Universitas Gadjah Mada (UGM) Yogyakarta (Indonesia)
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Introduction to analytic number theory

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First course : July 17, 2023 – blackboard, online.

- Statement of the Prime Number Theorem PNT
- Euler product formula

$$\sum_{n\geq 1} n^{-s} = \prod_p (1-p^{-s})^{-1}.$$

- Complex logarithm
- Infinite products
- Divergence of the harmonic series
- Dedekind eta function
- Acceleration of convergence
- Analytic continuation of the Riemann zeta function

Second course : July 18, 2023 – slides, online.

$\zeta(s)$ for s close to 1

Result :

$$\lim_{s\to 1}\left(\zeta(s)-\frac{1}{s-1}\right)=\gamma$$

where γ is *Euler constant* :

$$\gamma=\lim_{N\to\infty}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}-\log N.$$

Abel summation.

$$A_0 = 0, \quad A_n = \sum_{m=1}^{n} a_m, \quad a_n = A_n - A_{n-1} \quad (n \geq 1)$$

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N-1} A_n (b_n - b_{n+1}) + A_N b_N.$$

(telescopic sum)

Abel's Partial Summation Formula PSF.

Let a_n be a sequence of complex numbers and $A(x) = \sum_{n \leq x} a_n$. Let f be a function of class $\mathcal{C}^1$. Then

$$\sum_{y<n\leq x} a_n f(n) = A(x)f(x) - \int_y^x A(t)f'(t)\mathrm{d}t.$$

Abel summation : an example.

Take $a_n = 1$ for all n, so that $A(t) = \lfloor t \rfloor$. Then

$$\sum_{n=M+1}^{N} f(n) = \int_M^N f(t)dt + \int_M^N (t - \lfloor t \rfloor) f'(t)\mathrm{d}t.$$

Take $f(t) = 1/t$. Then

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + O(1/N)$$

where γ is *Euler constant*

$$\gamma = 1 - \int_1^\infty \{t\} \frac{\mathrm{d}t}{t^2} \qquad \text{where} \qquad \{t\} = t - \lfloor t \rfloor.$$

Abel summation for ζ.

Take $a_n = n^{-s}$ for all n; assume $\mathrm{Re}(s) > 1$. An approximation of the series

$$\sum_{n \geq 1} n^{-s}$$

is

$$\int_1^{\infty} t^{-s} \mathrm{d}t = \frac{1}{s-1}.$$

We proved yesterday that $(s-1)\zeta(s) \to 1$ as $s \to 1_+$.

If we remove this singularity of ζ at $s = 1$, the difference

$$\zeta(s) - \frac{1}{s-1}$$

becomes an entire function (analytic in $\mathbb{C}$).

Analytic continuation of $\zeta(s)$

The function $\zeta(s) - 1/(s-1)$ extends to an analytic function in $\mathrm{Re}(s) > 0$.

$$\frac{1}{n^s} = s\int_n^\infty t^{-s-1}\mathrm{d}t, \qquad \sum_{n=1}^{t} 1 = [t].$$

$$\zeta(s) = s\sum_{n\geq 1}\int_n^\infty t^{-s-1}\mathrm{d}t = s\int_1^\infty [t]t^{-s-1}\mathrm{d}t$$

$$\zeta(s) = s\int_1^\infty t^{-s}\mathrm{d}t + s\int_1^\infty ([t]-t)t^{-s-1}\mathrm{d}t.$$

$$s\int_1^\infty t^{-s}\mathrm{d}t = \frac{1}{s-1} + 1.$$

No zero of $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$

Recall, for $\mathrm{Re}(s) > 1$,

$$\log \zeta(s) = \sum_p \sum_{m\geq 1} \frac{1}{mp^{ms}}.$$

Hence

$$\log |\zeta(\sigma + \mathrm{i}t)| = \sum_p \sum_{m\geq 1} \frac{1}{mp^{m\sigma}} \cos(mt \log p).$$

