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Centre of Excellence in Mathematics (CEM)
Department of Mathematics, Mahidol University

Transcendental Number Theory: recent results and open problems.

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Abstract

An algebraic number is a complex number which is a root of a polynomial with rational coefficients. For instance $\sqrt{2}$, $i = \sqrt{-1}$, the Golden Ratio $(1 + \sqrt{5})/2$, the roots of unity $e^{2i\pi a/b}$, the roots of the polynomial $X^5 - 6X + 3$ are algebraic numbers. A **transcendental number** is a complex number which is not algebraic.

Abstract (continued)

The existence of transcendental numbers was proved in 1844 by J. Liouville who gave explicit ad-hoc examples. The transcendence of constants from analysis is harder ; the first result was achieved in 1873 by Ch. Hermite who proved the transcendence of e. In 1882, the proof by F. Lindemann of the transcendence of π gave the final (and negative) answer to the Greek problem of squaring the circle. The transcendence of $2^{\sqrt{2}}$ and e^{π}, which was included in Hilbert's seventh problem in 1900, was proved by Gel'fond and Schneider in 1934. During the last century, this theory has been extensively developed, and these developments gave rise to a number of deep applications. In spite of that, most questions are still open. In this lecture we survey the state of the art on know results and open problems.

http://www.math.jussieu.fr/∼miw/

Rational, algebraic irrational, transcendental

Goal : decide upon the arithmetic nature of "given" numbers : rational, algebraic irrational, transcendental.

Rational integers : $\mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$.

Rational numbers :

$$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q > 0, \gcd(p, q) = 1\}.$$

Algebraic number : root of a polynomial with rational coefficients.

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Goal : decide wether a “given” real number is rational, algebraic irrational or else transcendental.

- **Question** : what means ”given” ?

- Criteria for irrationality : development in a given basis (e.g. : decimal expansion, binary expansion), continued fraction.

- Analytic formulae, limits, sums, integrals, infinite products, any limiting process.

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Algebraic irrational numbers

Examples of algebraic irrational numbers :

- $\sqrt{2}$, $i = \sqrt{-1}$, the Golden Ratio $(1 + \sqrt{5})/2$,

- $\sqrt{d}$ for $d \in \mathbf{Z}$ not the square of an integer (hence not the square of a rational number),

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Rule and compass ; squaring the circle

Construct a square with the same area as a given circle by using only a finite number of steps with compass and straightedge.

Any constructible length is an algebraic number, though not every algebraic number is constructible
(for example $\sqrt[3]{2}$ is not constructible).

Pierre Laurent Wantzel (1814 – 1848)
Recherches sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas. Journal de Mathématiques Pures et Appliquées **1** (2), (1837), 366–372.

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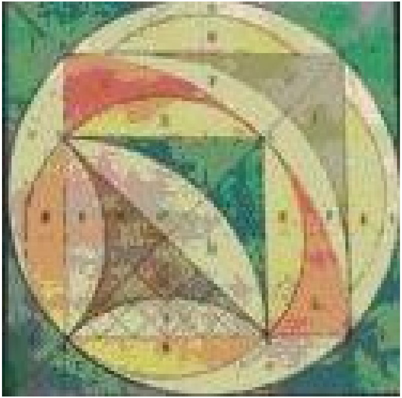
Quadrature of the circle

Marie Jacob
La quadrature du cercle
Un problème
à la mesure des Lumières
Fayard (2006).



Resolution of equations by radicals

The roots of the polynomial $X^5 - 6X + 3$ are algebraic numbers, and are not expressible by radicals.

Evariste Galois
(1811 – 1832)

Born 200 years ago.

Gottfried Wilhelm Leibniz

Introduction of the concept of the transcendental in mathematics by Gottfried Wilhelm Leibniz in 1684 : *"Nova methodus pro maximis et minimis itemque tangentibus, qua nec fractas, nec irrationales quantitates moratur, ..."*

Breger, Herbert. *Leibniz' Einführung des Transzendenten*, 300 Jahre "Nova Methodus" von G. W. Leibniz (1684-1984), p. 119-32. Franz Steiner Verlag (1986).

Serfati, Michel. *Quadrature du cercle, fractions continues et autres contes*, Editions APMEP, Paris (1992).

§1 Irrationality

Given a basis $b \geq 2$, a real number x is rational if and only if its expansion in basis b is ultimately periodic.

$b = 2$: binary expansion.

$b = 10$: decimal expansion.

