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# Two invariants related to two conjectures due to Nagata

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# Abstract

**Seshadri's** constant is related to a conjecture due to **Nagata**. Another conjecture, also due to **Nagata** and solved by **Bombieri** in 1970, is related with algebraic values of meromorphic functions. The main argument of **Bombieri's** proof leads to a **Schwarz** Lemma in several variables, the proof of which gives rise to another invariant associated with symbolic powers of the ideal of functions vanishing on a finite set of points. This invariant is an asymptotic measure of the least degree of a polynomial in several variables with given order of vanishing on a finite set of points. Recent works on the resurgence of ideals of points and the containment problem compare powers and symbolic powers of ideals.

# C.S. Seshadri memorial lectures — July 31, 2020

## Seshadri Memorial Lectures, 2020

Tata Institute of Fundamental Research, Mumbai

July 31, 2020

### Speakers

**J.P. Demailly**  
(On Seshadri Constant)

**P. Littelmann**  
(Standard Monomials)

**V. Balaji**  
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Conjeevaram Srirangachari Seshadri  
(1932-2020)



Jean-Pierre Demailly  
(1957-2022)

<http://conferences.math.tifr.res.in/lectures/seshadri-lectures/demailly.pdf>



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## A brief survey on Seshadri constants

Jean-Pierre Demailly

Institut Fourier, Université Grenoble Alpes & Académie des Sciences de Paris

Memorial Lectures for  
Professor Conjeevaram Srirangachari Seshadri  
organized by the TIFR School of Mathematics  
July 31, 2020, 4 PM IST

# The starting point of Seshadri constant theory

The paper that started the whole theory :

C.S. Seshadri, *Annals of Mathematics*, Vol. 95 (May 1972) 511–556

## Quotient spaces modulo reductive algebraic groups

By C. S. SESHADRI



### Introduction

In his book “Geometric invariant theory” [M], Mumford developed a theory of quotient spaces of algebraic schemes acted on by reductive algebraic groups when the ground field is of characteristic zero and showed how this can be used for several questions of moduli. In order to have this theory in arbitrary characteristic, he made the following conjecture:

(A): Let  $G$  be a reductive algebraic group (say over an algebraically closed field) and  $V$  a finite dimensional rational  $G$ -module. Then given a  $G$ -invariant point  $v$ ,  $v \neq 0$ , there is a  $G$ -invariant homogeneous polynomial  $F$  on  $V$  such that  $F(v) \neq 0$ .

# Seshadri's ampleness criterion

LEMMA 7.2. *Let  $X$  be a complete variety and  $x$  a point of  $X$  at which  $X$  is smooth. Let  $p: X' \rightarrow X$  be the BLOWING UP of  $X$  at the point  $x$  (i.e. blowing up of  $X$  with respect to the sheaf of ideals defining the reduced subscheme of  $X$  consisting of the one point  $x$ ). Let  $L$  be a line bundle on  $X$  which is PSEUDO-AMPLE i.e.  $\forall C \hookrightarrow X$  where  $C$  is a closed integral curve of  $X$ ,  $\deg(L|_C) \geq 0$ . Let  $Z = p^{-1}(x)$  and  $E$  the line bundle on  $X'$  defined by the effective divisor  $Z$  (so that  $-E$  i.e.  $E^{-1}$  is relatively ample with respect to  $p$ ). Suppose that  $aL - bE$  is also pseudo-ample for some  $a, b \in \mathbf{Z}$ ,  $a, b > 0$ . Then if  $n = \dim X$ , we have*

$$L^{(n)} = L \cdots L \text{ (} n\text{-fold intersection product)} > 0 .$$

⋮

Remark 7.1. The above lemma can be interpreted (as has been remarked by C.P. Ramanujam) to give a criterion of ampleness as follows: Let  $X$  be a complete algebraic scheme. To every closed integral curve  $C$ ,  $C \hookrightarrow X$ , define  $m(C)$  to be the maximum of the multiplicities at the different points of  $C$ . Let  $L$  be a line bundle on  $X$ . Then  $L$  is ample on  $X \Leftrightarrow \exists \varepsilon > 0$  such that  $\forall$  closed integral curve  $C \hookrightarrow X$ ,  $\deg(L|_C) \geq \varepsilon m(C)$ .

# Definition of the Seshadri constants

## Definition (Demailly, 1990)

Let  $X$  be a projective nonsingular variety and  $L$  a nef (or pseudo-ample) line bundle over  $X$ . Given a point  $x \in X$ , one defines the **Seshadri constant**  $\varepsilon(L, x)$  of  $L$  at  $x$  to be

$$\varepsilon(L, x) = \inf_{\text{all alg. curves } C \ni x} \frac{L \cdot C}{\text{mult}_x(C)}.$$

This is a very interesting numerical invariant that measures in a deep manner the “local positivity” of the line bundle  $L$  at point  $x$ .

**Equivalent definition (already observed in Seshadri’s paper!)**

Let  $\pi : \tilde{X} \rightarrow X$  be the blow-up of  $X$  at  $x \in X$ , and  $E$  the exceptional divisor in  $\tilde{X}$ . Then, for  $L \in \text{Pic}(X)$  assumed to be nef, one has

$$\varepsilon(L, x) = \sup\{\gamma \geq 0 / \pi^*L - \gamma E \text{ is nef on } \tilde{X}\}.$$



# Reformulation of Seshadri's ampleness criterion

## Reformulation of Seshadri's ampleness criterion

A nef line bundle  $L \in \text{Pic}(X)$  is ample if and only if one has

$$\varepsilon(L) := \inf_{x \in X} \varepsilon(L, x) > 0.$$

A direct consequence of the fact  $(\pi^*L - \gamma E)^n = L^n - \gamma^n \geq 0$  is that

$$\varepsilon(L, x) \leq (L^n)^{1/n}, \quad \forall x \in X.$$

A curve  $C$  is said to be **submaximal** if  $\frac{L \cdot C}{\text{mult}_x(C)} < (L^n)^{1/n}$ .

A large part of the investigations on Seshadri constants, especially in the case of surfaces, rests upon the study of submaximal curves.

Remark. In [D, 1990], over  $\mathbb{K} = \mathbb{C}$ , the Seshadri constant is related to more analytic invariants. For instance, if  $L$  is ample, it can be shown that  $\varepsilon(L, x)$  is the supremum of  $\gamma \geq 0$  for which  $L$  possesses a singular Hermitian metric  $h$  with  $\Theta_{L,h} \geq 0$ , that is smooth on  $X \setminus \{x\}$  with a **logarithmic pole of Lelong number  $\gamma$  at  $x$** .

# Relation to the Nagata conjecture

The concept of Seshadri constant is already highly non trivial on rational surfaces. For instance, the famous **Nagata conjecture**, has attracted lot of work by Hirschowitz, Harbourne, Biran, Bauer, Szemberg, Dumnicki and others. It can be reformulated :

## Nagata conjecture (1959), reformulated

Let  $x_1, \dots, x_p$  be  $p$  **very general** points in  $\mathbb{P}^2$ ,  $p \geq 9$ . Then the multipoint Seshadri constant of  $\mathcal{O}(1)$  on  $\mathbb{P}^2$  satisfies

$$\varepsilon(\mathcal{O}(1), x_1, \dots, x_p) = \frac{1}{\sqrt{p}}.$$

A simple counting argument implies that  $\varepsilon(\mathcal{O}(1), x_1, \dots, x_p) \leq \frac{1}{\sqrt{p}}$ , and the main difficulty is to find good configurations of points to get lower bounds. In case  $p = q^2$  is a perfect square, a square grid works, hence equality. For  $4 < p < 9$ , one is in the Del Pezzo case, and the equality turns out to be strict.

# Schneider – Lang Theorem (1949, 1966)



Theodor Schneider  
(1911 – 1988)



Serge Lang  
(1927 – 2005)

Let  $f_1, \dots, f_m$  be meromorphic functions in  $\mathbb{C}$ . Assume  $f_1$  and  $f_2$  are algebraically independent and of finite order. Let  $\mathbb{K}$  be a number field. Assume  $f_j'$  belongs to  $\mathbb{K}[f_1, \dots, f_m]$  for  $j = 1, \dots, m$ . Then the set

$$S = \{w \in \mathbb{C} \mid w \text{ not pole of } f_j, f_j(w) \in \mathbb{K} (j = 1, \dots, m)\}$$

is finite.

