October 12, 2020

Upstate New York Online Number Theory Colloquium

Some variants of Seshadri's constant.

Michel Waldschmidt

Sorbonne Université
Institut de Mathématiques de Jussieu Paris rive gauche

http://www.imj-prg.fr/~michel.waldschmidt/

Abstract

Seshadri's constant is related to a conjecture due to Nagata. Another conjecture, also due to Nagata and solved by Bombieri in 1970, is related with algebraic values of meromorphic functions. The main argument of Bombieri's proof leads to a Schwarz Lemma in several variables, the proof of which gives rise to another invariant associated with symbolic powers of the ideal of functions vanishing on a finite set of points. This invariant is an asymptotic measure of the least degree of a polynomial in several variables with given order of vanishing on a finite set of points. Recent works on the resurgence of ideals of points and the containment problem compare powers and symbolic powers of ideals.

C.S. Seshadri memorial lectures — July 31, 2020

Seshadri Memorial Lectures, 2020

Tata Institute of Fundamental Research, Mumbai

July 31, 2020

Speakers

J.P. Demailly
(On Seshadri Constant)

P. Littelmann
(Standard Monomials)

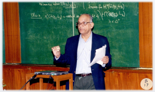
V. Balaji
(On GIT)

Poster | Videos | Slides

tifr — School of Mathematics

C. S. SESHADRI MEMORIAL LECTURES

July 31, 2020; 4 PM IST



Speakers

J. P. Demailly (On Seshadri Constant)
P. Littelmann (Standard Monomials)
V. Balaji (On GIT)

Duration of each talk will be 30 mins.
The lectures will be held via Zoom

For Zoom ID and Password please contact:
Milind Pilankar
(pilankar @ math.tifr.res.in)

http://conferences.math.tifr.res.in/lectures/seshadri-lectures/seshadri-lectures.html

Conjeevaram Srirangachari Seshadri

(1932-2020)

Jean–Pierre Demailly

http://conferences.math.tifr.res.in/lectures/seshadri-lectures/demailly.pdf

A brief survey on Seshadri constants

Jean-Pierre Demailly

Institut Fourier, Université Grenoble Alpes & Académie des Sciences de Paris

Memorial Lectures for
Professor Conjeevaram Srirangachari Seshadri
organized by the TIFR School of Mathematics
July 31, 2020, 4 PM IST

The starting point of Seshadri constant theory

The paper that started the whole theory :

C.S. Seshadri, *Annals of Mathematics*, Vol. 95 (May 1972) 511–556

Quotient spaces modulo reductive algebraic groups

By C. S. SESHADRI

Introduction

In his book "Geometric invariant theory" [M], Mumford developed a theory of quotient spaces of algebraic schemes acted on by reductive algebraic groups when the ground field is of characteristic zero and showed how this can be used for several questions of moduli. In order to have this theory in arbitrary characteristic, he made the following conjecture:

(A): Let G be a reductive algebraic group (say over an algebraically closed field) and V a finite dimensional rational G-module. Then given a G-invariant point v, $v \neq 0$, there is a G-invariant homogeneous polynomial F on V such that $F(v) \neq 0$.

Seshadri's ampleness criterion

LEMMA 7.2. *Let X be a complete variety and x a point of X at which X is smooth. Let $p\colon X' \to X$ be the* BLOWING UP *of X at the point x (i.e. blowing up of X with respect to the sheaf of ideals defining the reduced subscheme of X consisting of the one point x). Let L be a line bundle on X which is* PSEUDO-AMPLE *i.e.* $\forall\, C \hookrightarrow X$ *where C is a closed integral curve of X,* $\deg(L/C) \geqq 0$. *Let $Z = p^{-1}(x)$ and E the line bundle on X' defined by the effective divisor Z (so that $-E$ i.e. E^{-1} is relatively ample with respect to p). Suppose that $aL - bE$ is also pseudo-ample for some $a, b \in \mathbf{Z}$, $a, b > 0$. Then if $n = \dim X$, we have*

$$L^{(n)} = L \cdots L \ (n\text{-fold intersection product}) > 0 \,.$$

⋮

Remark 7.1. The above lemma can be interpreted (as has been remarked by C.P. Ramanujam) to give a criterion of ampleness as follows: Let X be a complete algebraic scheme. To every closed integral curve C, $C \hookrightarrow X$, define $m(C)$ to be the maximum of the multiplicities at the different points of C. Let L be a line bundle on X. Then L is ample on $X \Leftrightarrow \exists\, \varepsilon > 0$ such that $\forall$ closed integral curve $C \hookrightarrow X$, $\deg(L|_C) \geqq \varepsilon m(C)$.

Definition of the Seshadri constants

Definition (Demailly, 1990)

Let X be a projective nonsingular variety and L a nef (or pseudo-ample) line bundle over X. Given a point $x \in X$, one defines the Seshadri constant $\varepsilon(L, x)$ of L at x to be

$$\varepsilon(L, x) = \inf_{\text{all alg. curves } C \ni x} \frac{L \cdot C}{\operatorname{mult}_x(C)}.$$

This is a very interesting numerical invariant that measures in a deep manner the "local positivity" of the line bundle L at point x.

Equivalent definition (already observed in Seshadri's paper !)

Let $\pi : \tilde{X} \to X$ be the blow-up of X at $x \in X$, and E the exceptional divisor in $\tilde{X}$. Then, for $L \in \operatorname{Pic}(X)$ assumed to be nef, one has

$$\varepsilon(L, x) = \sup\{\gamma \geq 0 \,/\, \pi^* L - \gamma E \text{ is nef on } \tilde{X}\}.$$

Reformulation of Seshadri's ampleness criterion

Reformulation of Seshadri's ampleness criterion

A nef line bundle $L \in \mathrm{Pic}(X)$ is ample if and only if one has $\varepsilon(L) := \inf_{x\in X} \varepsilon(L,x) > 0$.
A direct consequence of the fact $(\pi^* L - \gamma E)^n = L^n - \gamma^n \geq 0$ is that

$$\varepsilon(L,x) \leq (L^n)^{1/n}, \qquad \forall x \in X.$$

A curve C is said to be submaximal if $\dfrac{L\cdot C}{\mathrm{mult}_x(C)} < (L^n)^{1/n}$.

A large part of the investigations on Seshadri constants, especially in the case of surfaces, rests upon the study of submaximal curves.

Remark. In [D, 1990], over $\mathbb{K} = \mathbb{C}$, the Seshadri constant is related to more analytic invariants. For instance, if L is ample, it can be shown that $\varepsilon(L,x)$ is the supremum of $\gamma \geq 0$ for which L possesses a singular Hermitian metric h with $\Theta_{L,h} \geq 0$, that is smooth on $X \smallsetminus \{x\}$ with a logarithmic pole of Lelong number γ at x.

Relation to the Nagata conjecture

The concept of Seshadri constant is already highly non trivial on rational surfaces. For instance, the famous Nagata conjecture, has attracted lot of work by Hirschowitz, Harbourne, Biran, Bauer, Szemberg, Dumnicki and others. It can be reformulated :

Nagata conjecture (1959), reformulated

Let $x_1, \ldots, x_p$ be p very general points in $\mathbb{P}^2$, $p \geq 9$. Then the multipoint Seshadri constant of $\mathcal{O}(1)$ on $\mathbb{P}^2$ satisfies

$$\varepsilon(\mathcal{O}(1), x_1, \ldots, x_p) = \frac{1}{\sqrt{p}}.$$

A simple counting argument implies that $\varepsilon(\mathcal{O}(1), x_1, \ldots, x_p) \leq \frac{1}{\sqrt{p}}$, and the main difficulty is to find good configurations of points to get lower bounds. In case $p = q^2$ is a perfect square, a square grid works, hence equality. For $4 < p < 9$, one is in the Del Pezzo case, and the equality turns out to be strict.