For instance the decimal number

$$0,123456789012345678901234567890\ldots$$

is rational :

$$= \frac{1\,234\,567\,890}{9\,999\,999\,999} = \frac{137\,174\,210}{1\,111\,111\,111}.$$

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First decimal digits of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.4142135623730950488016887242096980785696718753769480731766797 3
799073247846210703885038753432764157273501384623091229702492483
605585073721264412149709993583141322266592750559275579995050115
278206057147010955997160597027453459686201472851741864088919860
955232923048430871432145083976260362799525140798968725339654633
180882964062061525835239505474575028775996172983557522033753185
701135437460340849884716038689997069900481503054402779031645424
782306849293691862158057846311159666871301301561856898723723528
850926486124949771542183342042856860601468247207714358548741556
570696776537202264854470158588016207584749226572260020855844665
214583988939443709265918003113882464681570826301005948587040031
864803421948972782906410450726368813137398552561173220402450912
277002269411275736272804957381089675040183698683684507257993647
290607629969413804756548237289971803268024744206292691248590521
810044598421505911202494413417285314781058036033710773091828693
1471017111168391658172688941975871658215212822951848847 ...

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http://wims.unice.fr/wims/wims.cgi

1.0110101000001001111001100110011111110011101111001100100100001000
1000101100101111101100010011011001101110101010010101011111010011111
1111100011101011011110110000010111010100010010011101110101000010
1001100111011010001011110101100100000101100000110011001110011001100
1000101010100101011111100100000110000010000011101010111000101001000
0101100001110101000101100011111111001101111110111001000001111010
1101100111001000001111011101001010100000101111001000001110011100011
1111011010010100111000000000100100001110011011000111101111110111
0001001110110100011010010001000000000101110100001110100001010100101
1110001111101001110010100110000010110011100011000000000100011001101
1110000011001101111101111001010101100011011110010010001000101101
0001000001000101100010100100011000001010101111000111001000101111
1011111100010011100011001111000110110101011010100010100011100001
0111011011111101001110111001100101100101010011000110100001100011001
1000111110011110010000100110111111010100101111000100100000111111
0000011011011100101100000101110111010101010010010100000010001000100
1100100000010000000110010100100101010000000100111001010010101010 . . .

Computation of decimals of $\sqrt{2}$

1 542 decimals computed by hand by Horace Uhler in 1951

14 000 decimals computed in 1967

1 000 000 decimals in 1971

$137 \cdot 10^9$ decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

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Square root of 2 on the web

The first decimal digits of $\sqrt{2}$ are available on the web

$1,\ 4,\ 1,\ 4,\ 2,\ 1,\ 3,\ 5,\ 6,\ 2,\ 3,\ 7,\ 3,\ 0,\ 9,\ 5,\ 0,\ 4,\ 8,\ 8,\ 0,\ 1,$

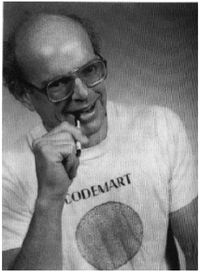
$6,\ 8,\ 8,\ 7,\ 2,\ 4,\ 2,\ 0,\ 9,\ 6,\ 9,\ 8,\ 0,\ 7,\ 8,\ 5,\ 6,\ 9,\ 6,\ 7,\ 1,\ 8,\ \ldots$

`http://oeis.org/A002193`

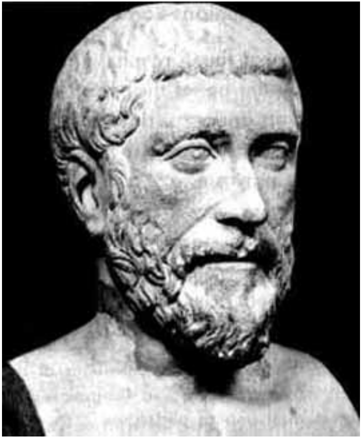
The On-Line Encyclopedia of Integer Sequences

`http://oeis.org/`

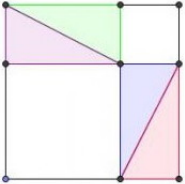
Neil J. A. Sloane

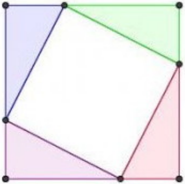


Pythagoras of Samos $\sim$ 569 BC – $\sim$ 475 BC

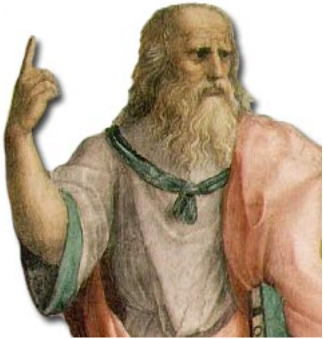


$$a^2 + b^2 = c^2 = (a+b)^2 - 2ab.$$





Irrationality in Greek antiquity



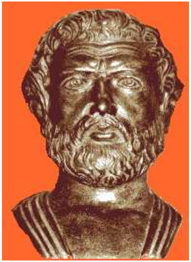
Platon, La République :
incommensurable lines,
irrational diagonals.

