Transcendental Number Theory: recent results and open problems.

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Abstract

This lecture will be devoted to a survey of transcendental number theory, including some history, the state of the art and some of the main conjectures.

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Extended abstract

An algebraic number is a complex number which is a root of a polynomial with rational coefficients. For instance $\sqrt{2}$, $i = \sqrt{-1}$, the Golden Ratio $(1 + \sqrt{5})/2$, the roots of unity $\mathrm{e}^{2\pi \mathrm{i} a/b}$, the roots of the polynomial $X^5 - 6X + 3$ are algebraic numbers. A **transcendental number** is a complex number which is not algebraic.

Extended abstract (continued)

The existence of transcendental numbers was proved in 1844 by J. Liouville who gave explicit ad-hoc examples. The transcendence of constants from analysis is harder ; the first result was achieved in 1873 by Ch. Hermite who proved the transcendence of e. In 1882, the proof by F. Lindemann of the transcendence of π gave the final (and negative) answer to the Greek problem of squaring the circle. The transcendence of $2^{\sqrt{2}}$ and e^{π}, which was included in Hilbert's seventh problem in 1900, was proved by Gel'fond and Schneider in 1934. During the last century, this theory has been extensively developed, and these developments gave rise to a number of deep applications. In spite of that, most questions are still open. In this lecture we survey the state of the art on known results and open problems.

Rational, algebraic irrational, transcendental

Goal : decide upon the arithmetic nature of "given" numbers : rational, algebraic irrational, transcendental.

Rational integers : $\mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$.

Rational numbers :

$$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q > 0, \ \gcd(p, q) = 1\}.$$

Algebraic number : root of a polynomial with rational coefficients.

A **transcendental number** is complex number which is not algebraic.

Rational, algebraic irrational, transcendental

Goal : decide wether a "given" real number is rational, algebraic irrational or else transcendental.

- **Question** : what means "given" ?

- Criteria for irrationality : development in a given basis (e.g. : decimal expansion, binary expansion), continued fraction.

- Analytic formulae, limits, sums, integrals, infinite products, any limiting process.

Algebraic irrational numbers

Examples of algebraic irrational numbers :

- $\sqrt{2}$, $i = \sqrt{-1}$, the Golden Ratio $(1 + \sqrt{5})/2$,

- $\sqrt{d}$ for $d \in \mathbf{Z}$ not the square of an integer (hence not the square of a rational number),

- the roots of unity $\mathrm{e}^{2\pi\mathrm{i}a/b}$, for $a/b \in \mathbf{Q}$,

- and, of course, any root of an irreducible polynomial with rational coefficients of degree > 1.

Rule and compass ; squaring the circle

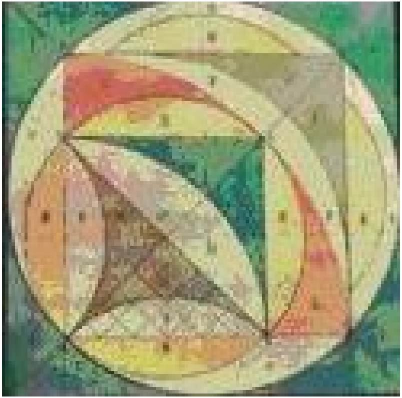
Construct a square with the same area as a given circle by using only a finite number of steps with compass and straightedge.

Any constructible length is an algebraic number, though not every algebraic number is constructible
(for example $\sqrt[3]{2}$ is not constructible).

Pierre Laurent Wantzel (1814 – 1848)
Recherches sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas. Journal de Mathématiques Pures et Appliquées **1** (2), (1837), 366–372.

Quadrature of the circle

Marie Jacob
La quadrature du cercle
Un problème
à la mesure des Lumières
Fayard (2006).



Resolution of equations by radicals

The roots of the polynomial $X^5 - 6X + 3$ are algebraic numbers, and are not expressible by radicals.

Evariste Galois
(1811 – 1832)

Gottfried Wilhelm Leibniz

Introduction of the concept of the transcendental in mathematics by Gottfried Wilhelm Leibniz in 1684 : *"Nova methodus pro maximis et minimis itemque tangentibus, qua nec fractas, nec irrationales quantitates moratur, . . ."*

Breger, Herbert. *Leibniz' Einführung des Transzendenten*, 300 Jahre "Nova Methodus" von G. W. Leibniz (1684-1984), p. 119-32. Franz Steiner Verlag (1986).

Serfati, Michel. *Quadrature du cercle, fractions continues et autres contes*, Editions APMEP, Paris (1992).

§1 Irrationality

Given a basis $b \geq 2$, a real number x is rational if and only if its expansion in basis b is ultimately periodic.

$b = 2$: binary expansion.

$b = 10$: decimal expansion.

For instance the decimal number

$$0.123456789012345678901234567890\dots$$

is rational :

$$= \frac{1\,234\,567\,890}{9\,999\,999\,999} = \frac{137\,174\,210}{1\,111\,111\,111}.$$

First decimal digits of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.4142135623730950488016887242096980785696718753769480731766797379
7990732478462107038850387534327641572735013846230912297024924836
0558507372126441214970999358314132226659275055927557999505011527
8206057147010955997160597027453459686201472851741864088919860955
2329230484308714321450839762603627995251407989687253396546331808
8296406206152583523950547457502877599617298355752203375318570113
5437460340849884716038689997069900481503054402779031645424782306
8492936918621580578463111596668713013015618568987237235288509264
8612494977154218334204285686060146824720771435854874155657069677
6537202264854470158588016207584749226572260020855844665214583988
9394437092659180031138824646815708263010059485870400318648034219
4897278290641045072636881313739855256117322040245091227700226941
1275736272804957381089675040183698683684507257993647290607629969
4138047565482372899718032680247442062926912485905218100445984215
0591120249441341728531478105803603371077309182869314710171111683
91658172688941975871658215212822951848847 ...

