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Elliptic Functions and Transcendence

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Abstract Transcendental numbers form a fascinating subject: so little is known about the nature of analytic constants that more research is needed in this area. Even when one is interested only in numbers like π and e^π which are related with the classical exponential function, it turns out that elliptic functions are required (so far – this should not last for ever!) to prove transcendence results and get a better understanding of the situation.

First we briefly recall some of the basic transcendence results related with the exponential function (section 1). Next, in section 2, we survey the main properties of elliptic functions that are involved in transcendence theory.

We survey transcendence theory of values of elliptic functions in section 3, linear independence in section 4, and algebraic independence in section 5. This splitting is somewhat artificial but convenient. Moreover, we restrict ourselves to elliptic functions, even when many results are only special cases of statements valid for abelian functions. A number of related topics are not considered here (e.g. heights, p-adic theory, theta functions, diophantine geometry on elliptic curves...).

Keywords Transcendental numbers, elliptic functions, elliptic curves, elliptic integrals, algebraic independence, transcendence measures, measures of algebraic independence, Diophantine approximation.

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1 Exponential Function and Transcendence

We start with a very brief list of some of the main transcendence results concerning numbers related with the exponential function. References are, for instance, [200, 86, 196, 115, 181, 12, 222, 80].

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Next, we point out some properties of the exponential function, the elliptic analog of which we shall consider later (§ 2.1).

1.1 Short survey on the transcendence of numbers related with the exponential function

1.1.1 Hermite, Lindemann and Weierstraß

The first transcendence result for a number related with the exponential function is Hermite's Theorem on the transcendence of e.

Theorem 1 (Hermite, 1873) *The number e is transcendental.*

This means that for any non-zero polynomial $P \in \mathbb{Z}[X]$, the number $P(e)$ is not zero. We denote by $\overline{\mathbb{Q}}$ the set of algebraic numbers. Hence Hermite's Theorem can be written: $e \notin \overline{\mathbb{Q}}$. A complex number is called *transcendental* if it is transcendental over $\mathbb{Q}$, or over $\overline{\mathbb{Q}}$, which is the same. Also we shall say that complex numbers $\theta_1, \ldots, \theta_n$ are *algebraically independent* if they are algebraically independent over $\mathbb{Q}$, which is the same as over $\overline{\mathbb{Q}}$: for any non-zero polynomial P in n variables (and coefficients in $\mathbb{Z}$, $\mathbb{Q}$ or $\overline{\mathbb{Q}}$), the number $P(\theta_1, \ldots, \theta_n)$ is not zero.

The second result in chronological order is Lindemann's Theorem on the transcendence of π.

Theorem 2 (Lindemann, 1882) *The number π is transcendental.*

The next result contains the transcendence of both numbers e and π:

Theorem 3 (Hermite-Lindemann, 1882) *For $\alpha \in \overline{\mathbb{Q}}^\times$, any non-zero logarithm $\log \alpha$ of α is transcendental.*

We denote by $\mathcal{L}$ the $\mathbb{Q}$-vector space of logarithms of algebraic numbers:

$$\mathcal{L} = \{\log \alpha \,;\, \alpha \in \overline{\mathbb{Q}}^\times\} = \{\ell \in \mathbb{C} \,;\, e^\ell \in \overline{\mathbb{Q}}^\times\} = \exp^{-1}(\overline{\mathbb{Q}}^\times).$$

Hence Theorem 3 means $\mathcal{L} \cap \overline{\mathbb{Q}} = \{0\}$. An alternate form is:

Theorem 4 (Hermite-Lindemann, 1882) *For any $\beta \in \overline{\mathbb{Q}}^\times$, the number e^β is transcendental.*

The first result of algebraic independence for the values of the exponential function goes back to the end of XIXth century.

Theorem 5 (Lindemann-Weierstraß, 1885) *Let $\beta_1, \ldots, \beta_n$ be $\mathbb{Q}$-linearly independent algebraic numbers. Then the numbers $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent.*

Again, there is an alternate form of Theorem 5: it amounts to a statement of linear independence.

Theorem 6 (Lindemann-Weierstraß, 1885) *Let $\gamma_1, \ldots, \gamma_m$ be distinct algebraic numbers. Then the numbers $e^{\gamma_1}, \ldots, e^{\gamma_m}$ are linearly independent over $\overline{\mathbb{Q}}$.*

It is not difficult to check that Theorem 6 is equivalent to Theorem 5 with the conclusion that $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent over $\overline{\mathbb{Q}}$; since it is equivalent to saying that $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent over $\mathbb{Q}$, one does not lose anything if one changes the conclusion of Theorem 6 by stating that the numbers $e^{\gamma_1}, \ldots, e^{\gamma_m}$ are linearly independent over $\mathbb{Q}$.

1.1.2 Hilbert's seventh problem, Gel'fond and Schneider

The solution of Hilbert's seventh problem on the transcendence of α^β was obtained by Gel'fond and Schneider in 1934 (see [86, 196]).

Theorem 7 (Gel'fond-Schneider, 1934) *For α and β algebraic numbers with $\alpha \neq 0$ and $\beta \notin \mathbb{Q}$ and for any choice of $\log \alpha \neq 0$, the number $\alpha^\beta = \exp(\beta \log \alpha)$ is transcendental.*

This means that the two algebraically independent functions e^z and $e^{\beta z}$ cannot take algebraic values at the points $\log \alpha$ (A.O. Gel'fond) and also that the two algebraically independent functions z and $\alpha^z = e^{z \log \alpha}$ cannot take algebraic values at the points $m + n\beta$ with $(m, n) \in \mathbb{Z}^2$ (Th. Schneider).

Examples (quoted by D. Hilbert in 1900) of numbers whose transcendence follows from Theorem 7 are $2^{\sqrt{2}}$ and e^π (recall $e^{i\pi} = -1$). The transcendence of e^π had already been proved in 1929 by A.O. Gel'fond.

Here is an equivalent statement to Gel'fond-Schneider Theorem 7:

Theorem 8 (Gel'fond-Schneider, 1934) *Let $\log \alpha_1, \log \alpha_2$ be two non-zero logarithms of algebraic numbers. Assume that the quotient $(\log \alpha_1)/(\log \alpha_2)$ is irrational. Then this quotient is transcendental.*

1.1.3 Linear independence of logarithms of algebraic numbers

The generalization of Theorem 8 to more than two logarithms, conjectured by A.O. Gel'fond [86], was proved by A. Baker in 1966. His results include not only Gel'fond-Scheider's Theorem 8 but also Hermite-Lindemann's Theorem 3.

Theorem 9 (Baker, 1966) *Let $\log \alpha_1, \ldots, \log \alpha_n$ be $\mathbb{Q}$–linearly independent logarithms of algebraic numbers. Then the numbers $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the field $\overline{\mathbb{Q}}$.*

1.1.4 The Six Exponentials Theorem and the Four Exponentials Conjecture

The next result, which does not follow from any of the previously mentioned results, was proved independently in the 1940's by C.L. Siegel (unpublished) and in the 1960's by S. Lang and K. Ramachandra (see [115, 180, 227]; see also Problem 1 in [196] for the Four Exponentials Conjecture). As suggested by K. Ramachandra (see [181] Chap. III § 1 Th. 2), Theorem 10 also follows from Schneider's criterion proved in 1949 [195].

Theorem 10 (Six Exponentials Theorem) *Let $x_1, \ldots, x_d$ be $\mathbb{Q}$-linearly independent complex numbers and let $y_1, \ldots, y_\ell$ be $\mathbb{Q}$-linearly independent complex numbers. Assume $\ell d > \ell + d$. Then at least one of the ℓd numbers*

$$e^{x_i y_j}, \qquad (1 \leq i \leq d,\ 1 \leq j \leq \ell)$$

is transcendental.

Notice that the condition $\ell d > \ell + d$ can be written ($\ell \geq 2$ and $d \geq 3$) or ($\ell \geq 3$ and $d \geq 2$); it suffices to consider the case $\ell d = 6$ (hence the name of the result). Therefore, Theorem 10 can be stated in an equivalent form:

Theorem 11 (Six Exponentials Theorem - logarithmic form) *Let*

$$M = \begin{pmatrix} \log \alpha_1 & \log \alpha_2 & \log \alpha_3 \\ \log \beta_1 & \log \beta_2 & \log \beta_3 \end{pmatrix}$$

be a 2 by 3 matrix whose entries are logarithms of algebraic numbers. Assume that the three columns are linearly independent over $\mathbb{Q}$ and the two rows are also linearly independent over $\mathbb{Q}$. Then the matrix M has rank 2.

It is expected that the condition $d\ell > d + \ell$ in Theorem 10 is too restrictive and that the same conclusion holds in case $d = \ell = 2$. We state this conjecture in the logarithmic form:

Conjecture 12 (Four Exponentials conjecture - logarithmic form) Let

$$M = \begin{pmatrix} \log \alpha_1 & \log \alpha_2 \\ \log \beta_1 & \log \beta_2 \end{pmatrix}$$

be a 2 by 2 matrix whose entries are logarithms of algebraic numbers. Assume that the two columns are linearly independent over $\mathbb{Q}$ and that the two rows are also linearly independent over $\mathbb{Q}$. Then the matrix M has rank 2.

1.1.5 Algebraic independence

In 1948 and 1949, A.O. Gel'fond extended his solution of Hilbert's seventh problem to a result of algebraic independence [86]. One of his theorems is that the two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent. His general statements can be seen as extensions of Theorem 10 into a result of algebraic independence (in spite of the fact that the Six Exponentials Theorem 10 was stated and proved only several years later). In his original work, Gel'fond needed a stronger assumption, namely a measure of linear independence of the x_i's as well as of the y_j's. This assumption was removed in the early 1970's by R. Tijdeman [206] (further references, especially to papers by A.A. Smelev and W.D. Brownawell, are given in [219]; see also [33, 224, 225, 168, 231]).

Theorem 13 *Let $x_1, \ldots, x_d$ be $\mathbb{Q}$-linearly independent complex numbers and let $y_1, \ldots, y_\ell$ be $\mathbb{Q}$-linearly independent complex numbers.*
1. If $d\ell \geq 2(d + \ell)$, then at least two of the $d\ell$ numbers

$$e^{x_i y_j} \qquad (1 \leq i \leq d,\ 1 \leq j \leq \ell)$$

are algebraically independent.
2. If $d\ell \geq d + 2\ell$, then at least two of the $d\ell + d$ numbers

$$x_i,\ e^{x_i y_j} \qquad (1 \leq i \leq d,\ 1 \leq j \leq \ell)$$

are algebraically independent.
3. If $d\ell > d + \ell$, then at least two of the $d\ell + d + \ell$ numbers

$$x_i,\ y_j,\ e^{x_i y_j} \qquad (1 \leq i \leq d,\ 1 \leq j \leq \ell)$$

are algebraically independent.
4. If $d = \ell = 2$ and if the two numbers $e^{x_1 y_1}$ and $e^{x_1 y_2}$ are algebraic, then at least two of the 6 numbers

$$x_1,\ x_2,\ \ y_1,\ y_2,\ \ e^{x_2 y_1},\ \ e^{x_2 y_2}$$

are algebraically independent.