Theodorus of Cyrene
(about 370 BC.)
irrationality of $\sqrt{3}, \ldots, \sqrt{17}$.

Theetetes : if an integer $n > 0$ is the square of a rational number, then it is the square of an integer.

Irrationality of $\sqrt{2}$

Pythagoreas school



Hippasus of Metapontum (around 500 BC).

Sulba Sutras, Vedic civilization in India, ∼800-500 BC.

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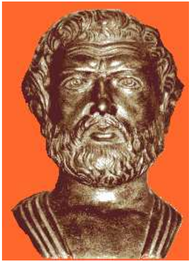

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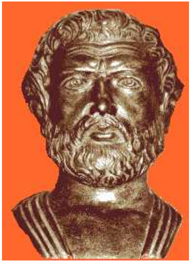

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Émile Borel : 1950

The sequence of decimal digits of $\sqrt{2}$ *should behave like a random sequence, each digit should be occurring with the same frequency* $1/10$*, each sequence of* 2 *digits occurring with the same frequency* $1/100$ *. . .*

Émile Borel (1871–1956)

- *Les probabilités dénombrables et leurs applications arithmétiques,*
  Palermo Rend. **27**, 247-271 (1909).
  Jahrbuch Database JFM 40.0283.01
  `http://www.emis.de/MATH/JFM/JFM.html`

- *Sur les chiffres décimaux de* $\sqrt{2}$ *et divers problèmes de probabilités en chaînes,*
  C. R. Acad. Sci., Paris **230**, 591-593 (1950).
  Zbl 0035.08302

Complexity of the b–ary expansion of an irrational algebraic real number

Let $b \geq 2$ be an integer.

• É. Borel (1909 and 1950) : *the b–ary expansion of an algebraic irrational number should satisfy some of the laws shared by almost all numbers (with respect to Lebesgue's measure).*

• **Remark** : no number satisfies **all** the laws which are shared by all numbers outside a set of measure zero, because the intersection of all these sets of full measure is empty !

$$\bigcap_{x \in \mathbf{R}} \mathbf{R} \setminus \{x\} = \emptyset.$$

• More precise statements by B. Adamczewski and Y. Bugeaud.

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Conjecture of Émile Borel

Conjecture (É. Borel). *Let* x *be an irrational algebraic real number,* $b \geq 3$ *a positive integer and* a *an integer in the range* $0 \leq a \leq b-1$*. Then the digit* a *occurs at least once in the* b*–ary expansion of* x*.*

Corollary. *Each given sequence of digits should occur infinitely often in the* b*–ary expansion of any real irrational algebraic number.*
(consider powers of b).

- An irrational number with a *regular* expansion in some basis b should be transcendental.

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The state of the art

There is no explicitly known example of a triple (b, a, x), where $b \geq 3$ is an integer, a is a digit in $\{0, \ldots, b-1\}$ and x is an algebraic irrational number, for which one can claim that the digit a occurs infinitely often in the b–ary expansion of x.

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The sequence of digits (in any basis) of $\sqrt{2}$ is not ultimately periodic

Among the decimal digits

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at least two of them occur infinitely often. Almost nothing else

is known.

Complexity of the expansion in basis b of a real irrational algebraic number

Theorem (B. Adamczewski, Y. Bugeaud 2005 ; conjecture of A. Cobham 1968).
If the sequence of digits of a real number x is produced by a finite automaton, then x is either rational or else transcendental.

§2 Irrationality of transcendental numbers

- The number e
- The number π
- Open problems

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Introductio in analysin infinitorum

Leonhard Euler (1737)

(1707 – 1783)

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Continued fraction of e :

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\ddots}}}}}}$$

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Joseph Fourier

Fourier (1815) : proof by means of the series expansion

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{N!} + r_N$$

with $r_N > 0$ and $N! r_N \to 0$ as $N \to +\infty$.

Course of analysis at the École Polytechnique Paris, 1815.