First binary digits of $\sqrt{2}$ http://wims.unice.fr/wims/wims.cgi

1.011010100000100111100110011001111111001110111100110010010000
1000101100101111101100010011011001101110101010010101011110100
11111000111010110111101100000101110101000100100111011101010000
100110011101101000101111010110010000101100000011001100111001100
10001010101001010111111001000001100000100001110101011100010100
01011000011101010001011000111111110011011111101110010000011110
11011001110010000111101110100101010000101111001000011100111000
11110110100101001111000000001001000011100110110001111011111101
000100111011010001101001000100000000101110100001110100001010101
111000111110100111001010011000001011001110001100000000010001101
11100001100110111101111001010101100011011110010010001000101101
00010000100010110001010010001100000101010111100011100100010111
10111110001001110001100111100011011010101101010001010001110001
01110110111111010011101110011001011001010100110001101000011001
10001111100111100100001001101111101010010111100010010000011111
000001101101110010110000010111011101010101001001010000001000100
11001000000100000011001010010010101000000010011100101001010 ...

Computation of decimals of $\sqrt{2}$

$1\,542$ decimals computed by hand by Horace Uhler in 1951

$14\,000$ decimals computed in 1967

$1\,000\,000$ decimals in 1971

$137 \cdot 10^9$ decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

- Motivation : computation of π.

Square root of 2 on the web

The first decimal digits of $\sqrt{2}$ are available on the web

$1,\ 4,\ 1,\ 4,\ 2,\ 1,\ 3,\ 5,\ 6,\ 2,\ 3,\ 7,\ 3,\ 0,\ 9,\ 5,\ 0,\ 4,\ 8,\ 8,\ 0,\ 1,$

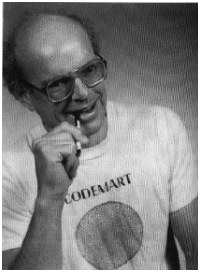
$6,\ 8,\ 8,\ 7,\ 2,\ 4,\ 2,\ 0,\ 9,\ 6,\ 9,\ 8,\ 0,\ 7,\ 8,\ 5,\ 6,\ 9,\ 6,\ 7,\ 1,\ 8,\ \dots$

http://oeis.org/A002193

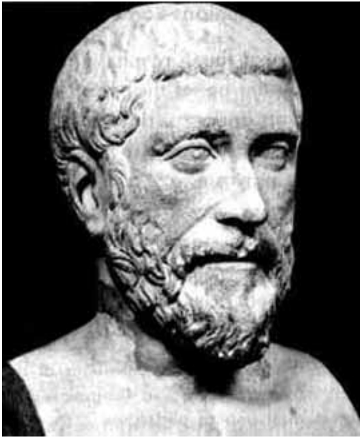
The On-Line Encyclopedia of Integer Sequences

http://oeis.org/

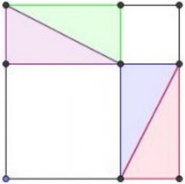
Neil J. A. Sloane

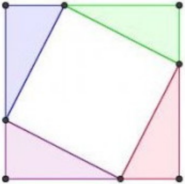


Pythagoras of Samos ∼ 569 BC – ∼ 475 BC

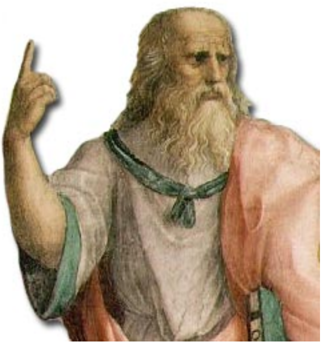


$$a^2 + b^2 = c^2 = (a+b)^2 - 2ab.$$





Irrationality in Greek antiquity



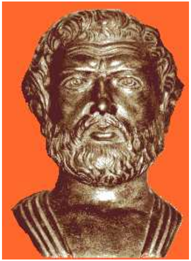
Platon, La République :
incommensurable lines,
irrational diagonals.

Theodorus of Cyrene
(about 370 BC.)
irrationality of $\sqrt{3}, \dots, \sqrt{17}$.

Theetetes : if an integer $n > 0$ is the square of a rational number, then it is the square of an integer.

Irrationality of $\sqrt{2}$

Pythagoreas school



Hippasus of Metapontum (around 500 BC).

Sulba Sutras, Vedic civilization in India, $\sim$800-500 BC.

Émile Borel : 1950

The sequence of decimal digits of $\sqrt{2}$ *should behave like a random sequence, each digit should be occurring with the same frequency* $1/10$*, each sequence of* 2 *digits occurring with the same frequency* $1/100$ *...*

Émile Borel (1871–1956)

- *Les probabilités dénombrables et leurs applications arithmétiques,*
  Palermo Rend. **27**, 247-271 (1909).
  Jahrbuch Database JFM 40.0283.01
  http://www.emis.de/MATH/JFM/JFM.html

- *Sur les chiffres décimaux de* $\sqrt{2}$ *et divers problèmes de probabilités en chaînes,*
  C. R. Acad. Sci., Paris **230**, 591-593 (1950).
  Zbl 0035.08302

Complexity of the b–ary expansion of an irrational algebraic real number

Let $b \geq 2$ be an integer.

