Conference "Modular Forms and Function Field Arithmetic"
in honor of ProfessorJing Yu's 60th birthday.
Taida Institute for Mathematical Sciences (TIMS)
National Taiwan University, Taipei, Taiwan.
http://www.tims.ntu.edu.tw/ May 19–22, 2009

Schanuel's Conjecture and Criteria for Algebraic Independence

Michel Waldschmidt

Institut de Mathématiques de Jussieu & Paris VI
http://www.math.jussieu.fr/∼miw/

Lecture given on May 22, 2009; updated: May 24, 2009

Abstract

One of the main open problems in transcendental number theory is Schanuel's Conjecture which was stated in the 1960's :

> *If $x_1, \ldots, x_n$ are $\mathbf{Q}$–linearly independent complex numbers, then among the $2n$ numbers $x_1, \ldots, x_n$, $e^{x_1}, \ldots, e^{x_n}$, at least n are algebraically independent.*

We first consider some of the consequences of this conjecture ; next we describe the transcendental approach which was initiated by A.O. Gel'fond in the 40's, and developed by a number of mathematicians including W.D. Brownawell, G.V. Chudnovsky, P. Philippon, Yu. Nesterenko and more recently D. Roy.

Dale Brownawell and Stephen Schanuel

Schanuel's Conjecture

Let $x_1, \ldots, x_n$ be $\mathbf{Q}$-linearly independent complex numbers. Then at least n of the $2n$ numbers $x_1, \ldots, x_n$, $e^{x_1}, \ldots, e^{x_n}$ are algebraically independent.

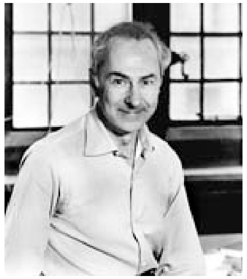
In other terms, the conclusion is

$$\operatorname{tr}\deg_{\mathbf{Q}}\mathbf{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n.$$

Remark : For almost all tuples (with respect to the Lebesgue measure) the transcendence degree is $2n$.

Origin of Schanuel's Conjecture

Course given by Serge Lang (1927–2005) at Columbia in the 60's

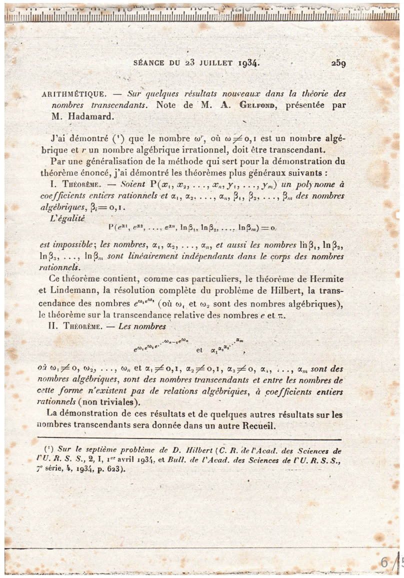


S. LANG – *Introduction to transcendental numbers*, Addison-Wesley 1966.

also attended by M. Nagata (1927–2008) (14*th* Problem of Hilbert).

Nagata's Conjecture solved by E. Bombieri.

A.O. Gel'fond CRAS 1934



SÉANCE DU 23 JUILLET 1934. 259

ARITHMÉTIQUE. — *Sur quelques résultats nouveaux dans la théorie des nombres transcendants.* Note de M. A. GELFOND, présentée par M. Hadamard.

J'ai démontré (1) que le nombre ω^r, où $\omega \neq 0, 1$ est un nombre algébrique et r un nombre algébrique irrationnel, doit être transcendant.

Par une généralisation de la méthode qui sert pour la démonstration du théorème énoncé, j'ai démontré les théorèmes plus généraux suivants :

I. THÉORÈME. — *Soient* $P(x_1, x_2, \dots, x_n, y_1, \dots, y_m)$ *un polynome à coefficients entiers rationnels et* $\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m$ *des nombres algébriques,* $\beta_i = 0, 1$.

L'égalité

$$P(e^{\alpha_1}, e^{\alpha_2}, \dots, e^{\alpha_n}, \ln\beta_1, \ln\beta_2, \dots, \ln\beta_m) = 0$$

est impossible; les nombres, $\alpha_1, \alpha_2, \dots, \alpha_n$, *et aussi les nombres* $\ln\beta_1, \ln\beta_2, \ln\beta_3, \dots, \ln\beta_m$ *sont linéairement indépendants dans le corps des nombres rationnels.*

Ce théorème contient, comme cas particuliers, le théorème de Hermite et Lindemann, la résolution complète du problème de Hilbert, la transcendance des nombres $e^{\omega_1 e^{\omega_2}}$ (où ω_1 et ω_2 sont des nombres algébriques), le théorème sur la transcendance relative des nombres e et π.