Schneider – Lang Theorem (1949, 1966)

Theodor Schneider
(1911 – 1988)

Serge Lang
(1927 – 2005)

Let $f_1, \ldots, f_m$ *be meromorphic functions in* $\mathbb{C}$. *Assume* f_1 *and* f_2 *are algebraically independent and of finite order. Let* $\mathbb{K}$ *be a number field. Assume* f_j' *belongs to* $\mathbb{K}[f_1, \ldots, f_m]$ *for* $j = 1, \ldots, m$. *Then the set*

$$S = \{w \in \mathbb{C} \mid w \text{ not pole of } f_j,\ f_j(w) \in \mathbb{K}\ (j = 1, \ldots, m)\}$$

is finite.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Schneider.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lang.html

Hermite – Lindemann Theorem (1882)

Charles Hermite
(1822 – 1901)

Carl Louis Ferdinand von Lindemann
(1852 – 1939)

Corollary. *If w is a nonzero complex number, one at least of the two numbers w, e^w is transcendental.*

Consequence : transcendence of e, π, $\log \alpha$, e^β, for algebraic α and β assuming $\alpha \neq 0$, $\log \alpha \neq 0$, $\beta \neq 0$.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hermite.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lindemann.html

Gel'fond – Schneider Theorem (1934)

Aleksandr Osipovich Gelfond

(1906 – 1968)

Theodor Schneider

(1911 – 1988)

Corollary (Hilbert's seventh problem). *If* β *is an irrational algebraic number and* w *a nonzero complex number, one at least of the two numbers* e^w, $e^{\beta w}$ *is transcendental.*

Consequence : transcendence of e^π, $2^{\sqrt{2}}$, α^β, $\log\alpha_1/\log\alpha_2$, for algebraic α, α_1, α_2 and β assuming $\alpha \neq 0$, $\log\alpha \neq 0$, $\beta \notin \mathbb{Q}$, $\log\alpha_1/\log\alpha_2 \notin \mathbb{Q}$.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Gelfond.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Schneider.html

Proofs of the corollaries

`Hermite - Lindemann`. Let $\mathbb{K} = \mathbb{Q}(w, e^w)$. The two functions $f_1(z) = z$, $f_2(z) = e^z$ are algebraically independent, of finite order, and satisfy the differential equations $f_1' = 1$, $f_2' = f_2$. The set S contains $\{\ell w \mid \ell \in \mathbb{Z}\}$. Since $w \neq 0$, this set is infinite ; it follows that $\mathbb{K}$ is not a number field. $\square$

`Gel'fond - Schneider`. Let $\mathbb{K} = \mathbb{Q}(\beta, e^w, e^{\beta w})$. The two functions $f_1(z) = e^z$, $f_2(z) = e^{\beta z}$ are algebraically independent, of finite order, and satisfy the differential equations $f_1' = f_1$, $f_2' = \beta f_2$. The set S contains $\{\ell w \mid \ell \in \mathbb{Z}\}$. Since $w \neq 0$, this set is infinite ; it follows that $\mathbb{K}$ is not a number field. $\square$

Schneider's Theorems on elliptic functions (1937)

Corollary (Schneider). *Let $\wp$ be an elliptic function of Weierstrass with algebraic invariants g_2, g_3. Let w be a complex number, not pole of $\wp$. Then one at least of the two numbers w, $\wp(w)$ is transcendental.*

`Proof.` Let $\mathbb{K} = \mathbb{Q}(g_2, w, \wp(w), \wp'(w))$. The two functions $f_1(z) = z$, $f_2(z) = \wp(z)$ are algebraically independent, of finite order. Set $f_3(z) = \wp'(z)$. From $\wp'^2 = 4\wp^3 - g_2\wp - g_3$ one deduces

$$f_1' = 1, \quad f_2' = f_3, \quad f_3' = 6f_2^2 - (g_2/2).$$

The set S contains

$$\{\ell w \mid \ell \in \mathbb{Z},\ \ell w \text{ not pole of } \wp\}$$

which is infinite. Hence $\mathbb{K}$ is not a number field. □

The transcendence machinery

The prototype of transcendence methods is Hermite's proof of the transcendence of e.

The proof of the Schneider – Lang Theorem follows the following scheme :

Step 1 Construct an auxiliary function f with many zeroes.

Step 2 Find a point z_0 where $f(z_0) \neq 0$.

Step 3 Give a lower bound for $|f(z_0)|$ using arithmetic arguments.

Step 4 Give an upper bound for $|f(z_0)|$ using analytic arguments.

We are interested here mainly (but not only) with the last part (step 4) which is of analytic nature.

Schwarz Lemma in one variable

Hermann Amandus Schwarz

(1843 – 1921)

Let f be an analytic function in a disc $|z| \leq R$ of $\mathbb{C}$, with at least N zeroes in a disc $|z| \leq r$ with $r < R$. Then

$$|f|_r \leq \left(\frac{3r}{R}\right)^N |f|_R.$$

We use the notation

$$|f|_r = \sup_{|z|=r} |f(z)|.$$

When $R > 3r$, this improves the maximum modulus bound $|f|_r \leq |f|_R$.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Schwarz.html

Schwarz Lemma in one variable : proof

Let $a_1, \ldots, a_N$ be zeroes of f in the disc $|z| \leq r$, counted with multiplicities. The function

$$g(z) = f(z) \prod_{j=1}^{N} (z - a_j)^{-1}$$

is analytic in the disc $|z| \leq R$. Using the maximum modulus principle, from $r \leq R$ we deduce $|g|_r \leq |g|_R$. Now we have

$$|f|_r \leq (2r)^N |g|_r \quad \text{and} \quad |g|_R \leq (R-r)^{-N} |f|_R.$$

Finally, assuming (wlog) $R > 3r$,

$$\frac{2r}{R-r} \leq \frac{3r}{R}.$$

□

Schneider – Lang Theorem in several variables : cartesian products (1941, 1966)

Let $f_1, \ldots, f_m$ *be meromorphic functions in* $\mathbb{C}^n$ *with* $m \geq n+1$. *Assume* $f_1, \ldots, f_{n+1}$ *are algebraically independent of finite order. Let* $\mathbb{K}$ *be a number field. Assume* $(\partial/\partial z_i) f_j'$ *belongs to* $\mathbb{K}[f_1, \ldots, f_m]$ *for* $j = 1, \ldots, m$ *and* $i = 1, \ldots, n$. *If* $e_1, \ldots, e_n$ *is a basis of* $\mathbb{C}^n$, *then the set*

$$S = \{w \in \mathbb{C}^n \mid w \text{ not pole of } f_j,\ f_j(w) \in \mathbb{K}\ (j = 1, \ldots, m)\}$$

does not contain a cartesian product

$$\{s_1 e_1 + \cdots + s_n e_n \mid (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n\}$$

where each S_i *is infinite.*

Schneider's Theorem on Euler's Beta function

Leonhard Euler
(1707 – 1783)

Let a, b be rational numbers, not integers. Then the number

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Euler.html

Further results by Th. Schneider and S. Lang on abelian functions and algebraic groups.

Schwarz lemma in several variables : cartesian products (grids)

Let f be an analytic function in a ball $|z| \leq R$ of $\mathbb{C}^n$. Assume f vanishes with multiplicity at least t on a set $S_1 \times \cdots \times S_n$ where each S_i is contained in a disc $|z| \leq r$ with $r < R$ and has at least s elements.
Then

$$|f|_r \leq \left(\frac{3r}{R}\right)^{st} |f|_R.$$

Cartesian products

Schwarz Lemma for Cartesian products can be proved by induction.

§4.3 of M.W.. *Diophantine Approximation on Linear Algebraic Groups.* Grund. Math. Wiss. **326** Springer-Verlag (2000).

Another proof, based on integral formulae, yields a weaker result : for $R > 3r$,

$$|f|_r \leq \left(\frac{R-3r}{2r}\right)^n \left(\frac{3r}{R}\right)^{st} |f|_R.$$

The conclusion follows from a homogeneity argument : replace f by f^N (and t by Nt) and let $N \to \infty$.

Chap. 7 of M.W.. *Nombres transcendants et groupes algébriques.* Astérisque, **69–70** (1979).

Nagata's suggestion (1966)

Masayoshi Nagata
(1927 – 2008)

In the conclusion of the Schneider – Lang Theorem, replace the fact that S does not contain a cartesian product $S_1 \times \cdots \times S_n$ where each S_i is infinite by the fact that S is contained in an algebraic hypersurface.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Nagata.html

Bombieri's Theorem (1970)

Let $f_1, \ldots, f_m$ *be meromorphic functions in* $\mathbb{C}^n$ *with* $m \geq n+1$. *Assume* $f_1, \ldots, f_{n+1}$ *are algebraically independent and of finite order. Let* $\mathbb{K}$ *be a number field. Assume* $(\partial/\partial z_i) f'_j$ *belongs to* $\mathbb{K}[f_1, \ldots, f_m]$ *for* $j = 1, \ldots, m$ *and* $i = 1, \ldots, n$.