• É. Borel (1909 and 1950) : *the b–ary expansion of an algebraic irrational number should satisfy some of the laws shared by almost all numbers (with respect to Lebesgue's measure).*

• **Remark** : no number satisfies **all** the laws which are shared by all numbers outside a set of measure zero, because the intersection of all these sets of full measure is empty !

$$\bigcap_{x \in \mathbf{R}} \mathbf{R} \setminus \{x\} = \emptyset.$$

• More precise statements by B. Adamczewski and Y. Bugeaud.

Conjecture of Émile Borel

Conjecture (É. Borel). *Let x be an irrational algebraic real number, $b \geq 3$ a positive integer and a an integer in the range $0 \leq a \leq b-1$. Then the digit a occurs at least once in the b–ary expansion of x.*

Corollary. *Each given sequence of digits should occur infinitely often in the b–ary expansion of any real irrational algebraic number.*
(consider powers of b).

- An irrational number with a *regular* expansion in some basis b should be transcendental.

The state of the art

There is no explicitly known example of a triple (b, a, x), where $b \geq 3$ is an integer, a is a digit in $\{0, \ldots, b-1\}$ and x is an algebraic irrational number, for which one can claim that the digit a occurs infinitely often in the b–ary expansion of x.

A stronger conjecture, also due to Borel, is that algebraic irrational real numbers are *normal* : each sequence of n digits in basis b should occur with the frequency $1/b^n$, for all b and all n.

What is known on the decimal expansion of $\sqrt{2}$?

The sequence of digits (in any basis) of $\sqrt{2}$ is not ultimately periodic

Among the decimal digits

$$\{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\},$$

at least two of them occur infinitely often. Almost nothing

else is known.

Complexity of the expansion in basis b of a real irrational algebraic number

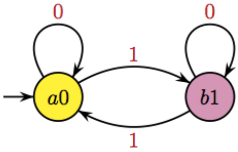
Theorem (B. Adamczewski, Y. Bugeaud 2005 ; conjecture of A. Cobham 1968).
If the sequence of digits of a real number x is produced by a finite automaton, then x is either rational or else transcendental.

Finite automata

The Prouhet – Thue – Morse sequence : A010060 OEIS

$$(t_n)_{n\geq 0} = (01101001100101101001011001101001\dots)$$

Write the number n in binary. If the number of ones in this binary expansion is odd then $t_n = 1$, if even then $t_n = 0$.



Fixed point of the morphism $0 \mapsto 01$, $1 \mapsto 10$.

Start with 0 and successively append the Boolean complement of the sequence obtained thus far.

$$t_0 = 0, \qquad t_{2n} = t_n, \qquad t_{2n+1} = 1 - t_n$$

Sequence without cubes XXX.

§2 Irrationality of transcendental numbers

- The number e
- The number π
- Open problems

Introductio in analysin infinitorum

Leonhard Euler (1737)

(1707 – 1783)

Introductio in analysin infinitorum

e is irrational.

Continued fraction of e :

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\ddots}}}}}}$$

Joseph Fourier

Fourier (1815) : proof by means of the series expansion

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{N!} + r_N$$

with $r_N > 0$ and $N! r_N \to 0$ as $N \to +\infty$.

Course of analysis at the École Polytechnique Paris, 1815.

Variant of Fourier's proof : e^{-1} is irrational

C.L. Siegel : Alternating series

For odd N,

$$1 - \frac{1}{1!} + \frac{1}{2!} - \cdots - \frac{1}{N!} < \mathrm{e}^{-1} < 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{(N+1)!}$$

$$\frac{a_N}{N!} < \mathrm{e}^{-1} < \frac{a_N}{N!} + \frac{1}{(N+1)!}, \quad a_N \in \mathbf{Z}$$

$$a_N < N!\mathrm{e}^{-1} < a_N + 1.$$

Hence $N!\mathrm{e}^{-1}$ is not an integer.

Irrationality of π

Āryabhaṭa, born 476 AD : $\pi \sim 3.1416$.

Nīlakaṇṭha Somayājī, born 1444 AD : *Why then has an approximate value been mentioned here leaving behind the actual value ? Because it (exact value) cannot be expressed.*

K. Ramasubramanian, *The Notion of Proof in Indian Science*, 13th World Sanskrit Conference, 2006.

Irrationality of π

Johann Heinrich Lambert (1728 – 1777)
Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques,
Mémoires de l'Académie des Sciences de Berlin, **17** (1761), p. 265-322 ;
lu en 1767 ; Math. Werke, t. II.

$\tan(v)$ is irrational when $v \neq 0$ is rational.
As a consequence, π is irrational, since $\tan(\pi/4) = 1$.

Lambert and Frederick II, King of Prussia

— Que savez vous, Lambert ?
— Tout, Sire.
— Et de qui le tenez–vous ?
— De moi-même !

Known and unknown transcendence results

Known :

$$\mathrm{e},\ \pi,\ \log 2,\ \mathrm{e}^{\sqrt{2}},\ \mathrm{e}^{\pi},\ 2^{\sqrt{2}},\ \Gamma(1/4).$$

Not known :

$$e+\pi,\ e\pi,\ \log\pi,\ \pi^{e},\ \Gamma(1/5),\ \zeta(3),\ \text{Euler constant}$$

Why is e^{π} known to be transcendental while π^{e} is not known to be irrational ?
Answer : $\mathrm{e}^{\pi} = (-1)^{-i}$.