II. THÉORÈME. — *Les nombres*

$$e^{\omega_1 e^{\omega_2 e^{\cdot^{\cdot^{\cdot \omega_{n-1} e^{\omega_n}}}}}} \quad \text{et} \quad \alpha_1^{\alpha_2^{\alpha_3^{\cdot^{\cdot^{\cdot \alpha_m}}}}},$$

où $\omega_1 \neq 0, \omega_2, \dots, \omega_n$ et $\alpha_1 \neq 0, 1$, $\alpha_2 \neq 0, 1$, $\alpha_3 \neq 0$, $\alpha_4, \dots, \alpha_m$ *sont des nombres algébriques, sont des nombres transcendants et entre les nombres de cette forme n'existent pas de relations algébriques, à coefficients entiers rationnels* (non triviales).

La démonstration de ces résultats et de quelques autres résultats sur les nombres transcendants sera donnée dans un autre Recueil.

(1) *Sur le septième problème de D. Hilbert* (*C. R. de l'Acad. des Sciences de l'U. R. S. S.*, 2, I, 1er avril 1934, et *Bull. de l'Acad. des Sciences de l'U. R. S. S.*, 7e série, 4, 1934, p. 623).

Statement by Gel'fond (1934)

Let $\beta_1, \dots, \beta_n$ be $\mathbf{Q}$-linearly independent algebraic numbers and let $\log\alpha_1, \dots, \log\alpha_m$ be $\mathbf{Q}$-linearly independent logarithms of algebraic numbers. Then the numbers

$$e^{\beta_1}, \dots, e^{\beta_n}, \ \log\alpha_1, \dots, \log\alpha_m$$

are algebraically independent over $\mathbf{Q}$.

Further statement by Gel'fond

Let $\beta_1, \dots, \beta_n$ be algebraic numbers with $\beta_1 \neq 0$ and let $\alpha_1, \dots, \alpha_m$ be algebraic numbers with $\alpha_1 \neq 0, 1$, $\alpha_2 \neq 0, 1$, $\alpha_i \neq 0$. Then the numbers

$$e^{\beta_1 e^{\beta_2 e^{\cdot^{\cdot^{\cdot \beta_{n-1} e^{\beta_n}}}}}} \quad \text{and} \quad \alpha_1^{\alpha_2^{\cdot^{\cdot^{\cdot \alpha_m}}}}$$

are transcendental, and there is no nontrivial algebraic relation between such numbers.

Remark : The condition on α_2 should be that it is irrational.

Easy consequence of Schanuel's Conjecture

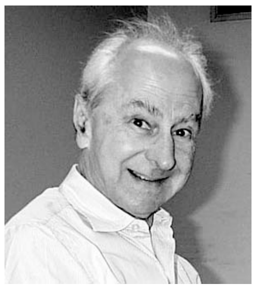
According to Schanuel's Conjecture, the following numbers are algebraically independent :

$$e+\pi,\ e\pi,\ \pi^e,\ e^e, e^{e^2}, \dots,\ e^{e^e}, \dots,\ \pi^\pi, \pi^{\pi^2}, \dots\ \pi^{\pi^\pi} \dots$$

$$\log\pi,\ \log(\log 2),\ \pi\log 2,\ (\log 2)(\log 3),\ 2^{\log 2},\ (\log 2)^{\log 3} \dots$$

Proof : Use Schanuel''s Conjecture several times.

Lang's exercise



Define $E_0 = \mathbf{Q}$. Inductively, for $n \geq 1$, define E_n as the algebraic closure of the field generated over E_{n-1} by the numbers $\exp(x) = e^x$, where x ranges over E_{n-1}. Let E be the union of E_n, $n \geq 0$. Then Schanuel's Conjecture implies that the number π does not belong to E.

More precisely : Schanuel's Conjecture implies that the numbers $\pi, \log\pi, \log\log\pi, \log\log\log\pi, \dots$ are algebraically independent over E.

A variant of Lang's exercise

Define $L_0 = \mathbf{Q}$. Inductively, for $n \geq 1$, define L_n as the algebraic closure of the field generated over L_{n-1} by the numbers y, where y ranges over the set of complex numbers such that $e^y \in L_{n-1}$. Let L be the union of L_n, $n \geq 0$. Then Schanuel's Conjecture implies that the number e does not belong to L.

