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Recent Diophantine results on zeta values: a survey

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Abstract

After the proof by R. Apéry of the irrationality of $\zeta(3)$ in 1976, a number of papers have been devoted to the study of Diophantine properties of values of the Riemann zeta function at positive integers.

A survey has been written by S. Fischler for the Bourbaki Seminar in November 2002.

We review more recent results including contributions by S. Fischler, M. Hata, C. Krattenthaler, R. Marcovecchio, R. Murty, G. Rhin, T. Rivoal, C. Viola, W. Zudilin.

We plan also to say a few words on the analog of this theory in finite characteristic, with works of W.D. Brownawell, M. Pappanikolas, D. Thakur, Chieh-Yu Chang, Jing Yu and others.

Zeta

- Riemann zeta values
- Multizeta values
- Weierstraß zeta function
- Fibonacci zeta values
- Hurwitz zeta function
- Carlitz zeta values
- (Other zeta functions : Dedekind, Hasse-Weil, Lerch, Selberg, Witten, Milnor, dynamical systems. . .)
- L– functions. . .

Riemann zeta function

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

Euler : $s \in \mathbf{R}$.

Riemann : $s \in \mathbf{C}$.

Special values of Riemann zeta function

Leonard Euler (1739) $\zeta(s)$ for $s \in \mathbf{Z}$

$$\zeta(2) = \frac{\pi^2}{6},$$

$$\zeta(4) = \frac{\pi^4}{90},$$

$$\zeta(6) = \frac{\pi^6}{945},$$

$$\zeta(8) = \frac{\pi^8}{9450}.$$

$$\pi^{-2k}\zeta(2k) \in \mathbf{Q} \quad \text{for } k \geq 1$$

Bernoulli numbers

Jacques Bernoulli (1654–1705),

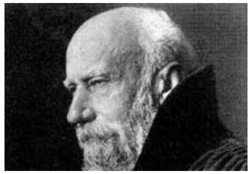
Bernoulli numbers :
$B_2 = 1/6$
$B_4 = -1/30$
$B_6 = 1/42$
$B_8 = -1/30$
$B_{10} = 5/66$

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!},$$

$$\boxed{\zeta(2k) = (-1)^{k-1} 2^{2k-1} \frac{B_{2k}}{(2k)!} \pi^{2k} \quad (k \geq 1).}$$

Transcendence of even zeta values

• F. Lindemann : *π is transcendental, hence $\zeta(2k)$ also for $k \geq 1$.*



Theorem (Hermite–Lindemann).
For any non-zero complex number z, one at least of the two numbers z and e^z is transcendental.
Corollaries. Transcendence of $\log \alpha$ and of e^β for α and β non-zero algebraic complex numbers, provided $\log \alpha \neq 0$.

Diophantine question

Odd positive integers : $\zeta(2k+1)$, $k \geq 1$?

Question. *For $n \geq 1$, is the kumber*

$$\frac{\zeta(2k+1)}{\pi^{2k+1}}$$

rational ?
Describe all algebraic relations among the numbers

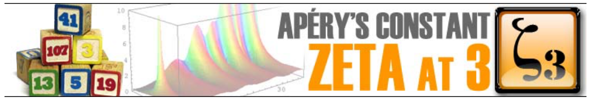
$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7), \ldots$$

Conjecture. *There is no relation at all : the numbers*

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7), \ldots$$

are algebraically independent.
In particular the numbers $\zeta(2k+1)$ and $\zeta(2k+1)/\pi^{2k+1}$ for $k \geq 1$ are conjectured to be transcendental.

Values of ζ at odd positive integers



- Apéry (1978) : *The number*

$$\zeta(3)=\sum_{n\geq 1}\frac{1}{n^3}=1.202\,056\,903\,159\,594\,285\,399\,738\,161\,511\,\dots$$

is irrational.