Trigonometric formula : $\cos(2x) = 2\cos^2 x - 1$. For $x \in \mathbb{R}$,

$$4\cos x + \cos(2x) + 3 = 2(1 + \cos x)^2 \geq 0.$$

Consequence : for $\sigma > 1$ and $t \in \mathbb{R}$,

$$\log \left(|\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \zeta(\sigma)^3\right)$$
$$= \sum_{m,p} \frac{1}{mp^{m\sigma}} (4\cos(mt \log p) + \cos(2mt \log p) + 3) \geq 0.$$

No zero of $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$

For $\sigma > 1$ and $t \in \mathbb{R}$,

$$|\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \zeta(\sigma)^3 \geq 1.$$

If ζ has a zero of order k in $1 + \mathrm{i}t$ and ℓ in $1 + 2\mathrm{i}t$, then for $\sigma > 1$, $\sigma \to 1$,

$$\begin{aligned} \zeta(\sigma + \mathrm{i}t) &\simeq a(\sigma - 1)^k, \\ \zeta(\sigma + 2\mathrm{i}t) &\simeq b(\sigma - 1)^\ell, \\ \zeta(\sigma) &\simeq (\sigma - 1)^{-1} \end{aligned}$$

with a and b not 0.
Hence $4k + \ell - 3 \leq 0$ and therefore $k = 0$.

Euler Gamma function

The integral

$$\Gamma(s) = \int_0^\infty \mathrm{e}^{-t} t^{s-1} \mathrm{d}t.$$

defines an analytic function on the half plane $\mathrm{Re}(s) > 0$ which satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s).$$

The Gamma function can be analytically continued to a meromorphic function in the complex plane $\mathbb{C}$ with a simple pole at any integer ≤ 0. The residue at $s = -k$ ($k \geq 0$) is $(-1)^k/k!$.

Euler Gamma function

Integrating by parts :

$$\Gamma(s) = \left[\frac{1}{s}\mathrm{e}^{-s}t^{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty \mathrm{e}^{-t}t^{s}\mathrm{d}t = \frac{1}{s}\Gamma(s+1).$$

By induction

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n)}.$$

The right hand side is analytic for $\mathrm{Re}(s) > -n-1$.

Remark. From $\Gamma(1) = 1$ we deduce $\Gamma(n+1) = n!$ for $n \geq 0$.

$\Gamma(1/2) = \sqrt{\pi}$

$$\Gamma(1/2) = \int_0^\infty e^{-t} t^{-1/2} \mathrm{d}t = 2\int_0^\infty e^{-x^2} \mathrm{d}x.$$

Hence

$$\begin{aligned}
\frac{1}{4}\Gamma(1/2)^2 &= \int_0^\infty \int_0^\infty e^{-x^2-y^2} \mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty \int_0^{\pi/2} e^{-r^2} r \mathrm{d}r\mathrm{d}\theta \\
&= \left[-\frac{1}{2}e^{-r^2}\right]_0^\infty \frac{\pi}{2} \\
&= \frac{\pi}{4}.
\end{aligned}$$

Euler Gamma function

Using

$$\int_0^1 \mathrm{e}^{-t} t^{s-1} \mathrm{d}t = \sum_{n=0}^{\infty} \int_0^1 \frac{(-t)^n}{n!} t^{s-1} \mathrm{d}t$$

we can write

$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+s)} + \int_1^{\infty} \mathrm{e}^{-t} t^{s-1} \mathrm{d}t.$$

The series in the right hand side defines a meromorphic function in $\mathbb{C}$ with simple poles at the negative integers, the residue at $-n$ is $(-1)^n/n!$.
The integral in the right hand side defines an entire function.

Euler Gamma function

Properties :

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\Gamma\left(\frac{1}{2}\right)\Gamma(s).$$

$$\Gamma(s) = \lim_{n\to\infty} \frac{n^s n!}{s(s+1)\cdots(s+n)}.$$

$$\frac{1}{\Gamma(s)} = s\mathrm{e}^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)\mathrm{e}^{-s/n}.$$

Analytic continuation of $\zeta(s)$ (continued)

For $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{\mathrm{e}^t - 1} t^s \frac{\mathrm{d}t}{t},$$

Proof

$$\frac{1}{\mathrm{e}^t - 1} = \sum_{n \geq 1} \mathrm{e}^{-nt},$$

$$\int_0^\infty \mathrm{e}^{-nt} t^s \frac{\mathrm{d}t}{t} = \frac{\Gamma(s)}{n^s}.$$

Analytic continuation

Lemma. Let $f \in \mathcal{C}^\infty(\mathbb{R}_{>0})$ be a fast decreasing function at infinity. Then the function defined for $\mathrm{Re}(s) > 0$ by the integral

$$L(f,s) = \frac{1}{\Gamma(s)} \int_0^\infty f(t) t^s \frac{\mathrm{d}t}{t}$$

has an analytic continuation to $\mathbb{C}$.