Variant of Fourier's proof : e^{-1} is irrational

F. Beukers : alternating series

For odd N,

$$1 - \frac{1}{1!} + \frac{1}{2!} - \cdots - \frac{1}{N!} < e^{-1} < 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{(N+1)!}$$

$$\frac{a_N}{N!} < e^{-1} < \frac{a_N}{N!} + \frac{1}{(N+1)!}, \quad a_N \in \mathbf{Z}$$

$$a_N < N!e^{-1} < a_N + 1.$$

Hence $N!e^{-1}$ is not an integer.

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Irrationality of π

Āryabhaṭa, born 476 AD : $\pi \sim 3.1416$.

Nīlakaṇṭha Somayājī, born 1444 AD : *Why then has an approximate value been mentioned here leaving behind the actual value ? Because it (exact value) cannot be expressed.*

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Johann Heinrich Lambert (1728 – 1777)
Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques,
Mémoires de l'Académie des Sciences de Berlin, **17** (1761), p. 265-322 ;
lu en 1767 ; Math. Werke, t. II.

$\tan(v)$ is irrational when $v \neq 0$ is rational.
As a consequence, π is irrational, since $\tan(\pi/4) = 1$.

Lambert and Frederick II, King of Prussia

— Que savez vous, Lambert ?
— Tout, Sire.
— Et de qui le tenez–vous ?
— De moi-même !

Known and unknown transcendence results

Known :

$$e,\ \pi,\ \log 2,\ e^{\sqrt{2}},\ e^{\pi},\ 2^{\sqrt{2}},\ \Gamma(1/4).$$

Not known :

$$e+\pi,\ e\pi,\ \log\pi,\ \pi^{e},\ \Gamma(1/5),\ \zeta(3),\ \text{Euler constant}$$

Why is e^{π} known to be transcendental while π^{e} is not known to be irrational ?
Answer : $e^{\pi}=(-1)^{-i}$.

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Catalan's constant

Is Catalan's constant

$$\sum_{n\geq 1} \frac{(-1)^n}{(2n+1)^2}$$

$= 0.915\,965\,594\,177\,219\,015\,0\ldots$

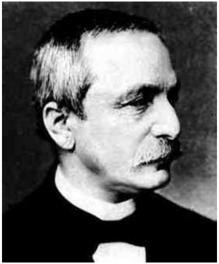
an irrational number ?

Catalan's constant, Dirichlet and Kronecker

Catalan's constant is the value at $s = 2$ of the Dirichlet L–function $L(s, \chi_{-4})$ associated with the Kronecker character

$$\chi_{-4}(n) = \left(\frac{n}{4}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv -1 \pmod 4. \end{cases}$$

Johann Peter Gustav Lejeune Dirichlet
1805 – 1859



Leopold Kronecker
1823 – 1891

Catalan's constant, Dedekind and Riemann

The Dirichlet L–function $L(s, \chi_{-4})$ associated with the Kronecker character χ_{-4} is the quotient of the Dedekind zeta function of $\mathbf{Q}(i)$ and the Riemann zeta function :

$$\zeta_{\mathbf{Q}(i)}(s) = L(s, \chi_{-4})\zeta(s)$$

Julius Wilhelm Richard Dedekind
1831 – 1916

Georg Friedrich Bernhard Riemann
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Riemann zeta function

The function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

was studied by Euler (1707– 1783)
for integer values of s
and by Riemann (1859) for complex values of s.

Euler : for any even integer value of $s \geq 2$, the number $\zeta(s)$ is a rational multiple of π^s.

Examples : $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450 \cdots$

Coefficients : Bernoulli numbers.

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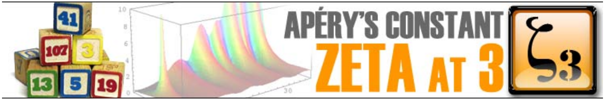


The number

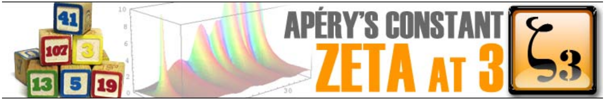
$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511\,\ldots$$

is irrational *(Apéry 1978)*.

Recall that $\zeta(s)/\pi^s$ is rational for any even value of $s \geq 2$.

Open question : Is the number $\zeta(3)/\pi^3$ irrational ?

Riemann zeta function



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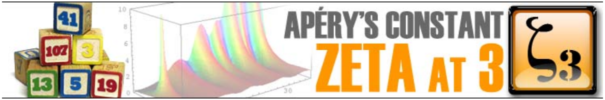

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T. Rivoal (2000) : infinitely many $\zeta(2n+1)$ are irrational.