- Rivoal (2000) & Ball, Zudilin... *Infinitely many* $\zeta(2k+1)$ *are irrational & lower bound for the dimension of the* $\mathbf{Q}$*-span.*

Infinitely many odd zeta are irrational

Tanguy Rivoal (2000)

Let $\epsilon>0$. *For any sufficiently large odd integer* a, *the dimension of the* $\mathbf{Q}$*–vector space spanned by the numbers* $1,\ \zeta(3),\ \zeta(5),\ \cdots,\zeta(a)$ *is at least*

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

Wadim Zudilin

- *At least one of the four numbers* $\zeta(5),\quad \zeta(7),\quad \zeta(9),\quad \zeta(11)$ *is irrational.*
- *There exists an odd integer* j *in the range* $[5,69]$ *such that the three numbers* $1,\ \zeta(3),\ \zeta(j)$ *are* $\mathbf{Q}$*–linearly independent.*

Zudilin's home page http://wain.mi.ras.ru/zw/index.html

References to works on zeta values by

2000 M. Hata, T. Rivoal
2001 K. Ball and T. Rivoal, L.A. Gutnik, G. Rhin and C. Viola, T. Vasilyev, W. Zudilin
2002 T. Rivoal, V.N. Sorokin, W. Zudilin
2003 Yu.V. Nesterenko, T. Rivoal, J. Sondow, C. Viola, W. Zudilin
2004 **S. Fischler**, W. Zudilin
2005 F. Calegari, S. Zlobin
2006 M. Huttner, C. Krattenthaler, T. Rivoal and Zudilin
2007 C. Krattenthaler and T. Rivoal
2008 F. Beukers
2009 S. Fischler and W. Zudilin

Last modified on September 19, 2009

Irrationality of zeta values

S. Fischler
Irrationalité de valeurs de zêta, (d'après Apéry, Rivoal, ...),
Sém. Nicolas Bourbaki, 2002-2003,
N° 910 (Novembre 2002).
Astérisque **294** (2004), 27-62
http://www.math.u-psud.fr/~fischler/publi.html

Christian Krattenthaler and Tanguy Rivoal

http://www-fourier.ujf-grenoble.fr/~rivoal/articles.html

C. Krattenthaler et T. Rivoal, *Hypergéométrie et fonction zêta de Riemann,* Mem. Amer. Math. Soc. **186** (2007), 93 p.

Irrationality measures : the state of the art

$\vartheta \in \mathbf{R}$

$$\left|\vartheta - \frac{p}{q}\right| \geq \frac{1}{q^{\mu+\epsilon}}$$

$\mu(\vartheta) < +\infty \iff \vartheta$ is not a Liouville number

ϑ	year	author	$\mu(\vartheta) <$
π	2008	V.Kh. Salikhov	7.6063085
$\zeta(2) = \pi^2/6$	1996	G. Rhin and C. Viola	5.441243
$\zeta(3)$	2001	G. Rhin and C. Viola	5.513891
$\log 2$	2008	R. Marcovecchio	3.57455391

Irrationality measure for π : history

K. Mahler 1953 : π is not a Liouville number and $\mu(\pi) \leq 30$
M. Mignotte 1974 : $\mu(\pi) \leq 20$
G.V. Chudnovsky 1984 : $\mu(\pi) \leq 14.5$
M. Hata 1992 : $\mu(\pi) \leq 8.0161$
V.Kh. Salikhov 2008 : $\mu(\pi) \leq 7.6063$

A bound $\mu(\vartheta^2) \leq \kappa$ for some $\vartheta \in \mathbf{R}$ implies $\mu(\vartheta) \leq 2\kappa$. Hence the result of Rhin and Viola $\mu(\zeta(2)) \leq 5.441\ldots$ implies only $\mu(\pi) \leq 11.882\ldots$.

Conversely, a bound for the irrationality exponent of ϑ does not yield any bound for $\mu(\vartheta^2)$!

Irrationality measure for $\zeta(2)$ and $\zeta(3)$: history

$\zeta(2)$

R. Apéry 1978, F. Beukers 1979	$\mu(\zeta(2)) < 11.85$
R. Dvornicich and C. Viola 1987	$\mu(\zeta(2)) < 10.02$
M. Hata 1990	$\mu(\zeta(2)) < 7.52$
G. Rhin and C. Viola 1993	$\mu(\zeta(2)) < 7.39$
G. Rhin and C. Viola 1996	$\mu(\zeta(2)) < 5.44$

$\zeta(3)$

R. Apéry 1978, F. Beukers 1979	$\mu(\zeta(3)) < 13.41$
R. Dvornicich and C. Viola 1987	$\mu(\zeta(3)) < 12.74$
M. Hata 1990	$\mu(\zeta(3)) < 8.83$
G. Rhin and C. Viola 2001	$\mu(\zeta(3)) < 5.51$