More precisely : Schanuel's Conjecture implies that the numbers $e, e^e, e^{e^e}, e^{e^{e^e}} \dots$ are algebraically independent over L.

Arizona Winter School AWS2008, Tucson

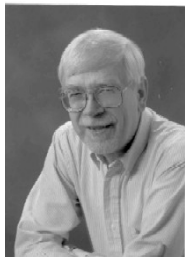
Theorem [Jonathan Bober, Chuangxun Cheng, Brian Dietel, Mathilde Herblot, Jingjing Huang, Holly Krieger, Diego Marques, Jonathan Mason, Martin Mereb and Robert Wilson.] *Schanuel's Conjecture implies that the fields E and L are linearly disjoint over $\overline{\mathbf{Q}}$.*

Definition Given a field extension F/K and two subextensions $F_1, F_2 \subseteq F$, we say F_1, F_2 are linearly disjoint over K when the following holds : *any set $\{x_1, \dots, x_n\} \subseteq F_1$ of $K-$ linearly independent elements is linearly independent over F_2.*

Reference : arXiv.0804.3550 [math.NT] 2008.

Formal analogs

W.D. Brownawell
(was a student of Schanuel)



J. Ax's Theorem (1968) : Version of Schanuel's Conjecture for power series over $\mathbf{C}$
(and R. Coleman for power series over $\overline{\mathbf{Q}}$)

Work by W.D. Brownawell and K. Kubota on the elliptic analog of Ax's Theorem.

Conjectures by A. Grothendieck and Y. André

Generalized Conjecture on Periods by Grothendieck : Dimension of the Mumford–Tate group of a smooth projective variety.

Generalization by Y. André to motives.

Case of 1–motives : Elliptico-Toric Conjecture of C. Bertolin.

Ubiquity of Schanuel's Conjecture

Other contexts : p–adic numbers, Leopoldt's Conjecture on the p–adic rank of the units of an algebraic number field
Non-vanishing of Regulators
Non–degenerescence of heights
Conjecture of B. Mazur on rational points
Diophantine approximation on tori

Dipendra Prasad

Gopal Prasad

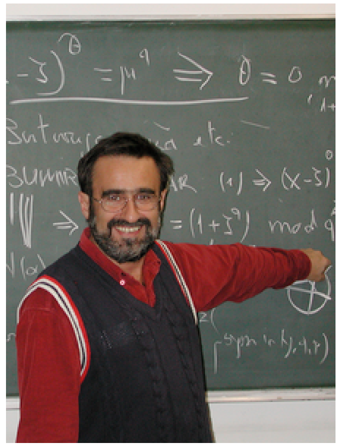
Preda Mihăilescu

arXiv:0905.1274

Date : Fri, 8 May 2009 14 :52 :57 GMT (16kb)

Title : *On Leopoldt's conjecture and some consequences*

Authors : Preda Mihăilescu



http ://arxiv.org/abs/0905.1274

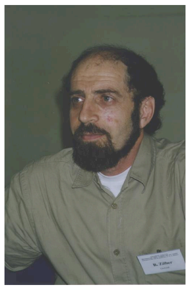
The conjecture of Leopoldt states that the p - adic regulator of a number field does not vanish. It was proved for the abelian case in 1967 by Brumer, using Baker theory. If the Leopoldt conjecture is false for a galois field $\mathbf{K}$, there is a *phantom* $\mathbf{Z}_p$ - extension of $\mathbf{K}_\infty$ arising. We show that this is strictly correlated to some infinite Hilbert class fields over $\mathbf{K}_\infty$, which are generated at intermediate levels by roots from units from the base fields. It turns out that the extensions of this type have bounded degree. This implies the Leopoldt conjecture for arbitrary finite number fields.

Preda Mihăilescu

Methods from logic

Ehud Hrushovski

Boris Zilber



Jonathan Kirby

Calculus of "predimension functions" (E. Hrushovski)

Zilber's construction of a "pseudoexponentiation"

Also : A. Macintyre, D.E. Marker, G. Terzo, A.J. Wilkie, D. Bertrand...

Methods from logic : Model theory

Exponential algebraicity in exponential fields
by
Jonathan Kirby

The dimension of the exponential algebraic closure operator in an exponential field satisfies a weak Schanuel property.

A corollary is that there are at most countably many essential counterexamples to Schanuel's Conjecture.

`arXiv :0810.4285v2`

Daniel Bertrand

Daniel Bertrand,

Schanuel's conjecture for non-isoconstant elliptic curves over function fields.