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Infinitely many odd zeta values are irrational

Tanguy Rivoal (2000)

Let $\epsilon > 0$. For any sufficiently large odd integer a, the dimension of the $\mathbf{Q}$–vector space spanned by the numbers $1,\ \zeta(3),\ \zeta(5),\ \cdots,\zeta(a)$ is at least

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

Euler–Mascheroni constant

Lorenzo Mascheroni
(1750 – 1800)

Euler's Constant is

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$
$$= 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots$$

Is it a rational number ?

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{1}{k}-\log\left(1+\frac{1}{k}\right)\right) = \int_1^{\infty}\left(\frac{1}{[x]}-\frac{1}{x}\right)dx$$
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Euler's constant

Recent work by *J. Sondow* inspired by the work of F. Beukers on Apéry's proof.

F. Beukers

Jonathan Sondow

http://home.earthlink.net/~jsondow/

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$$\gamma = \int_0^\infty \sum_{k=2}^\infty \frac{1}{k^2 \binom{t+k}{k}} dt$$

$$\gamma = \lim_{s \to 1+} \sum_{n=1}^\infty \left(\frac{1}{n^s} - \frac{1}{s^n} \right)$$

$$\gamma = \int_1^\infty \frac{1}{2t(t+1)} F \left(\begin{matrix} 1, & 2, & 2 \\ 3, & t+2 \end{matrix} \right) dt.$$

Euler Gamma function

Is the number

$$\Gamma(1/5) = 4.590\ 843\ 711\ 998\ 803\ 053\ 204\ 758\ 275\ 929\ 152\ \ldots$$

irrational ?

$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} = \int_0^{\infty} e^{-t} t^z \cdot \frac{dt}{t}$$

Here is the set of rational values for $z \in (0,1)$ for which the answer is known (and, for these arguments, the Gamma value is a transcendental number) :

$$r \in \left\{ \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \right\} \pmod 1.$$

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Georg Cantor (1845 - 1918)

The set of algebraic numbers is countable, not the set of real (or complex) numbers.

Cantor (1874 and 1891).

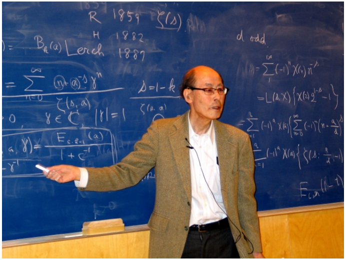
Henri Léon Lebesgue (1875 – 1941)

Almost all numbers for Lebesgue measure are transcendental numbers.

Most numbers are transcendental

Meta conjecture : *any number given by some kind of limit, which is not obviously rational (resp. algebraic), is irrational (resp. transcendental).*

Goro Shimura



Special values of hypergeometric series

Jürgen Wolfart

Frits Beukers

Sum of values of a rational function

Work by S.D. Adhikari, N. Saradha, T.N. Shorey and R. Tijdeman (2001),
Let P and Q be non-zero polynomials having rational coefficients and $\deg Q \geq 2 + \deg P$. Consider

$$\sum_{\substack{n\geq 0\\ Q(n)\neq 0}} \frac{P(n)}{Q(n)}.$$

Robert Tijdeman

Sukumar Das Adhikari

N. Saradha

Telescoping series

Examples

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1, \quad \sum_{n=0}^{\infty} \frac{1}{n^2-1} = \frac{3}{4},$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{5n+2} - \frac{3}{5n+7} + \frac{1}{5n-3} \right) = \frac{5}{6}$$

Transcendental values

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3}$$

are transcendental.

Transcendental values

$$\sum_{n=0}^{\infty} \frac{1}{(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(6n+6)}$$

$$= \frac{1}{4320}(192\log 2 - 81\log 3 - 7\pi\sqrt{3})$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}} = 2.076\,674\,047\,4\ldots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{2\pi}{e^{\pi}-e^{-\pi}} = 0.272\,029\,054\,982\ldots$$

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Leonardo Pisano (Fibonacci)

The Fibonacci sequence $(F_n)_{n\geq 0}$:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,$$

$$34,\ 55,\ 89,\ 144,\ 233\ldots$$

is defined by

$$F_0 = 0,\ F_1 = 1,$$

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$$

Leonardo Pisano (Fibonacci)
(1170–1250)

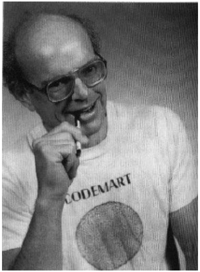
Encyclopedia of integer sequences (again)

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, ...