Special values :

Under the assumptions of the lemma, for $n \geq 0$ we have

$$L(f,-n) = (-1)^n f^{(n)}(0).$$

Bernoulli numbers

The function

$$f_0(t) = \frac{t}{\mathrm{e}^t - 1}.$$

satisfies the hypotheses of the Lemma. Define $(B_n)_{n\geq 0}$ by

$$f_0(t) = \sum_{n\geq 0} B_n \frac{t^n}{n!}.$$

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42},$$

$$B_3 = B_5 = B_7 = \cdots = 0.$$

http://www.bernoulli.org/

n	0	1	2	4	6	8	10	12	14	16	18	20
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$
$\frac{B_n}{n}$			$\frac{1}{12}$	$-\frac{1}{120}$	$\frac{1}{252}$	$-\frac{1}{240}$	$\frac{1}{132}$	$-\frac{691}{32760}$	$\frac{1}{12}$	$-\frac{3617}{8160}$	$\frac{43867}{14364}$	$-\frac{174611}{6600}$

$\zeta(s)$ as an integral

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$$

Poles :

For $\Gamma(s)$: $s = 0, -1, -2, \ldots$ residue $(-1)^n/n!$ at $s = -n$.

For $\int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$: $s = 1, 0, -1, -2, \ldots$ residue $B_{n+1}/(n+1)!$ at $s = -n$.

Hence the poles cancel except for $s = 1$.

Values at negative integers

The Riemann zeta function $\zeta(s)$ has a meromorphic continuation to $\mathbb{C}$ which is analytic in $\mathbb{C} \setminus \{1\}$, with a simple pole at $s = 1$ with residue 1.

For n a positive integer,

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}.$$

In particular $\zeta(-n) \in \mathbb{Q}$ for $n \geq 0$ and $\zeta(-2n) = 0$ for $n \geq 1$.

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12}, \quad \zeta(-3) = \frac{1}{120},$$

$$\zeta(-5) = \frac{1}{252}, \quad \zeta(-7) = \frac{1}{240}.$$

Values at positive even integers

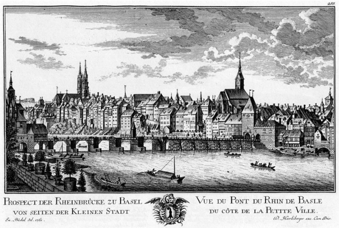
Euler : $\zeta(2n)/\pi^{2n}$ is a rational number.

Examples :
$\zeta(2) = \pi^2/6$ (The Basel problem).
$\zeta(4) = \pi^4/90$,
$\zeta(6) = \pi^6/945$,
$\zeta(8) = \pi^8/9450$.

The Basel Problem (1644) : $\sum_{n\geq 1} 1/n^2$

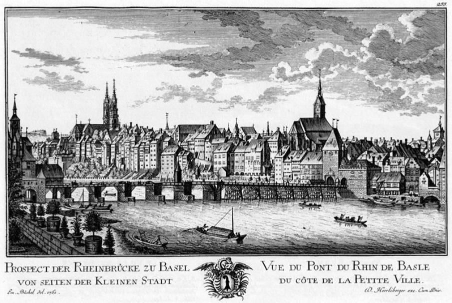
In 1644, Pietro Mengoli (1626 – 1686) asked the exact value of the sum

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots = 1.644934\ldots$$



Basel in 1761

The Bernoulli family was originally from Antwerp, at that time in the Spanish Netherlands, but emigrated to escape the Spanish persecution of the Huguenots. After a brief period in Frankfurt the family moved to Basel, in Switzerland.