Irrationality measure for $\log 2$: history

$\log 2$

Hermite–Lindemann, Mahler, Baker, Gel'fond, Feldman,. . . : transcendence measures

G. Rhin 1987	$\mu(\log 2) < 4.07$
E.A. Rukhadze 1987	$\mu(\log 2) < 3.89$
R. Marcovecchio 2008	$\mu(\log 2) < 3.57$

Reference : R. Marcovecchio, The Rhin-Viola method for log 2, Acta Arithmetica vol. **139** no.2 (2009), 147–184.

Georges Rhin and Carlo Viola

On a permutation group related to $\zeta(2)$, Acta Arith. **77** (1996), no.1, 23–56.
The group structure for $\zeta(3)$, Acta Arith. **97** (2001), no.3, 269–293.
The permutation group method for the dilogarithm, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (**5**) **4** (2005), no.3, 389–437.

Criterion of Yu. V. Nesterenko (qualitative)

Let $\vartheta_1, \dots, \vartheta_m$ be complex numbers.

Yu.V.Nesterenko (1985)

Let m be a positive integer and α a positive real number satisfying $\alpha > m - 1$. Assume there is a sequence $(L_n)_{n\geq 0}$ of linear forms in $\mathbf{Z}X_0 + \mathbf{Z}X_1 + \dots + \mathbf{Z}X_m$ of height $\leq e^n$ such that

$$|L_n(1, \vartheta_1, \dots, \vartheta_m)| = e^{-\alpha n + o(n)}.$$

Then $1, \vartheta_1, \dots, \vartheta_m$ are linearly independent over $\mathbf{Q}$.

Example : $m = 1$ – irrationality criterion.

Simplified proof of Nesterenko's Theorem

Francesco Amoroso

Pierre Colmez

Refinements : Raffaele Marcovecchio, Pierre Bel.

Recent developments

Stéphane Fischler and Wadim Zudilin, *A refinement of Nesterenko's linear independence criterion with applications to zeta values.* To appear in Math. Annalen.

Preprint MPIM 2009-35.

Fischler and Zudilin, 2009

There exist positive odd integers $i \le 139$ and $j \le 1961$ such that the numbers 1, $\zeta(3)$, $\zeta(i)$, $\zeta(j)$ are linearly independent over $\mathbf{Q}$.

There exist positive odd integers $i \le 93$ and $j \le 1151$ such that the numbers 1, $\log 2$, $\zeta(i)$, $\zeta(j)$ are linearly independent over $\mathbf{Q}$.

Multizeta values

For $s_1, \dots, s_k$ positive integers with $s_1 \ge 2$,

$$\zeta(s_1, \dots, s_k) = \sum_{n_1 > \cdots > n_k \ge 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

P. Cartier. –
Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents.
Sém. Bourbaki no. 885
Astérisque **282** (2002), 137-173.

M. Hoffman's web site

http://www.usna.edu/Users/math/meh/biblio.html

References on multizeta values and Euler sums

A	Double harmonic series	*48 references*
B	Triple harmonic series	*8 references*
C	Multiple harmonic series/multiple zeta values	*137 references*
D	Multiple zeta values over function fields	*5 references*
E	Alternating series	*16 references*
F	Multiple polylogarithms/nested sums	*46 references*
G	Finite multiple harmonic sums	*25 references*

In 2008 : 62 references
In 2009 : 30 references
+ preprints : 66 references

Last modified on August 18, 2009

EZFace calculator at CECM

http://oldweb.cecm.sfu.ca/projects/EZFace/

Centre for Experimental and Constructive Mathematics at Simon Fraser University

The calculator gives numerical values of MZVs with up to 100 decimal places accuracy.
The calculator also has a function to look for relations of linear dependence ;
$\mathrm{lindep}([a,b,c])$ looks for a vanishing linear combination of a, b, c with integer coefficients.
This makes it easy (EZ ?) to discover new identities !
J. Blümlein, D.J. Broadhurst, J.A.M. Vermaseren
The Multiple Zeta Value Data Mine
arXiv :0907.2557v1 [math-ph]