Catalan's constant

Is Catalan's constant

$$\sum_{n\geq 1} \frac{(-1)^n}{(2n+1)^2} = 0.915\,965\,594\,177\,219\,015\,0\dots$$

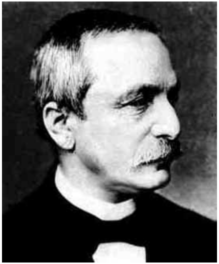
an irrational number ?

Catalan's constant, Dirichlet and Kronecker

Catalan's constant is the value at $s = 2$ of the Dirichlet L–function $L(s, \chi_{-4})$ associated with the Kronecker character

$$\chi_{-4}(n) = \left(\frac{n}{4}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv -1 \pmod 4. \end{cases}$$

Johann Peter Gustav Lejeune Dirichlet
1805 – 1859



Leopold Kronecker
1823 – 1891

Catalan's constant, Dedekind and Riemann

The Dirichlet L–function $L(s, \chi_{-4})$ associated with the Kronecker character χ_{-4} is the quotient of the Dedekind zeta function of $\mathbf{Q}(i)$ and the Riemann zeta function :

$$\zeta_{\mathbf{Q}(i)}(s) = L(s, \chi_{-4})\zeta(s)$$

Julius Wilhelm Richard Dedekind
1831 – 1916

Georg Friedrich Bernhard Riemann
1826 – 1866

Riemann zeta function

The function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

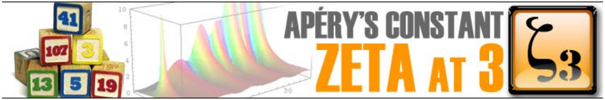
was studied by Euler (1707– 1783)
for integer values of s
and by Riemann (1859) for complex values of s.

Euler : for any even integer value of $s \geq 2$, the number $\zeta(s)$ is a rational multiple of π^s.

Examples : $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450 \cdots$

Coefficients : Bernoulli numbers.

Riemann zeta function



The number

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1.202\,056\,903\,159\,594\,285\,399\,738\,161\,511\ \ldots$$

is irrational *(Apéry 1978)*.

Recall that $\zeta(s)/\pi^s$ is rational for any even value of $s \geq 2$.

Open question : Is the number $\zeta(3)/\pi^3$ irrational ?

Riemann zeta function

Is the number

$$\zeta(5) = \sum_{n\geq 1} \frac{1}{n^5} = 1.036\,927\,755\,143\,369\,926\,331\,365\,486\,457\ldots$$

irrational ?

T. Rivoal (2000) : infinitely many $\zeta(2n+1)$ are irrational.

Infinitely many odd zeta values are irrational

Tanguy Rivoal (2000)

Let $\epsilon > 0$. For any sufficiently large odd integer a, the dimension of the $\mathbf{Q}$–vector space spanned by the numbers $1,\ \zeta(3),\ \zeta(5),\ \cdots,\zeta(a)$ is at least

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

Euler–Mascheroni constant

Lorenzo Mascheroni
(1750 – 1800)

Euler's Constant is

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$
$$= 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots$$

Is it a rational number ?

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{1}{k}-\log\left(1+\frac{1}{k}\right)\right) = \int_1^{\infty}\left(\frac{1}{[x]}-\frac{1}{x}\right)\mathrm{d}x$$
$$= -\int_0^1\int_0^1\frac{(1-x)\mathrm{d}x\mathrm{d}y}{(1-xy)\log(xy)}.$$

Euler's constant

Recent work by *J. Sondow* inspired by the work of F. Beukers on Apéry's proof.

F. Beukers

Jonathan Sondow

http://home.earthlink.net/~jsondow/

Jonathan Sondow http://home.earthlink.net/∼jsondow/

$$\gamma = \int_0^\infty \sum_{k=2}^{\infty} \frac{1}{k^2 \binom{t+k}{k}} \mathrm{d}t$$

$$\gamma = \lim_{s \to 1+} \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \frac{1}{s^n} \right)$$

$$\gamma = \int_1^\infty \frac{1}{2t(t+1)} F \left(\begin{matrix} 1, & 2, & 2 \\ 3, & t+2 \end{matrix} \right) \mathrm{d}t.$$

Euler Gamma function

Is the number

$$\Gamma(1/5) = 4.590\,843\,711\,998\,803\,053\,204\,758\,275\,929\,152\,\ldots$$

irrational ?

$$\Gamma(z) = \mathrm{e}^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} \mathrm{e}^{z/n} = \int_0^{\infty} \mathrm{e}^{-t} t^z \cdot \frac{\mathrm{d}t}{t}$$

Here is the set of rational values $r \in (0,1)$ for which the answer is known (and, for these arguments, the Gamma value $\Gamma(r)$ is a transcendental number) :

$$r \in \left\{ \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \right\} \pmod 1.$$

Georg Cantor (1845 - 1918)

The set of algebraic numbers is countable, not the set of real (or complex) numbers.

Cantor (1874 and 1891).

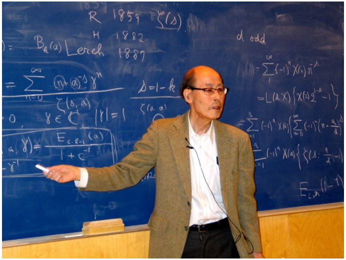
Henri Léon Lebesgue (1875 – 1941)

Almost all numbers for Lebesgue measure are transcendental numbers.