Enrico Bombieri

Then the set

$$S = \{w \in \mathbb{C}^n \mid w \text{ not pole of } f_j, \; f_j(w) \in \mathbb{K} \; (j = 1, \ldots, m)\}$$

is contained in an algebraic hypersurface.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Bombieri.html

Bombieri – Lang (1970)

Let f be an analytic function in a ball $|z| \leq R$ of $\mathbb{C}^n$. Assume f vanishes at N points z_i (counting multiplicities) in a ball $|z| \leq r$ with $r < R$. Assume $\min_{z_i \neq z_k} |z_i - z_k| \geq \delta$.

Then

$$|f|_r \leq \left(\frac{3r}{R}\right)^M |f|_R$$

with

$$M = N\left(\frac{\delta}{6r}\right)^{2n-2}.$$

Lelong number

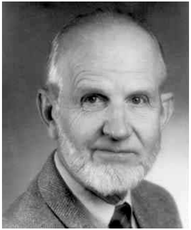
E. Bombieri. *Algebraic values of meromorphic maps.* Invent. Math. **10** (1970), 267–287.
E. Bombieri and S. Lang. *Analytic subgroups of group varieties.* Invent. Math. **11** (1970), 1–14.

Pierre Lelong
(1912 – 2011)

P. Lelong. *Intégration sur un ensemble analytique complexe*, Bulletin S.M.F. **85** (1957), 239–262,

https://fr.wikipedia.org/wiki/Pierre_Lelong

L^2- estimates of Hörmander



Lars Hörmander
(1931 – 2012)

Existence theorems for the $\overline{\partial}$ operator.

E. Bombieri. *Let* φ *be a plurisubharmonic function in* $\mathbb{C}^n$ *and* $z_0 \in \mathbb{C}^n$ *be such that* $e^{-\varphi}$ *is integrable near* z_0*. Then there exists a nonzero entire function* F *such that*

$$\int_{\mathbb{C}^n} |F(z)|^2 e^{-\varphi(z)} (1 + |z|^2)^{-3n} d\lambda(z) < \infty.$$

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hormander.html

Towards a Schwarz lemma in several variables

Let S be a finite subset of $\mathbb{C}^n$ and t a positive integer. Let M be a positive number with the following property.
There exists a real number r such that for $R > r$, if f is an analytic function in the ball $|z| \leq R$ of $\mathbb{C}^n$ which vanishes with multiplicity at least t at each point of S, then

$$|f|_r \leq \left(\frac{c(n)r}{R}\right)^M |f|_R,$$

where $c(n)$ depends only on the dimension n.

`Question`: what is the largest possible value for M ?

`Answer`: use the property for f a nonzero polynomial vanishing on S with multiplicity t. We deduce that f has degree at least M.

Degree of hypersurfaces

Let S be a finite set of $\mathbb{C}^n$ and t a positive integer.

Denote by $\omega_t(S)$ the smallest degree of a polynomial vanishing at each point of S with multiplicity $\geq t$.

M.W. *Propriétés arithmétiques de fonctions de plusieurs variables* (II). Sém. P. Lelong (Analyse), 16ème année, 1975/76 ; Lecture Notes in Math., **578** (1977), 274–292.
M.W. *Nombres transcendants et groupes algébriques.* Astérisque, **69–70**. Société Mathématique de France, Paris, 1979.

Schwarz lemma in several variables

Let S be a finite set of $\mathbb{C}^n$ and t a positive integer. There exists a real number r such that for $R > r$, if f is an analytic function in the ball $|z| \leq R$ of $\mathbb{C}^n$ which vanishes with multiplicity at least t at each point of S, then

$$|f|_r \leq \left(\frac{e^n r}{R}\right)^{\omega_t(S)} |f|_R.$$

This is a refined asymptotic version due to Jean-Charles Moreau.

The exponent $\omega_t(S)$ cannot be improved : take for f a non–zero polynomial of degree $\omega_t(S)$.

$\omega_t(S)$: examples

For $n = 1$ we have $\omega_t(S) = t|S|$:

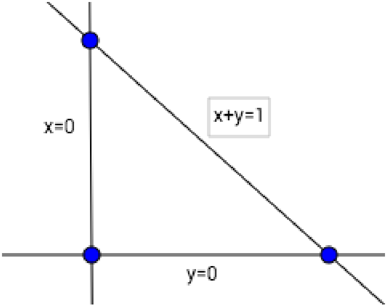
$$\prod_{s \in S} (z - s)^t.$$

More generally, for a Cartesian product $S = S_1 \times \cdots \times S_n$ in $\mathbb{K}^n$,

$$\omega_t(S) = t \min_{1 \leq i \leq n} |S_i|.$$

Proof by induction.

Generic $S \subset \mathbb{K}^2$ with $|S| = 3$



$$S = \{(0,0),(0,1),(1,0)\}.$$

$$P_1(X,Y) = XY$$

$$P_2(X,Y) = XY(X+Y-1)$$

$$\omega_1(S) = 2, \quad \omega_2(S) = 3.$$

With

$$P_{2m-1} = X^m Y^m (X+Y-1)^{m-1}, \; P_{2m} = X^m Y^m (X+Y-1)^m,$$

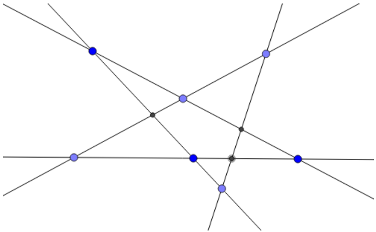
we deduce

$$\omega_{2m-1}(S) = 3m-1, \quad \omega_{2m}(S) = 3m.$$

Complete intersections of hyperplanes

Let $H_1, \dots, H_N$ be N hyperplanes in general position in $\mathbb{K}^n$ with $N \geq n$ and S the set of $\binom{N}{n}$ intersection points of any n of them. Then, for $t \geq 1$,

$$\omega_{nt}(S) = Nt.$$



$n = 2$, $N = 5$, $|S| = 10$.

Dirichlet's box principle

Johann Peter Gustav Lejeune Dirichlet
(1805 – 1859)

Given a finite subset S of $\mathbb{K}^n$ and a positive integer t, if D is a positive integer such that

$$|S|\binom{t+n-1}{n} < \binom{D+n}{n},$$

then

$$\omega_t(S) \leq D.$$

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Dirichlet.html

Properties of $\omega_t(S)$

Consequence of Dirichlet's box principle :

$$\omega_t(S) \leq (t+n-1)|S|^{1/n}.$$

Subadditivity :

$$\omega_{t_1+t_2}(S) \leq \omega_{t_1}(S) + \omega_{t_2}(S).$$

Therefore $\limsup_{t\to\infty} \omega_t(S)/t$ exists and is $\leq \omega_t(S)/t$ for all $t \geq 1$.

An asymptotic invariant

Theorem. *The sequence*

$$\left(\frac{1}{t}\omega_t(S)\right)_{t\geq 1}$$

has a limit $\Omega(S)$ *as* $t \to \infty$, *and*

$$\frac{1}{n}\omega_1(S) - 2 \leq \Omega(S) \leq \omega_1(S).$$

Further, for all $t \geq 1$ *we have*

$$\Omega(S) \leq \frac{\omega_t(S)}{t}.$$

Remark : $\Omega(S) \leq |S|^{1/n}$.

M.W. *Propriétés arithmétiques de fonctions de plusieurs variables* (II). Sém. P. Lelong (Analyse), 16è année, 1975/76 ; Lecture Notes in Math., **578** (1977), 274–292.

Improvement of L^2 estimate by Henri Skoda

Let φ be a plurisubharmonic function in $\mathbb{C}^n$ and $z_0 \in \mathbb{C}^n$ be such that $e^{-\varphi}$ is integrable near z_0. For any $\epsilon > 0$ there exists a nonzero entire function F such that

$$\int_{\mathbb{C}^n} |F(z)|^2 e^{-\varphi(z)}(1+|z|^2)^{-n-\epsilon} d\lambda(z) < \infty.$$

Corollary :

$$\frac{1}{n}\omega_1(S) \leq \Omega(S) \leq \omega_1(S).$$

H. Skoda. *Estimations L^2 pour l'opérateur $\overline{\partial}$ et applications arithmétiques.* Springer Lecture Notes in Math., **578** (1977), 314–323.