Model theory with applications to algebra and analysis. Vol. 1, 41–62, London Math. Soc. Lecture Note Ser., **349**, Cambridge Univ. Press, Cambridge, 2008.

Schanuel's Conjecture for $n = 1$

For $n = 1$, Schanuel's Conjecture is the Hermite–Lindemann Theorem :

If x is a non–zero complex numbers, then one at least of the 2 numbers x, e^x is transcendental.

Equivalently, *if x is a non–zero algebraic number, then e^x is a transcendental number.*

Another equivalent statement is that *if α is a non–zero algebraic number and $\log\alpha$ any non–zero logarithm of α, then $\log\alpha$ is a transcendental number.*

Consequence : transcendence of numbers like

$$e, \quad \pi, \quad \log 2, \quad e^{\sqrt{2}}.$$

Not known

For $n = 2$ Schanuel's Conjecture is not yet known :

? If x_1, x_2 are $\mathbf{Q}$–linearly independent complex numbers, then among the 4 numbers $x_1, x_2, e^{x_1}, e^{x_2}$, at least 2 are algebraically independent.

A few consequences :
With $x_1 = 1$, $x_2 = i\pi$: algebraic independence of e and π.
With $x_1 = 1$, $x_2 = e$: algebraic independence of e and e^e.
With $x_1 = \log 2$, $x_2 = (\log 2)^2$: algebraic independence of $\log 2$ and $2^{\log 2}$.
With $x_1 = \log 2$, $x_2 = \log 3$: algebraic independence of $\log 2$ and $\log 3$.

Not known

It is not known that *there exist two logarithms of algebraic numbers which are algebraically independent.*

Even the non–existence of non–trivial quadratic relations among logarithms of algebraic numbers is not yet established.

According to the *four exponentials Conjecture*, *any quadratic relation $(\log\alpha_1)(\log\alpha_4) = (\log\alpha_2)(\log\alpha_3)$ is trivial : either $\log\alpha_1$ and $\log\alpha_2$ are linearly dependent, or else $\log\alpha_1$ and $\log\alpha_3$ are linearly dependent.*

Known

Lindemann–Weierstraß Theorem = case where $x_1, \dots, x_n$ are algebraic.

Let $\beta_1, \dots, \beta_n$ be algebraic numbers which are linearly independent over $\mathbf{Q}$. Then the numbers $e^{\beta_1}, \dots, e^{\beta_n}$ are algebraically independent over $\mathbf{Q}$.

Hilbert's seventh problem

A.O. Gel'fond and Th. Schneider (1934).
Solution of Hilbert's seventh problem :
transcendence of α^β
and of $(\log\alpha_1)/(\log\alpha_2)$
for algebraic α, β, α_2 *and* α_2.

A. Baker, 1968. *Let* $\log\alpha_1,\dots,\log\alpha_n$ *be* $\mathbf{Q}$*–linearly independent logarithms of algebraic numbers. Then the numbers* $1,\log\alpha_1,\dots,\log\alpha_n$ *are linearly independent over the field* $\overline{\mathbf{Q}}$.

Problem of Gel'fond and Schneider

Raised by A.O. Gel'fond in 1948 and Th. Schneider in 1952.

Conjecture : *If* α *is an algebraic number,* $\alpha\neq 0$, $\alpha\neq 1$ *and if* β *is an irrational algebraic number of degree* d, *then the* $d-1$ *numbers*

$$\alpha^\beta,\ \alpha^{\beta^2},\ \dots,\alpha^{\beta^{d-1}}$$

are algebraically independent.

Special case of Schanuel's Conjecture : Take $x_i=\beta^{i-1}\log\alpha$, $n=d$, so that $\{x_1,\dots,x_n,\ e^{x_1},\dots,e^{x_n}\}$ is

$$\{\log\alpha,\ \beta\log\alpha,\ \dots,\ \beta^{d-1}\log\alpha,\quad \alpha,\ \alpha^\beta,\ \dots,\ \alpha^{\beta^{d-1}}\}.$$

The conclusion of Schanuel's Conjecture is

$$\mathrm{tr\,deg}_{\mathbf{Q}}\mathbf{Q}(\log\alpha,\alpha^\beta,\ \alpha^{\beta^2},\ \dots,\alpha^{\beta^{d-1}})=d.$$

Algebraic independence method of Gel'fond

A.O. Gel'fond (1948)
The two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent.
More generally, if α is an algebraic number, $\alpha\neq 0$, $\alpha\neq 1$ and if β is a algebraic number of degree $d\geq 3$, then two at least of the numbers

$$\alpha^\beta,\ \alpha^{\beta^2},\ \dots,\alpha^{\beta^{d-1}}$$

are algebraically independent.