Most numbers are transcendental

Meta conjecture : *any number given by some kind of limit, which is not obviously rational (resp. algebraic), is irrational (resp. transcendental).*

Goro Shimura



Special values of hypergeometric series

Jürgen Wolfart

Frits Beukers

Sum of values of a rational function

Work by S.D. Adhikari, N. Saradha, T.N. Shorey and R. Tijdeman (2001),

Let P and Q be non-zero polynomials having rational coefficients and $\deg Q \geq 2 + \deg P$. Consider

$$\sum_{\substack{n \geq 0 \\ Q(n) \neq 0}} \frac{P(n)}{Q(n)}.$$

Robert Tijdeman

Sukumar Das Adhikari

N. Saradha

Telescoping series

Examples

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1, \quad \sum_{n=0}^{\infty} \frac{1}{n^2-1} = \frac{3}{4},$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{5n+2} - \frac{3}{5n+7} + \frac{1}{5n-3} \right) = \frac{5}{6}$$

Transcendental values

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3}$$

are transcendental.

Transcendental values

$$\sum_{n=0}^{\infty} \frac{1}{(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(6n+6)}$$

$$= \frac{1}{4320}(192\log 2 - 81\log 3 - 7\pi\sqrt{3})$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}} = 2.076\,674\,047\,4\ldots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{2\pi}{e^{\pi}-e^{-\pi}} = 0.272\,029\,054\,982\ldots$$

Leonardo Pisano (Fibonacci)

The Fibonacci sequence $(F_n)_{n\geq 0}$:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,$$

$$34,\ 55,\ 89,\ 144,\ 233\dots$$

is defined by

$$F_0 = 0,\ F_1 = 1,$$

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$$

Leonardo Pisano (Fibonacci)
(1170–1250)

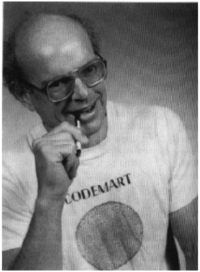
Encyclopedia of integer sequences (again)

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, \ldots$

The Fibonacci sequence is available online

The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane



http://oeis.org/A000045

Series involving Fibonacci numbers

The number

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1$$

is rational, while

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1-\sqrt{5}}{2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}+1} = \frac{\sqrt{5}}{2}$$

are irrational algebraic numbers.

Series involving Fibonacci numbers

The numbers

$$\sum_{n=1}^{\infty} \frac{1}{{F_n}^2}, \quad \sum_{n=1}^{\infty} \frac{1}{{F_n}^4}, \quad \sum_{n=1}^{\infty} \frac{1}{{F_n}^6},$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{{F_n}^2}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{2n}},$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2^n-1} + F_{2^n+1}}, \qquad \sum_{n=1}^{\infty} \frac{1}{F_{2^n+1}}$$

are all transcendental

Series involving Fibonacci numbers

Each of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n + F_{n+2}}$$

$$\sum_{n\geq 1} \frac{1}{F_1 F_2 \cdots F_n}$$

is irrational, but it is not known whether they are algebraic or transcendental.

The first challenge here is to formulate a conjectural statement which would give a satisfactory description of the situation.

The Fibonacci zeta function

For $\Re e(s) > 0$,

$$\zeta_F(s) = \sum_{n\geq 1} \frac{1}{{F_n}^s}$$

$\zeta_F(2)$, $\zeta_F(4)$, $\zeta_F(6)$ are algebraically independent.

Iekata Shiokawa, Carsten Elsner and Shun Shimomura (2006)

Iekata Shiokawa

§3 Transcendental numbers

- Liouville (1844)
- Hermite (1873)
- Lindemann (1882)
- Hilbert's Problem 7th (1900)
- Gel'fond–Schneider (1934)
- Baker (1968)
- Nesterenko (1995)

Existence of transcendental numbers (1844)

J. Liouville (1809 - 1882)

gave the first examples of transcendental numbers.
For instance

$$\sum_{n\geq 1} \frac{1}{10^{n!}} = 0.110\,001\,000\,000\,0\ldots$$

is a transcendental number.

Charles Hermite and Ferdinand Lindemann

Hermite (1873) :
Transcendence of e
$e = 2.718\,281\,828\,4\ldots$

Lindemann (1882) :
Transcendence of π
$\pi = 3.141\,592\,653\,5\ldots$

Hermite–Lindemann Theorem

For any non-zero complex number z, one at least of the two numbers z and e^z is transcendental.

Corollaries : Transcendence of $\log \alpha$ and of e^β for α and β non-zero algebraic complex numbers, provided $\log \alpha \neq 0$.

Transcendental functions

A complex function is called transcendental if it is transcendental over the field $\mathbf{C}(z)$, which means that the functions z and $f(z)$ are algebraically independent : if $P \in \mathbf{C}[X, Y]$ is a non-zero polynomial, then the function $P\big(z, f(z)\big)$ is not 0.

Exercise. *An entire function (analytic in $\mathbf{C}$) is transcendental if and only if it is not a polynomial.*

Example. *The transcendental entire function e^z takes an algebraic value at an algebraic argument z only for $z = 0$.*

Weierstrass question

Is it true that a transcendental entire function f takes usually *transcendental values at algebraic arguments ?*

Examples : for $f(z) = \mathrm{e}^z$, there is a single exceptional point α algebraic with e^α also algebraic, namely $\alpha = 0$.

For $f(z) = \mathrm{e}^{P(z)}$ where $P \in \mathbf{Z}[z]$ is a non–constant polynomial, there are finitely many exceptional points α, namely the roots of P.

The exceptional set of $\mathrm{e}^z + \mathrm{e}^{1+z}$ is empty (Lindemann–Weierstrass).