Henri Skoda

https://en.wikipedia.org/wiki/Henri_Skoda

Comparing $\omega_{t_1}(S)$ and $\omega_{t_2}(S)$

`Idea:` Let P be a polynomial of degree $\omega_{t_1}(S)$ vanishing on S with multiplicity $\geq t_1$. If the function P^{t_2/t_1} were an entire function, it would be a polynomial of degree $\frac{t_2}{t_1}\omega_{t_1}(S)$ vanishing on S with multiplicity $\geq t_2$, which would yield $\omega_{t_2}(S) \leq \frac{t_2}{t_1}\omega_{t_1}(S)$.

However P^{t_2/t_1} is usually not an entire function but $\varphi = \frac{t_2}{t_1}\log P$ is a plurisubharmonic function. By the L^2–estimates of Hörmander – Bombieri – Skoda, e^{φ} is well approximated by a nonzero entire function. This function is a polynomial vanishing on S with multiplicity $\geq t_2$.

Theorem. *For all* $t \geq 1$,

$$\frac{\omega_t}{t+n-1} \leq \Omega(S) \leq \frac{\omega_t}{t}\cdot$$

M.W. *Nombres transcendants et groupes algébriques.* Astérisque, **69–70** . Société Mathématique de France, Paris, 1979.

Hilbert's 14th problem

David Hilbert
(1862 – 1943)

Let k be a field and K a subfield of $k(X_1,\ldots,X_n)$ containing k. Is the k–algebra

$$K \cap k[X_1,\ldots,X_n]$$

finitely generated ?

Oscar Zariski (1954) : true for $n=1$ and $n=2$.
Counterexample by Masayoshi Nagata in 1959.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hilbert.html

http://www.clarku.edu/~djoyce/hilbert/

Hilbert's 14th problem : restricted case

Masayoshi Nagata
(1927 – 2008)

Original 14th problem :
Let G be a subgroup of the full linear group of the polynomial ring in indeterminate $X_1, \ldots, X_n$ over a field k, and let $\mathfrak{o}$ be the set of elements of $k[X_1, \ldots, X_n]$ which are invariant under G. Is $\mathfrak{o}$ finitely generated ?

M. Nagata. *On the 14-th Problem of Hilbert*. Amer. J. Math **81** (1959), 766–772.
http://www.jstor.org/stable/2372927

Fundamental Lemma of Nagata

Given 16 independent generic points of the projective plane over a prime field and a positive integer t, there is no curve of degree $4t$ which goes through each p_i with multiplicity at least t.

In other words for $|S| = 16$ generic in $\mathbb{K}^2$, we have $\omega_t(S) > 4t$.

Reference: M. Nagata. *On the fourteenth problem of Hilbert.* Proc. Internat. Congress Math. 1958, Cambridge University Press, pp. 459–462.

http://www.mathunion.org/ICM/ICM1958/Main/icm1958.0459.0462.ocr.pdf

Nagata' contribution

Masayoshi Nagata
(1927 – 2008)

Proposition. *Let* $p_1, \ldots, p_r$ *be independent generic points of the projective plane over the prime field. Let* C *be a curve of degree* d *passing through the* p_i*'s with multiplicities* $\geq m_i$. *Then*

$$m_1 + \cdots + m_r < d\sqrt{r}$$

for $r = s^2$, $s \geq 4$.

It is not known if $r > 9$ is sufficient to ensure the inequality of the Proposition.

M. Nagata. *Lectures on the fourteenth problem of Hilbert.* Tata Institute of Fundamental Research Lectures on Mathematics **31**, (1965), Bombay.

http://www.math.tifr.res.in/~publ/ln/tifr31.pdf

Reformulation of Nagata's Conjecture

By considering $\sum_\sigma C_\sigma$ where σ runs over the cyclic permutations of $\{1, \dots, r\}$, it is sufficient to consider the case $m_1 = \cdots = m_r$.

Conjecture. *Let S be a finite generic subset of the projective plane over the prime field with $|S| \geq 10$. Then*

$$\omega_t(S) > t\sqrt{|S|}.$$

Nagata :

- True for $|S|$ a square.
- False for $|S| \leq 9$.

Harbourne, Brian. *On Nagata's conjecture*, Journal of Algebra **236** 2 (2001), 692–702.
https://doi.org/10.1006/jabr.2000.8515

$|S| \leq 9$ in $\mathbb{K}^2$

Nagata : generic S in $\mathbb{K}^2$ with $|S| \leq 9$ have $\frac{\omega_t(S)}{t} \leq \sqrt{|S|}$.

$\lvert S\rvert$	$=$	1	2	3	4	5	6	7	8	9
$\omega_1(S)$	$=$	1	1	2	2	2	3	3	3	3
t	$=$	1	1	2	1	1	$\mathbf{5}$	$\mathbf{8}$	$\mathbf{17}$	1
$\omega_t(S)$	$=$	1	1	3	2	2	$\mathbf{12}$	$\mathbf{21}$	$\mathbf{48}$	3
$\frac{\omega_t(S)}{t}$	$=$	1	1	$\frac{3}{2}$	2	2	$\frac{\mathbf{12}}{\mathbf{5}}$	$\frac{\mathbf{21}}{\mathbf{8}}$	$\frac{\mathbf{48}}{\mathbf{17}}$	3
$\sqrt{\lvert S\rvert}$	$=$	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	$\sqrt{8}$	3

$|S| = 1$ or 2 in $\mathbb{K}^2$

$|S| = 1 : S = \{(0,0)\}$, $P_t(X,Y) = X^t$,
$\omega_t(S) = t$, $\Omega(S) = 1$.

$|S| = 2 : S = \{(0,0),(1,0)\}$, $P_t(X,Y) = Y^t$,
$\omega_t(S) = t$, $\Omega(S) = 1$.

Generic S with $|S| = 3$ in $\mathbb{K}^2$

Given a set S of 3 points in $\mathbb{K}^2$, not on a straight line, we have

$$\omega_t(S) = \begin{cases} \dfrac{3t+1}{2} & \text{for } t \text{ odd}, \\ \dfrac{3t}{2} & \text{for } t \text{ even}, \end{cases}$$

hence

$$\Omega(S) = \lim_{n\to\infty} \frac{\omega_t(S)}{t} = \frac{3}{2}.$$

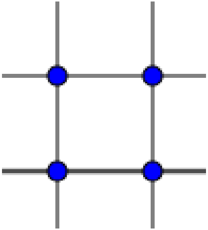
Since $\omega_1(S) = 2$ and $n = 2$, this is an example with

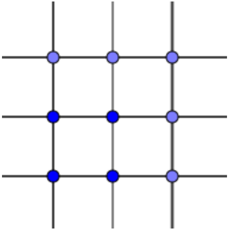
$$\frac{\omega_1(S)}{n} < \Omega(S) < \omega_1(S).$$

Generic $S \subset \mathbb{K}^2$ with $|S| = 4$

For a generic S in $\mathbb{K}^2$ with $|S| = 4$, we have $\omega_t(S) = 2t$, hence $\Omega(S) = 2$.

Easy for a Cartesian product $S_1 \times S_2$ with $|S_1| = |S_2| = 2$, also true for a generic S with $|S| = 4$.





More generally, for the same reason, when S is a Cartesian product $S_1 \times S_2$ with $|S_1| = |S_2| = m$, we have $\omega_t(S) = mt$ and $\Omega(S) = m = \sqrt{|S|}$. The inequality $\Omega(S) \geq \sqrt{|S|}$ for a generic S with $|S|$ a square follows (Chudnovsky).

Generic $S \subset \mathbb{K}^2$ with $|S| = 5$

Since 5 points in $\mathbb{K}^2$ lie on a conic, for a generic S with $|S| = 5$ we have $\omega_t(S) = 2t$ and $\Omega(S) = 2$.

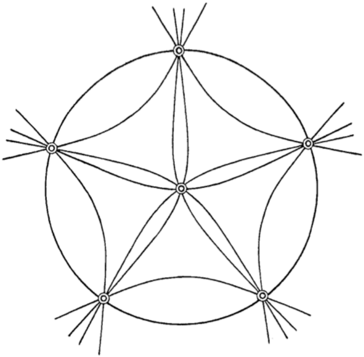
`Remark.` A polynomial in 2 variables of degree d has

$$\frac{(d+1)(d+2)}{2}$$

coefficients. Hence for $2|S| < (d+1)(d+2)$ we have $\omega_1(S) \leq d$.