Tools

Transcendence criterion : Replaces Liouville's inequality in transcendence proofs.
Liouville : A non–zero rational integer $n\in\mathbf{Z}$ satisfies $|n|\geq 1$.
Gel'fond : Needs to give a lower bound for $|P(\theta)|$ with $P\in\mathbf{Z}[X]\setminus\{0\}$ when θ is transcendental.

Zero estimate for exponential polynomials :
C. Hermite, P. Turan, K. Mahler, R. Tijdeman,...

Small transcendence degree :
A.O. Gel'fond, A.A. Smelev, R. Tijdeman, W.D. Brownawell...

Analytic zero estimates for exponential polynomials

C. Hermite, P. Turan, K. Mahler



Sketch of proof

Assume the transcendence degree over $k := \mathbf{Q}(\alpha, \beta)$ of the field

$$L = k\big(\alpha^{\beta},\ \alpha^{\beta^2},\ \ldots, \alpha^{\beta^{d-1}}\big)$$

is ≤ 1. By the Theorem of Gel'fond and Schneider (solution to Hilbert's seventh problem) we know that the transcendence degree is 1.

(As a matter of fact, the proof of algebraic independence will reprove it).

Consider the exponential functions

$$e^{z}, e^{\beta z}, \ldots, e^{\beta^{d-1} z}$$

which are algebraically independent and satisfy differential equations with coefficients in $\mathbf{Q}(\beta) \subset k \subset L$.

These functions take values in L when the variable z is in

$$\Gamma = \big(\mathbf{Z} + \mathbf{Z}\beta \cdots + \mathbf{Z}\beta^{d-1}\big) \log \alpha.$$

Gel'fond–Schneider Method

Following the approach of Gel'fond and Schneider, one constructs a non–zero polynomial $P \in L[X_0, \ldots, X_{d-1}]$ such that the exponential polynomial

$$F(z) = P\big(e^{z}, e^{\beta z}, \ldots, e^{\beta^{d-1} z}\big)$$

vanishes with some multiplicity at many points in Γ, say

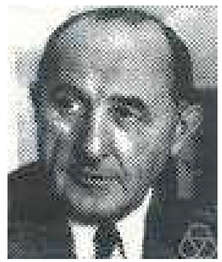
$$\left(\frac{d}{dz}\right)^{t} F\big(m_0 \log \alpha + m_1 \beta \log \alpha + \cdots + m_{d-1}\beta^{d-1} \log \alpha\big) = 0$$

for t, $m_0, \ldots, m_{d-1}$ non–negative integers in a certain range.

This is achieved by means of Dirichlet's Box Principle, hence one cannot get more such equations than there are unknowns (where unknowns are the coefficients of the auxiliary polynomial).

Pigeonhole principle (Dirichlet), Thue–Siegel Lemma

Lejeune-Dirichlet, C.L. Siegel

Extrapolation : Cauchy Schwarz

From Schwarz's Lemma we get a sharp upper bound for the maximum modulus of the auxiliary function F on some disc. Using Cauchy's inequalities, we deduce that many more values

$$\left(\frac{d}{dz}\right)^t F\left(m_0 \log\alpha + m_1\beta\log\alpha + \cdots + m_{d-1}\beta^{d-1}\log\alpha\right)$$

have a small modulus.
A zero estimate shows that these numbers cannot all vanish.
We endup with a non–zero number γ in L with a very small absolute value, for which we can also bound the *size*.

Size

Assume for simplicity that there is a transcendental number θ such that all the numbers β and α^{β^j} for $0 \le j \le d-1$ belong to $\mathbf{Z}[\theta]$. Then the number γ which is produced is just in $\mathbf{Z}[\theta]$, and the size of γ measures the degree and the height of this polynomial.

For a transcendence proof, one reaches the conclusion by means of Liouville's inequality. Here another argument is required. This is the transcendence criterion.

Transcendence criterion

Simple form : *Given a complex number ϑ, if there exists a sequence $(P_n)_{n\ge1}$ of non–zero polynomials in $\mathbf{Z}[X]$, with P_n of degree $\le n$ and height $\le e^n$, such that*

$$|P_n(\vartheta)| \le e^{-6n^2}$$

for all $n \ge 1$, then ϑ is algebraic and $P_n(\vartheta) = 0$ for all $n \ge 1$.

Simplification due to R. Tijdeman, W.D. Brownawell,. . .in the 70's and more recently M. Laurent and D. Roy.