From the last part of Theorem 13, taking $x_1 = y_1 = i\pi$ and $x_2 = y_2 = 1$, one deduces that at least one of the two following statements is true:
(i) *The number e^{π^2} is transcendental.*
(ii) *The two numbers e and π are algebraically independent.*

One expects that both statements are true.

If it were possible to prove that, under the assumptions of Theorem 13, at least two of the 8 numbers

$$x_1,\ x_2,\ \ y_1,\ y_2,\ \ e^{x_1 y_1},\ \ e^{x_1 y_2},\ \ e^{x_2 y_1},\ \ e^{x_2 y_2}$$

are algebraically independent, one would deduce the algebraic independence of the two numbers π and e^π (take $x_1 = 1$, $x_2 = i$, $y_1 = \pi$, $y_2 = i\pi$; see Corollary 48 below).

For results concerning *large transcendence degree*, see § 5.3 below.

1.2 The exponential function

The exponential function

$$\begin{aligned} \exp : \mathbb{C} &\to \mathbb{C}^\times \\ z &\mapsto e^z \end{aligned}$$

satisfies both a differential equation and an addition formula:

$$\frac{d}{dz}e^z = e^z, \qquad e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

It is a homomorphism of the additive group $\mathbb{C}$ of complex numbers onto the multiplicative group $\mathbb{C}^\times$ of non-zero complex numbers, with kernel

$$\ker \exp = 2i\pi\mathbb{Z},$$

hence it yields an isomorphism between the quotient additive group $\mathbb{C}/2i\pi\mathbb{Z}$ and the multiplicative group $\mathbb{C}^\times$.

The group $\mathbb{C}^\times$ is the group of complex points of the multiplicative group $\mathbb{G}_m$; $z \mapsto e^z$ is the exponential function of the multiplicative group $\mathbb{G}_m$. We shall replace this algebraic group by an elliptic curve. We could replace it also by other commutative algebraic groups. As a first example, the exponential function of the additive group $\mathbb{G}_a$ is

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z \end{aligned}$$

More general examples are commutative linear algebraic groups; over an algebraically closed field, these are nothing else than products of several copies of the additive and multiplicative group. Further examples of algebraic groups are abelian varieties. In full generality, algebraic groups are extensions of abelian varieties by commutative linear algebraic groups. See, for instance, [115, 222, 148].

2 Elliptic curves and elliptic functions

Among many references for this section are the books by S. Lang [121], K. Chandrasekharan [41] and J. Silverman [201, 202]. See also the book by M. Hindry and J. Silverman [95].

2.1 Basic concepts

An elliptic curve may be defined as

- $y^2 = C(x)$ for a squarefree cubic polynomial $C(x)$,
- a connected compact Lie group of dimension 1,
- a complex torus $\mathbb{C}/\Omega$ where Ω is a lattice in $\mathbb{C}$,
- a Riemann surface of genus 1,
- a non-singular cubic in $\mathbb{P}_2(\mathbb{C})$,
- an algebraic group of dimension 1, with underlying projective algebraic variety.

We shall use the Weierstraß form

$$E = \left\{(t : x : y)\ ;\ y^2t = 4x^3 - g_2xt^2 - g_3t^3\right\} \subset \mathbb{P}_2.$$

Here g_2 and g_3 are complex numbers, with the only assumption $g_2^3 \neq 27g_3^2$, which means that the discriminant of the polynomial $4X^3 - g_2X - g_3$ does not vanish.

An analytic parametrization of the complex points $E(\mathbb{C})$ of E is given by means of *Weierstraß elliptic function* $\wp$, which satisfies the differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3. \tag{1}$$

It has a double pole at the origin with principal part $1/z^2$ and also satisfies an addition formula

$$\wp(z_1+z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\cdot\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2. \tag{2}$$

The exponential map of the Lie group $E(\mathbb{C})$ is

$$\begin{aligned}\exp_E : \mathbb{C} &\to E(\mathbb{C})\\ z &\mapsto \big(1 : \wp(z) : \wp'(z)\big).\end{aligned}$$

The kernel of this map is a *lattice* in $\mathbb{C}$ (that is a discrete rank 2 subgroup),

$$\Omega = \ker\exp_E = \{\omega\in\mathbb{C}\ ;\ \wp(z+\omega)=\wp(z)\} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

Hence $\exp_E$ induces an isomorphism between the quotient additive group $\mathbb{C}/\Omega$ and $E(\mathbb{C})$ with the law given by (2). The elements of Ω are the *periods* of $\wp$. A pair (ω_1,ω_2) of fundamental periods is given by

$$\omega_i = 2\int_{e_i}^{\infty}\frac{dx}{\sqrt{4x^3-g_2x-g_3}}, \qquad (i=1,2),$$

where

$$4x^3 - g_2x - g_3 = 4(x-e_1)(x-e_2)(x-e_3).$$

Indeed, since $\wp'$ is periodic and odd, it vanishes at $\omega_1/2$, $\omega_2/2$ and $(\omega_1+\omega_2)/2$, hence the values of $\wp$ at these points are the three distinct complex numbers e_1, e_2 and e_3 (recall that the discriminant of $4x^3-g_2x-g_3$ is not 0).

Conversely, given a lattice Ω, there is a unique Weierstraß elliptic function $\wp_\Omega$ whose period lattice is Ω (see § 2.5). We denote its invariants in the differential equation (1) by $g_2(\Omega)$ and $g_3(\Omega)$.

We shall be interested mainly (but not only) with elliptic curves which are defined over the field of algebraic numbers: they have a Weierstraß equation with algebraic g_2 and g_3. However we shall also use the Weierstraß elliptic function associated with the lattice $\lambda\Omega$ where $\lambda\in\mathbb{C}^\times$ may be transcendental; the relations are

$$\wp_{\lambda\Omega}(\lambda z) = \lambda^{-2}\wp_\Omega(z), \qquad g_2(\lambda\Omega)=\lambda^{-4}g_2(\Omega), \qquad g_3(\lambda\Omega)=\lambda^{-6}g_3(\Omega). \tag{3}$$

The lattice $\Omega=\mathbb{Z}+\mathbb{Z}\tau$, where τ is a complex number with positive imaginary part, satisfies

$$g_2(\mathbb{Z}+\mathbb{Z}\tau) = 60G_2(\tau) \quad\text{and}\quad g_3(\mathbb{Z}+\mathbb{Z}\tau) = 140G_3(\tau),$$

where $G_k(\tau)$ are the Eisenstein series (see, for instance, p. 240 of [242], [197] Chap. VII, § 2.3 and [63])

$$G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}(m+n\tau)^{-2k}. \tag{4}$$

2.2 Morphisms between elliptic curves. The modular invariant

If Ω and Ω' are two lattices in $\mathbb{C}$ and if $f : \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$ is an analytic homomorphism, then the map $\mathbb{C} \to \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$ factors through a homothecy $\mathbb{C} \to \mathbb{C}$ given by some $\lambda \in \mathbb{C}$ such that $\lambda\Omega \subset \Omega'$:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/\Omega & \xrightarrow[f]{} & \mathbb{C}/\Omega' \end{array}$$

If $f \neq 0$, then $\lambda \in \mathbb{C}^\times$ and f is surjective.

Conversely, if there exists $\lambda \in \mathbb{C}$ such that $\lambda\Omega \subset \Omega'$, then $f_\lambda(x+\Omega) = \lambda x + \Omega'$ defines an analytic surjective homomorphism $f_\lambda : \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$. In this case $\lambda\Omega$ is a subgroup of finite index in Ω', hence the kernel of f_λ is finite and there exists $\mu \in \mathbb{C}^\times$ with $\mu\Omega' \subset \Omega$: the two elliptic curves $\mathbb{C}/\Omega$ and $\mathbb{C}/\Omega'$ are *isogeneous*.

If Ω and Ω^* are two lattices, $\wp$ and $\wp^*$ the associated Weierstraß elliptic functions and g_2, g_3 the invariants of $\wp$, the following statements are equivalent:
(i) There is a 2×2 matrix with rational coefficients which maps a basis of Ω to a basis of Ω^*.
(ii) There exists $\lambda \in \mathbb{Q}^\times$ such that $\lambda\Omega \subset \Omega^*$.
(iii) There exists $\lambda \in \mathbb{Z} \setminus \{0\}$ such that $\lambda\Omega \subset \Omega^*$.
(iv) The two functions $\wp$ and $\wp^*$ are algebraically dependent over the field $\mathbb{Q}(g_2, g_3)$.
(v) The two functions $\wp$ and $\wp^*$ are algebraically dependent over $\mathbb{C}$.

The map f_λ is an isomorphism if and only if $\lambda\Omega = \Omega'$.

The number

$$j = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}$$

is the *modular invariant* of the elliptic curve E. Two elliptic curves over $\mathbb{C}$ are isomorphic if and only if they have the same modular invariant.

Set $\tau = \omega_2/\omega_1$, $q = e^{2i\pi\tau}$ and $J(e^{2i\pi\tau}) = j(\tau)$. Then (see [132] § 4.12, [197] Chap. VII)

$$\begin{aligned} J(q) &= q^{-1}\left(1 + 240\sum_{m=1}^{\infty} m^3 \frac{q^m}{1-q^m}\right)^3 \prod_{n=1}^{\infty}(1-q^n)^{-24} \\ &= \frac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + \cdots \end{aligned}$$

2.3 Endomorphisms of an elliptic curve; complex multiplications

Let Ω be a lattice in $\mathbb{C}$. The set of analytic endomorphisms of $\mathbb{C}/\Omega$ is the subring

$$\mathrm{End}(\mathbb{C}/\Omega) = \left\{ f_\lambda \ ; \ \lambda \in \mathbb{C} \ \text{ with } \ \lambda\Omega \subset \Omega \right\}$$

of $\mathbb{C}$. We also call it the ring of endomorphisms of the associated elliptic curve, or of the corresponding Weierstraß $\wp$ function and we identify it with the subring

$$\left\{ \lambda \in \mathbb{C} \ ; \ \lambda\Omega \subset \Omega \right\}$$

of $\mathbb{C}$. The *field of endomorphisms* is the quotient field $\mathrm{End}(\mathbb{C}/\Omega) \otimes_{\mathbb{Z}} \mathbb{Q}$ of this ring.

