Lagrange (1770) : it follows that e is not a quadratic number.

Assume $ae^2 + be + c = 0$. Replacing e and e^2 by the series and truncating does not work : the denominator is too large and the *remainder* does not tend to zero.

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(1) **5** (1840), p. 192 and p. 193-194.

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The irrationality of e^4, hence of $e^{4/b}$ for b a positive integer, follows.

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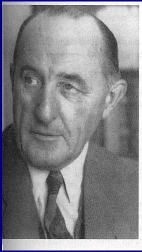
Irrationality of e^r and π *(Lambert, 1766)*

Charles Hermite (1873)

Carl Ludwig Siegel (1929, 1949)

Yuri Nesterenko (2005)



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We wish to prove the irrationality of e^a for a a positive integer.

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Substitute $z = a$, set $q = B_n(a)$, $p = A_n(a)$ and get

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Rational approximation to exp

Given $n_0 \geq 0$, $n_1 \geq 0$, *find* A *and* B *in* $\mathbf{R}[z]$ *of degrees* $\leq n_0$ *and* $\leq n_1$ *such that* $R(z) = B(z)e^z - A(z)$ *has a zero at the origin of multiplicity* $\geq N+1$ *with* $N = n_0 + n_1$.

Theorem *There is a non-trivial solution, it is unique with* B *monic. Further,* B *is in* $\mathbf{Z}[z]$ *and* $(n_0!/n_1!)A$ *is in* $\mathbf{Z}[z]$. *Furthermore* A *has degree* n_0, B *has degree* n_1 *and* R *has multiplicity exactly* $N+1$ *at the origin.*

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$B(z)e^z = A(z) + R(z)$

Proof. Unicity of R, hence of A and B.
Let $D = d/dz$. Since A has degree $\leq n_0$,

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is the product of e^z with a polynomial of the same degree as the degree of B and same leading coefficient.
Since $D^{n_0+1}R(z)$ has a zero of multiplicity $\geq n_1$ at the origin, $D^{n_0+1}R = z^{n_1}e^z$. Hence R is the unique function satisfying $D^{n_0+1}R = z^{n_1}e^z$ with a zero of multiplicity $\geq n_0$ at 0 and B has degree n_1.

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The same argument gives the irrationality of $\log(-1)$, meaning $\log(-1) = i\pi \notin \mathbf{Q}(i)$.

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Solve $D^{n_0+1}R(z) = z^{n_1}e^z$.

The operator $J\varphi = \int_0^z \varphi(t)dt$,

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Simultaneous approximation and transcendence

Irrationality proofs involve rational approximation to a single real number θ.

We wish to prove transcendence results.

A complex number θ is transcendental if and only if the numbers

$$1, \theta, \theta^2, \ldots, \theta^m, \ldots$$

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$L = a_0 + a_1 x_1 + \cdots + a_m x_m$

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is not zero. Approximate simultaneously $x_1, \ldots, x_m$ by rational numbers $b_1/b_0, \ldots, b_m/b_0$.
Let $b_0, b_1, \ldots, b_m$ be rational integers. For $1 \le k \le m$ set

$$\epsilon_k = b_0 x_k - b_k.$$

Then $b_0 L = A + R$ with

$$A = a_0 b_0 + \cdots + a_m b_m \in \mathbf{Z} \quad \text{and} \quad R = a_1 \epsilon_1 + \cdots + a_m \epsilon_m \in \mathbf{R}.$$

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Simultaneous approximation to the exponential function

Irrationality results follow from rational approximations $A/B \in \mathbf{Q}(x)$ to the exponential function e^x.

One of Hermite's ideas is to consider *simultaneous rational approximations to the exponential function*, in analogy with Diophantine approximation.

Let $B_0, B_1, \ldots, B_m$ be polynomials in $\mathbf{Z}[x]$. For $1 \le k \le m$ define

$$R_k(x) = B_0(x)e^{kx} - B_k(x).$$

Set $b_j = B_j(1)$, $0 \le j \le m$ and

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If $0 < |R| < 1$, then $a_0 + a_1 e + \cdots + a_m e^m \neq 0$.

Hermite–Lindemann Theorem

For any non-zero complex number z, one at least of the two numbers z and e^z is transcendental.

Hermite (1873) : transcendence of e.

Lindemann (1882) : transcendence of π.