Rob Tijdeman

http ://www.wiskundemeisjes.nl/20080830/ridder-tijdeman/

On the algebraic independence of certain numbers.
Nederl. Akad. Wetensch. Proc. Ser. A **74**=Indag. Math. **33** (1971), 146–162.

Gel'fond's transcendence criterion

First extension : Replace the upper bound for the degree by d_n, the upper bound for the height by e^{h_n}, and the upper bound for $|P_n(\vartheta)|$ by $e^{-\nu_n}$.

Assumptions on the sequences $(d_n)_{n\geq 1}$, $(h_n)_{n\geq 1}$ and $(\nu_n)_{n\geq 1}$:

$$d_n \leq d_{n+1} \leq \kappa d_n, \qquad d_n \leq h_n \leq h_{n+1} \leq \kappa h_n,$$

with some constant $\kappa > 0$ independent of n, and

$$\nu_n / d_n h_n \to \infty.$$

An equivalent statement

Let $m \geq 1$ and $(\vartheta_1, \ldots, \vartheta_m) \in \mathbf{C}^m$. Let $(d_n)_{n\geq 1}$, $(h_n)_{n\geq 1}$ and $(\nu_n)_{n\geq 1}$ satisfy :

$$d_n \leq d_{n+1} \leq \kappa d_n, \qquad d_n \leq h_n \leq h_{n+1} \leq \kappa h_n,$$

with some constant $\kappa > 0$ independent of n, and

$$\nu_n / d_n h_n \to \infty.$$

Assume that there exists a sequence $(P_n)_{n\geq 1}$ of non–zero polynomials in $\mathbf{Z}[X_1, \ldots, X_m]$, with P_n of degree $\leq d_n$ and height $\leq e^{h_n}$, such that

$$0 < |P_n(\vartheta_1, \ldots, \vartheta_m)| \leq e^{-\nu_n}$$

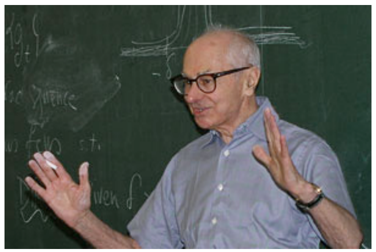
for all $n \geq 1$,
Then two at least of the numbers $\vartheta_1, \ldots, \vartheta_m$ are algebraically independent.
The conclusion is that *the transcendence degree of the field $\mathbf{Q}(\vartheta_1, \ldots, \vartheta_m)$ is at least 2.*

Criterion for large transcendence degree

It might seem natural to expect that the same statement with the stronger assumption $\nu_n / d_n^t h_n \to \infty$ in place of $\nu_n / d_n h_n \to \infty$ would yield the conclusion that the transcendence degree of the field $\mathbf{Q}(\vartheta_1, \ldots, \vartheta_m)$ is at least t.

A counterexample due to Khinchine (a reference is in Cassel's book on Diophantine Approximation) rules this out. Some further assumption is necessary.

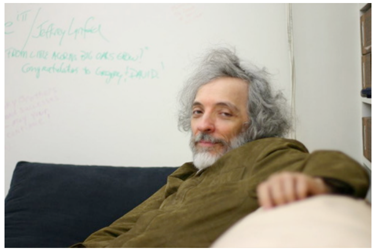
Lang's transcendence type



An inductive process has been suggested by S. Lang : at each step one produces a quantitative estimate (transcendence measure to start with, next measures of algebraic independence) which replace Liouville's inequality at the next stage.

Results produced by the method are rather weak and do not go further than small transcendence degree.

Large transcendence degree



G.V. Chudnovsky (1976)
Among the numbers

$$\alpha^\beta,\ \alpha^{\beta^2},\ \dots,\alpha^{\beta^{d-1}}$$

at least $[\log_2 d]$ are algebraically independent.

G.V. Chudnovsky – *On the path to Schanuel's Conjecture. Algebraic curves close to a point. I. General theory of colored sequences. II. Fields of finite transcendence type and colored sequences. Resultants.* Studia Sci. Math. Hungar. **12** (1977), 125–157 (1980).

Partial result on the problem of Gel'fond and Schneider

A.O. Gel'fond, G.V. Chudnovskii, P. Philippon, Yu.V. Nesterenko.

G. Diaz (1989) : If α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an irrational algebraic number of degree d, then

$$\mathrm{tr}\deg_{\mathbf{Q}}\mathbf{Q}(\alpha^\beta,\ \alpha^{\beta^2},\ \dots,\alpha^{\beta^{d-1}}) \geq \left[\frac{d+1}{2}\right].$$

How could we attack Schanuel's Conjecture ?