If $\lambda \in \mathbb{C}$ satisfies $\lambda\Omega \subset \Omega$, then λ is either a rational integer or else an algebraic integer in an imaginary quadratic field. For such a λ, $\wp_\Omega(\lambda z)$ is a rational function of $\wp_\Omega(z)$; the degree of the numerator is λ^2 if $\lambda \in \mathbb{Z}$ and $N(\lambda)$ otherwise (here, N is the norm of the imaginary quadratic field); the degree of the denominator is $\lambda^2 - 1$ if $\lambda \in \mathbb{Z}$ and $N(\lambda) - 1$ otherwise.

Let E be the elliptic curve attached to the Weierstraß $\wp$ function. The ring of endomorphisms $\mathrm{End}(E)$ of E is either $\mathbb{Z}$ or else an order in an imaginary quadratic field k. The latter case arises if

and only if the quotient $\tau = \omega_2/\omega_1$ of a pair of fundamental periods is a quadratic number; in this case the field of endomorphisms of E is $k = \mathbb{Q}(\tau)$ and the curve E has *complex multiplications*. This means also that the two functions $\wp(z)$ and $\wp(\tau z)$ are algebraically dependent. In this case, the value $j(\tau)$ of the modular invariant j is an algebraic integer whose degree is the class number of the quadratic field $k = \mathbb{Q}(\tau)$.

Remark 14 From Theorem 7 one deduces the transcendence of the number

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743.999\,999\,999\,999\,2\ldots.$$

If we set

$$\tau = \frac{1+i\sqrt{163}}{2}, \quad q = e^{2i\pi\tau} = -e^{-\pi\sqrt{163}}$$

then the class number of the imaginary quadratic field $\mathbb{Q}(\tau)$ is 1, we have $j(\tau) = -(640\,320)^3$ and

$$\left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}.$$

Let $\wp$ be a Weierstraß elliptic function with field of endomorphisms k. Hence $k = \mathbb{Q}$ if the associated elliptic curve has no complex multiplication, while in the other case k is an imaginary quadratic field, namely $k = \mathbb{Q}(\tau)$, where τ is the quotient of two linearly independent periods of $\wp$. Let $u_1, \ldots, u_d$ be non-zero complex numbers. Then the functions $\wp(u_1 z), \ldots, \wp(u_d z)$ are algebraically independent (over $\mathbb{C}$ or over $\mathbb{Q}(g_2, g_3)$, this is equivalent) if and only if the numbers $u_1, \ldots, u_d$ are linearly independent over k. This generalizes the fact that $\wp(z)$ and $\wp(\tau z)$ are algebraically dependent if and only if the elliptic curve has complex multiplications. Much more general and deeper results of algebraic independence of functions (exponential and elliptic functions, zeta functions...) was proved by W.D. Brownawell and K.K. Kubota [35].

If $\wp$ is a Weierstraß elliptic function with algebraic invariants g_2 and g_3, if E is the associated elliptic curve and if k denotes its field of endomorphisms, then the set

$$\mathcal{L}_E = \Omega \cup \left\{ u \in \mathbb{C} \setminus \Omega \; ; \; \wp(u) \in \overline{\mathbb{Q}} \right\}$$

is a k-vector subspace of $\mathbb{C}$: this is the set of *elliptic logarithms of algebraic points on* E. It plays a role with respect to E similar to the role of $\mathcal{L}$ for the multiplicative group $\mathbb{G}_m$.

Let $k = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field with class number $h(d) = h$. There are h non-isomorphic elliptic curves $E_1, \ldots, E_h$ with ring of endomorphisms the ring of integers of k. The numbers $j(E_i)$ are conjugate algebraic integers of degree h; each of them generates the Hilbert class field H of k (maximal unramified abelian extension of k). The Galois group of H/k is isomorphic to the ideal class group of the ring of integers of k.

Since the group of roots of units of an imaginary quadratic field is $\{-1, +1\}$ except for $\mathbb{Q}(i)$ and $\mathbb{Q}(\varrho)$, where $\varrho = e^{2i\pi/3}$, it follows that there are exactly two elliptic curves over $\mathbb{Q}$ (up to isomorphism) having an automorphism group bigger than $\{-1, +1\}$. They correspond to Weierstraß elliptic functions $\wp$ for which there exists a complex number $\lambda \neq \pm 1$ with $\lambda^2 \wp(\lambda z) = \wp(z)$.

The first one has $g_3 = 0$ and $j = 1728$. A pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4xt^2.$$

is given by (see, for instance, Appendice 1 of [230])

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}} = \frac{1}{2} B(1/4, 1/2) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}} \quad \text{and} \quad \omega_2 = i\omega_1. \tag{5}$$

The lattice $\mathbb{Z}[i]$ has $g_2 = 4\omega_1^4$, thus

$$\sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} (m+ni)^{-4} = \frac{\Gamma(1/4)^8}{2^6 \cdot 3 \cdot 5 \cdot \pi^2}.$$

The second one has $g_2 = 0$ and $j = 0$. A pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4t^3.$$

is

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3-1}} = \frac{1}{3}B(1/6,1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} \quad \text{and} \quad \omega_2 = \varrho\omega_1. \tag{6}$$

The lattice $\mathbb{Z}[\varrho]$ has $g_3 = 4\omega_1^6$, thus

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m+n\varrho)^{-6} = \frac{\Gamma(1/3)^{18}}{2^8\pi^6}.$$

These two examples involve special values of Euler's Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t}t^z \cdot \frac{dt}{t} = e^{-\gamma z}z^{-1}\prod_{n=1}^\infty \left(1+\frac{z}{n}\right)^{-1} e^{z/n}, \tag{7}$$

while Euler's Beta function is

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

More generally, the formula of Chowla and Selberg (1966) [42] (see also [91,112,111,92,223] for related results) expresses periods of elliptic curves with complex multiplications as products of Gamma values: *if k is an imaginary quadratic field and $\mathcal{O}$ an order in k, if E is an elliptic curve with complex multiplications by $\mathcal{O}$, then the corresponding lattice Ω determines a vector space $\Omega \otimes_{\mathbb{Z}} \mathbb{Q}$ which is invariant under the action of k and thus has the form $k \cdot \omega$ for some $\omega \in \mathbb{C}^\times$ defined up to elements in $k^\times$. In particular, if $\mathcal{O}$ is the ring of integers $\mathbb{Z}_k$ of k, then*

$$\omega = \alpha\sqrt{\pi}\prod_{\substack{0<a<d\\(a,d)=1}} \Gamma(a/d)^{w\epsilon(a)/4h},$$

where α is a non-zero algebraic number, w is the number of roots of unity in k, h is the class number of k, ϵ is the Dirichlet character modulo the discriminant d of k.

2.4 Standard relations among Gamma values

Euler's Gamma function satisfies the following relations:
(Translation)

$$\Gamma(z+1) = z\Gamma(z);$$

(Reflection)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)};$$

(Multiplication) For any positive integer n,

$$\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right) = (2\pi)^{(n-1)/2}n^{-nz+(1/2)}\Gamma(nz).$$

D. Rohrlich conjectured that any multiplicative relation among Gamma values is a consequence of these standard relations, while (see, for instance, [232]) S. Lang was more optimistic.

Conjecture 15 (D. Rohrlich) Any multiplicative relation

$$\pi^{b/2}\prod_{a\in\mathbb{Q}}\Gamma(a)^{m_a}\in\overline{\mathbb{Q}}$$

with b and m_a in $\mathbb{Z}$ is a consequence of the standard relations.

Conjecture 16 (S. Lang) Any algebraic dependence relation among the numbers $(2\pi)^{-1/2}\Gamma(a)$ with $a\in\mathbb{Q}$ is in the ideal generated by the standard relations.

2.5 Quasi-periods of elliptic curves and elliptic integrals of the second kind

Let $\Omega=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. The *Weierstraß canonical product* attached to this lattice is the entire function σ_Ω defined by

$$\sigma_\Omega(z)=z\prod_{\omega\in\Omega\setminus\{0\}}\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{z^2}{2\omega^2}}.$$

It has a simple zero at any point of Ω.

Hence the Weierstraß sigma function plays, for the lattice Ω, the role of the function

$$z\prod_{n\geq 0}\left(1-\frac{z}{n}\right)e^{z/n}=-e^{\gamma z}\Gamma(-z)^{-1}$$

for the set of non-negative integers $\mathbb{N}=\{0,1,2,\dots\}$ (see the infinite product (7) for Euler's Gamma function), and also the role of the function

$$\pi^{-1}\sin(\pi z)=z\prod_{n\in\mathbb{Z}\setminus\{0\}}\left(1-\frac{z}{n}\right)e^{z/n}$$

for the set $\mathbb{Z}$ of rational integers.

The Weierstraß sigma function σ associated with a lattice in $\mathbb{C}$ is an entire function of *order* 2:

$$\limsup_{r\to\infty}\frac{1}{\log r}\cdot\log\log\sup_{|z|=r}|\sigma(z)|=2;$$

the product $\sigma^2\wp$ is also an entire function of order 2 (this can be checked by using infinite products, but it is easier to use the quasi-periodicity of σ, see formula (8) below).

The logarithmic derivative of the sigma function is *Weierstraß zeta function* $\zeta=\sigma'/\sigma$ whose Laurent expansion at the origin is

$$\zeta(z)=\frac{1}{z}+\sum_{k\geq 2}s_k z^{2k-1},$$

where, for $k\in\mathbb{Z}$, $k\geq 2$,

$$s_k=s_k(\Omega)=\sum_{\substack{\omega\in\Omega\\ \omega\neq 0}}\omega^{-2k}=\omega_2^{-2k}G_k(\tau)$$

(recall (4)).

The derivative of ζ is $\wp$. From

$$\wp''=6\wp^2-(g_2/2)$$

one deduces that $s_k(\Omega)$ is a homogenous polynomial in $\mathbb{Q}[g_2,g_3]$ of weight $2k$ for the graduation of $\mathbb{Q}[g_2,g_3]$ determined by assigning to g_2 the degree 4 and to g_3 the degree 6.