Gamma and Beta values

$$\Gamma(z) = \int_0^\infty e^{-t}t^z \cdot \frac{dt}{t} = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n}.$$

$$\Gamma(n+1) = n!, \quad (n \geq 0); \qquad \Gamma(1/2) = \sqrt{\pi}, \qquad \Gamma'(1) = -\gamma.$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

Weierstraß functions

Let $\Omega = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ be a lattice in $\mathbf{C}$.
The *canonical product* attached to Ω is the Weierstraß sigma function

$$\sigma(z) = \sigma_\Omega(z) = z \prod_{\omega\in\Omega\setminus\{0\}} \left(1-\frac{z}{\omega}\right) e^{(z/\omega)+(z^2/2\omega^2)}$$

The logarithmic derivative of the sigma function is the *Weierstraß zeta function*

$$\frac{\sigma'}{\sigma} = \zeta$$

and the derivative of ζ is $-\wp$, where $\wp$ is the Weierstraß elliptic function :

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

$$\wp(z+\omega) = \wp(z), \qquad \zeta(z+\omega) = \zeta(z) + \eta.$$

Complex multiplication : $\mathbf{Q}(i)$

$$\wp'^2 = 4\wp^3 - 4\wp, \quad g_2 = 4, \quad g_3 = 0,$$

$$\omega_1 = \int_0^1 \frac{dx}{\sqrt{x - x^3}} = \frac{1}{2}\mathrm{B}(1/4, 1/2) = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.622\,057\,554\,2\dots$$

$$\eta_1 = \frac{\pi}{\omega_1} = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}, \qquad \eta_2 = -i\eta_1.$$

Complex multiplication : $\mathbf{Q}(\varrho)$

$\varrho = e^{2i\pi/3}$

$$\wp'^2 = 4\wp^3 - 4, \quad g_2 = 0, \quad g_3 = 4,$$

$$\omega_1 = \int_0^1 \frac{dx}{\sqrt{1 - x^3}} = \frac{1}{3}\mathrm{B}(1/3, 1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428\,650\,648\dots$$

$$\eta_1 = \frac{2\pi}{\sqrt{3}\omega_1} = \frac{2^{7/3}\pi^2}{3^{1/2}\Gamma(1/3)^3}, \qquad \eta_2 = \varrho^2\eta_1.$$

Transcendence of special values of Weierstraß functions

Th. Schneider (1934). *The numbers*

$$\Gamma(1/4)^4/\pi^3$$

and

$$\Gamma(1/3)^3/\pi^2$$

are transcendental.

Diophantine approximation

$\Gamma(1/4)^4/\pi^3$ and $\Gamma(1/3)^3/\pi^2$ are not Liouville numbers.

Lower bounds for linear combinations of elliptic logarithms : Baker, Coates, Anderson …in the CM case, Philippon-Waldschmidt in the general case, refinements by N. Hirata Kohno, S. David, É. Gaudron - use Arakhelov's Theory (J-B. Bost : *slopes inequalities*).

Motivation : method of S. Lang for solving Diophantine equations (integer points on elliptic curves).

Sinnou David and Noriko Hirata

David, Sinnou ; Hirata-Kohno, Noriko
Linear forms in elliptic logarithms.
J. Reine Angew. Math. **628** (2009), 37–89.

Abelian varieties

Th. Schneider (1948). *For* a *and* b *in* $\mathbf{Q}$ *with* a, b *and* $a+b$ *not in* $\mathbf{Z}$, *the number*

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental.

The proof involves Abelian integrals of higher genus, related with the Jacobian of a Fermat curve.

Chudnovsky's algebraic independence Theorem

G.V. Chudnovsky (1978)
Theorem *Two at least of the numbers*

$$g_2,\ g_3,\ \omega_1,\ \omega_2,\ \eta_1,\ \eta_2$$

are algebraically independent.