The Fibonacci sequence is available online

The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane



http://oeis.org/A000045

Series involving Fibonacci numbers

The number

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1$$

is rational, while

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1-\sqrt{5}}{2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}+1} = \frac{\sqrt{5}}{2}$$

are irrational algebraic numbers.

Series involving Fibonacci numbers

The numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^4}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^6},$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{2n}},$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2^n-1} + F_{2^n+1}}, \qquad \sum_{n=1}^{\infty} \frac{1}{F_{2^n+1}}$$

are all transcendental

Series involving Fibonacci numbers

Each of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n + F_{n+2}}$$

$$\sum_{n \geq 1} \frac{1}{F_1 F_2 \cdots F_n}$$

is irrational, but it is not known whether they are algebraic or transcendental.

The first challenge here is to formulate a conjectural statement which would give a satisfactory description of the situation.

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The first challenge here is to formulate a conjectural statement which would give a satisfactory description of the situation.

The Fibonacci zeta function

For $\Re e(s) > 0$,

$$\zeta_F(s) = \sum_{n\geq 1} \frac{1}{F_n^s}$$

$\zeta_F(2)$, $\zeta_F(4)$, $\zeta_F(6)$ are algebraically independent.

Iekata Shiokawa, Carsten Elsner and Shun Shimomura (2006)

Iekata Shiokawa

§3 Transcendental numbers

- Liouville (1844)
- Hermite (1873)
- Lindemann (1882)
- Hilbert's Problem 7th (1900)
- Gel'fond–Schneider (1934)
- Baker (1968)
- Nesterenko (1995)

Existence of transcendental numbers (1844)

J. Liouville (1809 - 1882)

gave the first examples of transcendental numbers.
For instance

$$\sum_{n\geq 1} \frac{1}{10^{n!}} = 0.110\,001\,000\,000\,0\ldots$$

is a transcendental number.

Charles Hermite and Ferdinand Lindemann

Hermite (1873) :
Transcendence of e
$e = 2.718\,281\,828\,4\ldots$

Lindemann (1882) :
Transcendence of π
$\pi = 3.141\,592\,653\,5\ldots$

Hermite–Lindemann Theorem

For any non-zero complex number z, one at least of the two numbers z and e^z is transcendental.

Corollaries : Transcendence of $\log \alpha$ and of e^β for α and β non-zero algebraic complex numbers, provided $\log \alpha \neq 0$.

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Transcendental functions

A complex function is called transcendental if it is transcendental over the field $\mathbf{C}(z)$, which means that the functions z and $f(z)$ are algebraically independent : if $P \in \mathbf{C}[X, Y]$ is a non-zero polynomial, then the function $P(z, f(z))$ is not 0.

Exercise. *An entire function (analytic in $\mathbf{C}$) is transcendental if and only if it is not a polynomial.*

Example. *The transcendental entire function e^z takes an algebraic value at an algebraic argument z only for $z = 0$.*

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Weierstrass question

Is it true that a transcendental entire function f takes usually *transcendental values at algebraic arguments ?*

Examples : for $f(z) = e^z$, there is a single exceptional point α algebraic with e^α also algebraic, namely $\alpha = 0$.

For $f(z) = e^{P(z)}$ where $P \in \mathbf{Z}[z]$ is a non–constant polynomial, there are finitely many exceptional points α, namely the roots of P.

The exceptional set of $e^z + e^{1+z}$ is empty (Lindemann–Weierstrass).

The exceptional set of functions like 2^z or $e^{i\pi z}$ is $\mathbf{Q}$, (Gel'fond and Schneider).

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Exceptional sets

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain...

If S is a countable subset of $\mathbf{C}$ and T is a dense subset of $\mathbf{C}$, there exist transcendental entire functions f mapping S into T, as well as all its derivatives.

Any set of algebraic numbers is the exceptional set of some transcendental entire function.
Also multiplicities can be included.

van der Poorten : *there are transcendental entire functions f such that $D^k f(\alpha) \in \mathbf{Q}(\alpha)$ for all $k \geq 0$ and all algebraic α.*

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Integer valued entire functions

An integer valued entire function is a function f, which is analytic in $\mathbf{C}$, and maps $\mathbf{N}$ into $\mathbf{Z}$.