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Schneider.html>

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lang.html>



# Hermite – Lindemann Theorem (1882)



Charles Hermite  
(1822 – 1901)



Carl Louis Ferdinand von Lindemann  
(1852 – 1939)

**Corollary.** *If  $w$  is a nonzero complex number, one at least of the two numbers  $w$ ,  $e^w$  is transcendental.*

**Consequence :** transcendence of  $e$ ,  $\pi$ ,  $\log \alpha$ ,  $e^\beta$ , for algebraic  $\alpha$  and  $\beta$  assuming  $\alpha \neq 0$ ,  $\log \alpha \neq 0$ ,  $\beta \neq 0$ .

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hermite.html>

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lindemann.html>

# Gel'fond – Schneider Theorem (1934)



Aleksandr Osipovich Gelfond  
(1906 – 1968)



Theodor Schneider  
(1911 – 1988)

**Corollary (Hilbert's seventh problem).** *If  $\beta$  is an irrational algebraic number and  $w$  a nonzero complex number, one at least of the two numbers  $e^w$ ,  $e^{\beta w}$  is transcendental.*

**Consequence :** transcendence of  $e^\pi$ ,  $2^{\sqrt{2}}$ ,  $\alpha^\beta$ ,  $\log \alpha_1 / \log \alpha_2$ , for algebraic  $\alpha$ ,  $\alpha_1$ ,  $\alpha_2$  and  $\beta$  assuming  $\alpha \neq 0$ ,  $\log \alpha \neq 0$ ,  $\beta \notin \mathbb{Q}$ ,  $\log \alpha_1 / \log \alpha_2 \notin \mathbb{Q}$ .

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Gelfond.html>

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Schneider.html>

# Proofs of the corollaries

**Hermite - Lindemann.** Let  $\mathbb{K} = \mathbb{Q}(w, e^w)$ . The two functions  $f_1(z) = z$ ,  $f_2(z) = e^z$  are algebraically independent, of finite order, and satisfy the differential equations  $f_1' = 1$ ,  $f_2' = f_2$ . The set  $S$  contains  $\{\ell w \mid \ell \in \mathbb{Z}\}$ . Since  $w \neq 0$ , this set is infinite; it follows that  $\mathbb{K}$  is not a number field.  $\square$

**Gel'fond - Schneider.** Let  $\mathbb{K} = \mathbb{Q}(\beta, e^w, e^{\beta w})$ . The two functions  $f_1(z) = e^z$ ,  $f_2(z) = e^{\beta z}$  are algebraically independent, of finite order, and satisfy the differential equations  $f_1' = f_1$ ,  $f_2' = \beta f_2$ . The set  $S$  contains  $\{\ell w \mid \ell \in \mathbb{Z}\}$ . Since  $w \neq 0$ , this set is infinite; it follows that  $\mathbb{K}$  is not a number field.  $\square$

# Schneider's Theorems on elliptic functions (1937)

**Corollary (Schneider).** Let  $\wp$  be an elliptic function of Weierstrass with algebraic invariants  $g_2, g_3$ . Let  $w$  be a complex number, not pole of  $\wp$ . Then one at least of the two numbers  $w, \wp(w)$  is transcendental.

Proof. Let  $\mathbb{K} = \mathbb{Q}(g_2, w, \wp(w), \wp'(w))$ . The two functions  $f_1(z) = z, f_2(z) = \wp(z)$  are algebraically independent, of finite order. Set  $f_3(z) = \wp'(z)$ . From  $\wp'^2 = 4\wp^3 - g_2\wp - g_3$  one deduces

$$f_1' = 1, \quad f_2' = f_3, \quad f_3' = 6f_2^2 - (g_2/2).$$

The set  $S$  contains

$$\{lw \mid l \in \mathbb{Z}, lw \text{ not pole of } \wp\}$$

which is infinite. Hence  $\mathbb{K}$  is not a number field.  $\square$

# The transcendence machinery

The prototype of transcendence methods is **Hermite's** proof of the transcendence of  $e$ .

The proof of the **Schneider – Lang** Theorem follows the following scheme :

**Step 1** Construct an auxiliary function  $f$  with many zeroes.

**Step 2** Find a point  $z_0$  where  $f(z_0) \neq 0$ .

**Step 3** Give a lower bound for  $|f(z_0)|$  using arithmetic arguments.

**Step 4** Give an upper bound for  $|f(z_0)|$  using analytic arguments.

We are interested here mainly (but not only) with the last part (step 4) which is of analytic nature.



# Schwarz Lemma in one variable



Hermann Amandus Schwarz  
(1843 – 1921)

Let  $f$  be an analytic function in a disc  $|z| \leq R$  of  $\mathbb{C}$ , with at least  $N$  zeroes in a disc  $|z| \leq r$  with  $r < R$ . Then

$$|f|_r \leq \left(\frac{3r}{R}\right)^N |f|_R.$$

We use the notation

$$|f|_r = \sup_{|z|=r} |f(z)|.$$

When  $R > 3r$ , this improves the maximum modulus bound  $|f|_r \leq |f|_R$ .

## Schwarz Lemma in one variable : proof

Let  $a_1, \dots, a_N$  be zeroes of  $f$  in the disc  $|z| \leq r$ , counted with multiplicities. The function

$$g(z) = f(z) \prod_{j=1}^N (z - a_j)^{-1}$$

is analytic in the disc  $|z| \leq R$ . Using the maximum modulus principle, from  $r \leq R$  we deduce  $|g|_r \leq |g|_R$ . Now we have

$$|f|_r \leq (2r)^N |g|_r \quad \text{and} \quad |g|_R \leq (R - r)^{-N} |f|_R.$$

Finally, assuming (wlog)  $R > 3r$ ,

$$\frac{2r}{R - r} \leq \frac{3r}{R}.$$



# Schneider – Lang Theorem in several variables : cartesian products (1941, 1966)

Let  $f_1, \dots, f_m$  be meromorphic functions in  $\mathbb{C}^n$  with  $m \geq n + 1$ . Assume  $f_1, \dots, f_{n+1}$  are algebraically independent of finite order. Let  $\mathbb{K}$  be a number field. Assume  $(\partial/\partial z_i)f_j$  belongs to  $\mathbb{K}[f_1, \dots, f_m]$  for  $j = 1, \dots, m$  and  $i = 1, \dots, n$ . If  $e_1, \dots, e_n$  is a basis of  $\mathbb{C}^n$ , then the set

$$S = \{w \in \mathbb{C}^n \mid w \text{ not pole of } f_j, f_j(w) \in \mathbb{K} (j = 1, \dots, m)\}$$

does not contain a cartesian product

$$\{s_1 e_1 + \dots + s_n e_n \mid (s_1, \dots, s_n) \in S_1 \times \dots \times S_n\}$$

where each  $S_i$  is infinite.

# Schneider's Theorem on Euler's Beta function



Leonhard Euler  
(1707 – 1783)

Let  $a, b$  be rational numbers, not integers. Then the number

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental.

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Euler.html>

Further results by Th. Schneider and S. Lang on abelian functions and algebraic groups.

## Schwarz lemma in several variables : cartesian products (grids)

Let  $f$  be an analytic function in a ball  $|z| \leq R$  of  $\mathbb{C}^n$ . Assume  $f$  vanishes with multiplicity at least  $t$  on a set  $S_1 \times \cdots \times S_n$  where each  $S_i$  is contained in a disc  $|z| \leq r$  with  $r < R$  and has at least  $s$  elements.

Then

$$|f|_r \leq \left(\frac{3r}{R}\right)^{st} |f|_R.$$

# Cartesian products

**Schwarz** Lemma for Cartesian products can be proved by induction.

§4.3 of [M.W.](#). *Diophantine Approximation on Linear Algebraic Groups*. Grund. Math. Wiss. **326** Springer-Verlag (2000).

Another proof, based on integral formulae, yields a weaker result : for  $R > 3r$ ,

$$|f|_r \leq \left( \frac{R - 3r}{2r} \right)^n \left( \frac{3r}{R} \right)^{st} |f|_R.$$

The conclusion follows from a homogeneity argument : replace  $f$  by  $f^N$  (and  $t$  by  $Nt$ ) and let  $N \rightarrow \infty$ .

Chap. 7 of [M.W.](#). *Nombres transcendants et groupes algébriques*. Astérisque, **69–70** (1979).

# Nagata's suggestion (1966)



Masayoshi Nagata  
(1927 – 2008)

In the conclusion of the Schneider – Lang Theorem, replace the fact that  $S$  does not contain a cartesian product  $S_1 \times \cdots \times S_n$  where each  $S_i$  is infinite by the fact that  $S$  is contained in an algebraic hypersurface.