The Bernoulli family

Jacob Bernoulli (1654–1705 ; also known as James or Jacques) Mathematician after whom Bernoulli numbers are named.

Johann Bernoulli (1667–1748 ; also known as Jean) Mathematician and early adopter of infinitesimal calculus.

The Bernoulli family (continued)

Nicolaus II Bernoulli (1695–1726) Mathematician ;
: worked on curves, differential equations, and probability.

Daniel Bernoulli (1700–1782) Developer of
: Bernoulli's principle and *St. Petersburg paradox*.

Johann II Bernoulli (1710–1790 ; also known as Jean)
: Mathematician and physicist.

Johann III Bernoulli (1744–1807 ; also known as Jean)
: Astronomer, geographer, and mathematician.

Jacob II Bernoulli (1759–1789 ; also known as Jacques)
: Physicist and mathematician.

Nicolaus II

Daniel

Johan III

Jacob II

Similar series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots=\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots=1.$$

Telescoping series :

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1.$$

Known by Gottfried Wilhelm von Leibniz (1646 – 1716) and Johann Bernoulli (1667–1748)

Another similar series

Example

$$\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}\cdots=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\log 2.$$

$$\log(1+t)=\sum_{n\geq 1}(-1)^{n-1}\frac{t^n}{n} \qquad -1<t\leq 1.$$

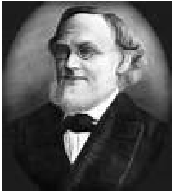
$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)(2n+2)}=\log 2.$$

The Basel Problem : $\sum_{n \geq 1} 1/n^2$

1728 Daniel Bernoulli : approximate value $8/5 = 1.6$

1728 Christian Goldbach : 1.6445 ± 0.0008

1731 Leonard Euler : $1.644934 \cdots$



$\zeta(2) = \pi^2/6$ by L. Euler (1707 – 1783)

The Basel problem, first posed by Pietro Mengoli in 1644, was solved by Leonhard Euler in 1735, when he was 28 only.

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n \geq 1} \frac{1}{n^2}.$$

$$\zeta(2) = \frac{\pi^2}{6}.$$

“Proof” of $\zeta(2) = \pi^2/6$, following Euler

The sum of the inverses of the roots of a polynomial f with $f(0) = 1$ is $-f'(0)$: for

$$1 + a_1 z + a_2 z^2 + \cdots + a_n z^n = (1 - \alpha_1 z) \cdots (1 - \alpha_n z)$$

we have $\alpha_1 + \cdots + \alpha_n = -a_1$.
Write

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

Set $z = x^2$. The zeroes of the function

$$\frac{\sin \sqrt{z}}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \cdots$$

are π^2, $4\pi^2$, $9\pi^2$, ... *hence* the sum of the inverses of these numbers is

$$\sum_{n \geq 1} \frac{1}{n^2 \pi^2} = \frac{1}{6}.$$

Remark

Let $\lambda \in \mathbb{C}$. The functions

$$f(z) = 1 + a_1 z + a_2 z^2 + \cdots$$

and

$$e^{\lambda z} f(z) = 1 + (a_1 + \lambda) z + \cdots$$

have the same zeroes, say $1/\alpha_i$.

The sum $\sum_i \alpha_i$ cannot be at the same time $-a_1$ and $-a_1 - \lambda$.

Completing Euler's proof

$$\frac{\sin x}{x} = \prod_{n \geq 1} \left(1 - \frac{x^2}{n^2 \pi^2}\right).$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \cdots \implies \sum_{n \geq 1} \frac{1}{n^2 \pi^2} = \frac{1}{6}.$$

http://en.wikipedia.org/wiki/Basel_problem

Evaluating $\zeta(2)$. Fourteen proofs compiled by Robin Chapman.

The cotangent function $\cot x = (\cos x)/\sin x$

Proposition : for $x \in \mathbb{C} \setminus \pi\mathbb{Z}$,

$$\cot\, x = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2\pi^2}.$$

Proof. The function $\cot z = (\cos z)/\sin z$ is meromorphic in $\mathbb{C}$, odd, periodic of period 2π, with simple poles at $n\pi$, $n \in \mathbb{Z}$.
The residue of $z \mapsto \cos z/(z - x)\sin z$

- at $z = x$ is $(\cos x)/\sin x$,
- at $z = 0$ is $-1/x$,
- at $z = n > 0$ is $-1/(x - n\pi)$,
- at $z = n < 0$ is $-1/(x + |n|\pi)$.