Let $x_1,\dots,x_n$ be $\mathbf{Q}$–linearly independent complex numbers. Following the transcendence methods of Hermite, Gel'fond, Schneider..., one may start by introducing an auxiliary function

$$F(z) = P(z, e^z)$$

where $P \in \mathbf{Z}[X_0, X_1]$ is a non–zero polynomial. One considers the derivatives of F

$$F^{(k)} = \left(\frac{d}{dz}\right)^k F$$

at the points

$$m_1x_1 + \cdots + m_nx_n$$

for various values of $(m_1,\dots,m_n) \in \mathbf{Z}^n$.

The derivation

Let $\mathcal{D}$ denote the derivation

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1\frac{\partial}{\partial X_1}$$

over the ring $\mathbf{C}[X_0, X_1]$, so that for $P \in \mathbf{C}[X_0, X_1]$ the derivatives of the function

$$F(z) = P(z, e^z)$$

are given by

$$\left(\frac{d}{dz}\right)^k F = (\mathcal{D}^k P)(z, e^z).$$

Auxiliary function

Recall that $x_1, \dots, x_n$ are $\mathbf{Q}$–linearly independent complex numbers. Let $\alpha_1, \dots, \alpha_n$ be non–zero complex numbers. The transcendence machinery produces a sequence $(P_N)_{N\geq 0}$ of polynomials with integer coefficients satisfying

$$\left|(\mathcal{D}^k P_N)\Big(\sum_{j=1}^n m_j x_j, \prod_{j=1}^n \alpha_j^{m_j}\Big)\right| \leq \exp(-N^u)$$

for any non-negative integers k, $m_1, \dots, m_n$ with $k \leq N^{s_0}$ and $\max\{m_1, \dots, m_n\} \leq N^{s_1}$.

Roy's approach to Schanuel's Conjecture (1999)

If the number of equations we produce is too small, such a set of relations does not contain any information : the existence of a sequence of non–trivial polynomials $(P_N)_{N\geq 0}$ follows from linear algebra.

On the other hand, following D. Roy, one may expect that the existence of a sequence $(P_N)_{N\geq 0}$ producing sufficiently many such equations will yield the conclusion :

$$\operatorname{tr}\deg_{\mathbf{Q}}\mathbf{Q}(x_1, \dots, x_n, \alpha_1, \dots, \alpha_n) \geq n.$$

A remarquable result of D. Roy is that such equations imply $\alpha_j^d = e^{dx_j}$ for some positive integer d, and this enables him to show that Schanuel's Conjecture is *equivalent* to the existence of sufficiently many small values.

Roy's Conjecture (1999)

Let s_0, s_1, t_0, t_1, u positive real numbers satisfying

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}$$

and

$$\max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

Assume that, for any sufficiently large positive integer N, there exists a non–zero polynomial $P_N \in \mathbf{Z}[X_0, X_1]$ with partial degree $\leq N^{t_0}$ in X_0, partial degree $\leq N^{t_1}$ in X_1 and height $\leq e^N$ which satisfies

$$\left|(\mathcal{D}^k P_N)\Big(\sum_{j=1}^n m_j x_j, \prod_{j=1}^n \alpha_j^{m_j}\Big)\right| \leq \exp(-N^u)$$

for any non-negative integers k, $m_1, \dots, m_n$ with $k \leq N^{s_0}$ and $\max\{m_1, \dots, m_n\} \leq N^{s_1}$. Then

$$\operatorname{tr}\deg_{\mathbf{Q}}\mathbf{Q}(x_1, \dots, x_n, \alpha_1, \dots, \alpha_n) \geq n.$$

Roy's Theorem (1999)

Roy's Conjecture is equivalent to Schanuel's Conjecture.

More precisely, if Schanuel's Conjecture is true, then Roy's Conjecture holds for any set of parameters s_0, s_1, t_0, t_1, u satisfying

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}$$

and

$$\max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

Conversely, if Roy's Conjecture holds for one set of parameters s_0, s_1, t_0, t_1, u satisfying these conditions, then Schanuel's Conjecture is true.

Extending the range

Recently Nguyen Ngoc Ai Van succeeded to extend slightly the range of the admissible values of the parameters s_0, s_1, t_0, t_1, u.

Such an extension is interesting for both implications of the equivalence between Schanuel's Conjecture and Roy's Conjecture.