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Nagata.html>

# Bombieri's Theorem (1970)

Let  $f_1, \dots, f_m$  be meromorphic functions in  $\mathbb{C}^n$  with  $m \geq n + 1$ . Assume  $f_1, \dots, f_{n+1}$  are algebraically independent and of finite order. Let  $\mathbb{K}$  be a number field. Assume  $(\partial/\partial z_i)f_j$  belongs to  $\mathbb{K}[f_1, \dots, f_m]$  for  $j = 1, \dots, m$  and  $i = 1, \dots, n$ .



Enrico Bombieri

Then the set

$$S = \{w \in \mathbb{C}^n \mid$$

$w$  not pole of  $f_j$ ,

$$f_j(w) \in \mathbb{K} \ (j = 1, \dots, m)\}$$

is contained in an algebraic hypersurface.



# Bombieri – Lang (1970)



Let  $f$  be an analytic function in a ball  $|z| \leq R$  of  $\mathbb{C}^n$ . Assume  $f$  vanishes at  $N$  points  $z_i$  (counting multiplicities) in a ball  $|z| \leq r$  with  $r < R$ . Assume  $\min_{z_i \neq z_k} |z_i - z_k| \geq \delta$ .

Then

$$|f|_r \leq \left(\frac{3r}{R}\right)^M |f|_R$$

with

$$M = N \left(\frac{\delta}{6r}\right)^{2n-2}.$$

# Lelong number

E. Bombieri. *Algebraic values of meromorphic maps*. Invent. Math. **10** (1970), 267–287.

E. Bombieri and S. Lang. *Analytic subgroups of group varieties*. Invent. Math. **11** (1970), 1–14.

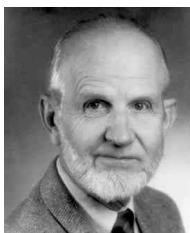


Pierre Lelong  
(1912 – 2011)

P. Lelong. *Intégration sur un ensemble analytique complexe*, Bulletin S.M.F. **85** (1957), 239–262,

[https://fr.wikipedia.org/wiki/Pierre\\_Lelong](https://fr.wikipedia.org/wiki/Pierre_Lelong)

# $L^2$ - estimates of Hörmander



Lars Hörmander  
(1931 – 2012)

Existence theorems for the  $\bar{\partial}$  operator.

**E. Bombieri.** Let  $\varphi$  be a plurisubharmonic function in  $\mathbb{C}^n$  and  $z_0 \in \mathbb{C}^n$  be such that  $e^{-\varphi}$  is integrable near  $z_0$ .

Then there exists a nonzero entire function  $F$  such that

$$\int_{\mathbb{C}^n} |F(z)|^2 e^{-\varphi(z)} (1 + |z|^2)^{-3n} d\lambda(z) < \infty.$$

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hormander.html>

# Towards a Schwarz lemma in several variables

Let  $S$  be a finite subset of  $\mathbb{C}^n$  and  $t$  a positive integer. Let  $M$  be a positive number with the following property.

*There exists a real number  $r$  such that for  $R > r$ , if  $f$  is an analytic function in the ball  $|z| \leq R$  of  $\mathbb{C}^n$  which vanishes with multiplicity at least  $t$  at each point of  $S$ , then*

$$|f|_r \leq \left( \frac{c(n)r}{R} \right)^M |f|_R,$$

where  $c(n)$  depends only on the dimension  $n$ .

Question: what is the largest possible value for  $M$ ?

Answer: use the property for  $f$  a nonzero polynomial vanishing on  $S$  with multiplicity  $t$ . We deduce that  $f$  has degree at least  $M$ .

# Degree of hypersurfaces

Let  $S$  be a finite set of  $\mathbb{C}^n$  and  $t$  a positive integer.

Denote by  $\omega_t(S)$  the smallest degree of a polynomial vanishing at each point of  $S$  with multiplicity  $\geq t$ .

*M.W. Propriétés arithmétiques de fonctions de plusieurs variables (II). Sémin. P. Lelong (Analyse), 16ème année, 1975/76 ; Lecture Notes in Math., **578** (1977), 274–292.*

*M.W. Nombres transcendants et groupes algébriques. Astérisque, **69–70**. Société Mathématique de France, Paris, 1979.*

# Schwarz lemma in several variables

Let  $S$  be a finite set of  $\mathbb{C}^n$  and  $t$  a positive integer. There exists a real number  $r$  such that for  $R > r$ , if  $f$  is an analytic function in the ball  $|z| \leq R$  of  $\mathbb{C}^n$  which vanishes with multiplicity at least  $t$  at each point of  $S$ , then

$$|f|_r \leq \left( \frac{e^n r}{R} \right)^{\omega_t(S)} |f|_R.$$

This is a refined asymptotic version due to [Jean-Charles Moreau](#).

The exponent  $\omega_t(S)$  cannot be improved : take for  $f$  a non-zero polynomial of degree  $\omega_t(S)$ .

## $\omega_t(S)$ : examples

For  $n = 1$  we have  $\omega_t(S) = t|S|$  :

$$\prod_{s \in S} (z - s)^t.$$

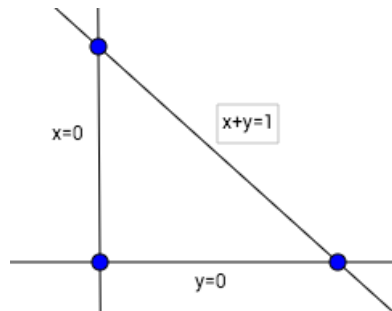
More generally, for a Cartesian product (grid)

$S = S_1 \times \cdots \times S_n$  in  $\mathbb{K}^n$ ,

$$\omega_t(S) = t \min_{1 \leq i \leq n} |S_i|.$$

Proof by induction.

Generic  $S \subset \mathbb{C}^2$  with  $|S| = 3$



$$S = \{(0, 0), (0, 1), (1, 0)\}.$$

$$P_1(X, Y) = XY$$

$$P_2(X, Y) = XY(X + Y - 1)$$

$$\omega_1(S) = 2, \quad \omega_2(S) = 3.$$

With

$$P_{2m-1} = X^m Y^m (X + Y - 1)^{m-1}, \quad P_{2m} = X^m Y^m (X + Y - 1)^m,$$

we deduce

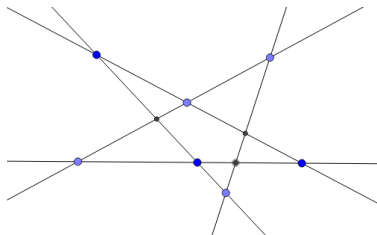
$$\omega_{2m-1}(S) = 3m - 1, \quad \omega_{2m}(S) = 3m.$$



# Complete intersections of hyperplanes

Let  $H_1, \dots, H_N$  be  $N$  hyperplanes in general position in  $\mathbb{C}^n$  with  $N \geq n$  and  $S$  the set of  $\binom{N}{n}$  intersection points of any  $n$  of them. Then, for  $t \geq 1$ ,

$$\omega_{nt}(S) = Nt.$$



$$n = 2, N = 5, |S| = 10.$$

# Dirichlet's box principle



Johann Peter Gustav Lejeune  
Dirichlet  
(1805 – 1859)

Given a finite subset  $S$  of  $\mathbb{C}^n$  and a positive integer  $t$ , if  $D$  is a positive integer such that

$$|S| \binom{t+n-1}{n} < \binom{D+n}{n},$$

then

$$\omega_t(S) \leq D.$$

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Dirichlet.html>

# Properties of $\omega_t(S)$

Consequence of Dirichlet's box principle :

$$\omega_t(S) \leq (t + n - 1)|S|^{1/n}.$$

Subadditivity :

$$\omega_{t_1+t_2}(S) \leq \omega_{t_1}(S) + \omega_{t_2}(S).$$

Therefore  $\limsup_{t \rightarrow \infty} \omega_t(S)/t$  exists and is  $\leq \omega_t(S)/t$  for all  $t \geq 1$ .

# An asymptotic invariant

**Theorem.** *The sequence*

$$\left( \frac{1}{t} \omega_t(S) \right)_{t \geq 1}$$

*has a limit  $\Omega(S)$  as  $t \rightarrow \infty$ , and*

$$\frac{1}{n} \omega_1(S) - 2 \leq \Omega(S) \leq \omega_1(S).$$

*Further, for all  $t \geq 1$  we have*

$$\Omega(S) \leq \frac{\omega_t(S)}{t}.$$

Remark :  $\Omega(S) \leq |S|^{1/n}$ .