Hence for $R = (2N+1)\pi/2$ with $N \in \mathbb{Z}$, $N \to \infty$

$$\frac{1}{2\pi \mathrm{i}} \int_{|z|=R} \frac{\cos z}{(z-x)\sin z} \mathrm{d}z = \frac{\cos x}{\sin x} - \frac{1}{x} - 2x \sum_{1 \le n < R} \frac{1}{x^2 - n^2\pi^2}.$$

$\sin z$ as an infinite product

The logarithmic derivative of the function

$$h(z) := z \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2 \pi^2}\right)$$

is

$$\frac{h'(z)}{h(z)} = \frac{1}{z} + \sum_{n \geq 1} \frac{2z}{z^2 - n^2 \pi^2} = \cot z.$$

The logarithmic derivative at $z \in \mathbb{C} \setminus \pi\mathbb{Z}$ of the function

$$\frac{h(z)}{\sin z}$$

is 0, hence this function is a constant ; from

$$\lim_{z \to 0} \frac{h(z)}{z} = 1 = \lim_{z \to 0} \frac{\sin z}{z}$$

we deduce $h(z) = \sin z$.

Another proof of $\zeta(2)=\pi^2/6$ (Calabi)

P. Cartier. – *Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents.* Sém. Bourbaki no. 885 Astérisque **282** (2002), 137-173.

Another proof (Calabi)

$$\frac{1}{1-x^2y^2} = \sum_{n\geq 0} x^{2n}y^{2n}.$$

$$\int_0^1 x^{2n}\mathrm{d}x = \frac{1}{2n+1}.$$

$$\int_0^1\int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-x^2y^2} = \sum_{n\geq 0}\frac{1}{(2n+1)^2}.$$

$$x = \frac{\sin u}{\cos v}, \qquad y = \frac{\sin v}{\cos u},$$

$$\int_0^1\int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-x^2y^2} = \int_{0\leq u\leq\pi/2,\ 0\leq v\leq\pi/2,\ u+v\leq\pi/2} \mathrm{d}u\mathrm{d}v = \frac{\pi^2}{8}.$$

Completing Calabi's proof of $\zeta(2) = \pi^2/6$

From

$$\sum_{n\geq 0} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

one deduces

$$\sum_{n\geq 1} \frac{1}{n^2} = \sum_{n\geq 1} \frac{1}{(2n)^2} + \sum_{n\geq 0} \frac{1}{(2n+1)^2}.$$

$$\sum_{n\geq 1} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n\geq 1} \frac{1}{n^2}.$$

$$\zeta(2) = \sum_{n\geq 1} \frac{1}{n^2} = \frac{4}{3} \sum_{n\geq 0} \frac{1}{(2n+1)^2} = \frac{\pi^2}{6}.$$

$\zeta(2)$ is a period

$$\begin{aligned}
\int_{1>t_1>t_2>0} \frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{1-t_2} &= \int_0^1 \left(\int_0^{t_1} \frac{\mathrm{d}t_2}{1-t_2}\right)\frac{\mathrm{d}t_1}{t_1} \\
&= \int_0^1 \left(\int_0^{t_1} \sum_{n\geq 0} t_2^n \mathrm{d}t_2\right)\frac{\mathrm{d}t_1}{t_1} \\
&= \int_0^1 \left(\sum_{n\geq 0} \frac{t_1^{n+1}}{n+1}\right)\frac{\mathrm{d}t_1}{t_1} \\
&= \sum_{n\geq 0} \frac{1}{n+1}\int_0^1 t_1^n \mathrm{d}t_1 \\
&= \sum_{n\geq 0} \frac{1}{(n+1)^2} = \zeta(2)
\end{aligned}$$