The exceptional set of functions like 2^z or $\mathrm{e}^{\pi \mathrm{i} z}$ is $\mathbf{Q}$, (Gel'fond and Schneider).

Exceptional sets

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain...
If S is a countable subset of $\mathbf{C}$ and T is a dense subset of $\mathbf{C}$, there exist transcendental entire functions f mapping S into T, as well as all its derivatives.

Any set of algebraic numbers is the exceptional set of some transcendental entire function.
Also multiplicities can be included.

van der Poorten : *there are transcendental entire functions f such that $D^k f(\alpha) \in \mathbf{Q}(\alpha)$ for all $k \geq 0$ and all algebraic α.*

Integer valued entire functions

An integer valued entire function is a function f, which is analytic in $\mathbf{C}$, and maps $\mathbf{N}$ into $\mathbf{Z}$.

Example : 2^z *is an integer valued entire function, not a polynomial.*

Question : *Are there integer valued entire function growing slower than* 2^z *without being a polynomial ?*

Let f be a transcendental entire function in $\mathbf{C}$. For $R > 0$ set

$$|f|_R = \sup_{|z|=R} |f(z)|.$$

Integer valued entire functions

G. Pólya (1914) :
if f *is not a polynomial*
and $f(n) \in \mathbf{Z}$ *for* $n \in \mathbf{Z}_{\geq 0}$, *then*

$$\limsup_{R\to\infty} 2^{-R}|f|_R \geq 1.$$

Further works on this topic by G.H. Hardy, G. Pólya, D. Sato, E.G. Straus, A. Selberg, Ch. Pisot, F. Carlson, F. Gross,...

Integer valued entire function on $\mathbf{Z}[i]$

A.O. Gel'fond (1929) : growth of entire functions mapping the Gaussian integers into themselves.
Newton interpolation series at the points in $\mathbf{Z}[i]$.

An entire function f *which is not a polynomial and satisfies* $f(a+ib) \in \mathbf{Z}[i]$ *for all* $a+ib \in \mathbf{Z}[i]$ *satisfies*

$$\limsup_{R\to\infty} \frac{1}{R^2} \log |f|_R \geq \delta.$$

F. Gramain (1981) : $\delta = \pi/(2e) = 0.577\,863\,674\,8\ldots$
This is best possible : *D.W. Masser (1980).*

Transcendence of e^{π}

A.O. Gel'fond (1929).

If
$$\mathrm{e}^{\pi} = 23.140\,692\,632\,779\,269\,005\,729\,086\,367\ \ldots$$
is rational, then the function $\mathrm{e}^{\pi z}$ takes values in $\mathbf{Q}(i)$ when the argument z is in $\mathbf{Z}[i]$.

Expand $\mathrm{e}^{\pi z}$ into an interpolation series at the Gaussian integers.

Hilbert's Problems

August 8, 1900

Second International Congress of Mathematicians in Paris.

David Hilbert (1862 - 1943)

Twin primes,

Goldbach's Conjecture,

Riemann Hypothesis

Transcendence of e^{π} and $2^{\sqrt{2}}$

A.O. Gel'fond and Th. Schneider

Solution of Hilbert's seventh problem (1934) : *Transcendence of* α^β *and of* $(\log \alpha_1)/(\log \alpha_2)$ *for algebraic* α, β, α_1 *and* α_2.

Transcendence of α^β and $\log\alpha_1/\log\alpha_2$: examples

The following numbers are transcendental :

$$2^{\sqrt{2}} = 2.665\,144\,142\,6\ldots$$

$$\frac{\log 2}{\log 3} = 0.630\,929\,753\,5\ldots$$

$$\mathrm{e}^\pi = 23.140\,692\,632\,7\ldots \qquad\qquad (\mathrm{e}^\pi = (-1)^{-i})$$

$$\mathrm{e}^{\pi\sqrt{163}} = 262\ 537\ 412\ 640\ 768\ 743.999\ 999\ 999\ 999\ 25\ldots$$

$e^\pi = (-1)^{-i}$

Example : Transcendence of the number

$$e^{\pi\sqrt{163}} = 262\ 537\ 412\ 640\ 768\ 743.999\ 999\ 999\ 999\ 2\ldots.$$

Remark. For

$$\tau = \frac{1 + i\sqrt{163}}{2}, \quad q = e^{2\pi i\tau} = -e^{-\pi\sqrt{163}}$$

we have $j(\tau) = -640\ 320^3$ and

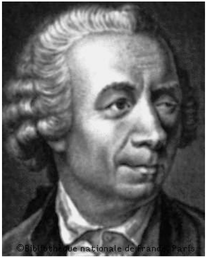
$$\left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}.$$

Beta values : Th. Schneider 1948

Euler Gamma and Beta functions

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x.$$

$$\Gamma(z)=\int_0^\infty \mathrm{e}^{-t}t^z\cdot\frac{\mathrm{d}t}{t}$$



$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Algebraic independence : A.O. Gel'fond 1948

The two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent.

More generally, if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an algebraic number of degree $d \geq 3$, then two at least of the numbers

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \ldots,\alpha^{\beta^{d-1}}$$

are algebraically independent.

Alan Baker 1968

Transcendence of numbers like

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

or

$$\mathrm{e}^{\beta_0} {\alpha_1}^{\beta_1} \cdots {\alpha_1}^{\beta_1}$$

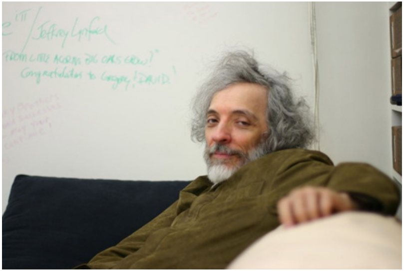
for algebraic α_i's and β_j's.