Corollary : π *and* $\Gamma(1/4) = 3.625\,609\,908\,2\ldots$ *are algebraically independent. Also* π *and* $\Gamma(1/3) = 2.678\,938\,534\,7\ldots$ *are algebraically independent.*

Diophantine approximation

Transcendence measures for $\Gamma(1/4)$
(P. Philippon, S. Bruiltet)
For $P \in \mathbf{Z}[X,Y]$ *with degree* d *and height* H,

$$\log\left|P\bigl(\pi,\Gamma(1/4)\bigr)\right| > -10^{326}\bigl((\log H + d\log(d+1)\bigr) \cdot d^2\bigl(\log(d+1)\bigr)^2$$

Corollary : $\Gamma(1/4)$ *is not a Liouville number :*

$$\left|\Gamma(1/4) - \frac{p}{q}\right| > \frac{1}{q^{10^{330}}}.$$

Chudnovsky's method

(K.G. Vasil′ev 1996, P. Grinspan 2002). Two at least of the three numbers π, $\Gamma(1/5)$ and $\Gamma(2/5)$ are algebraically independent.

The proof involves a simple factor of dimension 2 of the Jacobian of the Fermat curve

$$X^5 + Y^5 = Z^5$$

which is an Abelian variety of dimension 6.

Ramanujan Functions

$$P(q) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},$$

$$Q(q) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n},$$

$$R(q) = 1 - 504\sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n}.$$

Eisenstein Series

$$E_{2k}(z) = 1 + (-1)^k\frac{4k}{B_k}\sum_{n=1}^{\infty} \frac{n^{2k-1}z^n}{1-z^n}.$$

F. G. M. Eisenstein
(1823 - 1852)

$$P(z) = E_2(z),$$
$$Q(z) = E_4(z),$$
$$R(z) = E_6(z).$$

Special values

$$\tau = i, \quad q = e^{-2\pi}, \quad \omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.622\,057\,554\,2\dots$$

$$P(q) = \frac{3}{\pi}, \quad Q(q) = 3\left(\frac{\omega_1}{\pi}\right)^4, \quad R(q) = 0.$$

$$\tau = \varrho, \quad q = -e^{-\pi\sqrt{3}}, \quad \omega_1 = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428\,650\,648\dots$$

$$P(q) = \frac{2\sqrt{3}}{\pi}, \quad Q(q) = 0, \quad R(q) = \frac{27}{2}\left(\frac{\omega_1}{\pi}\right)^6.$$

Yu. V. Nesterenko

Theorem (Nesterenko, 1996).
For any $q \in \mathbf{C}$ with $0 < |q| < 1$, three at least of the four numbers
$q, \quad P(q), \quad Q(q), \quad R(q)$
are algebraically independent.

`Tools` : The functions P, Q, R are algebraicaly independent over $\mathbf{C}(q)$ (K. Mahler) and satisfy a system of differential equations for $D = q\, d/dq$:

$$12\frac{DP}{P} = P - \frac{Q}{P}, \qquad 3\frac{DQ}{Q} = P - \frac{R}{Q}, \qquad 2\frac{DR}{R} = P - \frac{Q^2}{R}.$$

Consequences of Nesterenko's Theorem

The three numbers

$$\pi, \quad e^{\pi}, \quad \Gamma(1/4)$$

are algebraically independent.

The three numbers

$$\pi, \quad e^{\pi\sqrt{3}}, \quad \Gamma(1/3)$$

are algebraically independent.

Special values of Weierstraß sigma functions

The number

$$\sigma_{\mathbf{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}e^{\pi/8}\Gamma(1/4)^{-2}$$

is transcendental.

Fibonacci zeta values

$F_0 = 0, \; F_1 = 1, \; F_{n+1} = F_n + F_{n-1}$

$$\zeta_F(s) = \sum_{n \geq 1} \frac{1}{F_n^s}$$

Iekata Shiokawa (joint works with Carsten Elsner and Shun Shimomura, 2006)

$\zeta_F(2)$, $\zeta_F(4)$, $\zeta_F(6)$ are algebraically independent.
Consequence of Nesterenko's Theorem.

Fibonacci zeta values $\zeta_F(s) = \sum_{n\ge 1} \frac{1}{F_n^s}$

$$u = \zeta_F(2), \qquad v = \zeta_F(4)$$

$\zeta_F(4s+2) \in \mathbf{Q}(u,v)$ for $s \ge 0$, $s \in \mathbf{Z}$.

$\zeta_F(4s) - r_s\zeta_F(4) \in \mathbf{Q}(u,v)$ for $s \ge 2$, $s \in \mathbf{Z}$, with some $r_s \in \mathbf{Q} \setminus \{0\}$.