As a side remark, we notice that for any $u \in \mathbb{C} \setminus \Omega$ we have

$$\mathbb{Q}(g_2, g_3) \subset \mathbb{Q}\big(\wp(u), \wp'(u), \wp''(u)\big).$$

Since its derivative is periodic, the function ζ is *quasi-periodic*: for each $\omega \in \Omega$ there is a complex number $\eta = \eta(\omega)$ such that

$$\zeta(z + \omega) = \zeta(z) + \eta.$$

These numbers η are the *quasi-periods* of the elliptic curve. If (ω_1, ω_2) is a pair of fundamental periods and if we set $\eta_1 = \eta(\omega_1)$ and $\eta_2 = \eta(\omega_2)$, then, for $(a, b) \in \mathbb{Z}^2$,

$$\eta(a\omega_1 + b\omega_2) = a\eta_1 + b\eta_2.$$

Coming back to the sigma function, one deduces that

$$\sigma(z + \omega_i) = -\sigma(z) \exp\Big(\eta_i\big(z + (\omega_i/2)\big)\Big) \qquad (i = 1, 2). \tag{8}$$

The zeta function also satisfies an addition formula:

$$\zeta(z_1 + z_2) = \zeta(z_1) + \zeta(z_2) + \frac{1}{2} \cdot \frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}.$$

The Legendre relation relating the periods and the quasi-periods

$$\omega_2\eta_1 - \omega_1\eta_2 = \pm 2i\pi$$

can be obtained by integrating $\zeta(z)$ along the boundary of a fundamental parallelogram.

In the case of complex multiplication, if τ is the quotient of a pair of fundamental periods of $\wp$, then the function $\zeta(\tau z)$ is algebraic over the field $\mathbb{Q}\big(g_2, g_3, z, \wp(z), \zeta(z)\big)$.

Examples. For the curve $y^2t = 4x^3 - 4xt^2$ the quasi-periods attached to the above mentioned pair of fundamental periods (5) are

$$\eta_1 = \frac{\pi}{\omega_1} = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}, \qquad \eta_2 = -i\eta_1; \tag{9}$$

it follows that the fields $\mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2)$ and $\mathbb{Q}\big(\pi, \Gamma(1/4)\big)$ have the same algebraic closure over $\mathbb{Q}$, hence the same transcendence degree. For the curve $y^2t = 4x^3 - 4t^3$ with periods (6) they are

$$\eta_1 = \frac{2\pi}{\sqrt{3}\omega_1} = \frac{2^{7/3}\pi^2}{3^{1/2}\Gamma(1/3)^3}, \qquad \eta_2 = \varrho^2\eta_1. \tag{10}$$

In this case the fields $\mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2)$ and $\mathbb{Q}\big(\pi, \Gamma(1/3)\big)$ have the same algebraic closure over $\mathbb{Q}$, hence the same transcendence degree.

2.6 Elliptic integrals

Let

$$\mathcal{E} = \{(t : x : y) \in \mathbb{P}_2; y^2 t = 4x^3 - g_2 x t^2 - g_3 t^3\}$$

be an elliptic curve. The field of rational (meromorphic) functions on $\mathcal{E}$ over $\mathbb{C}$ is $\mathbb{C}(\mathcal{E}) = \mathbb{C}(\wp, \wp') = \mathbb{C}(x, y)$ where x and y are related by the cubic equation $y^2 = 4x^3 - g_2 x - g_3$. Under the isomorphism $\mathbb{C}/\Omega \to \mathcal{E}(\mathbb{C})$ given by $(1 : \wp : \wp')$, the differential form dz is mapped to dx/y. The holomorphic differential forms on $\mathbb{C}/\Omega$ are λdz with $\lambda \in \mathbb{C}$.

The differential form $d\zeta = \zeta'/\zeta$ is mapped to $-xdx/y$. The differential forms of second kind on $\mathcal{E}(\mathbb{C})$ are $adz + bd\zeta + d\chi$, where a and b are complex numbers and $\chi \in \mathbb{C}(x, y)$ is a meromorphic function on $\mathcal{E}$.

Assume that the elliptic curve $\mathcal{E}$ is defined over $\overline{\mathbb{Q}}$: the invariants g_2 and g_3 are algebraic. We shall be interested with differential forms which are defined over $\overline{\mathbb{Q}}$. Those of second kind are $adz + bd\zeta + d\chi$, where a and b are algebraic numbers and $\chi \in \overline{\mathbb{Q}}(x, y)$.

An elliptic integral ([233] Chap. 22, § 7 and [201] Chap. VI, § 1) is an integral

$$\int R(x, y)dx$$

where R is a rational function of x and y, while y^2 is a polynomial in x of degree 3 or 4 without multiple roots. One may transform this integral as follows: one reduces it to an integral of $dx/\sqrt{P(x)}$ where P is a polynomial of 3rd or 4th degree; in case P has degree 4 one replaces it with a degree 3 polynomial by sending one root to infinity; finally one reduces it to a Weierstraß equation by means of a birational transformation. The value of the integral is not modified.

For transcendence purposes, if the initial differential form is defined over $\overline{\mathbb{Q}}$, then all these transformations involve only algebraic numbers.

3 Transcendence results of numbers related with elliptic functions

3.1 Elliptic analog of Lindemann's Theorem on the transcendence of π and of Hermite-Lindemann Theorem on the transcendence of $\log \alpha$.

The first transcendence result on periods of elliptic functions was proved by C.L. Siegel [199] as early as 1932.

Theorem 17 (Siegel, 1932) *Let $\wp$ be a Weierstraß elliptic function with period lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Assume that the invariants g_2 and g_3 of $\wp$ are algebraic. Then at least one of the two numbers ω_1, ω_2 is transcendental.*

One main feature of Siegel's proof is that he used Dirichlet's box principle (the so-called Thue-Siegel Lemma which is included in his 1929 paper) to construct an auxiliary function. This idea turned out to be of fundamental importance for the solution of Hilbert's seventh problem by Gel'fond and Schneider two years later.

In the case of complex multiplication, it follows from Theorem 17 that *any non-zero period of $\wp$ is transcendental.*

From formulae (5) and (6) it follows as a consequence of Siegel's 1932 result [199] that both numbers $\Gamma(1/4)^4/\pi$ and $\Gamma(1/3)^3/\pi$ are transcendental.

Other consequences of Siegel's result concern the transcendence of the length of an arc of an ellipse [200, 196]:

$$2\int_{-b}^{b} \sqrt{1 + \frac{a^2x^2}{b^4 - b^2x^2}}\, dx$$

for algebraic a and b, as well as the transcendence of an arc of the lemniscate $(x^2+y^2)^2 = 2a^2(x^2-y^2)$ with a algebraic.

A further example of application of Siegel's Theorem [200] is the transcendence of values of hypergeometric series related with elliptic integrals

$$\begin{aligned} K(z) &= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-z^2x^2)}} \\ &= \frac{\pi}{2} \cdot {}_2F_1\left(1/2,\ 1/2\ ;\ 1 \,\middle|\, z^2\right), \end{aligned}$$

where ${}_2F_1$ denotes Gauss hypergeometric series

$${}_2F_1\left(a,\ b\ ;\ c \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

with $(a)_n = a(a+1)\cdots(a+n-1)$.

Further results on this topic were obtained by Th. Schneider [192] in 1934 and in a joint work by K. Mahler and J. Popken [179] in 1935 using Siegel's method. These results were superseded by Th. Schneider's fundamental memoir [193] in 1936 where he proved a number of definitive results on the subject, including:

Theorem 18 (Schneider, 1936) *Assume that the invariants g_2 and g_3 of $\wp$ are algebraic. Then for any non-zero period ω of $\wp$, the numbers ω and $\eta(\omega)$ are transcendental.*

It follows from Theorem 18 that any non-zero period of an elliptic integral of the first or second kind is transcendental:

Corollary 19 *Let $\mathcal{E}$ be an elliptic curve over $\overline{\mathbb{Q}}$, p_1 and p_2 two algebraic points on $\mathcal{E}(\overline{\mathbb{Q}})$, w a differential form of first or second kind on $\mathcal{E}$ which is defined over $\overline{\mathbb{Q}}$, holomorphic at p_1 and p_2 and which is not the differential of a rational function. Let γ be a path on $\mathcal{E}$ from p_1 to p_2. In case $p_1 = p_2$ one assumes that γ is not homologous to 0. Then the number*

$$\int_\gamma w$$

is transcendental.

Examples: Using Corollary 19 and formulae (9) and (10), one deduces that the numbers

$$\Gamma(1/4)^4/\pi^3 \quad \text{and} \quad \Gamma(1/3)^3/\pi^2$$

are transcendental.

The main results of Schneider's 1936 paper [193] are as follows (see also [196]):

Theorem 20 (Schneider, 1936)
1. *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3. Let β be a non-zero algebraic number. Then β is not a pole of $\wp$ and $\wp(\beta)$ is transcendental.*
More generally, if a and b are two algebraic numbers with $(a,b) \neq (0,0)$, then for any $u \in \mathbb{C} \setminus \Omega$ at least one of the two numbers $\wp(u)$, $au + b\zeta(u)$ is transcendental.
2. *Let $\wp$ and $\wp^*$ be two algebraically independent elliptic functions with algebraic invariants g_2, g_3, g_2^*, g_3^*. If $t \in \mathbb{C}$ is not a pole of $\wp$ or of $\wp^*$, then at least one of the two numbers $\wp(t)$ and $\wp^*(t)$ is transcendental.*
3. *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3. Then for any $t \in \mathbb{C} \setminus \Omega$, at least one of the two numbers $\wp(t)$, e^t is transcendental.*

It follows from Theorem 20.2 that the quotient of an elliptic integral of the first kind (between algebraic points) by a non-zero period is either in the field of endomorphisms (hence a rational number, or a quadratic number in the field of complex multiplications), or a transcendental number.

Here is another important consequence of Theorem 20.2.

Corollary 21 (Schneider, 1936) *Let $\tau \in \mathcal{H}$ be a complex number in the upper half plane $\Im m(\tau) > 0$ such that $j(\tau)$ is algebraic. Then τ is algebraic if and only if τ is imaginary quadratic.*

In this connection we quote Schneider's second problem in [196], which is still open (see Wakabayashi's papers [215, 216, 217]):

Open problem: *Prove Corollary 21 without using elliptic functions.*

Sketch of proof of Corollary 21 as a consequence of part 2 of Theorem 20.