M.W. *Propriétés arithmétiques de fonctions de plusieurs variables* (II). Sémin. P. Lelong (Analyse), 16<sup>e</sup> année, 1975/76 ; Lecture Notes in Math., **578** (1977), 274–292.

# Improvement of $L^2$ estimate by Henri Skoda

Let  $\varphi$  be a plurisubharmonic function in  $\mathbb{C}^n$  and  $z_0 \in \mathbb{C}^n$  be such that  $e^{-\varphi}$  is integrable near  $z_0$ . For any  $\epsilon > 0$  there exists a nonzero entire function  $F$  such that

$$\int_{\mathbb{C}^n} |F(z)|^2 e^{-\varphi(z)} (1 + |z|^2)^{-n-\epsilon} d\lambda(z) < \infty.$$

**Corollary :**

$$\frac{1}{n} \omega_1(S) \leq \Omega(S) \leq \omega_1(S).$$

H. Skoda. *Estimations  $L^2$  pour l'opérateur  $\bar{\partial}$  et applications arithmétiques*. Springer Lecture Notes in Math., **578** (1977), 314–323.



Henri Skoda

[https://en.wikipedia.org/wiki/Henri\\_Skoda](https://en.wikipedia.org/wiki/Henri_Skoda)

## Comparing $\omega_{t_1}(S)$ and $\omega_{t_2}(S)$

Idea: Let  $P$  be a polynomial of degree  $\omega_{t_1}(S)$  vanishing on  $S$  with multiplicity  $\geq t_1$ . If the function  $P^{t_2/t_1}$  were an entire function, it would be a polynomial of degree  $\frac{t_2}{t_1}\omega_{t_1}(S)$  vanishing on  $S$  with multiplicity  $\geq t_2$ , which would yield  $\omega_{t_2}(S) \leq \frac{t_2}{t_1}\omega_{t_1}(S)$ .

However  $P^{t_2/t_1}$  is usually not an entire function but  $\varphi = \frac{t_2}{t_1} \log P$  is a plurisubharmonic function. By the  $L^2$ -estimates of Hörmander – Bombieri – Skoda,  $e^\varphi$  is well approximated by a nonzero entire function. This function is a polynomial vanishing on  $S$  with multiplicity  $\geq t_2$ .

**Theorem.** For all  $t \geq 1$ ,

$$\frac{\omega_t}{t+n-1} \leq \Omega(S) \leq \frac{\omega_t}{t}.$$

M.W. *Nombres transcendants et groupes algébriques*. Astérisque, **69–70**. Société Mathématique de France, Paris, 1979.

# Hilbert's 14th problem



David Hilbert  
(1862 – 1943)

Let  $k$  be a field and  $K$  a subfield of  $k(X_1, \dots, X_n)$  containing  $k$ . Is the  $k$ -algebra

$$K \cap k[X_1, \dots, X_n]$$

*finitely generated?*

Oscar Zariski (1954) : true for  $n = 1$  and  $n = 2$ .

Counterexample by Masayoshi Nagata in 1959.

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hilbert.html>

<http://www.clarku.edu/~djoyce/hilbert/>

# Hilbert's 14th problem : restricted case



Masayoshi Nagata  
(1927 – 2008)

Original 14th problem :  
Let  $G$  be a subgroup of the full linear group of the polynomial ring in indeterminate  $X_1, \dots, X_n$  over a field  $k$ , and let  $\mathfrak{o}$  be the set of elements of  $k[X_1, \dots, X_n]$  which are invariant under  $G$ . Is  $\mathfrak{o}$  finitely generated ?

M. Nagata. *On the 14-th Problem of Hilbert*. Amer. J. Math **81** (1959), 766–772.

<http://www.jstor.org/stable/2372927>



# Fundamental Lemma of Nagata

Given 16 independent generic points of the projective plane over a prime field and a positive integer  $t$ , there is no curve of degree  $4t$  which goes through each  $p_i$  with multiplicity at least  $t$ .

In other words for  $|S| = 16$  generic in  $\mathbb{C}^2$ , we have  $\omega_t(S) > 4t$ .

Reference: M. Nagata. *On the fourteenth problem of Hilbert*. Proc. Internat. Congress Math. 1958, Cambridge University Press, pp. 459–462.

<http://www.mathunion.org/ICM/ICM1958/Main/icm1958.0459.0462.ocr.pdf>

# Nagata' contribution



Masayoshi Nagata  
(1927 – 2008)

**Proposition.** Let  $p_1, \dots, p_r$  be independent generic points of the projective plane over the prime field. Let  $C$  be a curve of degree  $d$  passing through the  $p_i$ 's with multiplicities  $\geq m_i$ . Then

$$m_1 + \dots + m_r < d\sqrt{r}$$

for  $r = s^2$ ,  $s \geq 4$ .

It is not known if  $r > 9$  is sufficient to ensure the inequality of the Proposition.

M. Nagata. *Lectures on the fourteenth problem of Hilbert*. Tata Institute of Fundamental Research Lectures on Mathematics **31**, (1965), Bombay.

# Reformulation of Nagata's Conjecture

By considering  $\sum_{\sigma} C_{\sigma}$  where  $\sigma$  runs over the cyclic permutations of  $\{1, \dots, r\}$ , it is sufficient to consider the case  $m_1 = \dots = m_r$ .

**Conjecture.** Let  $S$  be a finite generic subset of the projective plane over the prime field with  $|S| \geq 10$ . Then

$$\omega_t(S) > t\sqrt{|S|}.$$

Nagata :

- True for  $|S|$  a square.
- False for  $|S| \leq 9$ .

Harbourne, Brian. *On Nagata's conjecture*, Journal of Algebra **236** 2 (2001), 692–702.

<https://doi.org/10.1006/jabr.2000.8515>

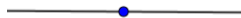
$$|S| \leq 9 \text{ in } \mathbb{C}^2$$

Nagata : generic  $S$  in  $\mathbb{C}^2$  with  $|S| \leq 9$  have  $\frac{\omega_t(S)}{t} \leq \sqrt{|S|}$ .

$ S $	=	1	2	3	4	5	6	7	8	9
$\omega_1(S)$	=	1	1	2	2	2	3	3	3	3
$t$	=	1	1	2	1	1	5	8	17	1
$\omega_t(S)$	=	1	1	3	2	2	12	21	48	3
$\frac{\omega_t(S)}{t}$	=	1	1	$\frac{3}{2}$	2	2	$\frac{12}{5}$	$\frac{21}{8}$	$\frac{48}{17}$	3
$\sqrt{ S }$	=	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	$\sqrt{8}$	3

$|S| = 1$  or  $2$  in  $\mathbb{C}^2$

$|S| = 1 : S = \{(0, 0)\}, P_t(X, Y) = X^t,$   
 $\omega_t(S) = t, \Omega(S) = 1.$



$|S| = 2 : S = \{(0, 0), (1, 0)\}, P_t(X, Y) = Y^t,$   
 $\omega_t(S) = t, \Omega(S) = 1.$



## Generic $S$ with $|S| = 3$ in $\mathbb{C}^2$

Given a set  $S$  of 3 points in  $\mathbb{C}^2$ , not on a straight line, we have

$$\omega_t(S) = \begin{cases} \frac{3t+1}{2} & \text{for } t \text{ odd,} \\ \frac{3t}{2} & \text{for } t \text{ even,} \end{cases}$$

hence

$$\Omega(S) = \lim_{n \rightarrow \infty} \frac{\omega_t(S)}{t} = \frac{3}{2}.$$

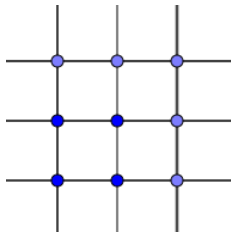
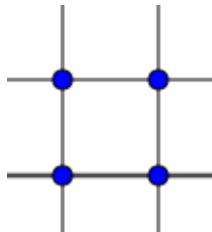
Since  $\omega_1(S) = 2$  and  $n = 2$ , this is an example with

$$\frac{\omega_1(S)}{n} < \Omega(S) < \omega_1(S).$$

## Generic $S \subset \mathbb{C}^2$ with $|S| = 4$

For a generic  $S$  in  $\mathbb{C}^2$  with  $|S| = 4$ , we have  $\omega_t(S) = 2t$ , hence  $\Omega(S) = 2$ .