For s_1, s_2, s_3, distinct positive integers, the numbers $\zeta_F(2s_1), \zeta_F(2s_2), \zeta_F(2s_3)$ are algebraically dependent if and only if the three integers s_i are odd.

Standard relations among Gamma values

Translation :

$$\Gamma(a+1) = a\Gamma(a)$$

Reflexion :

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

Multiplication : for any positive integer n,

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)} \Gamma(na).$$

Rohrlich's Conjecture

Conjecture (D. Rohrlich) *Any multiplicative relation*

$$\pi^{b/2} \prod_{a\in\mathbf{Q}} \Gamma(a)^{m_a} \in \overline{\mathbf{Q}}$$

with b and m_a in $\mathbf{Z}$ lies in the ideal generated by the standard relations.

Examples :

$$\Gamma\left(\frac{1}{14}\right)\Gamma\left(\frac{9}{14}\right)\Gamma\left(\frac{11}{14}\right) = 4\pi^{3/2}$$

$$\prod_{\substack{1\le k\le n \\ (k,n)=1}} \Gamma(k/n) = \begin{cases} (2\pi)^{\varphi(n)/2}/\sqrt{p} & \text{if } n = p^r \text{ is a prime power,} \\ (2\pi)^{\varphi(n)/2} & \text{otherwise.} \end{cases}$$

Small Gamma Products with Simple Values

The two previous examples are due respectively to

Albert Nijenhuis, *Small Gamma products with Simple Values* http ://arxiv.org/abs/0907.1689, July 9, 2009.

and to

Greg Martin, *A product of Gamma function values at fractions with the same denominator* http ://arxiv.org/abs/0907.4384, July 24, 2009.

Lang's Conjecture

Conjecture (S. Lang) *Any algebraic dependence relation among the numbers* $(2\pi)^{-1/2}\Gamma(a)$ *with* $a \in \mathbf{Q}$ *lies in the ideal generated by the standard relations.* (Universal odd distribution).

Consequence of the Rohrlich–Lang Conjecture

As an example, the Rohrlich–Lang Conjecture implies that for any $q > 1$, the transcendence degree of the field generated by numbers

$$\pi, \quad \Gamma(a/q) \quad 1 \le a \le q, \ (a,q) = 1$$

is $1 + \varphi(q)/2$.

Variant of the Rohrlich–Lang Conjecture

Conjecture of S. Gun, R. Murty, P. Rath (2009) : for any $q > 1$, the numbers

$$\log \Gamma(a/q) \quad 1 \le a \le q, \ (a,q) = 1$$

are linearly independent over the field $\overline{\mathbf{Q}}$ of algebraic numbers.

A consequence is that for any $q > 1$, there is at most one primitive odd character χ modulo q for which

$$L'(1,\chi) = 0.$$

Ram and Kumar Murty (2009)

Ram Murty

Kumar Murty

Transcendental values of class group L–functions.

Peter Bundschuh (1979)

.

For $p/q \in \mathbf{Q}$ with $0 < |p/q| < 1$,

$$\sum_{n=2}^{\infty} \zeta(n)(p/q)^n$$

is transcendental.
For $p/q \in \mathbf{Q} \setminus \mathbf{Z}$,

$$\frac{\Gamma'}{\Gamma}\left(\frac{p}{q}\right) + \gamma$$

is transcendental

Peter Bundschuh (1979)

(P. Bundschuh) : *As a consequence of Nesterenko's Theorem, the number*

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}} = 2.076\,674\,047\,4\ldots$$

is transcendental, while

$$\sum_{n=0}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$$

(telescoping series).
Hence the number

$$\sum_{n=2}^{\infty} \frac{1}{n^s-1}$$

is transcendental over $\mathbf{Q}$ for $s = 4$. The transcendence of this number for even integers $s \geq 4$ would follow as a consequence of Schanuel's Conjecture.

$\sum_{n \geq 1} A(n)/B(n)$

Arithmetic nature of

$$\sum_{n \geq 1} \frac{A(n)}{B(n)}$$

where

$$A/B \in \mathbf{Q}(X).$$

In case B has distinct zeroes, by decomposing A/B in simple fractions one gets linear combinations of logarithms of algebraic numbers (Baker's method).
The example $A(X)/B(X) = 1/X^3$ shows that the general case is hard.
Work by S.D. Adhikari, N. Saradha, T.N. Shorey and R. Tijdeman (2001),
Sanoli Gun, Ram Murty and Purusottam Rath (2009).