For $|S| = 1, 2$ we have $\omega_1(S) = 1$,
for $|S| = 3, 4, 5$ we have $\omega_1(S) \leq 2$,
for $|S| = 6, 7, 8, 9$ we have $\omega_1(S) \leq 3$.

Generic $S \subset \mathbb{K}^2$ with $|S| = 6$ (Nagata)



Given 6 generic points $s_1, \dots, s_6$ in $\mathbb{K}^2$, consider 6 conics $C_1, \dots, C_6$ where S_i passes through the 5 points s_j for $j \neq i$. This produces a polynomial of degree 12 with multiplicity ≥ 5 at each s_i. Hence $\omega_5(S) \leq 12$.

In fact $\omega_{5t}(S) = 12t$, $\Omega(S) = 12/5$.

Generic $S \subset \mathbb{K}^2$ with $|S| = 7$ (Nagata)

Given 7 points in $\mathbb{K}^2$, there is a cubic passing through these 7 points with a double point at one of them.

Number of coefficients of a cubic polynomial : 10.

Number of conditions : 6 for the simple zeros, 3 for the double zero.

This gives a polynomial of degree $7 \times 3 = 21$ with the 7 assigned zeroes of multiplicities 8.

In fact $\omega_{8t}(S) = 21t$, $\Omega(S) = 21/8$.

Generic $S \subset \mathbb{K}^2$ with $|S| = 8$ (Nagata)

Given 8 points in $\mathbb{K}^2$, there is a sextic with a double point at 7 of them and a triple point at 1 of them.

Number of coefficients of a sextic polynomial : $(6+1)(6+2)/2 = 28$.

Number of conditions : $3 \times 7 = 21$ for the double zeros, 6 for the triple zero.

This gives a polynomial of degree $8 \times 6 = 48$ with the 8 assigned zeroes of multiplicities $2 \times 8 + 1 = 17$.

In fact $\omega_{17t}(S) = 48t$, $\Omega(S) = 47/17$.

G.V. Chudnovsky

Gregory Chudnovsky

G.V. Chudnovsky. *Singular points on complex hypersurfaces and multidimensional Schwarz Lemma*. M.-J. Bertin (Ed.), Séminaire de Théorie des Nombres Delange-Pisot-Poitou, Paris, 1979–80, Prog. Math., vol. **12**, Birkhäuser.

Conjecture :

$$\frac{\omega_1 + n - 1}{n} \leq \frac{\omega_t}{t}.$$

True for $n = 2$ (J-P. Demailly).

https://fr.wikipedia.org/wiki/David_et_Gregory_Chudnovsky

Chudnovsky : $n = 2,\ |S| = 2, 3, 4, 5$

APPENDIX 1

$|S| = 1$

1 g

$\Omega = 1$

$\Omega_0 = 1$

$\hat{\Omega}_0 = 1$

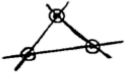
$|S| = 2$

2 g

$\Omega = 1$

$\Omega_0 = 1$

$\hat{\Omega}_0 = 1$



$|S| = 3$

3 g

$\Omega = 2$

$\Omega_0 = 3/2$

$\hat{\Omega}_0 = 3/2$

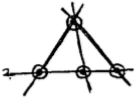
$|S| = 4$

4 g

$\Omega = 2$

$\Omega_0 = 2$

$\hat{\Omega}_0 = 2$



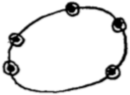
$|S| = 4$

5 g

$= 2$

$_0 = 5/3$

$_0 = 4/3$



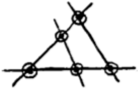
$|S| = 5$

6 g

$\Omega = 2$

$\Omega_0 = 2$

$\hat{\Omega}_0 = 2$



$|S| = 5$

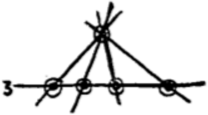
7 g

$\Omega = 2$

$\Omega_0 = 2$

$\hat{\Omega}_0 = 5/3$

Chudnovsky : $n = 2$, $|S| = 5, 6$



$|S| = 5$

8_g

$\Omega = 2$

$\Omega_0 = 7/4$

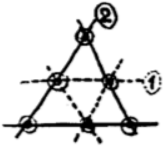
$\hat{\Omega}_0 = 5/4$

For generic S, $|S| = 6$ (see 5_g)

$\Omega(S) = 3$

$\Omega_0(S) = 12/5$

$\hat{\Omega}_0(S) = 12/5$



$|S| = 6$

10_g

$\Omega = 3$

$\Omega_0 = 9/4$

$\hat{\Omega}_0 = 2$

$|S| = 6$

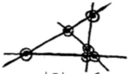
11_g

$\Omega = 3$

$\Omega_0 = 12/5$

$\hat{\Omega}_0 = 2$

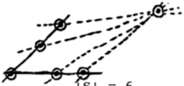
Chudnovsky : $n = 2,\ |S| = 6, 7$



$|S| = 6$

12_g

$\Omega = 3$
$\Omega_0 = 2$
$\hat{\Omega}_0 = 2$



$|S| = 6$

13_g

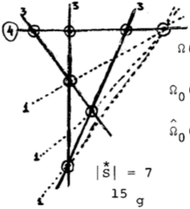
$\Omega = 3$
$\Omega_0 = 7/3$
$\hat{\Omega}_0 = 2$

$|S| = 7$

14_g

For generic S, $|S| = 7$,
7 cubics with double point
for generic case,
$\Omega(S) = 3$, $\Omega_0(S) = 21/8$,
$\hat{\Omega}_0(S) = 21/8$.

Particular Cases:



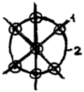
$\Omega(\overset{*}{S}) = 3$

$\Omega_0(\overset{*}{S}) = 16/7$

$\hat{\Omega}_0(\overset{*}{S}) = 7/4$

$|\overset{*}{S}| = 7$

15_g



$\Omega = 3$

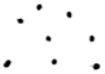
$\Omega_0 = 7/3$

$\hat{\Omega}_0 = 7/3$

$|S| = 7$

16_g

Chudnovsky : $n = 2$, $|S| = 8$



$|S| = 8$
(See 18 - 19)

For generic S, $|S| = 8$,
8 sextics with 7 double points,
1 triple point. For generic
case, $\Omega(S) = 3$, $\Omega_0(S) = 48/17$,
$\hat{\Omega}_0(S) = 48/17$.

Chudnovsky : $n = 2$, $|S| = 9$

65

$|S| = 9$

20_g

For generic S, $|S| = 9$, 1 cubic and $\Omega(S) = 3$, $\Omega_0(S) = 3$, $\hat{\Omega}_0(S) = 3$.