Corollaries : transcendence of $\log \alpha$ and of e^β for α and β non-zero algebraic complex numbers, with $\log \alpha \neq 0$.

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Hermite : approximation to the functions $1, e^{\alpha_1 x}, \dots, e^{\alpha_m x}$

Let $\alpha_1, \dots, \alpha_m$ be pairwise distinct complex numbers and $n_0, \dots, n_m$ be rational integers, all ≥ 0. Set $N = n_0 + \cdots + n_m$.

Hermite constructs explicitly polynomials B_0, $B_1, \dots$, B_m with B_j of degree $N - n_j$ such that each of the functions

$$B_0(z)e^{\alpha_k z} - B_k(z), \quad (1 \leq k \leq m)$$

has a zero at the origin of multiplicity at least N.

Approximants de Padé

Henri Eugène Padé (1863 - 1953)
Approximation of complex analytic functions by rational functions.

Transcendental functions

A complex function is called transcendental if it is transcendental over the field $\mathbf{C}(z)$, which means that the functions z and $f(z)$ are algebraically independent : if $P \in \mathbf{C}[X,Y]$ is a non-zero polynomial, then the function $P\bigl(z, f(z)\bigr)$ is not 0.

Exercise. *An entire function (analytic in $\mathbf{C}$) is transcendental if and only if it is not a polynomial.*

Example. *The transcendental entire function e^z takes an algebraic value at an algebraic argument z only for $z=0$.*

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Weierstrass question

Is–it true that a transcendental entire function f takes usually *transcendental values at algebraic arguments ?*

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain...

If S is a countable subset of $\mathbf{C}$ and T is a dense subset of $\mathbf{C}$, there exist transcendental entire functions f mapping S into T, as well as all its derivatives.

Also there are transcendental entire functions f such that $D^k f(\alpha) \in \mathbf{Q}(\alpha)$ for all $k \geq 0$ and all algebraic α.

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An integer valued entire function is a function f, which is analytic in $\mathbf{C}$, and maps $\mathbf{N}$ into $\mathbf{Z}$.

Example : 2^z *is an integer valued entire function, not a polynomial.*

Question : *Are-there integer valued entire function growing slower than* 2^z *without being a polynomial ?*

Let f be a transcendental entire function in $\mathbf{C}$. For $R > 0$ set

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if f is not a polynomial
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Pólya's proof starts by expanding the function f into a *Newton interpolation series* at the points $0, 1, 2, \ldots$:

$$f(z) = a_0 + a_1 z + a_2 z(z-1) + a_3 z(z-1)(z-2) + \cdots$$

Since $f(n)$ is an integer for all $n \geq 0$, the coefficients a_n are rational and one can bound the denominators. If f does not grow fast, one deduces that these coefficients vanish for sufficiently large n.

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Newton interpolation series

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$$f(z) = f(\alpha_1) + (z - \alpha_1) f_1(z), \quad f_1(z) = f_1(\alpha_2) + (z - \alpha_2) f_2(z), \ldots$$

we deduce

$$f(z) = a_0 + a_1(z - \alpha_1) + a_2(z - \alpha_1)(z - \alpha_2) + \cdots$$

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An identity due to Ch. Hermite

$$\frac{1}{x-z} = \frac{1}{x-\alpha} + \frac{z-\alpha}{x-\alpha} \cdot \frac{1}{x-z}.$$

Repeat :

$$\frac{1}{x-z} = \frac{1}{x-\alpha_1} + \frac{z-\alpha_1}{x-\alpha_1} \cdot \left(\frac{1}{x-\alpha_2} + \frac{z-\alpha_2}{x-\alpha_2} \cdot \frac{1}{x-z} \right).$$

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An identity due to Ch. Hermite

Inductively we deduce the next formula due to Hermite :

$$\frac{1}{x-z} = \sum_{j=0}^{n-1} \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_j)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})}$$
$$+ \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)} \cdot \frac{1}{x-z}.$$

Newton interpolation expansion

Application. Multiply by $(1/2i\pi)f(z)$ and integrate :

$$f(z) = \sum_{j=0}^{n-1} a_j(z-\alpha_1)\cdots(z-\alpha_j) + R_n(z)$$

with

$$a_j = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{F(x)dx}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})} \quad (0 \le j \le n-1)$$

and

$$R_n(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)\cdot \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{F(x)dx}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)(x-z)}.$$

Integer valued entire function on $\mathbf{Z}[i]$

A.O. Gel'fond (1929) : growth of entire functions mapping the Gaussian integers into themselves.
Newton interpolation series at the points in $\mathbf{Z}[i]$.