Example : 2^z *is an integer valued entire function, not a polynomial.*

Question : *Are there integer valued entire function growing slower than* 2^z *without being a polynomial ?*

Let f be a transcendental entire function in $\mathbf{C}$. For $R > 0$ set

$$|f|_R = \sup_{|z|=R} |f(z)|.$$

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if f is not a polynomial
and $f(n) \in \mathbf{Z}$ for $n \in \mathbf{Z}_{\geq 0}$, then

$$\limsup_{R\to\infty} 2^{-R}|f|_R \geq 1.$$

Further works on this topic by G.H. Hardy, G. Pólya, D. Sato, E.G. Straus, A. Selberg, Ch. Pisot, F. Carlson, F. Gross,...

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Integer valued entire function on $\mathbf{Z}[i]$

A.O. Gel'fond (1929) : growth of entire functions mapping the Gaussian integers into themselves.
Newton interpolation series at the points in $\mathbf{Z}[i]$.

An entire function f which is not a polynomial and satisfies $f(a+ib) \in \mathbf{Z}[i]$ for all $a+ib \in \mathbf{Z}[i]$ satisfies

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F. Gramain (1981) : $\gamma = \pi/(2e) = 0.106\,172\,888\ldots$.
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Transcendence of e^π

A.O. Gel'fond (1929).

If
$$e^\pi = 23.140\,692\,632\,779\,269\,005\,729\,086\,367\ \ldots$$
is rational, then the function $e^{\pi z}$ takes values in $\mathbf{Q}(i)$ when the argument z is in $\mathbf{Z}[i]$.

Expand $e^{\pi z}$ into an interpolation series at the Gaussian integers.

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Hilbert's Problems

August 8, 1900

Second International Congress of Mathematicians in Paris.

David Hilbert (1862 - 1943)

Twin primes,

Goldbach's Conjecture,

Riemann Hypothesis

Transcendence of e^{π} and $2^{\sqrt{2}}$

A.O. Gel'fond and Th. Schneider

Solution of Hilbert's seventh problem (1934) : *Transcendence of* α^β *and of* $(\log \alpha_1)/(\log \alpha_2)$ *for algebraic* α, β, α_1 *and* α_2.

Transcendence of α^β and $\log \alpha_1 / \log \alpha_2$: examples

The following numbers are transcendental :

$$2^{\sqrt{2}} = 2.665\,144\,142\,6\ldots$$

$$\frac{\log 2}{\log 3} = 0.630\,929\,753\,5\ldots$$

$$e^{\pi} = 23.140\,692\,632\,7\ldots \qquad (e^{\pi} = (-1)^{-i})$$

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743.999\,999\,999\,999\,25\ldots$$

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Example : Transcendence of the number

$$e^{\pi\sqrt{163}} = 262\;537\;412\;640\;768\;743.999\;999\;999\;999\;2\ldots.$$

Remark. For

$$\tau = \frac{1 + i\sqrt{163}}{2}, \quad q = e^{2i\pi\tau} = -e^{-\pi\sqrt{163}}$$

we have $j(\tau) = -640\;320^3$ and

$$\left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}.$$

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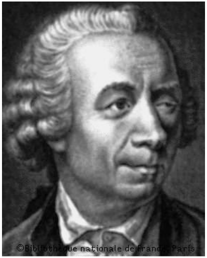
Beta values : Th. Schneider 1948

Euler Gamma and Beta functions

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

$$\Gamma(z) = \int_0^\infty e^{-t}t^z \cdot \frac{dt}{t}$$



$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Algebraic independence : A.O. Gel'fond 1948

The two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent.

More generally, if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an algebraic number of degree $d \geq 3$, then two at least of the numbers

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \ldots,\alpha^{\beta^{d-1}}$$

are algebraically independent.

Alan Baker 1968

Transcendence of numbers like

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

or

$$e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_1^{\beta_1}$$

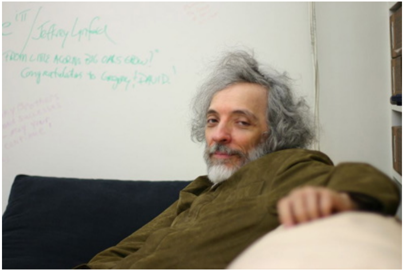
for algebraic α_i's and β_j's.

Example (Siegel) :

$$\int_0^1 \frac{dx}{1+x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right) = 0.835\,648\,848\,\ldots$$

is transcendental.

Gregory V. Chudnovsky



G.V. Chudnovsky (1976)

Algebraic independence of the numbers π and $\Gamma(1/4)$.

Also : algebraic independence of the numbers π and $\Gamma(1/3)$.