$|S| = 9$

21_g

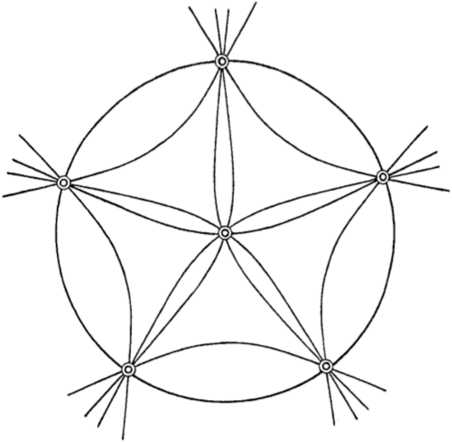
$\Omega = 3$

$\Omega_0 = 5/2$

$\hat{\Omega}_0 = 9/4$

$n = 2$, $|S| = 6$, $t = 5$, $\omega_t = 12$, $\Omega = 12/5$ generic

APPENDIX 2



|S| = 6

9 g

Generic S, |S| = 6

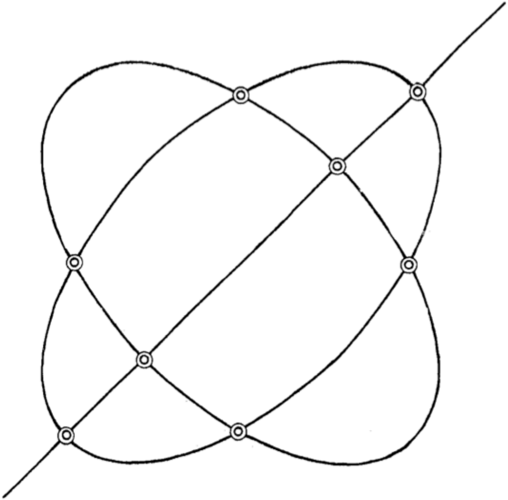
6 conics

$\Omega(S) = 3$

$\Omega_0(S) = 12/5$

$\Omega_0(S) = 12/5$

$n = 2$, $|S| = 8$, $t = 2$, $\omega_t = 5$, $\Omega = 5/2$



$|S| = 8$

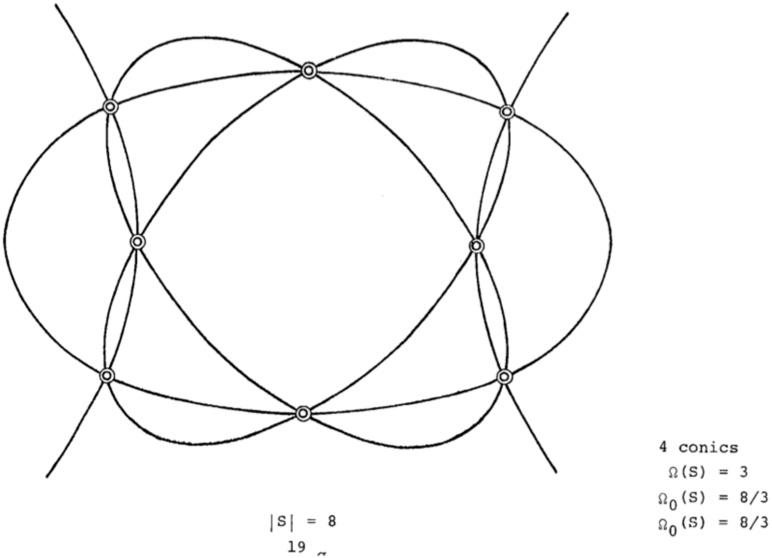
18 g

$\Omega(S) = 3$

$\Omega_0(S) = 5/2$

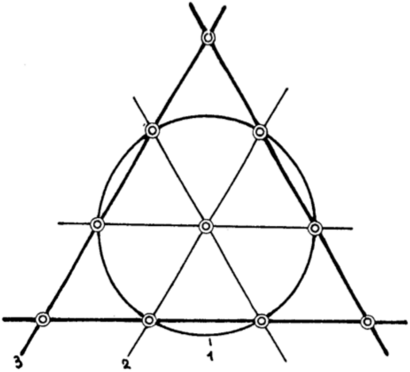
$\hat{\Omega}_0(S) = 3/2$

$n = 2$, $|S| = 8$, $t = 3$, $\omega_t = 8$, $\Omega = 8/3$



$n = 2$, $|S| = 10$, $t = 6$, $\omega_t = 17$, $\Omega = 17/6$

Three sides : multiplicity 3.
Three concurrent lines : multiplicity 2.



$|S_c| = 10$

22 g

$\Omega(S_c) = 4$
$\Omega_0(S_c) = 17/6$
$\hat{\Omega}_0(S_c) = 5/2$

Hélène Esnault and Eckart Viehweg

Hélène Esnault

Eckart Viehweg

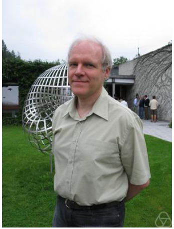
(1948 - 2010)

H. Esnault and E. Viehweg *Sur une minoration du degré d'hypersurfaces s'annulant en certains points.* Math. Ann. **263** (1983), 75 – 86

Methods of projective geometry : for $n \geq 2$,

$$\Omega(S) \geq \frac{\omega_t + 1}{t + n - 1}.$$

Jean-Pierre Demailly



Jean-Pierre Demailly

Using an appropriate generalization of the Poisson–Jensen formula, proves a new variant of the Schwarz lemma in $\mathbb{C}^n$.

Consequence :

$$\Omega(S) \geq \frac{\omega_1(S)\big(\omega_1(S)+1\big)\cdots\big(\omega_1(S)+n-1\big)}{n!\,\omega_1(S)^{n-1}}$$

Corollary : For $n = 1$ or 2,

$$\Omega(S) \geq \frac{\omega_1(S)+n-1}{n}.$$

Demailly's Conjecture

Recall the Conjecture of Chudnovsky and the Theorem of Esnault and Viehweg :

$$\Omega(S) \geq \frac{\omega_1 + n - 1}{n}, \qquad \Omega(S) \geq \frac{\omega_t + 1}{t + n - 1}.$$

Conjecture of Demailly :

$$\Omega(S) \geq \frac{\omega_t(S) + n - 1}{t + n - 1}.$$

J–P. Demailly. *Formules de Jensen en plusieurs variables et applications arithmétiques.* Bull. Soc. Math. France **110** (1982), 75–102.

`https://de.wikipedia.org/wiki/Jean-Pierre_Demailly`

Abdelhak Azhari

Abdelhak Azhari

A. Azhari. *Démonstration analytique d'un lemme de multiplicités.* C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), no. 7, 269–272.

A. Azhari. *Sur la conjecture de Chudnovsky – Demailly et les singularités des hypersurfaces algébriques.* Ann. Inst. Fourier **40** (1990), no. 1, 103–116.
http://www.numdam.org/item?id=AIF_1990__40_1_103_0

Connection with C.S. Seshadri constant

Conjeevaram Srirangachari Seshadri
(1932-2020)

For a generic set S of s points in $\mathbb{P}^n$, Seshadri's constant $\epsilon(S)$ is related to $\Omega(S)$ by

$$\epsilon(S)^{n-1} = \frac{\Omega(S)}{s}.$$

C.S. Seshadri's criterion

Let X be a smooth projective variety and L a line bundle on X. Then L is ample if and only if there exists a positive number ϵ such that for all points x on X and all irreducible curves C passing through x one has

$$L \cdot C \leq \epsilon \, \mathrm{mult}_x C.$$

Robin Hartshorne

R. Hartshorne. *Ample subvarieties of algebraic varieties.* Springer Lecture Notes in Math., vol. **156**, Springer (1970).

C.S. Seshadri constant at a point

Let X be a smooth projective variety and L a nef line bundle on X. For a fixed point $x \in X$ the real number

$$\epsilon(X, L; x) := \inf \frac{L \cdot C}{\text{mult}_x C}$$

is the Seshadri constant of L at x.
The infimum is over all irreducible curves passing through x.

Jean–Pierre Demailly

J.-P. Demailly. *Singular Hermitian metrics on positive line bundles*. Complex algebraic varieties (Bayreuth, 1990), Lecture Notes Math. **1507**, Springer-Verlag, (1992) 87–104.

Seshadri's constant and Nagata's Conjecture

Beata Maria Strycharz-Szemberg

Tomasz Szemberg

Beata Strycharz-Szemberg & Tomasz Szemberg. *Remarks on the Nagata conjecture*, Serdica Mathematical Journal **30** 2-3 (2004), 405– 430.

`http://www.math.bas.bg/serdica/2004/2004-405-430.pdf`

The Nagata – Biran Conjecture

Masayoshi Nagata
(1927 – 2008)

Paul Biran

Let X be a smooth algebraic surface and L an ample line bundle on X of degree d. For sufficiently large r, the Seshadri constant of a generic set $Z = \{p_1, \ldots, p_r\}$ satisfies

$$\epsilon(X, L; Z) = \frac{d}{\sqrt{r}}.$$

Zero estimates

Recall step 2 of the transcendence machinery :

Step 2 Find a point z_0 where $f(z_0) \neq 0$.

`Context` : zero estimates, multiplicity estimates, interpolation estimates on an algebraic group.

Results of C Hermite, C.L. Siegel, Th. Schneider, K. Mahler, A.O. Gel'fond, R. Tijdeman, W.D. Brownawell, D.W. Masser, G. Wüstholz, P. Philippon, J-C. Moreau, D. Roy, M. Nakamaye, N. Rattazzi, S. Fischler...