Example (Siegel) :

$$\int_0^1 \frac{\mathrm{d}x}{1+x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right) = 0.835\,648\,848\,\ldots$$

is transcendental.

Gregory V. Chudnovsky



G.V. Chudnovsky (1976)
Algebraic independence of the numbers π and $\Gamma(1/4)$.
Also : algebraic independence of the numbers π and $\Gamma(1/3)$.

Corollaries : *Transcendence of* $\Gamma(1/4) = 3.625\,609\,908\,2\ldots$ *and* $\Gamma(1/3) = 2.678\,938\,534\,7\ldots$.

Yuri V. Nesterenko

Yu.V.Nesterenko (1996)
Algebraic independence of
$\Gamma(1/4)$, π and e^{π}.
Also : Algebraic independence of
$\Gamma(1/3)$, π and $e^{\pi\sqrt{3}}$.

Corollary : *The numbers* $\pi = 3.141\,592\,653\,5\ldots$ *and* $e^{\pi} = 23.140\,692\,632\,7\ldots$ *are algebraically independent.*

Transcendence of values of Dirichlet's L–functions :
Sanoli Gun, Ram Murty and Purusottam Rath (2009).

Eisenstein series, modular functions,

$$E_2(q) = 1 - 24 \sum_{k=1}^{\infty} \sigma_1(k) q^{2k},$$

$$E_4(q) = 1 + 240 \sum_{k=1}^{\infty} \sigma_3(k) q^{2k}, \qquad \sigma_t(n) = \sum_{d|n} d^t.$$

$$E_6(q) = 1 - 504 \sum_{k=1}^{\infty} \sigma_5(k) q^{2k},$$

Ramanujan notation : $P = E_2$, $Q = E_4$, $R = E_6$.

Yu.V.Nesterenko (1996) : For $0 < |q| < 1$, at least three of the four numbers q, $P(q)$, $Q(q)$, $R(q)$ are algebraically independent.

E. Grosswald (1976)

$$F_k(z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{-k} \mathrm{e}^{2\pi \mathrm{i} mnz}.$$

For k odd ≥ 3,

$$F_k(z) = \sum_{n=1}^{\infty} \sigma_{-k}(n) \mathrm{e}^{2\pi \mathrm{i} nz} = -\zeta(k) - \sum_{n=1}^{\infty} \frac{1}{n^k(\mathrm{e}^{2\pi \mathrm{i} nz} - 1)}.$$

Lambert type series

$$\zeta(3) + 2\sum_{n=1}^{\infty} \frac{1}{n^3(\mathrm{e}^{2\pi n} - 1)} = \frac{7\pi^3}{180}.$$

For $k \geq 0$, one at least of the two numbers

$$\zeta(4k+3), \qquad \sum_{n=1}^{\infty} \frac{1}{n^{4k+3}(\mathrm{e}^{2\pi n} - 1)}$$

is transcendental.

Eichler integrals

$q = \mathrm{e}^{2\pi \mathrm{i} z}$:

$$\mathrm{E}_k(z) = \gamma_k \sum_{n=1}^{\infty} \sigma_{k-1}(n) \mathrm{e}^{2\pi \mathrm{i} n z}, \quad \gamma_k = -\frac{B_k}{2k}.$$

Eichler integrals

$$F_k(z) = \frac{(-2\pi \mathrm{i})^k}{(k-1)!} \int_{i\infty}^{z} \left[\mathrm{E}_{k+1}(\tau) - \gamma_k\right] (\tau - z)^{k-1} \mathrm{d}\tau.$$

S. Gun, R. Murty, P. Rath (2011) : for k a non negative integer, with at most $2k+5$ exceptions, the number

$$F_{2k+1}(\alpha) - \alpha^{2k} F_{2k+1}(-1/\alpha)$$

is transcendental for every algebraic α in the upper half plane.

Purusottam Rath, Ram Murty, Sanoli Gun

S. Gun, R. Murty, P. Rath, *Transcendental values of certain Eichler integrals*, Bull. London Math. Soc. **43** (2011), 939–952.

R. Murty, C. Smyth, R. Wang, *Zeroes of Ramanujan polynomials*, J. Ramanujan Math. Soc. **26** (2011), 107–125.

Weierstraß sigma function

Let $\Omega = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ be a lattice in $\mathbf{C}$. The canonical product attached to Ω is the *Weierstraß sigma function*

$$\sigma(z) = \sigma_\Omega(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \mathrm{e}^{(z/\omega)+(z^2/2\omega^2)}.$$

The number

$$\sigma_{\mathbf{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}\mathrm{e}^{\pi/8}\Gamma(1/4)^{-2}$$

is transcendental.

§4 : Conjectures

Borel 1909, 1950

Schanuel 1964

Grothendieck 1960's

Rohrlich and Lang 1970's

André 1990's

Kontsevich and Zagier 2001.

Periods : Maxime Kontsevich and Don Zagier

Periods, *Mathematics unlimited—2001 and beyond*, Springer 2001, 771–808.

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbf{R}^n$ given by polynomial inequalities with rational coefficients.