Easy for a Cartesian product (square grid)  $S_1 \times S_2$  with  $|S_1| = |S_2| = 2$ , also true for a generic  $S$  with  $|S| = 4$ .



More generally, for the same reason, when  $S$  is a Cartesian product  $S_1 \times S_2$  with  $|S_1| = |S_2| = m$ , we have  $\omega_t(S) = mt$  and  $\Omega(S) = m = \sqrt{|S|}$ . The inequality  $\Omega(S) \geq \sqrt{|S|}$  for a generic  $S$  with  $|S|$  a square follows (Chudnovsky).

# Generic $S \subset \mathbb{C}^2$ with $|S| = 5$

Since 5 points in  $\mathbb{C}^2$  lie on a conic, for a generic  $S$  with  $|S| = 5$  we have  $\omega_t(S) = 2t$  and  $\Omega(S) = 2$ .

Remark. A polynomial in 2 variables of degree  $d$  has

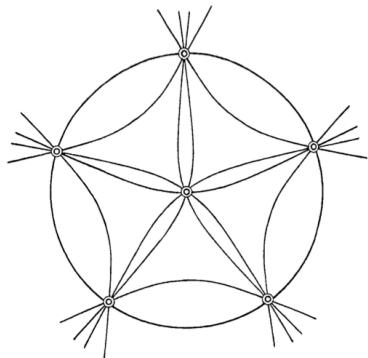
$$\frac{(d+1)(d+2)}{2}$$

coefficients. Hence for  $2|S| < (d+1)(d+2)$  we have  $\omega_1(S) \leq d$ .

For  $|S| = 1, 2$  we have  $\omega_1(S) = 1$ ,  
for  $|S| = 3, 4, 5$  we have  $\omega_1(S) \leq 2$ ,  
for  $|S| = 6, 7, 8, 9$  we have  $\omega_1(S) \leq 3$ .



# Generic $S \subset \mathbb{C}^2$ with $|S| = 6$ (Nagata)



Given 6 generic points  $s_1, \dots, s_6$  in  $\mathbb{C}^2$ , consider 6 conics  $C_1, \dots, C_6$  where  $S_i$  passes through the 5 points  $s_j$  for  $j \neq i$ . This produces a polynomial of degree 12 with multiplicity  $\geq 5$  at each  $s_i$ . Hence  $\omega_5(S) \leq 12$ .

In fact  $\omega_{5t}(S) = 12t$ ,  
 $\Omega(S) = 12/5$ .

## Generic $S \subset \mathbb{C}^2$ with $|S| = 7$ (Nagata)

Given 7 points in  $\mathbb{C}^2$ , there is a cubic passing through these 7 points with a double point at one of them.

Number of coefficients of a cubic polynomial : 10.

Number of conditions : 6 for the simple zeros, 3 for the double zero.

This gives a polynomial of degree  $7 \times 3 = 21$  with the 7 assigned zeroes of multiplicities 8.

In fact  $\omega_{8t}(S) = 21t$ ,  $\Omega(S) = 21/8$ .

## Generic $S \subset \mathbb{C}^2$ with $|S| = 8$ (Nagata)

Given 8 points in  $\mathbb{C}^2$ , there is a sextic with a double point at 7 of them and a triple point at 1 of them.

Number of coefficients of a sextic polynomial :

$$(6 + 1)(6 + 2)/2 = 28.$$

Number of conditions :  $3 \times 7 = 21$  for the double zeros, 6 for the triple zero.

This gives a polynomial of degree  $8 \times 6 = 48$  with the 8 assigned zeroes of multiplicities  $2 \times 8 + 1 = 17$ .

In fact  $\omega_{17t}(S) = 48t$ ,  $\Omega(S) = 48/17$ .

# G.V. Chudnovsky



Gregory Chudnovsky

**Conjecture :**

$$\frac{\omega_1 + n - 1}{n} \leq \frac{\omega_t}{t}.$$

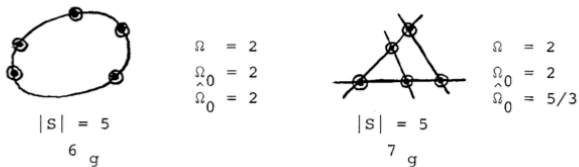
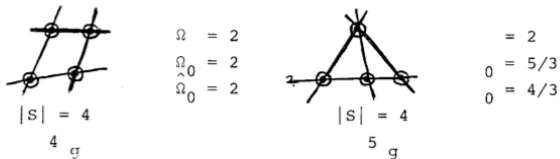
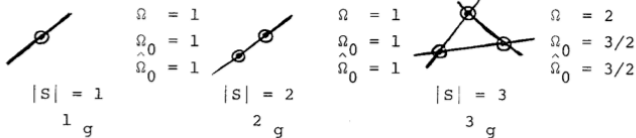
True for  $n = 2$  (J-P. Demailly).

G.V. Chudnovsky. *Singular points on complex hypersurfaces and multidimensional Schwarz Lemma*. M.-J. Bertin (Ed.), Séminaire de Théorie des Nombres *Delange-Pisot-Poitou*, Paris, 1979–80, Prog. Math., vol. **12**, Birkhäuser.

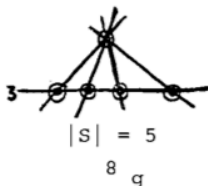
[https://fr.wikipedia.org/wiki/David\\_et\\_Gregory\\_Chudnovsky](https://fr.wikipedia.org/wiki/David_et_Gregory_Chudnovsky)

# Chudnovsky : $n = 2, |S| = 2, 3, 4, 5$

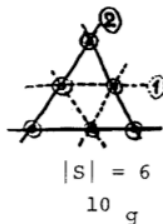
## APPENDIX 1



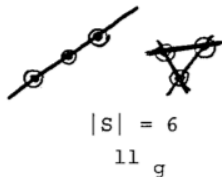
# Chudnovsky : $n = 2$ , $|S| = 5, 6$



$$\begin{array}{ll} \Omega = 2 & \text{For generic } S, \quad \Omega(S) = 3 \\ \hat{\Omega}_0 = 7/4 & |S| = 6 \quad \hat{\Omega}_0(S) = 12/5 \\ \hat{\Omega}_0 = 5/4 & (\text{see } 5_g) \quad \hat{\Omega}_0(S) = 12/5 \end{array}$$



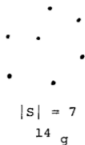
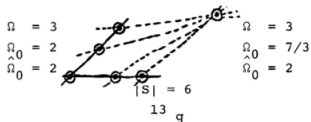
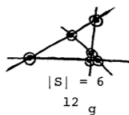
$$\begin{array}{l} \Omega = 3 \\ \hat{\Omega}_0 = 9/4 \\ \hat{\Omega}_0 = 2 \end{array}$$



$$\begin{array}{l} \Omega = 3 \\ \hat{\Omega}_0 = 12/5 \\ \hat{\Omega}_0 = 2 \end{array}$$

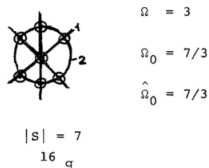
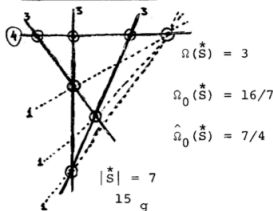
# Chudnovsky : $n = 2$ , $|S| = 6, 7$

64

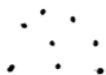


For generic  $S$ ,  $|S| = 7$ ,  
7 cubics with double point  
for generic case,  
 $\Omega(S) = 3$ ,  $\hat{\Omega}_0(S) = 21/8$ ,  
 $\hat{\Omega}_0(S) = 21/8$ .

## Particular Cases:



# Chudnovsky : $n = 2$ , $|S| = 8$



$$|S| = 8$$

(See 18 - 19 )

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For generic  $S$ ,  $|S| = 8$ ,  
8 sextics with 7 double points,  
1 triple point. For generic  
case,  $\Omega(S) = 3$ ,  $\Omega_0(S) = 48/17$ ,  
 $\hat{\Omega}_0(S) = 48/17$ .



# Chudnovsky : $n = 2, |S| = 9$

65



$$|S| = 9$$
$$20 \text{ g}$$

For generic  $S$ ,  $|S| = 9$ , 1 cubic  
and  $\Omega(S) = 3$ ,  $\Omega_0(S) = 3$ ,  
 $\hat{\Omega}_0(S) = 3$ .