Michael Nakamaye and Nicolas Ratazzi

Michael Nakamaye

Nicolas Ratazzi

M. Nakamaye and N. Ratazzi. *Lemmes de multiplicités et constante de Seshadri.* Math. Z. **259**, No. 4, 915-933 (2008).

http://www.math.unm.edu/research/faculty_hp.php?d_id=96

http://www.math.u-psud.fr/~ratazzi/

Stéphane Fischler and Michael Nakamaye

Stéphane Fischler

Michael Nakamaye

S. Fischler and M. Nakamaye. *Seshadri constants and interpolation on commutative algebraic groups.* Ann. Inst. Fourier **64**, No. 3, 1269-1289 (2014).

http://www.math.u-psud.fr/~fischler/

http://www.math.unm.edu/research/faculty_hp.php?d_id=96

S. David, M. Nakamaye, P. Philippon

Bornes uniformes pour le nombre de points rationnels de certaines courbes, Diophantine geometry, 143–164, CRM Series, 4, Ed. Norm., Pisa, 2007.

http://www.math.unm.edu/~nakamaye/Pisa.pdf

S. David, M. Nakamaye, P. Philippon

Au Professeur C. S. Seshadri à l'occasion de son 75ème anniversaire.

Nous commençons par une étude indépendante des jets des sections de fibrés amples sur une surface lisse, puis sur le carré d'une courbe elliptique, utile pour le Théorème 4.2. Ceci nous permet en particulier d'introduire dans le présent contexte les constantes de Seshadri, dont l'utilisation en géométrie diophantienne nous semble devoir être positivement stimulée.

`http://www.math.unm.edu/~nakamaye/Pisa.pdf`

Homogeneous ideals of $\mathbb{K}[X_0, \ldots, X_n]$

For $p = (\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}^n(\mathbb{K})$, denote by $I(p)$ the homogeneous ideal generated by the polynomials $\alpha_i X_j - \alpha_j X_i$ $(0 \leq i, j \leq n)$ in the polynomial ring $R = \mathbb{K}[X_0, \ldots, X_n]$.

For $S = \{p_1, \ldots, p_s\} \subset \mathbb{P}^n(\mathbb{K})$, set

$$I(S) = I(p_1) \cap \cdots \cap I(p_s).$$

This is the ideal of forms vanishing on S. The least degree of a polynomial in $I(S)$ is $\omega_1(S)$.

Conjecture of André Hirschowitz

André Hirschowitz

Denote by $\omega_t(n, m)$ the maximum of $\omega_t(S)$ over all finite sets S in $\mathbb{K}^n$ with m elements.

Conjecture : $\omega_t(n, m)$ is as large as possible.

For every $n \geq 1$ *there is an integer* $c(n)$ *such that, for every* $m \geq c(n)$ *and, for all* t, $\omega_t(n, m)$ *is the smallest integer* d *such that*

$$\binom{d+n}{n} > m\binom{t+n-1}{n}.$$

True for $t = 2$ and $n = 2$ and 3, and for $t = 3$ and $n = 2$.

A. Hirschowitz. *La méthode d'Horace pour l'interpolation à plusieurs variables.* Manuscripta Math. **50** (1985), 337–388.

Initial degree

Generally, when J is a nonzero homogeneous ideal of R, define $\omega(J)$ as the least degree of a polynomial in J.

Since J is homogeneous,

$$J = \bigoplus_{m \geq 0} J_m$$

we have

$$\omega(J) = \min\{m \geq 0 \mid J_m \neq 0\}$$

and $\omega(J)$ is also called the *initial degree* of J.

Since $J_1 J_2$ is generated by the products $P_1 P_2$ with $P_i \in J_i$, it is plain that $\omega(J_1 J_2) = \omega(J_1) + \omega(J_2)$, hence

$$\omega(J^t) = t\omega(J).$$

Symbolic powers

For $t \geq 1$, define the symbolic power $I^{(t)}(S)$ by

$$I^{(t)}(S) = I(p_1)^t \cap \cdots \cap I(p_s)^t.$$

This is the ideal of forms vanishing on S with multiplicities $\geq t$. Hence

$$\omega(I^{(t)}(S)) = \omega_t(S).$$

We have $I(S)^t \subset I(p_i)^t$ for all i, hence

$$I(S)^t \subset \bigcap_{i=1}^{s} I(p_i)^t = I^{(t)}(S).$$

From $I(S)^t \subset I^{(t)}(S)$ we deduce $\omega_t(S) \leq t\omega_1(S)$.

Alternate proof of $\Omega(S) \geq \frac{\omega_1(S)}{n}$ (2001, 2002)

Theorem (Ein-Lazarfeld-Smith, Hochster-Huneke). *Let J be a homogeneous ideal in $\mathbb{K}[X_0, \ldots, X_n]$ and $t \geq 1$. Then*

$$J^{(tn)} \subset J^t.$$

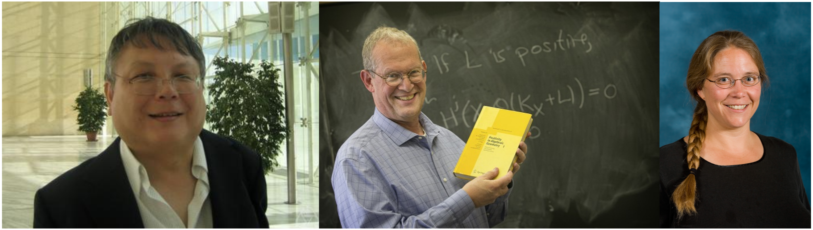
Consequence : From $I(S)^t \supset I(S)^{(tn)}$ we deduce

$$t\omega_1(S) \leq \omega_{tn}(S)$$

and

$$\frac{\omega_1(S)}{n} \leq \frac{\omega_{tn}(S)}{tn} \to \Omega(S) \quad \text{as} \quad t \to \infty.$$

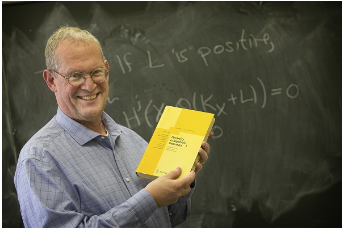
$J^{(tn)} \subset J^t$ by L. Ein, R. Lazarfeld and K.E. Smith



The proof by Lawrence Ein, Robert Lazarfeld and Karen E. Smith uses multiplier ideals.

L. Ein, R. Lazarfeld and K.E. Smith. *Uniform behavior of symbolic powers of ideals.* Invent. Math. **144** (2001), 241–252.

$J^{(tn)} \subset J^t$



R. Lazarfeld. *Positivity in algebraic geometry* I – II. Ergeb. Math. **48–49**, Springer, Berlin (2004).

$J^{(tn)} \subset J^t$ by M. Hochster and C. Huneke

The proof by Melvin Hochster and Craig Huneke uses Frobenius powers and tight closure.

Melvin Hochster

Craig Huneke

M. Hochster and C. Huneke. *Comparison of symbolic and ordinary powers of ideals.* Invent. Math. **147** (2002), 349–369.

Briançon-Skoda Theorem

Melvin Hochster

Craig Huneke

For an m–generated ideal $\mathfrak{a}$ in the ring of germs of analytic functions at $0 \in \mathbb{C}^n$, the ν–th power of its integral closure is contained in $\mathfrak{a}$, where $\nu = \min\{m, n\}$.

M. Hochster and C. Huneke. *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116.

Symbolic powers

For a homogeneous ideal J in the ring $R = \mathbb{K}[X_0, \ldots, X_n]$ and $m \geq 1$, define the symbolic power $J^{(m)}$ as follows. Write primary decompositions of J and J^m as

$$J = \bigcap_i \mathfrak{Q}_i, \qquad J^m = \bigcap_j \mathfrak{Q}'_j,$$

where $\mathfrak{Q}_i$ is homogeneous and $\mathfrak{P}_i$ primary, $\mathfrak{Q}'_j$ is homogeneous and $\mathfrak{P}'_j$ primary. We set

$$J^{(m)} = \bigcap_j \mathfrak{Q}'_j$$

where the intersection is over the j with $\mathfrak{P}'_j$ contained in some $\mathfrak{P}_i$.

Symbolic powers

Notice that $J^m \subset J^{(m)}$.