$\zeta(s)$ is a period

For s integer ≥ 2,

$$\zeta(s) = \int_{1>t_1>t_2\cdots>t_s>0} \frac{\mathrm{d}t_1}{t_1} \cdots \frac{\mathrm{d}t_{s-1}}{t_{s-1}} \cdot \frac{\mathrm{d}t_s}{1-t_s}.$$

Induction :

$$\int_{t_1>t_2\cdots>t_s>0} \frac{\mathrm{d}t_2}{t_2} \cdots \frac{\mathrm{d}t_{s-1}}{t_{s-1}} \cdot \frac{\mathrm{d}t_s}{1-t_s} = \sum_{n\geq 1} \frac{t_1^{n-1}}{n^{s-1}}.$$

The function $\pi \cot(\pi z)$ (continued)

Proposition : recall, for $z \in \mathbb{C} \setminus \mathbb{Z}$,

$$\pi \cot \pi z = \frac{1}{z} + \sum_{m=1}^{\infty} \left(\frac{1}{z+m} + \frac{1}{z-m} \right) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{2z}{z^2 - m^2}.$$

Notice that

$$\frac{2z}{z^2 - m^2} = -\frac{2z}{m^2} \cdot \frac{1}{1 - \dfrac{z^2}{m^2}} = -2 \sum_{k \geq 0} \frac{z^{2k+1}}{m^{2k+2}}$$

hence

$$\sum_{m=1}^{\infty} \frac{2z}{z^2 - m^2} = -2 \sum_{n=1}^{\infty} \zeta(2n) z^{2n-1}.$$

We also have

$$\pi \cot \pi z = \pi \frac{e^{2i\pi z} + 1}{e^{2i\pi z} - 1} = i\pi + \frac{2i\pi}{e^{2i\pi z} - 1} = i\pi + \frac{1}{z} \sum_{n=0}^{\infty} B_n \frac{(2i\pi z)^n}{n!}.$$

Values at positive even integers

Theorem. Let $n \geq 1$ be a positive integer. Then

$$\zeta(2n) = -\frac{1}{2} B_{2n} \frac{(2i\pi)^{2n}}{(2n)!}.$$

In particular $\zeta(2n)/\pi^{2n}$ is a rational number.

Examples :
$\zeta(2) = \pi^2/6$ (The Basel problem).
$\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450$.

Functional equation of the Riemann zeta function

An entire function (analytic in $\mathbb{C}$) is defined by

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

$$\xi(0) = \xi(1) = 1.$$

Theorem (Riemann) :

$$\xi(s) = \xi(1-s).$$

Non trivial zeroes

Denote by Z the multiset of zeroes (counting multiplicities) of $\zeta(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$ and by Z_+ the multiset of zeroes (counting multiplicities) of $\zeta(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$ with positive imaginary part.
Then

$$Z_+ = Z \cup \{1 - \rho \mid \rho \in Z\}.$$

Hadamard product expansion

Explicit formula :

$$s(1-s)\pi^{-s/2}\Gamma(s/2)\zeta(s) = -\mathrm{e}^{Bs}\prod_{\rho\in Z}\left(1-\frac{s}{\rho}\right)\mathrm{e}^{s/\rho}$$

with

$$B = -\frac{1}{2}\sum_{\rho\in Z}\frac{1}{\rho(1-\rho)} = -\frac{\gamma}{2} - 1 + \frac{1}{2}\log(4\pi) = -0.023095\ldots$$

We can write

$$\mathrm{e}^{Bs}\prod_{\rho\in Z}\left(1-\frac{s}{\rho}\right)\mathrm{e}^{s/\rho} = \prod_{\rho\in Z_+}\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right).$$

Explicit formula for the logarithmic derivative

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)} - \frac{1}{s} - \frac{1}{s-1} + \sum_{\rho \in Z} \frac{1}{s-\rho}.$$

Poisson formula

For $f \in L^1(\mathbb{R})$ let $\hat{f}$ be its Fourier transform :

$$\widehat{f}(y) = \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{2i\pi xy}\mathrm{d}x.$$

Assume that the function $x \mapsto \sum_{n\in\mathbb{Z}} f(x+n)$ is continuous with bounded variation on $[0,1]$; then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m).$$

Poisson formula

Corollary. The *theta series*

$$\theta(u) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\pi u n^2}$$

satisfies the functional equation , for $u > 0$:

$$\theta(1/u) = \sqrt{u}\theta(u).$$

For $\mathrm{Re}(s) > 1$,

$$\xi(s) = s(s-1)\int_0^\infty \frac{(\theta(u)-1)u^{s/2}}{2u}\mathrm{d}u.$$

The Riemann Memoir (1859).