Assume that both $\tau \in \mathcal{H}$ and $j(\tau)$ are algebraic. There exists an elliptic function with algebraic invariants g_2, g_3 and periods ω_1, ω_2 such that

$$\tau = \frac{\omega_2}{\omega_1} \quad \text{and} \quad j(\tau) = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}.$$

Set $\wp^*(z) = \tau^2 \wp(\tau z)$. Then $\wp^*$ is a Weierstraß function with algebraic invariants g_2^*, g_3^*. For $u = \omega_1/2$ the two numbers $\wp(u)$ and $\wp^*(u)$ are algebraic. Hence the two functions $\wp(z)$ and $\wp^*(z)$ are algebraically dependent. It follows that the corresponding elliptic curve has non-trivial endomorphisms, therefore τ is quadratic. □

A quantitative refinement of Schneider's Theorem on the transcendence of $j(\tau)$ given by A. Faisant and G. Philibert in 1984 [71] became useful 10 years later in connection with Nesterenko's result (see § 5). See also [72].

We will not review the results related with abelian integrals, but only quote the first result on this topic, which involves the Jacobian of a Fermat curve: in 1941 Schneider [194] proved that *for a and b in $\mathbb{Q}$ with a, b and $a+b$ not in $\mathbb{Z}$, the number*

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental. We notice that in his 1932 paper [199], C.L. Siegel had already announced partial results on the values of the Euler Gamma function (see also [19]).

Schneider's above mentioned results deal with elliptic (and abelian) integrals of the first or second kind. His method can be extended to deal with elliptic (and abelian) integrals of the third kind (this is Schneider's third problem in [196]).

As pointed out by J-P. Serre in 1979 [222], it follows from the quasi-periodicity of Weierstraß sigma function (8) that the function

$$F_u(z) = \frac{\sigma(z+u)}{\sigma(z)\sigma(u)} e^{-z\zeta(u)}$$

satisfies

$$F_u(z+\omega_i) = F_u(z) e^{\eta_i u - \omega_i \zeta(u)}.$$

Theorem 22 *Let u_1 and u_2 be two non-zero complex numbers. Assume that g_2, g_3, $\wp(u_1)$, $\wp(u_2)$, β are algebraic and $\mathbb{Z}u_1 \cap \Omega = \{0\}$. Then the number*

$$\frac{\sigma(u_1+u_2)}{\sigma(u_1)\sigma(u_2)} e^{\big(\beta - \zeta(u_1)\big)u_2}$$

is transcendental.

From the next corollary, one can deduce that non-zero periods of elliptic integrals of the third kind are transcendental (see [221]).

Corollary 23 *For any non-zero period ω and for any $u \in \mathbb{C} \setminus \Omega$ the number $e^{\omega\zeta(u)-\eta u+\beta\omega}$ is transcendental.*

Further results on elliptic integrals are due to M. Laurent [122]. See also his papers [124, 125, 126, 127].

Ya. M. Kholyavka wrote several papers devoted to the approximation of transcendental numbers related with elliptic functions [110, 109, 108, 107, 106, 105, 103, 104, 102]

Quantitative estimates (measures of transcendence) related with the results of this section were derived by N.I. Fel'dman [73, 74, 75, 76, 77]. See also the papers by S. Lang [114], N.D. Nagaev [155], N. Hirata [97], E. Reyssat [184, 185, 187, 188], M. Laurent [123], R. Tubbs [208], G. Diaz [61], N. Saradha [191], P. Grinspan [90].

3.2 Elliptic analogs of the Six Exponentials Theorem

Elliptic analogs of the Six Exponentials Theorem 10 were considered by S. Lang [115] and K. Ramachandra [180] in the 1960's.

Let d_1, d_2 be non-negative integers and m a positive integer, let $x_1, \dots, x_{d_1}$ be complex numbers which are linearly independent over $\mathbb{Q}$, let $y_1, \dots, y_m$ be complex numbers which are linearly independent over $\mathbb{Q}$ and let $u_1, \dots, u_{d_2}$ be non-zero complex numbers. We consider Weierstraß elliptic functions $\wp_1, \dots, \wp_{d_2}$ and we denote by K_0 the field generated over $\mathbb{Q}$ by their invariants $g_{2,k}$ and $g_{3,k}$ ($1 \le k \le d_2$). We assume that the d_2 functions $\wp_1(u_1 z), \dots, \wp_{d_2}(u_{d_2} z)$ are algebraically independent. We denote by K_1 the field generated over K_0 by the numbers $\exp(x_i y_j)$, ($1 \le i \le d_1$, $1 \le j \le m$) together with the numbers $\wp_k(u_k y_j)$, ($1 \le k \le d_2$, $1 \le j \le m$). Next, define

$$K_2 = K_1(y_1, \dots, y_m), \qquad K_3 = K_1\big(x_1, \dots, x_{d_1},\ u_1, \dots, u_{d_2}\big),$$

and let K_4 be the compositum of K_2 and K_3:

$$K_4 = K_1\big(y_1, \dots, y_m,\ x_1, \dots, x_{d_1},\ u_1, \dots, u_{d_2}\big).$$

The Theorems of Hermite-Lindemann (Theorem 3), Gel'fond-Schneider (Theorem 7), the Six Exponentials Theorem 10 and their elliptic analogs due to Schneider, Lang and Ramachandra can be stated as follows.

Any one of the four assumptions below will imply $d_1 + d_2 > 0$, the case where d_1 (resp. d_2) vanishes means that one considers only elliptic (resp. exponential) functions.

Theorem 24
1. *Assume $(d_1 + d_2)m > m + d_1 + 2d_2$. Then the field K_1 has transcendence degree ≥ 1 over $\mathbb{Q}$.*
2. *Assume either $d_1 \ge 1$ and $m \ge 2$, or $d_2 \ge 1$ and $m \ge 3$. Then K_2 has transcendence degree ≥ 1 over $\mathbb{Q}$.*
3. *Assume $d_1 + d_2 \ge 2$. Then K_3 has transcendence degree ≥ 1 over $\mathbb{Q}$.*
4. *Assume $d_1 + d_2 \ge 1$. Then K_4 has transcendence degree ≥ 1 over $\mathbb{Q}$.*

Parts 3 and 4 of Theorem 24 are consequences of the Schneider-Lang criterion [115], which deals with meromorphic functions satisfying differential equations, while parts 1 and 2 follow from a criterion which involves no differential equations. Such criteria were given by Schneider [195, 196], Lang [115] and Ramachandra [180] (see also [217] and [218]).

Theorem 24 also includes Theorem 20 apart from the case $b \neq 0$ in part 1 of that statement. However there are extensions of Theorem 24 which include results on Weierstraß zeta function (and also on Weierstraß sigma function in connection with elliptic integrals of the third kind). See [221, 122, 124, 125, 126, 222, 188, 127].

Here is a corollary of part 1 of Theorem 24.

Corollary 25 *Let E be an elliptic curve with algebraic invariants g_2, g_3. Assume E has complex multiplications. Let*

$$M = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

be a 2×3 matrix whose entries are elliptic logarithms of algebraic numbers, i. e. u_i and v_i ($i = 1, 2, 3$) are in $\mathcal{L}_E$. Assume that the three columns are linearly independent over $\mathrm{End}(E)$ and the two rows are also linearly independent over $\mathrm{End}(E)$. Then the matrix M has rank 2.

In the case where the curve has no complex multiplication, a similar statement holds for 2×5 matrices. Also in the non-CM case, one deduces from Theorem 24 that such 3×4 matrices $\big(u_{ij}\big)$ (where $\wp(u_{ij})$ are algebraic numbers) have rank ≥ 2.

Lower bounds better than 2 for the rank of matrices of larger sizes are known, but we will not discuss this question here. We just mention the fact that higher dimensional considerations are relevant to a problem discussed by B. Mazur on the density of rational points on varieties [229].

4 Linear independence of numbers related with elliptic functions

From Schneider's Theorem 20 part 1, one deduces the linear independence over the field of algebraic numbers of the three numbers 1, ω and η, when ω is a non-zero period of a Weierstraß elliptic function (with algebraic invariants g_2 and g_3) and $\eta = \eta(\omega)$ is the associated quasi-period of the corresponding Weierstraß zeta function. However, the Gel'fond-Schneider method in one variable alone does not yield strong results of linear independence. Baker's method is better suited for this purpose.

4.1 Linear independence of periods and quasi-periods

Baker's method of proof for his Theorem 9 on linear independence of logarithms of algebraic numbers was used as early as 1969 and 1970 by A. Baker himself [11,9] when he proved the transcendence of linear combinations with algebraic coefficients of the numbers ω_1, ω_2, η_1 and η_2 associated with an elliptic curve having algebraic invariants g_2 and g_3. His method is effective: it provides quantitative Diophantine estimates [10].

In 1971 J. Coates [50] proved the transcendence of linear combinations with algebraic coefficients of $\omega_1,\ \omega_2,\ \eta_1,\ \eta_2$ and $2i\pi$. Moreover, he proved in [49,51,52,53] that in the non-CM case, the three numbers ω_1, ω_2 and $2i\pi$ are $\overline{\mathbb{Q}}$-linearly independent. Further results are due including usual logarithms of algebraic numbers are due to T. Harase in 1974 and 1976 [94,93].

The final result on the question of linear dependence of periods and quasi-periods for a single elliptic function was given by D.W. Masser in 1975 [133,134].

Theorem 26 (Masser, 1975) *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2 and g_3, denote by ζ the corresponding Weierstraß zeta function, let ω_1, ω_2 be a basis of the period lattice of $\wp$ and let η_1, η_2 be the associated quasi-periods of ζ. Then the six numbers 1, ω_1, ω_2, η_1, η_2 and $2i\pi$ span a $\overline{\mathbb{Q}}$-vector space of dimension 6 in the non-CM case, 4 in the CM case:*

$$\dim_{\overline{\mathbb{Q}}}\{1, \omega_1, \omega_2, \eta_1, \eta_2, 2i\pi\} = 2 + 2\dim_{\overline{\mathbb{Q}}}\{\omega_1, \omega_2\}.$$

The fact that the dimension is 4 in the CM case means that there are two independent linear relations among these 6 numbers. One of them is $\omega_2 = \tau\omega_1$ with $\tau \in \overline{\mathbb{Q}}$. The second one (see [134]; see also [35]) can be written

$$C^2\tau\eta_2 - AC\eta_1 + \gamma\omega_1 = 0$$

where $A + BX + CX^2$ is the minimal polynomial of τ over $\mathbb{Z}$ and γ is an element in $\mathbb{Q}(g_2, g_3, \tau)$.

In [134], D.W. Masser also produces quantitative estimates (measures of linear independence). In 1976, R. Franklin and D.W. Masser [82,141] produced an extension involving a logarithm of an algebraic number.