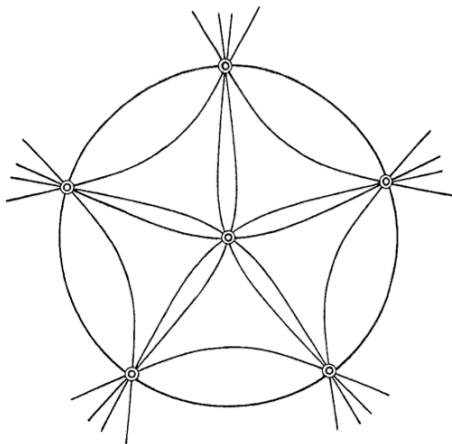
$$|S| = 9$$
$$21 \text{ g}$$

---

$$\Omega = 3$$
$$\Omega_0 = 5/2$$
$$\hat{\Omega}_0 = 9/4$$

$n = 2, |S| = 6, t = 5, \omega_t = 12, \Omega = 12/5$  generic

APPENDIX 2



$|S| = 6$   
9  
g

Generic  $S, |S| = 6$

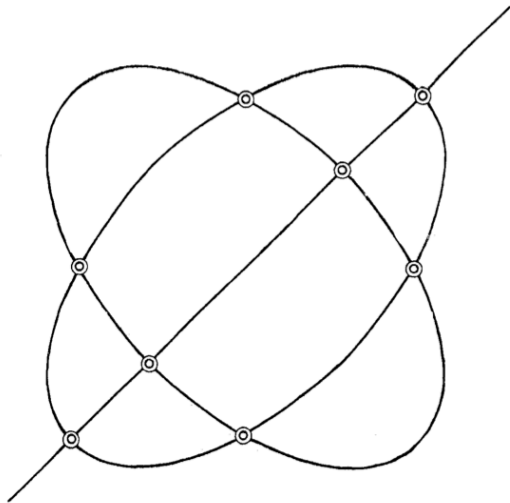
6 conics

$\Omega(S) = 3$

$\Omega_0(S) = 12/5$

$\Omega_\theta(S) = 12/5$

$$n = 2, |S| = 8, t = 2, \omega_t = 5, \Omega = 5/2$$



$$|S| = 8$$

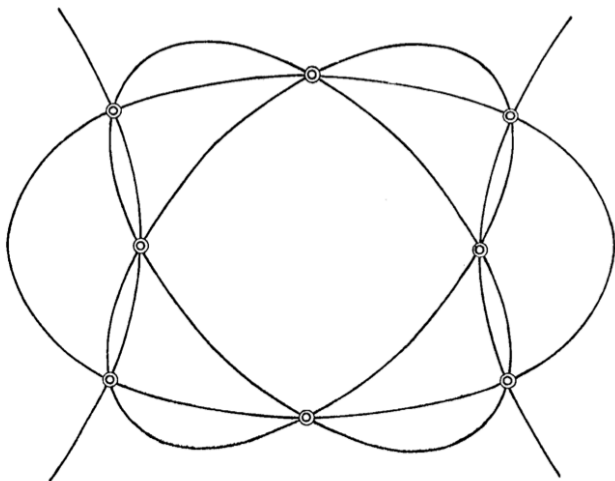
$$18_g$$

$$\Omega(S) = 3$$

$$\hat{\Omega}_0(S) = 5/2$$

$$\hat{\Omega}_0(S) = 3/2$$

$$n = 2, |S| = 8, t = 3, \omega_t = 8, \Omega = 8/3$$



$$|S| = 8$$

19 ~

4 conics

$$\Omega(S) = 3$$

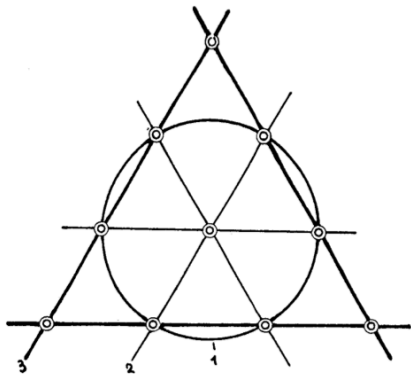
$$\Omega_0(S) = 8/3$$

$$\Omega_0(S) = 8/3$$

$$n = 2, |S| = 10, t = 6, \omega_t = 17, \Omega = 17/6$$

Three sides : multiplicity 3.

Three concurrent lines : multiplicity 2.



$$|S_c| = 10$$

$$22_g$$

$$\Omega(S_c) = 4$$

$$\hat{\Omega}_0(S_c) = 17/6$$

$$\hat{\hat{\Omega}}_0(S_c) = 5/2$$

# Hélène Esnault and Eckart Viehweg



Hélène Esnault



Eckart Viehweg

(1948 - 2010)

H. Esnault and E. Viehweg *Sur une minoration du degré d'hypersurfaces s'annulant en certains points*. Math. Ann. **263** (1983), 75 – 86

Methods of projective geometry : for  $n \geq 2$ ,

$$\Omega(S) \geq \frac{\omega_t + 1}{t + n - 1}.$$

# Jean-Pierre Demailly



Jean-Pierre Demailly

Using an appropriate generalization of the **Poisson–Jensen** formula, proves a new variant of the **Schwarz** lemma in  $\mathbb{C}^n$ .

**Consequence :**

$$\Omega(S) \geq \frac{\omega_1(S)(\omega_1(S) + 1) \cdots (\omega_1(S) + n - 1)}{n! \omega_1(S)^{n-1}}$$

**Corollary :** For  $n = 1$  or  $2$ ,

$$\Omega(S) \geq \frac{\omega_1(S) + n - 1}{n}.$$

# Demailly's Conjecture

Recall the Conjecture of **Chudnovsky** and the Theorem of **Esnault** and **Viehweg** :

$$\Omega(S) \geq \frac{\omega_1 + n - 1}{n}, \quad \Omega(S) \geq \frac{\omega_t + 1}{t + n - 1}.$$

**Conjecture of Demailly** :

$$\Omega(S) \geq \frac{\omega_t(S) + n - 1}{t + n - 1}.$$

**J-P. Demailly**. *Formules de Jensen en plusieurs variables et applications arithmétiques*. Bull. Soc. Math. France **110** (1982), 75–102.

[https://de.wikipedia.org/wiki/Jean-Pierre\\_Demailly](https://de.wikipedia.org/wiki/Jean-Pierre_Demailly)



# Conjecture of André Hirschowitz



André Hirschowitz

Denote by  $\omega_t(n, m)$  the maximum of  $\omega_t(S)$  over all finite sets  $S$  in  $\mathbb{C}^n$  with  $m$  elements.

**Conjecture :**  $\omega_t(n, m)$  is as large as possible.

For every  $n \geq 1$  there is an integer  $c(n)$  such that, for every  $m \geq c(n)$  and, for all  $t$ ,  $\omega_t(n, m)$  is the smallest integer  $d$  such that

$$\binom{d+n}{n} > m \binom{t+n-1}{n}.$$

True for  $t = 2$  and  $n = 2$  and  $3$ , and for  $t = 3$  and  $n = 2$ .

A. Hirschowitz. *La méthode d'Horace pour l'interpolation à plusieurs variables*. Manuscripta Math. **50** (1985), 337–388.

# Abdelhak Azhari



Abdelhak Azhari

A. Azhari. *Démonstration analytique d'un lemme de multiplicités*. C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), no. 7, 269–272.

A. Azhari. *Sur la conjecture de Chudnovsky – Demailly et les singularités des hypersurfaces algébriques*. Ann. Inst. Fourier **40** (1990), no. 1, 103–116.

[http://www.numdam.org/item?id=AIF\\_1990\\_\\_40\\_1\\_103\\_0](http://www.numdam.org/item?id=AIF_1990__40_1_103_0)

# Connection with C.S. Seshadri constant



Conjeevaram Srirangachari Seshadri  
(1932-2020)

For a generic set  $S$  of  $s$  points in  $\mathbb{P}^n$ , Seshadri's constant  $\epsilon(S)$  is related to  $\Omega(S)$  by

$$\epsilon(S)^{n-1} = \frac{\Omega(S)}{s}.$$

Duality between the cone of effective curves and the cone of ample divisors.

## C.S. Seshadri's criterion

Let  $X$  be a smooth projective variety and  $L$  a line bundle on  $X$ . Then  $L$  is ample if and only if there exists a positive number  $\epsilon$  such that for all points  $x$  on  $X$  and all irreducible curves  $C$  passing through  $x$  one has

$$L \cdot C \leq \epsilon \operatorname{mult}_x C.$$



Robin Hartshorne

R. Hartshorne. *Ample subvarieties of algebraic varieties*. Springer Lecture Notes in Math., vol. **156**, Springer (1970).