On the number of primes less than a given magnitude (9p.)

- The function $\zeta(s)$ defined by the Dirichlet series $\sum_{n\geq 1} n^{-s}$ has an analytic continuation to the whole complex plane where it is holomorphic except a simple pole at $s = 1$ with residue 1.
- The following functional equation holds

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

- The Riemann zeta function $\zeta(s)$ has simple zeroes at $s = -2, -4, , -6, \ldots$ which are called the trivial zeroes, and infinitely many non-trivial zeroes in the critical strip of the form $\rho = \beta + i\gamma$ with $0 \leq \beta \leq 1$ and $\gamma \in \mathbb{R}$.

http://www.claymath.org/sites/default/files/ezeta.pdf

The Riemann Memoir (1859) (continued).

- The following product formula holds

$$s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) = \prod_{\rho}\left(1 - \frac{s}{\rho}\right)$$

- The following prime number formula holds

$$\psi^{\flat}(x) = {\sum_{n\leq x}}^{\flat}\Lambda(n) = x - \sum_{\rho}\frac{x^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right).$$

- The Riemann Hypothesis. Every non-trivial zero of $\zeta(s)$ is on the critical line $\mathrm{Re}(s) = 1/2$:

$$\rho = \frac{1}{2} + i\gamma.$$

The Riemann Hypothesis.

The complex zeroes of the Riemann zeta function $\zeta(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$ lie on the critical line $\mathrm{Re}(s) = 1/2$:

$$s \in \mathbb{C},\ 0 < \mathrm{Re}(s) < 1 \text{ and } \zeta(s) = 0 \quad \Longrightarrow \quad \mathrm{Re}(s) = 1/2.$$

Equivalent statement involving the logarithmic integral $\mathrm{Li}(x) = \displaystyle\int_2^x \frac{\mathrm{d}t}{\log t}$ *:*

$$\pi(x) = \mathrm{Li}(x) + O(x^{1/2} \log x)$$

as $x \to \infty$.

Asymptotic expansion :

$$\mathrm{Li}(x) \simeq \frac{x}{\log x} \sum_{n \geq 0} \frac{n!}{(\log x)^n} \simeq \frac{x}{\log x} + \frac{x}{(\log x)^2} + \cdots$$

Notes by Riemann

Non–trivial zeroes :

$$\gamma_1 = 14.134725\ldots$$
$$\gamma_2 = 21.022039\ldots$$
$$\gamma_3 = 25.01085\ldots$$
$$\gamma_4 = 30.42487\ldots$$

http://oeis.org/wiki/Riemann_zeta_function

Table of nontrivial zeros[5]

n	Imaginary part (base 10) of nth nontrivial zero (above the real axis)	OEIS
1	14.134725141734693790457251983562470270784257115699243175685567460149...	A058303
2	21.022039638771554992628479593896902777334340524902781754629520403587...	A065434
3	25.010857580145688763213790992562821818659549672557996672496542006745...	A065452
4	30.424876125859513210311897530584091320181560023715440180962146036993...	A065453
5	32.935061587739189690662368964074903488812715603517039009280003440784...	A192492
6	37.586178158825671257217763480705332821405597350830793218333001113622...	
7	40.918719012147495187398126914633254395726165962777279536161303667253...	
8	43.327073280914999519496122165406805782645668371836871446878893685521...	
9	48.005150881167159727942472749427516041686844001144425117775312519814...	
10	49.773832477672302181916784678563724057723178299676662100781955750433...	

Non trivial zeroes of $\zeta(s)$

Hardy (1914) : infinitely many non-trivial zeroes of $\zeta(s)$ are on the critical line.

Levinson proved in 1974 that at least $\geq 1/3$ of the non-trivial zeroes of $\zeta(s)$ are on the critical line.

