RUPP Master in Mathematics
Algebra and Geometry

Tests

Michel Waldschmidt

April 21, 2015

1. Let G be an additive group. Recall that G is abelian. Let $S = \{x_1, \dots, x_m\}$ be a finite subset of G. Show that the subgroup of G generated by S is

$$\{a_1x_1 + \cdots + a_mx_m \mid a_1, \dots, a_m \in \mathbf{Z}\}.$$

2. Let G be an abelian multiplicative group. Let $S = \{y_1, \dots, y_m\}$ be a finite subset of G. What is the subgroup of G generated by S?

April 22, 2015

Let n be an integer, $n \geq 2$. Give a necessary and sufficient condition on n for the ring $\mathbf{Z}/n\mathbf{Z}$ to be a field. Prove the result.

April 24, 2015

1. Let $n \geq 2$ be an integer. Denote by $\varphi(n)$ the number of integers a in the range $1 \leq a \leq n$ such that $\gcd(a, n) = 1$. Prove that

$$a^{\varphi(n)} \equiv 1 \mod n \quad \text{for all } a \text{ in } \mathbf{Z} \text{ with } \gcd(a, n) = 1.$$

2. Let $\mathbf{F}$ be a finite field with q elements. Prove

$$x^{q-1} = 1 \quad \text{for all } x \in \mathbf{F}^\times.$$

Deduce

$$x^q = x \quad \text{for all } x \in \mathbf{F}.$$

April 25, 2015

Let $\mathbf{F}$ be a field, V a $\mathbf{F}$–vector space and f, g two endomorphisms of V.
1. Prove that $\ker f \subset \ker(f \circ g)$.
2. Prove that $\ker f = \ker(f \circ g)$ if and only if $\mathrm{im} f \cap \ker g = \{0\}$.

April 28, 2015

Let $\mathbf{F}$ be a field, f the endomorphism of $\mathbf{F}^3$ which maps $(a_1, a_2, a_3) \in \mathbf{F}^3$ to $(a_1 - a_2, a_2 - a_3, a_3 - a_1)$.
Give a basis for $\ker f$ and for $\mathrm{im} f$.

April 30, 2015

Show that the two linear forms $f_1(x, y) = x + y$ and $f_2(x, y) = x - y$ give a basis of the dual $(\mathbf{R}^2)^* = \mathcal{L}_{\mathbf{R}}(\mathbf{R}^2, \mathbf{R})$ of the $\mathbf{R}$-vector space $\mathbf{R}^2$.

Give the dual basis.

May 7, 2015

Set

$$\mathcal{H} = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 1\}.$$

Prove that $\mathcal{H}$ is an affine subspace of the affine space $\mathbf{R}^4$.
What is the underlying vector subspace H of the vector space $\mathbf{R}^4$?
Give a frame of $\mathcal{H}$.

http://webusers.imj-prg.fr/~michel.waldschmidt/enseignement.html

4, Place Jussieu Bureau 15–25 417
Tél. : [33] (0)1 44 27 54 41
http://www.imj-prg.fr/∼michel.waldschmidt/
Secr. : [33] (0)1 44 27 38 08
Fax : [33] (0)1 44 27 63 25
michel.waldschmidt@imj-prg.fr