Adolf Hurwitz (1859 - 1919)

Hurwitz zeta function :
for $z \in \mathbf{C}$ $z \neq 0$ and $\Re e(s) > 1$,

$$\zeta(s,z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s}$$

$$\zeta(s,1) = \zeta(s)$$

(Riemann zeta function)

Conjecture of Chowla and Milnor

Sarvadaman Chowla (1907 - 1995)

John Willard Milnor (1931 -)

For k and q integers > 1, the $\varphi(q)$ numbers

$$\zeta(k, a/q), \quad 1 \leq a \leq q, \quad (a,q) = 1$$

are linearly independent over $\mathbf{Q}$.

Sanoli Gun, Ram Murty and Purusottam Rath

The Chowla-Milnor Conjecture for $q = 4$ implies the irrationality of the numbers $\zeta(2n+1)/\pi^{2n+1}$ for $n \geq 1$.
Strong Chowla-Milnor Conjecture (2009) : *For k and q integers > 1, the $1 + \varphi(q)$ numbers*

$$1 \quad \textit{and} \quad \zeta(k, a/q), \quad 1 \leq a \leq q, \quad (a,q) = 1$$

are linearly independent over $\mathbf{Q}$.
For $k > 1$ odd, the number $\zeta(k)$ is irrational if and only if the strong Chowla-Milnor Conjecture holds for this value of k and for at least one of the two values $q = 3$ and $q = 4$.
Hence the strong Chowla-Milnor Conjecture holds for $k = 3$ (Apéry) and also for infinitely many k (Rivoal).

Linear independence of polylogarithms

For $k \geq 1$ and $|z| < 1$,

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

Thus $\mathrm{Li}_1(z) = \log(1-z)$ and $\mathrm{Li}_k(1) = \zeta(k)$ for $k \geq 2$.

Polylog Conjecture of S. Gun, R. Murty, P. Rath : *Let $k > 1$ be an integer and $\alpha_1, \ldots, \alpha_n$ algebraic numbers such that $\mathrm{Li}_k(\alpha_1), \ldots, \mathrm{Li}_k(\alpha_n)$ are linearly independent over $\mathbf{Q}$. Then these numbers $\mathrm{Li}_k(\alpha_1), \ldots, \mathrm{Li}_k(\alpha_n)$ are linearly independent over the field $\overline{\mathbf{Q}}$ of algebraic numbers.*

S. Gun, R. Murty, P. Rath : *if the polylog Conjecture is true, then the Chowla-Milnor Conjecture is true for all k and all q.*

The digamma function

For $z \in \mathbf{C} \setminus \{0, -1, -2, \ldots\}$,

$$\psi(x) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

$$\psi(z) = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right)$$

$$\psi(1+z) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}.$$

Special values of the digamma function

$$\psi(1) = -\gamma, \quad \psi\left(\frac{1}{2}\right) = -2\log(2) - \gamma,$$

$$\psi\left(2k - \frac{1}{2}\right) = -2\log(2) - \gamma + \sum_{n=1}^{2k-1} \frac{1}{n+1/2},$$

$$\psi\left(\frac{1}{4}\right) = -\frac{\pi}{2} - 3\log(2) - \gamma,$$

$$\psi\left(\frac{3}{4}\right) = \frac{\pi}{2} - 3\log(2) - \gamma.$$

Hence

$$\psi(1) + \psi(1/4) - 3\psi(1/2) + \psi(3/4) = 0.$$

Ram Murty and N. Saradha

Conjecture (2007) : Let K be a number field over which the q-th cyclotomic polynomial is irreducible. Then the $\varphi(q)$ numbers $\psi(a/q)$ with $1 \le a \le q$ and $(a,q) = 1$ are linearly independent over K.

Ram Murty and N. Saradha

Baker periods : elements of the $\overline{\mathbf{Q}}$–vector space spanned by the logarithms of algebraic numbers.
A Baker period is a period in the sense of Kontsevich and Zagier, and is either zero or else transcendental, by Baker's Theorem.
Murty and Saradha : one at least of the two following statements is true :
• Euler's Constant γ is not a Baker period
• the $\varphi(q)$ numbers $\psi(a/q)$ with $1 \le a \le q$ and $(a,q) = 1$ are linearly independent over K, whenever K be a number field over which the q-th cyclotomic polynomial is irreducible.