## C.S. Seshadri constant at a point

Let  $X$  be a smooth projective variety and  $L$  a nef line bundle on  $X$ . For a fixed point  $x \in X$  the real number

$$\epsilon(X, L; x) := \inf \frac{L \cdot C}{\text{mult}_x C}$$

is the **Seshadri** constant of  $L$  at  $x$ .

The infimum is over all irreducible curves passing through  $x$ .



Jean-Pierre Demailly

J.-P. Demailly. *Singular Hermitian metrics on positive line bundles*. Complex algebraic varieties (Bayreuth, 1990), Lecture Notes Math. **1507**, Springer-Verlag, (1992) 87–104.

# Seshadri's constant and Nagata's Conjecture



Beata Maria Strycharz-Szemberg



Tomasz Szemberg

Beata Strycharz-Szemberg & Tomasz Szemberg. *Remarks on the Nagata conjecture*, Serdica Mathematical Journal **30** 2-3 (2004), 405– 430.

<http://www.math.bas.bg/serdica/2004/2004-405-430.pdf>

# The Nagata – Biran Conjecture



Masayoshi Nagata  
(1927 – 2008)



Paul Biran

Let  $X$  be a smooth algebraic surface and  $L$  an ample line bundle on  $X$  of degree  $d$ . For sufficiently large  $r$ , the Seshadri constant of a generic set  $Z = \{p_1, \dots, p_r\} \subset X$  satisfies

$$\epsilon(X, L; Z) = \frac{d}{\sqrt{r}}.$$

# Zero estimates

Recall step 2 of the transcendence machinery :

**Step 2** Find a point  $z_0$  where  $f(z_0) \neq 0$ .

Context : zero estimates, multiplicity estimates, interpolation estimates on an algebraic group.

Results of C Hermite, C.L. Siegel, Th. Schneider, K. Mahler, A.O. Gel'fond, R. Tijdeman, W.D. Brownawell, D.W. Masser, G. Wüstholz, P. Philippon, J-C. Moreau, D. Roy, M. Nakamaye, N. Rattazzi, S. Fischler. . .



# Michael Nakamaye and Nicolas Ratazzi



Michael Nakamaye



Nicolas Ratazzi

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# Stéphane Fischler and Michael Nakamaye



Stéphane Fischler



Michael Nakamaye

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# S. David, M. Nakamaye, P. Philippon



*Bornes uniformes pour le nombre de points rationnels de certaines courbes*, Diophantine geometry, 143–164, CRM Series, 4, Ed. Norm., Pisa, 2007.

<http://www.math.unm.edu/~nakamaye/Pisa.pdf>

# S. David, M. Nakamaye, P. Philippon

*Au Professeur C. S. Seshadri à l'occasion de son 75ème anniversaire.*

Nous commençons par une étude indépendante des jets des sections de fibrés amples sur une surface lisse, puis sur le carré d'une courbe elliptique, utile pour le Théorème 4.2. Ceci nous permet en particulier d'introduire dans le présent contexte les constantes de Seshadri, dont l'utilisation en géométrie diophantienne nous semble devoir être positivement stimulée.

<http://www.math.unm.edu/~nakamaye/Pisa.pdf>

# Homogeneous ideals of $R = \mathbb{K}[X_0, \dots, X_n]$

Let  $\mathbb{K}$  be a field of zero characteristic.

For  $p = (\alpha_0 : \dots : \alpha_n) \in \mathbb{P}^n(\mathbb{K})$ , denote by  $I(p)$  the homogeneous ideal generated by the polynomials  $\alpha_i X_j - \alpha_j X_i$  ( $0 \leq i, j \leq n$ ) in the polynomial ring  $R = \mathbb{K}[X_0, \dots, X_n]$ .

For  $S = \{p_1, \dots, p_s\} \subset \mathbb{P}^n(\mathbb{K})$ , set

$$I(S) = I(p_1) \cap \dots \cap I(p_s).$$

This is the ideal of forms vanishing on  $S$ . The least degree of a polynomial in  $I(S)$  is  $\omega_1(S)$ .

# Initial degree

Generally, when  $J$  is a nonzero homogeneous ideal of  $R$ , define  $\omega(J)$  as the least degree of a polynomial in  $J$ .

Since  $J$  is homogeneous,

$$J = \bigoplus_{m \geq 0} J_m$$

we have

$$\omega(J) = \min\{m \geq 0 \mid J_m \neq 0\}$$

and  $\omega(J)$  is also called the *initial degree* of  $J$ .

Since  $J_1 J_2$  is generated by the products  $P_1 P_2$  with  $P_i \in J_i$ , it is plain that  $\omega(J_1 J_2) = \omega(J_1) + \omega(J_2)$ , hence

$$\omega(J^t) = t\omega(J).$$

# Symbolic powers

For  $t \geq 1$ , define the symbolic power  $I^{(t)}(S)$  by

$$I^{(t)}(S) = I(p_1)^t \cap \cdots \cap I(p_s)^t.$$

This is the ideal of forms vanishing on  $S$  with multiplicities  $\geq t$ . Hence

$$\omega(I^{(t)}(S)) = \omega_t(S).$$

We have  $I(S)^t \subset I(p_i)^t$  for all  $i$ , hence

$$I(S)^t \subset \bigcap_{i=1}^s I(p_i)^t = I^{(t)}(S).$$

From  $I(S)^t \subset I^{(t)}(S)$  we deduce  $\omega_t(S) \leq t\omega_1(S)$ .

# Symbolic powers

For a homogeneous ideal  $J$  in the ring  $R = \mathbb{K}[X_0, \dots, X_n]$  and  $m \geq 1$ , define the symbolic power  $J^{(m)}$  as follows. Write primary decompositions of  $J$  and  $J^m$  as

$$J = \bigcap_i \mathfrak{Q}_i, \quad J^m = \bigcap_j \mathfrak{Q}'_j,$$

where  $\mathfrak{Q}_i$  is homogeneous and  $\mathfrak{P}_i$  primary,  $\mathfrak{Q}'_j$  is homogeneous and  $\mathfrak{P}'_j$  primary. We set

$$J^{(m)} = \bigcap_j \mathfrak{Q}'_j$$

where the intersection is over the  $j$  with  $\mathfrak{P}'_j$  contained in some  $\mathfrak{P}_i$ .



# Symbolic powers

Notice that  $J^m \subset J^{(m)}$ .

Example of a *fat points* ideal. For  $J = \bigcap_j I(p_j)^{m_j}$ ,

$$J^{(m)} = \bigcap_j I(p_j)^{mm_j}.$$

# The containment problem

Find all  $m, t$  with

$$J^{(m)} \subset J^t.$$

**Brian Harbourne.** *Asymptotic invariants of ideals of points.* (2009).  
Special Session on Geometry, Syzygies and Computations  
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Slides.

[www.math.unl.edu/~bharbourne1/KSSNoPauseRev.pdf](http://www.math.unl.edu/~bharbourne1/KSSNoPauseRev.pdf)

# $\Omega(J)$ for a homogeneous ideal $J$

Cristiano Bocchi



Brian Harbourne



For a homogeneous ideal  $J$  of  $\mathbb{K}[X_0, \dots, X_n]$ , and  $t \geq 1$ , define  $\omega_t(J) = \omega(J^{(t)})$ . Then

$$\Omega(J) = \lim_{t \rightarrow \infty} \frac{\omega_t(J)}{t}$$

exists and satisfies

$$\Omega(J) \leq \frac{\omega_t(J)}{t} \quad \text{for all } t \geq 1.$$

Alternate proof of  $\Omega(S) \geq \frac{\omega_1(S)}{n}$  (2001, 2002)

**Theorem (Ein-Lazarfeld-Smith, Hochster-Huneke).** Let  $J$  be a homogeneous ideal in  $\mathbb{K}[X_0, \dots, X_n]$  and  $t \geq 1$ . Then

$$J^{(tn)} \subset J^t.$$

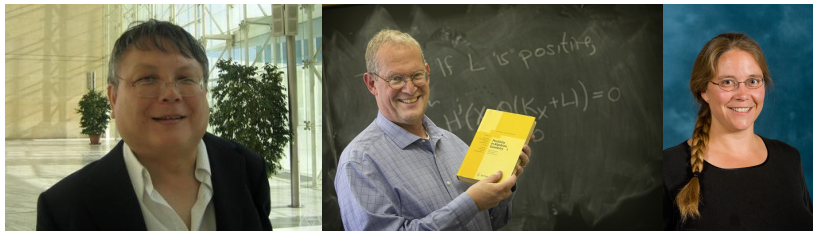
Consequence : From  $I(S)^t \supset I(S)^{(tn)}$  we deduce

$$t\omega_1(S) \leq \omega_{tn}(S)$$

and

$$\frac{\omega_1(S)}{n} \leq \frac{\omega_{tn}(S)}{tn} \rightarrow \Omega(S) \quad \text{as } t \rightarrow \infty.$$

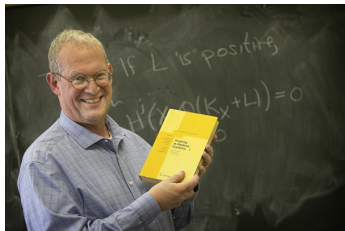
$J^{(tn)} \subset J^t$  by L. Ein, R. Lazarfeld and K.E. Smith



The proof by Lawrence Ein, Robert Lazarfeld and Karen E. Smith uses multiplier ideals.

