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A course on linear recurrent sequences
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Tutorial 3

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• **1.** Let d be a positive integer which is not the square of an integer. Let (x_1, y_1) satisfy $x_1^2 - dy_1^2 = 1$. Define the sequence $(x_n, y_n)_{n\geq 0}$ by

$$x_n + \sqrt{d}y_n = (x_1 + \sqrt{d}y_1)^n$$

for $n \geq 0$. Check that the sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ satisfy the linear recurrence relation

$$u_{n+2} = 2x_1 u_{n+1} - u_n.$$

• **2.** Set $u_0 = 1$, $u_1 = 4$, and, for $n \geq 2$, $u_n = 4u_{n-1} - 4u_{n-2}$.
(a) The generating series $\sum_{n\geq 0} u_n z^n$ is the Taylor expansion of a rational fraction : which one ?
(b) The exponential generating series $\sum_{n\geq 0} u_n \frac{z^n}{n!}$ is a solution of a differential equation : which one ?

• **3.** A word on the alphabet with two letters $\{a, b\}$ is a finite sequence of letters, like *aaba*, *abab*.

Let α and β be two positive integers. The weight of the letter a is α, the weight of the letter b is β. The weight of a word is the sum of the weights of its letters. Given a positive integer n, denote by u_n the number of words of weight n.
Write the linear recurrence formula satisfied by the sequence $(u_n)_{n\geq 0}$.

Write $\sum_{n\geq 0} u_n z^n$ as a rational fraction in the following cases
(a) $\alpha = \beta$.
(b) $\alpha = 1$, $\beta = 2$.

(c) $\alpha = 1$, $\beta = 3$.
(d) $\alpha = 2$, $\beta = 3$.

• **4.** Let u and d be two real numbers with $d > u > 0$. Let $(p_n)_{n\geq 0}$ be a sequence of real numbers in the interval $(0,1)$ satisfying

$$(u+d)p_n = up_{n-1} + dp_{n+1} \quad (n \geq 1) \quad \text{and} \quad \sum_{n\geq 0} p_n = 1.$$

Compute p_n.

`Remark`. This is a toy version of Ising model in *statistical mechanics*. There is a ball on a vibrating stair with levels $0, 1, 2, \ldots,$
• for $n \geq 0$, p_n is the probability that the ball reaches the level n,
• for $n \geq 0$, up_n the probability that the ball leaves level n, goes up and reaches level $n+1$,
• for $n \geq 1$, dp_n the probability that the ball leaves level n, goes down and reaches level $n-1$.
The *temperature* is $T = (\log \frac{d}{u})^{-1}$, the level n is the *energy*, the probability that the ball has energy n is

$$p_n = \frac{1}{Z}\mathrm{e}^{-n/T},$$

where

$$Z = \frac{1}{1 - e^{-1/T}}.$$

If T is small, that is if u is small, then $p_0 = 1/Z$ is close to 1, the ball is likely to be at level 0, the noise is low. If T is large, that is if u is large, then p_0 is small, there are many levels where the ball is likely to be, the noise is high.
`Reference:` Vincent Beffara "J. W. Gibbs : les mathématiques du hasard au cœur de la physique ?" Conférence donnée dans le cadre du cycle « Un texte, un mathématicien ».
`https://smf.emath.fr/evenements-smf/conference-bnf-v-beffara-2021`