Further results can be found in papers by P. Bundschuh [38], S. Lang [119] (see also his surveys [116,117]), D.W. Masser [144,142], M. Anderson [4] and in the joint paper [5] by M. Anderson and D.W. Masser.

4.2 Elliptic analog of Baker's Theorem

The elliptic analog of Baker's Theorem 9 on linear independence of logarithms was proved by D.W. Masser in 1974 [133,134] in the CM case.

His proof also yields quantitative estimates (measures of linear independence of elliptic logarithms of algebraic points on an elliptic curve). Such estimates have a number of applications: this was shown by A.O. Gel'fond for usual logarithms of algebraic numbers [86] and further consequences of such lower bounds in the case of elliptic curves for solving Diophantine equations (integer points on elliptic curves) were derived by S. Lang [120].

Lower bounds for linear combinations of elliptic logarithms in the CM case were obtained by several mathematicians including J. Coates [50], D.W. Masser [135,139,140], J. Coates and S. Lang [54], M. Anderson [4]. The work of Yu Kunrui [241] yields similar estimates, but his method is not that of Baker-Masser: instead of using a generalization of Gel'fond's solution to Hilbert's seventh problem, Yu Kunrui uses a generalization in several variables of Schneider's solution to the same problem. Again, this method is restricted to the CM case.

The question of linear independence of elliptic logarithms in the non-CM case was settled only in 1980 by D. Bertrand and D.W. Masser [29,30]. They found a new proof of Baker's Theorem 9 using functions of several variables and they succeeded to extend this argument to the situation of elliptic functions, either with or without complex multiplication. The criterion they use is the one that Schneider established in 1949 [194] for his proof of the transcendence of Beta values. This criterion (revisited by S. Lang in [115]) deals with Cartesian products. From the several variables point of view, this is a rather degenerate situation; much deeper results are available, including Bombieri's solution in 1970 of Nagata's Conjecture [115,222], which involves Hörmander L^2-estimates for analytic functions of several variables. However Bombieri's Theorem does not seem to yield new transcendence results, so far.

But so far these deeper results do not give further transcendence results in our context.

Theorem 27 (D.W. Masser 1974 for the CM case, D. Bertrand and D.W. Masser 1980 for the non-CM case) *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3 and field of endomorphisms k. Let $u_1, \ldots, u_n$ be k-linearly independent complex numbers. Assume, for $1 \le i \le n$, that either $u_i \in \Omega$ or else $\wp(u_i) \in \overline{\mathbb{Q}}$. Then the numbers $1, u_1, \ldots, u_n$ are linearly independent over the field $\overline{\mathbb{Q}}$.*

This means that *for an elliptic curve E which is defined over $\overline{\mathbb{Q}}$, if $u_1, \ldots, u_n$ are elements in $\mathcal{L}_E$ which are linearly independent over the field of endomorphisms of E, then the numbers $1, u_1, \ldots, u_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

The method of Bertrand-Masser yields only weak Diophantine estimates (measures of linear independence of logarithms).

4.3 Further results of linear independence

Theorem 26 deals only with periods and quasi-periods associated with one lattice, Theorem 27 deals only with elliptic logarithms of algebraic points on one elliptic curve. A far reaching generalization

of both results was achieved by G. Wüstholz in 1987 [238, 239, 240] when he succeeded in extending Baker's Theorem to abelian varieties and integrals, and, more generally, to commutative algebraic groups. If we restrict his general result to products of a commutative linear group, of copies of elliptic curves as well as of extensions of elliptic curves by the additive or the multiplicative group, the resulting statement settles the questions of linear independence of logarithms of algebraic numbers, of elliptic logarithms of algebraic points, including periods, quasi-periods, elliptic integrals of the first, second or third kind. This is a main step towards an answer to the questions of M. Kontsevich and D. Zagier on periods [113].

Wüstholz's method can be extended to yield measures of linear independence of logarithms of algebraic points on an algebraic group. The first effective such lower bounds were given in 1989 [177, 178]. As a special case, they provide the first measures of linear independence for elliptic logarithms which is also valid in the non-CM case. More generally, they give effective lower bounds for any non-vanishing linear combination of logarithms of algebraic points on algebraic groups (including usual logarithms, elliptic logarithms, elliptic integrals of any kind).

Refinements were obtained by N. Hirata Kohno [96, 97, 98, 99, 100], S. David [57], N. Hirata Kohno and S. David [59], M. Ably [2, 3] and É. Gaudron [84, 83, 85] who uses not only Hirata's reduction argument, but also the work of J-B. Bost [31] (slope inequalities) involving Arakelov's Theory. For instance, thanks to the recent work of David and Hirata-Kohno on the one hand, of Gaudron on the other, one knows that the above mentioned non-vanishing linear combinations of logarithms of algebraic points are not Liouville numbers.

In the p-adic case there is a paper of G. Rémond and F. Urfels [183] dealing with two elliptic logarithms.

Further applications to elliptic curves of the Baker-Masser-Wüstholz method were derived by D.W. Masser and G.Wüstholz [153, 154].

A survey on questions related with the isogeny Theorem is [168]. Other surveys dealing with the questions of *small points*, Bogomolov Conjecture and the André Oort Conjecture are [56, 58]. We do not cover these aspects of the theory in the present paper. Other related topics which would deserve more attention are the theory of height and theta functions as well as ultrametric questions.

Extensions of the above mentioned results to abelian varieties were considered by D.W. Masser [135, 136, 137, 138, 139, 140, 143, 145, 146, 147], S. Lang [118], J. Coates and S. Lang [54], D. Bertrand and Y.Z. Flicker [27], Y.Z. Flicker [81], D. Bertrand [24, 25]. For instance, J. Wolfart and G. Wüstholz [234] have shown that the only linear dependence relations with algebraic coefficients between the values $B(a, b)$ of the Euler Beta function at points $(a, b) \in \mathbb{Q}^2$ are those which follow from the Deligne-Koblitz-Ogus relations (see further references in [232]).

5 Algebraic independence of numbers related with elliptic functions

5.1 Small transcendence degree

We keep the notations and assumptions of section 3.2.

The following extension of Theorem 24 to a result of algebraic independence containing Gel'fond's 1948 results on the exponential function (see § 1.1.5) is a consequence of the works of many a mathematician, including A.O. Gel'fond [86], A.A. Šmelev [203, 204], W.D. Brownawell [33], W.D. Brownawell and K.K. Kubota [35], G. Wüstholz [235], D.W. Masser and G. Wüstholz [150] and others (further references are given in [224, 225, 168]).

Theorem 28
1. *Assume* $(d_1 + d_2)m \geq 2(m + d_1 + 2d_2)$. *Then the field* K_1 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.
2. *Assume* $(d_1 + d_2)m \geq m + 2(d_1 + 2d_2)$. *Then* K_2 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.
3. *Assume* $(d_1 + d_2)m \geq 2m + d_1 + 2d_2$. *Then* K_3 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.
4. *Assume* $(d_1 + d_2)m > m + d_1 + 2d_2$. *Then* K_4 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.

Quantitative estimates (measures of algebraic independence) exist (R. Tubbs [209] and E.M. Jabbouri [101]).

Further related results are due to N.I. Fel'dman [78, 79], R. Tubbs [208, 209, 210, 211, 212, 213], É. Reyssat [190], M. Toyoda and T. Yasuda [207]. See also the measure of algebraic independence given by M. Ably in [1] and by S.O. Shestakov in [198].

A survey on results related with small transcendence degree is given in [225] (see also Chapter 13 of [167]).

Again, like for Theorem 24, there are extensions of Theorem 28 which include results on Weierstraß zeta function as well as on functions of several variables, having a number of consequences related with abelian functions [226].

5.2 Algebraic independence of periods and quasi-periods

In the 1970's G.V. Chudnovsky proved strong results of algebraic independence (small transcendence degree) related with elliptic functions. One of his most spectacular contributions was obtained in 1976 [43] (see also [46] and [48]) :

Theorem 29 (G.V. Chudnovsky, 1976) *Let $\wp$ be a Weierstraß elliptic function with invariants g_2, g_3. Let (ω_1, ω_2) be a basis of the lattice period of $\wp$ and $\eta_1 = \eta(\omega_1)$, $\eta_2 = \eta(\omega_2)$ the associated quasi-periods of the associated Weierstraß zeta function. Then at least two of the numbers $g_2,\ g_3,\ \omega_1,\ \omega_2,\ \eta_1,\ \eta_2$ are algebraically independent.*

A more precise result ([48] Chap. 7, Th. 3.1) is that, for any non-zero period ω, at least two of the four numbers $g_2,\ g_3,\ \omega/\pi,\ \eta/\omega$ (with $\eta = \eta(\omega)$) are algebraically independent.

In the case where g_2 and g_3 are algebraic one deduces from Theorem 29 that two among the four numbers $\omega_1,\ \omega_2,\ \eta_1,\ \eta_2$ are algebraically independent; this statement is also a consequence of the next result [46, 48] (see also [224]):

Theorem 30 (G.V. Chudnovsky, 1981) *Assume that g_2 and g_3 are algebraic. Let ω be a non-zero period of $\wp$, set $\eta = \eta(\omega)$ and let u be a complex number which is not a period such that u and ω are $\mathbb{Q}$-linearly independent: $u \notin \mathbb{Q}\omega \cup \Omega$. Assume $\wp(u) \in \overline{\mathbb{Q}}$. Then the two numbers*

$$\zeta(u) - \frac{\eta}{\omega}u, \quad \frac{\eta}{\omega}$$

are algebraically independent.

From Theorem 29 or Theorem 30 one deduces:

Corollary 31 *Let ω be a non-zero period of $\wp$ and $\eta = \eta(\omega)$. If g_2 and g_3 are algebraic, then the two numbers π/ω and η/ω are algebraically independent.*

The following consequence of Corollary 31 shows that in the CM case, Chudnovsky's results are sharp:

Corollary 32 *Assume that g_2 and g_3 are algebraic and the elliptic curve has complex multiplications. Let ω be a non-zero period of $\wp$. Then the two numbers ω and π are algebraically independent.*

As a consequence of formulae (5) and (6), one deduces:

Corollary 33 *The numbers π and $\Gamma(1/4)$ are algebraically independent. Also the numbers π and $\Gamma(1/3)$ are algebraically independent.*

In connexion with these result let us quote a conjecture of S. Lang from 1971 [116].

Conjecture 34 If $j(\tau)$ is algebraic with $j'(\tau) \neq 0$, then $j'(\tau)$ is transcendental.