Pratt, Robles, Zaharescu and Zeindler proved in 2020 that at least $5/12 (= 41,66\%)$ of the non trivial zeroes are on the critical line.

Infinitely many non trivial zeroes

The function

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is an entire function of growth order 1 :

$$\limsup_{R\to\infty} \frac{1}{\log R} \log\log \sup_{|z|=R} |f(z)| = 1,$$

its zeroes are the non trivial zeroes of ζ, and $\xi(z) = \xi(1-z)$. Therefore the function $f(z) = f(\frac{1}{2}+z)$ is even, $f(-z) = f(z)$, and there exists an entire function $g(z)$, of order $1/2$, such that $f(z) = g(z^2)$. Since g is not a polynomial, Hadamard factorisation Theorem

$$g(z) = cz^k \prod_{g(z_i)=0} \left(1 - \frac{z}{z_i}\right)$$

implies that g has infinitely many zeroes, hence ξ also.

The asymptotic formula for $N(T)$

Let $N(T)$ be the number of zeroes $\rho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle

$$0 < \beta < 1, \qquad 0 < \gamma \leq T.$$

Then

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$$

for $T \geq 2$.

Zero free region for $\zeta(s)$

De la Vallée Poussin (1896) :

$$\sigma > 1 - \frac{c}{\log(2 + |t|)}$$

for an absolute constant $c > 0$.

$$\psi(x) = x + O\big(x \exp(-c\sqrt{\log x})\big).$$

Korobov and Vinogradov (1957)

$$\sigma > 1 - c(\log t)^{-2/3}, \qquad t \geq 2$$

$$\psi(x) = x + O\big(\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5})\big), \qquad x \geq 3$$

Diophantine problem

Conjecture. The numbers

$$\pi, \zeta(3), \zeta(5), \dots, \zeta(2n+1), \dots$$

are algebraically independent.

Apéry (1978) : $\zeta(3) \notin \mathbb{Q}$

Rivoal (2000) : infinitely many $\zeta(2n+1)$ are irrational ; the numbers $\zeta(2n+1)$ span a $\mathbb{Q}$–vector space of infinite dimension.

Zudilin (2004) : At least one of the 4 numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.

Multizeta values (MZV)

Euler

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

for $s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$.

MZV are periods

$$\zeta(2,1) = \int_{1>t_1>t_2>t_3>0} \frac{\mathrm{d}t_1}{t_1} \cdot \frac{\mathrm{d}t_2}{1-t_2} \cdot \frac{\mathrm{d}t_3}{1-t_3}.$$

Proof.
We have

$$\int_0^{t_2} \frac{\mathrm{d}t_3}{1-t_3} = \sum_{n\geq 1} \frac{t_2^{n-1}}{n}, \quad \text{next} \quad \int_0^{t_1} \frac{t_2^{n-1}\mathrm{d}t_2}{t_2-1} = \sum_{m>n} \frac{t_1^m}{m},$$

and

$$\int_0^1 t_1^{m-1}\mathrm{d}t_1 = \frac{1}{m},$$

hence

$$\int_{1>t_1>t_2>t_3>0} \frac{\mathrm{d}t_1}{t_1} \cdot \frac{\mathrm{d}t_2}{1-t_2} \cdot \frac{\mathrm{d}t_3}{1-t_3} = \sum_{m>n\geq 1} \frac{1}{m^2 n} = \zeta(2,1)$$

Linear relations among MZV

As a consequence, multiple zeta values satisfy a lot of independent linear relations with integer coefficients.

Example

Product of series :

$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$$

Product of integrals :

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1)$$

Hence

$$\zeta(4) = 4\zeta(3,1).$$

Conjecture Rohrlich–Lang

Any algebraic dependence relation among the numbers $(2\pi)^{-1/2}\Gamma(a)$ with $a \in \mathbb{Q}$ lies in the ideal generated by the standard relations :

(1) Translation :

$$\Gamma(a+1) = a\Gamma(a),$$

(2) Reflexion :

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}.$$

(3) Multiplication : for any positive integer n, we have

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)} \Gamma(na).$$

(Universal odd distribution).

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Introduction to analytic number theory

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