The number π

Basic example of a *period* :

$$e^{z+2\pi i} = e^z$$

$$2\pi i = \int_{|z|=1} \frac{dz}{z}$$

$$\begin{aligned} \pi &= \int\int_{x^2+y^2\leq 1} dx dy &&= 2\int_{-1}^{1} \sqrt{1-x^2} dx \\ &= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} &&= \int_{-\infty}^{\infty} \frac{dx}{1-x^2}. \end{aligned}$$

Further examples of periods

$$\sqrt{2} = \int_{2x^2 \leq 1} \mathrm{d}x$$

and all algebraic numbers.

$$\log 2 = \int_{1<x<2} \frac{\mathrm{d}x}{x}$$

and all logarithms of algebraic numbers.

$$\pi = \int_{x^2+y^2 \leq 1} \mathrm{d}x\mathrm{d}y,$$

A product of periods is a period (subalgebra of $\mathbf{C}$), but $1/\pi$ is expected not to be a period.

Numbers which are not periods

Problem (Kontsevich–Zagier) : *To produce an explicit example of a number which is not a period.*

Several levels :

1 *analog of Cantor* : the set of periods is countable. Hence there are real and complex numbers which are not periods ("most" of them).

Numbers which are not periods

2 *analog of Liouville*

Find a property which should be satisfied by all periods, and construct a number which does not satisfies that property.

Masahiko Yoshinaga, Periods and elementary real numbers
arXiv:0805.0349

Compares the periods with hierarchy of real numbers induced from computational complexities.
In particular, he proves that periods can be effectively approximated by elementary rational Cauchy sequences.

As an application, he exhibits a computable real number which is not a period.

Numbers which are not periods

3 *analog of Hermite*

Prove that given numbers are not periods

Candidates : $1/\pi$, e, Euler constant.

M. Kontsevich : exponential periods

"The last chapter, which is at a more advanced level and also more speculative than the rest of the text, is by the first author only."

Relations among periods

1 Additivity

(in the integrand and in the domain of integration)

$$\int_a^b \big(f(x)+g(x)\big)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x,$$

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$$

2 Change of variables :

if $y = f(x)$ is an invertible change of variables, then

$$\int_{f(a)}^{f(b)} F(y)\mathrm{d}y = \int_a^b F\big(f(x)\big) f'(x)\mathrm{d}x.$$

Relations among periods (continued)



3 Newton–Leibniz–Stokes Formula

$$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a).$$

Conjecture of Kontsevich and Zagier

A widely-held belief, based on a judicious combination of experience, analogy, and wishful thinking, is the following

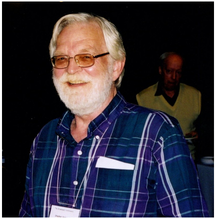
Conjecture (Kontsevich–Zagier). *If a period has two integral representations, then one can pass from one formula to another by using only rules* $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ *in which all functions and domains of integration are algebraic with algebraic coefficients.*

Conjecture of Kontsevich and Zagier (continued)

In other words, we do not expect any miraculous coïncidence of two integrals of algebraic functions which will not be possible to prove using three simple rules.
This conjecture, which is similar in spirit to the Hodge conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.

Advice : *if you wish to prove a number is transcendental, first prove it is a period.*

Conjectures by S. Schanuel and A. Grothendieck



• Schanuel : *if* $x_1, \ldots, x_n$ *are* $\mathbf{Q}$*–linearly independent complex numbers, then* n *at least of the* $2n$ *numbers* $x_1, \ldots, x_n$*,* $e^{x_1}, \ldots, e^{x_n}$ *are algebraically independent.*

• Periods conjecture by Grothendieck : Dimension of the Mumford–Tate group of a smooth projective variety.

Consequences of Schanuel's Conjecture

Ram Murty

Kumar Murty

N. Saradha

Purusottam Rath, Ram Murty, Sanoli Gun

Ram and Kumar Murty (2009)

Ram Murty

Kumar Murty

Transcendental values of class group L–functions.

Motives

Y. André : generalization of Grothendieck's conjecture to motives.

Case of 1–motives : Elliptico-Toric Conjecture of C. Bertolin.

Francis Brown

arXiv:1412.6508 Irrationality proofs for zeta values, moduli spaces and dinner parties
Date : Fri, 19 Dec 2014 20 :08 :31 GMT (50kb)

A simple geometric construction on the moduli spaces $\mathcal{M}_{0,n}$ of curves of genus 0 with n ordered marked points is described which gives a common framework for many irrationality proofs for zeta values. This construction yields Apéry's approximations to $\zeta(2)$ and $\zeta(3)$, and for larger n, an infinite family of small linear forms in multiple zeta values with an interesting algebraic structure. It also contains a generalisation of the linear forms used by Ball and Rivoal to prove that infinitely many odd zeta values are irrational.

Francis Brown

For k, $s_1, \dots, s_k$ positive integers with $s_1 \geq 2$, we set $\underline{s} = (s_1, \dots, s_k)$ and

$$\zeta(\underline{s}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{{n_1}^{s_1} \cdots {n_k}^{s_k}}.$$

The $\mathbf{Q}$–vector space $\mathfrak{Z}$ spanned by the numbers $\zeta(\underline{s})$ is also a $\mathbf{Q}$–algebra. For $n \geq 2$, denote by $\mathfrak{Z}_n$ the $\mathbf{Q}$-subspace of $\mathfrak{Z}$ spanned by the real numbers $\zeta(\underline{s})$ where $\underline{s}$ has weight $s_1 + \cdots + s_k = n$.

The numbers $\zeta(s_1, \dots, s_k)$, $s_1 + \cdots + s_k = n$, where each s_i is 2 or 3, span $\mathfrak{Z}_n$ over $\mathbf{Q}$.

Transcendental Number Theory: recent results and open problems.

Michel Waldschmidt

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