Since

$$j'(\tau) = 18\frac{\omega_1^2}{2i\pi}\cdot\frac{g_2}{g_3}j(\tau),$$

Conjecture 34 amounts to the transcendence of ω^2/π. Hence Corollary 32 implies that Conjecture 34 is true at least in the CM case (see [21]):

Corollary 35 *If $\tau \in \mathcal{H}$ is quadratic and $j'(\tau) \neq 0$, then π and $j'(\tau)$ are algebraically independent.*

A quantitative refinement (measure of algebraic independence) of Corollary 32 due to G. Philibert [170] turns out to be useful in connexion with Nesterenko's work in 1996 (further references on this topic are given in [228]).

A transcendence measure for $\Gamma(1/4)$ was obtained by P. Philippon [175, 176] and S. Bruiltet [37]:

Theorem 36 *For $P \in \mathbb{Z}[X,Y]$ with degree d and height H,*

$$\log|P(\pi,\Gamma(1/4)| > -10^{326}\big((\log H + d\log(d+1)\big)d^2\big(\log(d+1)\big)^2.$$

Corollary 37 *The number $\Gamma(1/4)$ is not a Liouville number:*

$$\left|\Gamma(1/4) - \frac{p}{q}\right| > \frac{1}{q^{10^{330}}}.$$

Further related references are papers by Chudnovsky [47], D. Bertrand [18] and E. Reyssat [186, 189] (see also the Bourbaki lecture [220] and the book of E.B. Burger and R. Tubbs [40]). Among Chudnovsky's other contributions are results dealing with G-functions (see [48]; see also Y. André's work [6, 7]).

We conclude this section by the following open problem, which simultaneously generalizes Theorems 29 and 30 of G.V. Chudnovsky.

Conjecture 38 Let $\wp$ be a Weierstraß elliptic function with invariants g_2, g_3, let ω be a non-zero period of $\wp$, set $\eta = \eta(\omega)$ and let $u \in \mathbb{C} \setminus \{\mathbb{Q}\omega \cup \Omega\}$. Then at least two of the five numbers

$$g_2, \quad g_3, \quad \wp(u), \quad \zeta(u) - \frac{\eta}{\omega}u, \quad \frac{\eta}{\omega}$$

are algebraically independent.

Chudnovsky's method was extended by K.G. Vasil'ev [214] and P. Grinspan [90], who proved that at least two of the three numbers π, $\Gamma(1/5)$ and $\Gamma(2/5)$ are algebraically independent. Their proof involves the Jacobian of the Fermat curve $X^5+Y^5 = Z^5$, which contains an abelian variety of dimension 2 as a factor. See also Pellarin's paper [169].

5.3 Large transcendence degree

Another important (and earlier) contribution of G.V. Chudnovsky goes back to 1974 when he worked on extending Gel'fond's method in order to prove results on large transcendence degree (see references in [48, 220]).

Chudnovsky proved that three of the numbers

$$\alpha^\beta, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}} \tag{11}$$

are algebraically independent if α is a non-zero algebraic number, $\log \alpha$ a non-zero logarithm of α and β an algebraic number of degree $d \geq 7$. The same year, with a much more difficult proof, he made the first substantial progress towards a proof that there exist at least n algebraically independent numbers in the set (11), provided that $d \geq 2^n - 1$. This was a remarkable achievement since no such result providing a lower bound for the transcendence degree was known. Later, thanks to the work of several mathematicians, including P. Philippon (see [171] for his trick involving the introduction of redundant variables) and Yu. V. Nesterenko [156,157,158], the proof was completed and the exponential lower bound for d was reduced to a polynomial bound, until G. Diaz [60] obtained the best known results so far: the transcendence degree is at least $[(d+1)/2]$.

During a short time, thanks to the work of Philippon, the elliptic results dealing with large transcendence degree where stronger than the exponential ones (see [224] p. 561).

Further results of algebraic independence related to elliptic functions are given in [186,220,44,46, 159,48,160,161].

In 1980, G.V. Chudnovsky [45] proved the Lindemann-Weierstraß Theorem 5 for $n = 2$ and $n = 3$ (small transcendence degree) by means of a clever variation of Gel'fond's method. This method was extended to large transcendence degree by P. Philippon [172,173,174] and G. Wüstholz [236,237,237] who also succeeded in 1982 to prove the elliptic analog of Lindemann Weierstraß Theorem on the algebraic independence of $e^{\alpha_1}, \ldots, e^{\alpha_n}$ in the CM case:

Theorem 39 *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3 and complex multiplications. Let $\alpha_1, \ldots, \alpha_m$ be algebraic numbers which are linearly independent over the field of endomorphisms of E . Then the numbers $\wp(\alpha_1), \ldots, \wp(\alpha_n)$ are algebraically independent.*

The same conclusion should also hold in the non-CM case – so far only the algebraic independence of at least $n/2$ of these numbers is known.

Further results on large transcendence degree are due to D.W. Masser and G. Wüstholz [151,152] W.D. Brownawell [34], W.D. Brownawell and R. Tubbs [36], M. Takeuchi [205] (for surveys on this topic, see [224,225,168]; see also [167] Chap. 14).

5.4 Modular functions and Ramanujan functions

Ramanujan [182] introduced the following functions

$$P(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \quad Q(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \quad R(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}.$$

They are special cases of Fourier expansions of Eisenstein series. Recall the Bernoulli numbers B_k defined by:

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{z^{2k}}{(2k)!},$$

$$B_1 = 1/6, \quad B_2 = 1/30, \quad B_3 = 1/42.$$

For $k \geq 1$ the Eisenstein series of weight k is ([197] Chap. VII, § 4.2; compare with (4))

$$E_{2k}(z) = 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{\infty} \frac{n^{2k-1} z^n}{1-z^n}.$$

The connection with Ramanujan's notation is

$$P(z) = E_2(z), \quad Q(z) = E_4(z), \quad R(z) = E_6(z).$$

The discriminant Δ and the modular invariant J are related with these functions by

$$\Delta = 12^{-3}(Q^3 - R^2) = q\prod_{n=1}^{\infty}(1-q^n)^{24} \quad \text{and} \quad J = Q^3/\Delta.$$

Let q be a complex number, $0 < |q| < 1$. There exists τ in the upper half plane $\mathcal{H}$ such that $q = e^{2i\pi\tau}$. Select any twelfth root of $\Delta(q)$ and set $\omega = 2\pi\Delta(q)^{1/12}$. The invariants g_2 and g_3 of the Weierstraß $\wp$ function attached to the lattice $(\mathbb{Z}+\mathbb{Z}\tau)\omega$ satisfy $g_2^3 - 27g_3^2 = 1$ and (see [121], Chap. 4, §2, Prop. 4 and Chap. 18, §3)

$$P(q) = 3\frac{\omega}{\pi}\cdot\eta\pi, \quad Q(q) = \frac{3}{4}\left(\frac{\omega}{\pi}\right)^4 g_2, \quad R(q) = \frac{27}{8}\left(\frac{\omega}{\pi}\right)^6 g_3.$$

According to formulae (5) and (6), here are a few special values (see, for instance, [230]).
For $\tau = i$, $q = e^{-2\pi}$,

$$P(e^{-2\pi}) = \frac{3}{\pi}, \quad Q(e^{-2\pi}) = 3\left(\frac{\omega_1}{\pi}\right)^4, \quad R(e^{-2\pi}) = 0 \quad \text{and} \quad \Delta(e^{-2\pi}) = \frac{1}{2^6}\left(\frac{\omega_1}{\pi}\right)^{12}, \tag{12}$$

with

$$\omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.6220575542\ldots$$

For $\tau = \varrho$, $q = -e^{-\pi\sqrt{3}}$,

$$P(-e^{-\pi\sqrt{3}}) = \frac{2\sqrt{3}}{\pi},\ Q(-e^{-\pi\sqrt{3}}) = 0,\ R(-e^{-\pi\sqrt{3}}) = \frac{27}{2}\left(\frac{\omega_1}{\pi}\right)^6,\ \Delta(-e^{-\pi\sqrt{3}}) = -\frac{27}{256}\left(\frac{\omega_1}{\pi}\right)^{12}, \tag{13}$$

with

$$\omega_1 = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428650648\ldots$$

5.5 Mahler-Manin problem on $J(q)$

After Schneider's Theorem (Corollary 21) on the transcendence of the values of the modular function $j(\tau)$, the first results on Eisenstein series (cf. § 5.6) go back to D. Bertrand's 1977 paper [19]. See also his papers [17, 18, 20, 22, 23] and his work with M. Laurent on values of theta functions : [28]. Another related reference is Chudnovsky's lecture at the Helsinki ICM in 1978 [46].

The first transcendence proof using modular forms is due to a team from St Étienne (K. Barré-Sirieix, G. Diaz, F. Gramain and G. Philibert) — hence the nickname *théorème stéphanois* for the next result, from [15] (see also [87, 88] and Chap. 2 of [167]), which answers a conjecture of K. Mahler [130, 129] in the complex case and of Yu. V. Manin [132] in the p-adic case as well as in the complex case (see also Conjecture 43 below). We state the result only in the complex case — the paper [15] solves both cases.

Theorem 40 (K. Barré, G. Diaz, F. Gramain, G. Philibert, 1996) *Let $q \in \mathbb{C}$, $0 < |q| < 1$. If q is algebraic, then $J(q)$ is transcendental.*

The solution of Manin's problem in the p-adic case has several consequences. It is a tool both for R. Greenberg in his study of zeroes of p-adic L functions, and for H. Hida, J. Tilouine and É. Urban in their solution of the main Conjecture for the Selmer group of the symmetric square of an elliptic curve with multiplicative reduction at p (references are given in [228]).

The proof of Theorem 40 involves upper bounds for the growth of the coefficients of the modular function $J(q)$. Such estimates were produced first by K. Mahler [131]. A refined estimate, due to

N. Brisebarre and G. Philibert [32], for the coefficients $c_k(m)$ (which are non-negative rational integers) in

$$\big(zJ(z)\big)^k = \sum_{m=0}^{\infty} c_k(m) z^m,$$

is

$$c_k(m) \le e^{4\sqrt{km}}.$$

As pointed out by D. Bertrand [26], the upper bound

$$|\tilde{c}_{N,k}(m)| \le C^N m^{12N}$$

($0 \le k \le N$, $N \ge 1$, $m \ge 1$, with an absolute constant C) for the coefficients in the Taylor development at the origin of $\Delta^{2N} J^k$:

$$\Delta(z)^{2N} J(z)^k = \sum_{m=1}^{\infty} \tilde{c}_{Nk}(m) z^m$$

is sufficient for the proof of Theorem 40 and is an easy consequence of a Theorem of Hecke together with the fact that Δ^2 and $\Delta^2 J$ are parabolic modular forms of weight 24.