Example of a *fat points* ideal. For $J = \cap_j I(p_j)^{m_j}$,

$$J^{(m)} = \bigcap_j I(p_j)^{mm_j}.$$

The containment problem

Find all m, t *with*

$$J^{(m)} \subset J^t.$$

Brian Harbourne. *Asymptotic invariants of ideals of points.* (2009). Special Session on Geometry, Syzygies and Computations Organized by Professors S. Kwak and J. Weyman KMS-AMS joint meeting, December 16–20, 2009.
Slides.
www.math.unl.edu/~bharbourne1/KSSNoPauseRev.pdf

$\Omega(J)$ for a homogeneous ideal J

Cristiano Bocci

Brian Harbourne

For a homogeneous ideal J of $\mathbb{K}[X_0, \ldots, X_n]$, and $t \geq 1$, define $\omega_t(J) = \omega(J^{(t)})$. Then

$$\Omega(J) = \lim_{t\to\infty} \frac{\omega_t(J)}{t}$$

exists and satisfies

$$\Omega(J) \leq \frac{\omega_t(J)}{t} \quad \text{for all} \quad t \geq 1.$$

The resurgence of Bocci and Harbourne

Define

$$\varrho(J) = \sup\left\{\frac{m}{r} \;\middle|\; J^{(m)} \not\subset J^r\right\}.$$

Hence, if $\frac{m}{r} > \varrho(J)$, then $J^{(m)} \subset J^r$.

By L. Ein, R. Lazarfeld and K.E. Smith, $\varrho(J) \leq n$.

C. Bocci and B. Harbourne. *Comparing powers and symbolic powers of ideals.* J. Algebraic Geom. **19** (2010), no. 3, 399–417.

C. Bocci and B. Harbourne. *The resurgence of ideals of points and the containment problem.* Proc. Amer. Math. Soc. **138** (2010), no. 4, 1175–1190

The resurgence of Bocci and Harbourne

Denote by $\operatorname{reg}(J)$ the Castelnuovo– Mumford regularity of J.

Theorem (Bocci, Harbourne). We have

$$\frac{\omega(J)}{\Omega(J)} \leq \varrho(J) \leq \frac{\operatorname{reg}(J)}{\Omega(J)}.$$

Further, if $\omega(J) = \operatorname{reg}(J)$, then

$$J^{(m)} \subset J^t \iff t\omega(J) \leq \omega(J^{(m)}).$$

Optimality

Following Bocci and Harbourne, we have

$$\sup_{|S|<\infty} \frac{\omega_1(S)}{\Omega(S)} = n.$$

Conjecture of Cristiano Bocci and Brian Harbourne

Let S be a finite subset of $\mathbb{P}^2$. Define $\varrho(S) = \varrho(J)$ for $J = I(S)$.

Conjecture.

$$\varrho(S) \leq 2\frac{\omega_1(S)}{\omega_1(S)+1}.$$

This conjecture implies Chudnovsky's conjecture : from

$$\frac{\omega_1(S)}{\Omega(S)} \leq \varrho(S) \leq 2\frac{\omega_1(S)}{\omega_1(S)+1}$$

one deduces

$$\frac{\omega_1(S)+1}{2} \leq \Omega(S).$$

The containment problem (continued)

Let $\mathfrak{M}$ be the homogeneous ideal $(X_0, \dots, X_n)$ in R.

Fact. In characteristic zero, the ideal $J = I(S)$ satisfies $J^{(2)} \subset \mathfrak{M}J$.

Proof. Let $P \in J^{(2)}$. Hence $\frac{\partial}{\partial X_i} P \in J$. Use Euler's formula

$$(\deg P)P = \sum_{i=0}^{n} X_i \frac{\partial}{\partial X_i} P.$$

Question. *For which m, t, j do we have $J^{(m)} \subset \mathfrak{M}^j J^t$?*

Remark. Since $\mathfrak{M}^j J^t \subset J^t$, the condition $J^{(m)} \subset \mathfrak{M}^j J^t$ implies $J^{(m)} \subset J^t$.

B. Harbourne and C. Huneke. *Are symbolic powers highly evolved ?* J. Ramanujan Math. Soc. **28**A (2013), 247–266. arxiv:1103.5809.

Conjecture of Brian Harbourne and Craig Huneke

Chudnovsky's result

$$\frac{\omega_1 + n - 1}{n} \le \frac{\omega_t}{t}.$$

for $n = 2$ follows from

$$J^{(2t)} \subset \mathfrak{M}^t J^t$$

for any homogeneous ideal of points $J = I(S)$ in $\mathbb{K}[X_0, X_1, X_2]$.

Generalization for $n \ge 2$.

Conjecture of Brian Harbourne and Craig Huneke

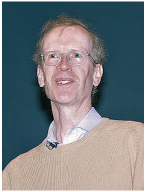
Let $J = \cap_j I(p_j)^{m_j}$ be a fat points ideal in R.

Conjecture (Harbourne and Huneke). For all $t > 0$,

$$J^{(tn)} \subset \mathfrak{M}^{t(n-1)} J^t.$$

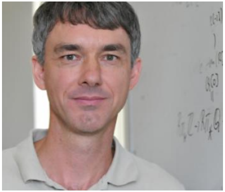
Evolutions

Andrew Wiles



Richard Taylor

Matthias Flach

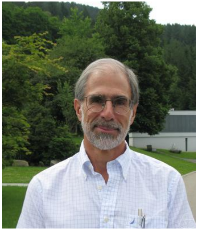


Evolutions are certain kinds of ring homomorphisms that arose in proving Fermat's last Theorem (A. Wiles, R. Taylor, M. Flach).

An important step in the proof was to show that in certain cases only trivial evolutions occurred.

Evolutions

D. Eisenbud and B. Mazur showed the question of triviality could be translated into a statement involving symbolic powers. They then made the following conjecture in characteristic 0 :



Conjecture (Eisenbud–Mazur) *Let* $\mathfrak{P} \subset \mathbb{C}[[x_1, \ldots, x_d]]$ *be a prime ideal. Let* $\mathfrak{M} = (x_1, \ldots, x_d)$*. Then* $\mathfrak{P}^{(2)} \subset \mathfrak{M}\mathfrak{P}$*.*

Evolutions

Heuristically, the main conjecture of Harbourne and Huneke can be thought of as a generalization of the conjecture of Eisenbud and Mazur.

B. Harbourne and C. Huneke. *Are symbolic powers highly evolved ?* J. Ramanujan Math. Soc. **28**, (2011) `arxiv:1103.5809`.

Brian Harbourne

Brian Harbourne, Sandra Di Rocco, Tomasz Szemberg, Thomas Bauer

Oberwolfach
Linear Series on Algebraic Varieties : 2010-10-03 – 2010-10-09

M. Dumnicki, B. Harbourne, T. Szemberg and H. Tutaj-Gasińska. *Linear subspaces, symbolic powers and Nagata type conjectures.* Adv. Math. **252** (2014), 471–491.

https://owpdb.mfo.de/detail?photo_id=13201

Marcin Dumnicki

Chudnovsky's conjecture $\Omega(S) \geq \frac{\omega_1(S) + n - 1}{n}$ holds for generic finite subsets in $\mathbb{P}^3$.

M. Dumnicki. *Symbolic powers of ideal of generic points in* $\mathbb{P}^3$*.* J. Pure Applied Algebra **216** (2012), 1410–1417.

Thomas Bauer and Thomasz Szemberg

Th. Bauer and T. Szemberg. *The effect of points fattening in dimension three*. Recent advances in Algebraic Geometry. A volume in honor of Rob Lazasfeld's 60th Birthday LMS, Cambridge University Press 2015.

Further references

M. Baczyńska, M. Dumnicki, A. Habura, G. Malara, P. Pokora, T. Szemberg, J. Szpond and H. Tutaj-Gasińska. *Points fattening on* $\mathbb{P}^1 \times \mathbb{P}^1$ *and symbolic powers of bi–homogeneous ideals.* J. Pure Applied Algebra **218** (2014), 1555–1562.

C. Bocci, S.M. Cooper and B. Harbourne. *Containment results for ideals of various configurations of points in* $\mathbb{P}^N$. J. Pure Applied Algebra **218** (2014), 65–75.

Further references

Giuliana Fatabbi, Brian Harbourne, Anna Lorenzini,
Inclics, galaxies, star configurations and Waldschmidt constants (2013)

Cristiano Bocci, Susan Cooper, Elena Guardo,
The Waldschmidt constant for squarefree monomial ideals (2015)

Marcin Dumnicki, Lucja Farnik, Halszka Tutaj-Gasinska,
Asymptotic Hilbert Polynomial and a bound for Waldschmidt constants (2015)

October 12, 2020

Upstate New York Online Number Theory Colloquium

Some variants of Seshadri's constant.

Michel Waldschmidt

Sorbonne Université
Institut de Mathématiques de Jussieu Paris rive gauche

http://www.imj-prg.fr/~michel.waldschmidt/