L. Ein, R. Lazarfeld and K.E. Smith. *Uniform behavior of symbolic powers of ideals*. *Invent. Math.* **144** (2001), 241–252.

$$J^{(tn)} \subset J^t$$



R. Lazarfeld. *Positivity in algebraic geometry I – II*.  
Ergeb. Math. **48–49**,  
Springer, Berlin (2004).

# $J^{(tn)} \subset J^t$ by M. Hochster and C. Huneke

The proof by Melvin Hochster and Craig Huneke uses Frobenius powers and tight closure.

Melvin Hochster



Craig Huneke



M. Hochster and C. Huneke. *Comparison of symbolic and ordinary powers of ideals*. *Invent. Math.* **147** (2002), 349–369.

# Briançon-Skoda Theorem

Melvin Hochster



Craig Huneke



For an  $m$ -generated ideal  $\mathfrak{a}$  in the ring of germs of analytic functions at  $0 \in \mathbb{C}^n$ , the  $\nu$ -th power of its integral closure is contained in  $\mathfrak{a}$ , where  $\nu = \min\{m, n\}$ .

M. Hochster and C. Huneke. *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116.



# The resurgence of Bocci and Harbourne

Define

$$\varrho(J) = \sup \left\{ \frac{m}{r} \mid J^{(m)} \not\subset J^r \right\}.$$

Hence, if  $\frac{m}{r} > \varrho(J)$ , then  $J^{(m)} \subset J^r$ .

By L. Ein, R. Lazarfled and K.E. Smith,  $\varrho(J) \leq n$ .

C. Bocci and B. Harbourne. *Comparing powers and symbolic powers of ideals*. J. Algebraic Geom. **19** (2010), no. 3, 399–417.

C. Bocci and B. Harbourne. *The resurgence of ideals of points and the containment problem*. Proc. Amer. Math. Soc. **138** (2010), no. 4, 1175–1190

# The resurgence of Bocchi and Harbourne

Denote by  $\text{reg}(J)$  the Castelnuovo–Mumford regularity of  $J$ .

**Theorem (Bocchi, Harbourne).** We have

$$\frac{\omega(J)}{\Omega(J)} \leq \varrho(J) \leq \frac{\text{reg}(J)}{\Omega(J)}.$$

Further, if  $\omega(J) = \text{reg}(J)$ , then

$$J^{(m)} \subset J^t \iff t\omega(J) \leq \omega(J^{(m)}).$$

# Optimality

Following Bocci and Harbourne, we have

$$\sup_{|S| < \infty} \frac{\omega_1(S)}{\Omega(S)} = n.$$

# Conjecture of Cristiano Bocchi and Brian Harbourne

Let  $S$  be a finite subset of  $\mathbb{P}^2$ . Define  $\varrho(S) = \varrho(J)$  for  $J = I(S)$ .

**Conjecture.**

$$\varrho(S) \leq 2 \frac{\omega_1(S)}{\omega_1(S) + 1}.$$

This conjecture implies Chudnovsky's conjecture : from

$$\frac{\omega_1(S)}{\Omega(S)} \leq \varrho(S) \leq 2 \frac{\omega_1(S)}{\omega_1(S) + 1}$$

one deduces

$$\frac{\omega_1(S) + 1}{2} \leq \Omega(S).$$

# The containment problem (continued)

Let  $\mathfrak{M}$  be the homogeneous ideal  $(X_0, \dots, X_n)$  in  $R$ .

Fact. In characteristic zero, the ideal  $J = I(S)$  satisfies  $J^{(2)} \subset \mathfrak{M}J$ .

Proof. Let  $P \in J^{(2)}$ . Hence  $\frac{\partial}{\partial X_i} P \in J$ . Use Euler's formula

$$(\deg P)P = \sum_{i=0}^n X_i \frac{\partial}{\partial X_i} P.$$

**Question.** For which  $m, t, j$  do we have  $J^{(m)} \subset \mathfrak{M}^j J^t$ ?

Remark. Since  $\mathfrak{M}^j J^t \subset J^t$ , the condition  $J^{(m)} \subset \mathfrak{M}^j J^t$  implies  $J^{(m)} \subset J^t$ .

B. Harbourne and C. Huneke. *Are symbolic powers highly evolved?*

J. Ramanujan Math. Soc. **28A** (2013), 247–266.

arxiv:1103.5809.

# Conjecture of Brian Harbourne and Craig Huneke

Chudnovsky's result

$$\frac{\omega_1 + n - 1}{n} \leq \frac{\omega_t}{t}.$$

for  $n = 2$  follows from

$$J^{(2t)} \subset \mathfrak{M}^t J^t$$

for any homogeneous ideal of points  $J = I(S)$  in  $\mathbb{K}[X_0, X_1, X_2]$ .

Generalization for  $n \geq 2$ .

# Conjecture of Brian Harbourne and Craig Huneke



Let  $J = \bigcap_j I(p_j)^{m_j}$  be a fat points ideal in  $R$ .

**Conjecture (Harbourne and Huneke).** For all  $t > 0$ ,

$$J^{(tn)} \subset \mathfrak{M}^{t(n-1)} J^t.$$

# Evolutions

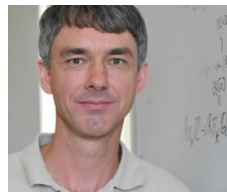
Andrew Wiles



Richard Taylor



Matthias Flach



*Evolutions* are certain kinds of ring homomorphisms that arose in proving **Fermat's last Theorem** (**A. Wiles**, **R. Taylor**, **M. Flach**).

An important step in the proof was to show that in certain cases only trivial evolutions occurred.



# Evolutions

D. Eisenbud and B. Mazur showed the question of triviality could be translated into a statement involving symbolic powers. They then made the following conjecture in characteristic 0 :



**Conjecture (Eisenbud–Mazur)** Let  $\mathfrak{P} \subset \mathbb{C}[[x_1, \dots, x_d]]$  be a prime ideal. Let  $\mathfrak{M} = (x_1, \dots, x_d)$ . Then  $\mathfrak{P}^{(2)} \subset \mathfrak{M}\mathfrak{P}$ .

# Evolutions

Heuristically, the main conjecture of [Harbourne](#) and [Huneke](#) can be thought of as a generalization of the conjecture of [Eisenbud](#) and [Mazur](#).

[B. Harbourne](#) and [C. Huneke](#). *Are symbolic powers highly evolved?*  
J. Ramanujan Math. Soc. **28**, (2011)  
[arxiv:1103.5809](#).

# Brian Harbourne



Brian Harbourne, Sandra Di Rocco, Tomasz Szemberg, Thomas Bauer

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M. Dumnicki, B. Harbourne, T. Szemberg and H. Tutaj-Gasińska.  
*Linear subspaces, symbolic powers and Nagata type conjectures.*  
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# Marcin Dumnicki



Chudnovsky's conjecture

$\Omega(S) \geq \frac{\omega_1(S) + n - 1}{n}$  holds  
for generic finite subsets in  $\mathbb{P}^3$ .

M. Dumnicki. *Symbolic powers of ideal of generic points in  $\mathbb{P}^3$* .  
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# Thomas Bauer and Tomasz Szemberg



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C. Bocci, S.M. Cooper and B. Harbourne. *Containment results for ideals of various configurations of points in  $\mathbb{P}^N$* . J. Pure Applied Algebra **218** (2014), 65–75.

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# Two invariants related to two conjectures due to Nagata

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