An entire function f which is not a polynomial and satisfies $f(a+ib) \in \mathbf{Z}[i]$ for all $a+ib \in \mathbf{Z}[i]$ satisfies

$$\limsup_{R\to\infty} \frac{1}{R^2} \log |f|_R \geq \gamma.$$

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Transcendence of e^π

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If

$$e^\pi = 23,140\,692\,632\,779\,269\,005\,729\,086\,367\ \ldots$$

is rational, then the function $e^{\pi z}$ takes values in $\mathbf{Q}(i)$ when the argument z is in $\mathbf{Z}[i]$.

Expand $e^{\pi z}$ into an interpolation series at the Gaussian integers.

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Hilbert's seventh problem

A.O. Gel'fond and Th. Schneider (1934).
Solution of Hilbert's seventh problem :
transcendence of α^β
and of $(\log \alpha_1)/(\log \alpha_2)$
for algebraic α, β, α_2 *and* α_2.

Dirichlet's box principle

Gel'fond and Schneider
use an *auxiliary function*,
the existence of which follows
from Dirichlet's box principle
(pigeonhole principle,
Thue-Siegel Lemma).

Auxiliary functions

C.L. Siegel (1929) :
Hermite's explicit formulae
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Slope inequalities in Arakelov theory

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matrices and determinants require choices of bases.
Arakelov's Theory produces *slope inequalities* which avoid the need of bases.

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Rational interpolation

René Lagrange (1935).

$$\frac{1}{x-z} = \frac{\alpha-\beta}{(x-\alpha)(x-\beta)} + \frac{x-\beta}{x-\alpha} \cdot \frac{z-\alpha}{z-\beta} \cdot \frac{1}{x-z}.$$

Iterating and integrating yield

$$f(z) = \sum_{n=0}^{N-1} B_n \frac{(z-\alpha_1)\cdots(z-\alpha_n)}{(z-\beta_1)\cdots(z-\beta_n)} + \tilde{R}_N(z).$$

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Hurwitz zeta function

T. Rivoal (2006) : consider Hurwitz zeta function

$$\zeta(s,z)=\sum_{k=1}^{\infty}\frac{1}{(k+z)^s}\cdot$$

Expand $\zeta(2,z)$ as a series in

$$\frac{z^2(z-1)^2\cdots(z-n+1)^2}{(z+1)^2\cdots(z+n)^2}\cdot$$

The coefficients of the expansion belong to $\mathbf{Q}+\mathbf{Q}\zeta(3)$. This produces a new proof of Apéry's Theorem on the irrationality of $\zeta(3)$.

In the same way : new proof of the irrationality of $\log 2$ by expanding

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Mixing C. Hermite and R. Lagrange

T. Rivoal (2006) : new proof of the irrationality of $\zeta(2)$ by expanding

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+z}\right)$$

as a Hermite–Lagrange series in

$$\frac{\bigl(z(z-1)\cdots(z-n+1)\bigr)^2}{(z+1)\cdots(z+n)}.$$

Taylor series and interpolation series

Taylor series are the special case of Hermite's formula with a single point and multiplicities — they give rise to Padé approximants.

Multiplicities can also be introduced in René Lagrange interpolation.

There is another duality between the methods of Gel'fond and Schneider : Fourier-Borel transform.

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Measures : transcendence, linear independence, algebraic independence…

Finite characteristic :

Federico Pellarin - *Aspects de l'indépendance algébrique en caractéristique non nulle [d'après Anderson, Brownawell, Denis, Papanikolas, Thakur, Yu,…]*
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Further develoments

Transcendence and algebraic independence of values of modular functions (*méthode stéphanoise* and work of Yu.V. Nesterenko).

Measures : transcendence, linear independence, algebraic independence...

Finite characteristic :

Federico Pellarin - *Aspects de l'indépendance algébrique en caractéristique non nulle [d'après Anderson, Brownawell, Denis, Papanikolas, Thakur, Yu,...]*
Séminaire Nicolas Bourbaki, Dimanche 18 mars 2007.
http://www.bourbaki.ens.fr/seminaires/2007/Prog_mars.07.html

History of irrational and transcendental numbers

Michel Waldschmidt

http://www.math.jussieu.fr/∼miw/