Equivalence between Schanuel and Roy

Let $(x, \alpha) \in \mathbf{C} \times \mathbf{C}^\times$, and let s_0, s_1, t_0, t_1, u be positive real numbers satisfying the inequalities of Roy's Conjecture. *Then the following conditions are equivalent :*
(a) The number αe^{-x} is a root of unity.
(b) For any sufficiently large positive integer N, there exists a non–zero polynomial $Q_N \in \mathbf{Z}[X_0, X_1]$ with partial degree $\le N^{t_0}$ in X_0, partial degree $\le N^{t_1}$ in X_1 and height $\mathrm{H}(Q_N) \le e^N$ such that

$$\left|(\mathcal{D}^k Q_N)(mx, \alpha^m)\right| \le \exp(-N^u).$$

for any $k, m \in \mathbf{N}$ with $k \le N^{s_0}$ and $m \le N^{s_1}$.

Roy's program towards Schanuel's Conjecture

In Gel'fond's transcendence criterion,

- replace a single variable by two variables X, Y
- introduce several points $(m_1x_1 + \cdots + m_\ell x_\ell, \alpha_1^{m_1} \cdots, \alpha_\ell^{m_\ell})$
- introduce multiplicity involving the derivative

$$D = (\partial/\partial X) + Y(\partial/\partial Y),$$

- get *large transcendence degree.*

Transcendence criterion with multiplicities

With derivatives : *Given a complex number ϑ, assume that there exists a sequence $(P_n)_{n\ge1}$ of non–zero polynomials in $\mathbf{Z}[X]$, with P_n of degree $\le d_n$ and height $\le e^{h_n}$, such that*

$$\max\left\{\left|P_n^{(j)}(\vartheta)\right| \; ; \; 0 \le j < t_n\right\} \le e^{-\nu_n}$$

for all $n \ge 1$. Assume $\nu_n t_n / d_n h_n \to \infty$. Then ϑ is algebraic.
Due to M. Laurent and D. Roy, applications to algebraic independence with interpolation determinants.

Criterion with several points

Goal : *Given a sequence of complex numbers $(\vartheta_i)_{i\geq 1}$, assume that there exists a sequence $(P_n)_{n\geq 1}$ of non–zero polynomials in $\mathbf{Z}[X]$, with P_n of degree $\leq d_n$ and height $\leq e^{h_n}$, such that*

$$\max\left\{\left|P_n^{(j)}(\vartheta_i)\right| \;;\; 0\leq j<t_n,\ 1\leq i\leq s_n\right\}\leq e^{-\nu_n}$$

for all $n\geq 1$. Assume $\nu_n t_n s_n/d_n h_n\to\infty$.
We wish to deduce that the numbers ϑ_i are algebraic.

D. Roy : Not true in general, but true in some special cases with a structure on the sequence $(\vartheta_i)_{i\geq 1}$.
Combines the elimination arguments used for criteria of algebraic independence and for zero estimates.

Small value estimates for the additive group

D. Roy. *Small value estimates for the additive group.* Intern. J. Number Theory, to appear.

Let ξ be a transcendental complex number, let h, σ, τ and ν be non-negative real numbers, let n_0 be a positive integer, and let $(P_n)_{n\geq n_0}$ be a sequence of non–zero polynomials in $\mathbf{Z}[X]$ satisfying $\deg(P_n)\leq n$ and $H(P_n)\leq\exp(n^h)$ for each $n\geq n_0$. Suppose that $h>1$, $(3/4)\sigma+\tau<1$ and $\nu>1+h-(3/4)\sigma-\tau$.
Then for infinitely many n, we have

$$\max\left\{|P_n^{(j)}(i\xi)| \;;\; 0\leq i\leq n^\sigma,\ 0\leq j\leq n^\tau\right\}>\exp(-n^\nu).$$

Small value estimates for the multiplicative group

D. Roy. *Small value estimates for the multiplicative group.* Acta Arith., to appear.

Let $\xi_1,\dots,\xi_m$ be multiplicatively independent complex numbers in a field of transcendence degree 1. Under suitable assumptions on the parameters h,σ,τ,ν, for infinitely many positive integers n, there exists no non–zero polynomial $P\in\mathbf{Z}[Y]$ satisfying $\deg(P)\leq n$, $H(P)\leq\exp(n^h)$ and

$$\max\left\{|P^{(j)}(\xi_1^{i_1}\cdots\xi_m^{i_m})| \;;\; 0\leq i_1,\dots,i_m\leq n^\sigma,\ 0\leq j\leq n^\tau\right\}$$
$$>\exp(-n^\nu).$$

The end

Happy Birthday, Jing Yu,

Heureux Anniversaire,

And lot of thanks to all those who contributed to the organization of this exceedingly interesting conference