Corollaries : *Transcendence of* $\Gamma(1/4) = 3.625\,609\,908\,2\ldots$ *and* $\Gamma(1/3) = 2.678\,938\,534\,7\ldots$.

Yuri V. Nesterenko

Yu.V.Nesterenko (1996)
Algebraic independence of
$\Gamma(1/4)$, π and e^π.
Also : Algebraic independence of
$\Gamma(1/3)$, π and $e^{\pi\sqrt{3}}$.

Corollary : *The numbers* $\pi = 3.141\,592\,653\,5\ldots$ *and* $e^\pi = 23.140\,692\,632\,7\ldots$ *are algebraically independent.*

Transcendence of values of Dirichlet's L–functions :
Sanoli Gun, Ram Murty and Purusottam Rath (2009).

Weierstraß sigma function

Let $\Omega = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ be a lattice in $\mathbf{C}$. The canonical product attached to Ω is the *Weierstraß sigma function*

$$\sigma(z) = \sigma_\Omega(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{(z/\omega) + (z^2/2\omega^2)}.$$

The number

$$\sigma_{\mathbf{Z}[i]}(1/2) = 2^{5/4} \pi^{1/2} e^{\pi/8} \Gamma(1/4)^{-2}$$

is transcendental.

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§4 : Conjectures

Borel 1909, 1950

Schanuel 1964

Grothendieck 1960's

Rohrlich and Lang 1970's

André 1990's

Kontsevich and Zagier 2001.

Periods : Maxime Kontsevich and Don Zagier

Periods,
Mathematics unlimited—2001 and beyond,
Springer 2001,
771–808.

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients , over domains in $\mathbf{R}^n$ given by polynomial inequalities with rational coefficients.

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The number π

Basic example of a *period* :

$$e^{z+2i\pi} = e^z$$

$$2i\pi = \int_{|z|=1} \frac{dz}{z}$$

$$\begin{aligned} \pi &= \iint_{x^2+y^2\leq 1} dxdy &&= 2\int_{-1}^{1} \sqrt{1-x^2}dx \\ &= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} &&= \int_{-\infty}^{\infty} \frac{dx}{1-x^2}. \end{aligned}$$

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The number π

Basic example of a *period* :

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Further examples of periods

$$\sqrt{2} = \int_{2x^2 \leq 1} dx$$

and all algebraic numbers.

$$\log 2 = \int_{1<x<2} \frac{dx}{x}$$

and all logarithms of algebraic numbers.

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Relations among periods

$\boxed{1}$ Additivity

(in the integrand and in the domain of integration)

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx,$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

$\boxed{2}$ Change of variables :

if $y=f(x)$ is an invertible change of variables, then

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Relations among periods (continued)

  

$\boxed{3}$ Newton–Leibniz–Stokes Formula

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Conjecture of Kontsevich and Zagier

A widely-held belief, based on a judicious combination of experience, analogy, and wishful thinking, is the following

Conjecture (Kontsevich–Zagier). *If a period has two integral representations, then one can pass from one formula to another by using only rules* $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ *in which all functions and domains of integration are algebraic with algebraic coefficients.*

Conjecture of Kontsevich and Zagier (continued)

In other words, we do not expect any miraculous coïncidence of two integrals of algebraic functions which will not be possible to prove using three simple rules.

This conjecture, which is similar in spirit to the Hodge conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.

Advice : *if you wish to prove a number is transcendental, first prove it is a period.*

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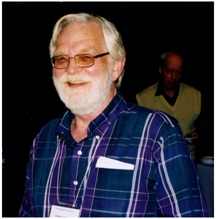
Conjectures by S. Schanuel and A. Grothendieck



• Schanuel : *if* $x_1, \ldots, x_n$ *are* $\mathbf{Q}$*–linearly independent complex numbers, then* n *at least of the* $2n$ *numbers* $x_1, \ldots, x_n$, $e^{x_1}, \ldots, e^{x_n}$ *are algebraically independent.*

• Periods conjecture by Grothendieck : Dimension of the Mumford–Tate group of a smooth projective variety.

Motives

Y. André : generalization of Grothendieck's conjecture to motives.

Case of 1–motives : Elliptico-Toric Conjecture of C. Bertolin.

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Centre of Excellence in Mathematics (CEM)
Department of Mathematics, Mahidol University

Transcendental Number Theory: recent results and open problems.

Michel Waldschmidt

Institut de Mathématiques de Jussieu
http://www.math.jussieu.fr/∼miw/