One of the main tools involved in the proof of Theorem 40 is an estimate for the degrees and height of $J(q^n)$ in terms of $J(q)$ (which is assumed to be algebraic) and $n \ge 1$. There exists a symmetric polynomial $\Phi_n \in \mathbb{Z}[X,Y]$, of degree

$$\psi(n) = n \prod_{p \mid n} \left(1 + \frac{1}{p}\right)$$

in each variable, such that $\Phi_n\big(J(q), J(q^n)\big) = 0$. Again, K. Mahler [130, 131] was the first to investigate the coefficients of the polynomial $\Phi_n(X,Y)$: he proved that its length (sum of the absolute values of the coefficients) satisfies

$$L(\Phi_n) \le e^{cn^{3/2}}$$

with an absolute constant c. In the special case $n = 2^m$ he improved his result to

$$L(\Phi_n) \le 2^{57n} n^{36n}$$

and claimed (see [130] p. 97) that if the sharper upper bound

$$L(\Phi_n) \le 2^{Cn}$$

with a positive absolute constant $C > 0$, were true for $n = 2^m$, he could prove Theorem 40. However in 1984 P. Cohen [55] produced asymptotic estimates which show that Mahler's expectation was too optimistic:

$$\lim_{\substack{n=2^m \\ m\to\infty}} \frac{1}{n \log n} \log L(\Phi_n) = 9.$$

In fact she proved more precise results, without the condition $n = 2^m$, which imply, for instance, $\log L(\Phi_n) \sim 6\psi(n) \log n$ for $n \to \infty$.

Further related results are given in [64] (G. Diaz and G. Philibert) for the j-function and [149] (D.W. Masser) for $\wp$-function.

The proof of [15] can be adapted to yield quantitative estimates [13, 14].

A reformulation of Theorem 40 on the transcendence of $J(q)$ is the following mixed analog of the Four Exponentials Conjecture 12:

Corollary 41 *Let $\log\alpha$ be a logarithm of a non-zero algebraic number. Let $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ be a lattice with algebraic invariants g_2, g_3. Then the determinant*

$$\begin{vmatrix} \omega_1 & \log\alpha \\ \omega_2 & 2i\pi \end{vmatrix}$$

does not vanish.

The Four Exponentials Conjecture for the product of an elliptic curve by the multiplicative group is the following more general open problem:

Conjecture 42 Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3. Let E be the corresponding elliptic curve, u_1 and u_2 be two elements in $\mathcal{L}_E$ and $\log\alpha_1$, $\log\alpha_2$ be two logarithms of algebraic numbers. Assume further that the two rows of the matrix

$$M = \begin{pmatrix} u_1 & \log\alpha_1 \\ u_2 & \log\alpha_2 \end{pmatrix}$$

are linearly independent over $\mathbb{Q}$. Then the determinant of M does not vanish.

Another special case of Conjecture 42, stronger than Corollary 41, is the next question of Yu. V. Manin [132] :

Conjecture 43 (Yu.V. Manin) Let $\log\alpha_1$ and $\log\alpha_2$ be two non-zero logarithms of algebraic numbers and let $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ be a lattice with algebraic invariants g_2 and g_3. Then

$$\frac{\omega_1}{\omega_2} \neq \frac{\log\alpha_1}{\log\alpha_2}.$$

In this direction let us quote some of the open problems raised by G. Diaz [62,63].

Conjecture 44 (G. Diaz)
1. For any $z\in\mathbb{C}$ with $|z|=1$ and $z\neq\pm1$, the number $e^{2i\pi z}$ is transcendental.
2. If q is an algebraic number with $0<|q|<1$ such that $J(q)\in[0,1728]$, then $q\in\mathbb{R}$.
3. The function J is injective on the set of algebraic numbers α with $0<|\alpha|<1$.

Remark (G. Diaz). Part 3 of Conjecture 44 implies the other two and also follows from the Four Exponentials Conjecture 12. It also follows from the next Conjecture of D. Bertrand.

Conjecture 45 (D. Bertrand) If α_1 and α_2 are two multiplicatively independent algebraic numbers in the domain $\{z\in\mathbb{C}\,;\,0<|z|<1\}$, then the two numbers $J(\alpha_1)$ and $J(\alpha_2)$ are algebraically independent.

This Conjecture 45 implies the special case of the Four Exponentials Conjecture 12, where two of the algebraic numbers are roots of unity and the two others have modulus $\neq 1$.

5.6 Nesterenko's Theorem

In 1976 [18], D. Bertrand pointed out that Schneider's Theorem 20 implies:
For any $q \in \mathbb{C}$ with $0 < |q| < 1$, at least one of the two numbers $Q(q),\ R(q)$ is transcendental.

Two years later [20], he noticed that G.V. Chudnovsky's Theorem 29 yields:
For any $q \in \mathbb{C}$ with $0 < |q| < 1$, at least two of the numbers $P(q),\ Q(q),\ R(q)$ are algebraically independent.

The following result of Yu.V. Nesterenko [162, 163] (see also [228, 165, 230, 166] as well as Chap. 3 and 4 of [167]) goes one step further:

Theorem 46 (Nesterenko, 1996) *For any $q \in \mathbb{C}$ with $0 < |q| < 1$, three of the four numbers q, $P(q)$, $Q(q)$, $R(q)$ are algebraically independent.*

Among the tools used by Nesterenko in his proof is the following result due to K. Mahler [128] (see also Chap. 1 of [167]):

The functions P, Q, R are algebraically independent over $\mathbb{C}(q)$.

Also he uses the fact (see again Chap. 1 of [167]) that they satisfy a system of differential equations for $D = q\ d/dq$:

$$12\frac{DP}{P} = P - \frac{Q}{P}, \quad 3\frac{DQ}{Q} = P - \frac{R}{Q}, \quad 2\frac{DR}{R} = P - \frac{Q^2}{R}\cdot$$

One of the main steps in his original proof [163] is his following zero estimate:

Theorem 47 (Nesterenko's zero estimate) *Let L_0 and L be positive integers, $A \in \mathbb{C}[z, X_1, X_2, X_3]$ a non-zero polynomial in four variables of degree $\leq L_0$ in z and $\leq L$ in each of the three other variables X_1, X_2, X_3. Then the multiplicity at the origin of the analytic function $A\big(z, P(z), Q(z), R(z)\big)$ is at most $2 \cdot 10^{45} L_0 L^3$.*

In the special case where $J(q)$ is algebraic, P. Philippon [176] produced an alternative proof for Nesterenko's result where this zero estimate 47 is not used; in place of it, he uses Philibert's measure of algebraic independence for ω/π and η/π (see [170] and § 5.2 above). However Philibert's proof requires a zero estimate for algebraic groups.

Using (12) one deduces from Theorem 46 (see [167] Chap. 3 § 1 Cor. 1.2)

Corollary 48 *The three numbers π, e^π, $\Gamma(1/4)$ are algebraically independent.*

while using (13) one deduces (see [167] Chap. 1 § 3.1 Cor. 3.2 Remark ii))

Corollary 49 *The three numbers π, $e^{\pi\sqrt{3}}$, $\Gamma(1/3)$ are algebraically independent.*

Consequences of Corollary 48 are the transcendence of the numbers

$$\sigma_{\mathbb{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}e^{\pi/8}\Gamma(1/4)^{-2}$$

and (P. Bundschuh [39])

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}}\cdot$$

D. Duverney, K. and K. Nishioka, and I. Shiokawa [65, 66, 68, 67, 69, 70] as well as D. Bertrand [26] derived from Nesterenko's Theorem 46 a number of interesting corollaries, including the following ones.

Corollary 50 *Rogers-Ramanujan continued fraction:*

$$RR(\alpha) = 1 + \cfrac{\alpha}{1 + \cfrac{\alpha^2}{1 + \cfrac{\alpha^3}{1 + \ddots}}}$$

is transcendental for any algebraic α *with* $0 < |\alpha| < 1$.

Corollary 51 *Let* $(F_n)_{n\geq 0}$ *be the Fibonacci sequence:* $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. *Then the number*

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2}$$

is transcendental.

Jacobi Theta Series are defined by

$$\theta_2(q) = 2q^{1/4} \sum_{n\geq 0} q^{n(n+1)} = 2q^{1/4} \prod_{n=1}^{\infty} (1-q^{4n})(1+q^{2n}),$$

$$\theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2} = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1})^2,$$

$$\theta_4(q) = \theta_3(-q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{2n-1})^2.$$

Corollary 52 . *Let* i, j *and* $k \in \{2,3,4\}$ *with* $i \neq j$. *Let* $q \in \mathbb{C}$ *satisfy* $0 < |q| < 1$. *Then each of the two fields*

$$\mathbb{Q}\big(q, \theta_i(q), \theta_j(q), D\theta_k(q)\big) \quad \textit{and} \quad \mathbb{Q}\big(q, \theta_k(q), D\theta_k(q), D^2\theta_k(q)\big)$$

has transcendence degree ≥ 3 *over* $\mathbb{Q}$.

As an example, *for an algebraic number* $q \in \mathbb{C}$ *with* $0 < |q| < 1$, *the number*

$$\theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2}$$

is transcendental. The number $\theta_3(q)$ was explicitly considered by Liouville in his 1844 memoir (see [167] p.30).

The proof of Yu.V. Nesterenko is effective and yields quantitative refinements (measures of algebraic independence): [164, 176, 89].

5.7 Further open problems

Among many open problems, we mention

– the algebraic independence of the three numbers π, $\Gamma(1/3)$, $\Gamma(1/4)$.
– the algebraic independence of three numbers among π, $\Gamma(1/5)$, $\Gamma(2/5)$, $e^{\pi\sqrt{5}}$.
– the algebraic independence of the four numbers e, π, e^{π} and $\Gamma(1/4)$.

The main conjectures in this domain are due to S. Schanuel, A. Grothendieck, Y. André [8] and C. Bertolin [16]. Chudnovsky's proof of the algebraic independence of π and $\Gamma(1/4)$ involves elliptic functions, Nesterenko's proof of the algebraic independence of π and e^{π} requires modular functions. One may expect that higher dimensional objects (abelian varieties, motives) may be required in order to go further. In this respect we conclude by alluding to the remarkable progress which were achieved recently in finite characteristic (after the works by Jing Yu, G.W. Anderson and D. Thakur, L. Denis, W.D. Brownawell, J.F. Voloch, M. Papanikolas among others).

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