Euler constant

Euler–Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.577\,215\,664\,9\ldots$$

Neil J. A. Sloane's encyclopaedia
http ://www.research.att.com/∼njas/sequences/A001620

Jonathan Sondow http://home.earthlink.net/~jsondow/

$$\gamma = \int_0^\infty \sum_{k=2}^{\infty} \frac{1}{k^2\binom{t+k}{k}} dt$$

$$\gamma = \lim_{s\to 1+} \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \frac{1}{s^n} \right)$$

$$\gamma = \int_1^\infty \frac{1}{2t(t+1)} F \begin{pmatrix} 1, & 2, & 2 \\ 3, & t+2 & \end{pmatrix} dt.$$

Euler's constant γ

A.I. Aptekarev (2007) : Approximation to Euler's constant.

Tanguy Rivoal (2009) : Approximation to the function $\gamma + \log x$.
Consequence : approximation to γ and to $\zeta(2) - \gamma^2$.

Open Problems.
- *Is the Euler constant γ irrational ?*
- *Is γ transcendental ?*
- Kontsevich – Zagier : *γ is not a period*

Carlitz zeta values

Leonard Carlitz (1907 - 1999)

$A = \mathbf{F}_q[t]$,

$A_+ \subset A$ monic polynomials,

$P =$ prime polynomials in A_+,

$K = \mathbf{F}_q(t)$,

$K_\infty = \mathbf{F}_q((1/t))$,

Carlitz zeta values : for $s \in \mathbf{Z}$,

$$\zeta_A(s) = \sum_{a\in A_+} \frac{1}{a^s} = \prod_{p\in P} (1 - p^{-s})^{-1} \in K_\infty.$$

Thakur Gamma function

Dinesh Thakur

$$\Gamma(z) = \frac{1}{z} \prod_{a\in A_+} \left(1 + \frac{z}{a}\right)$$

Thakur Gamma values

Independence of Gamma values in positive characteristic :
Linear relations (W.D. Brownawell and M. Papanikolas, 2002) and algebraic relations (with G. Anderson, 2004).

Dale Brownawell

Matt Papanikolas

Carlitz zeta values at even A–integers

Define

$$\widetilde{\pi} = (t - t^q)^{1/(q-1)} \prod_{n=1}^{\infty} \left(1 - \frac{t^{q^n} - t}{t^{q^{n+1}} - t}\right)$$

For m a multiple of $q-1$,

$$\widetilde{\pi}^{-m} \zeta_A(m) \in A$$

Carlitz – Bernoulli numbers.

Greg Anderson, Dinesh Thakur, Jing Yu

For m a positive integer, $\zeta_A(m)$ is transcendental over K.
For m a positive integer not a multiple of $q-1$, $\zeta_A(m)/\widetilde{\pi}^m$ is transcendental over K.

Dinesh Thakur

Jing Yu

Bourbaki Seminar

Federico PELLARIN

Aspects de l'indépendance algébrique en caractéristique non nulle

Aspects of algebraic independence in non–zero characteristic

Séminaire Bourbaki - Volume 2006/2007 - Exposés 967-981

Astérisque **317** (2008), 205–242

Chieh-Yu Chang, Matthew A. Papanikolas, Jing Yu

Title : *Geometric Gamma values and zeta values in positive characteristic* arXiv :0905.2876

Abstract : In analogy with values of the classical Euler Gamma-function at rational numbers and the Riemann zeta-function at positive integers, we consider Thakur's geometric Gamma-function evaluated at rational arguments and Carlitz zeta-values at positive integers. We prove that, when considered together, all of the algebraic relations among these special values arise from the standard functional equations of the Gamma-function and from the Euler-Carlitz relations and Frobenius p-th power relations of the zeta-function.

Chieh-Yu Chang, Matthew A. Papanikolas, Dinesh S. Thakur, Jing Yu

Title : *Algebraic independence of arithmetic gamma values and Carlitz zeta values* arXiv :0909.0096

Abstract : We consider the values at proper fractions of the arithmetic gamma function and the values at positive integers of the zeta function for $\mathbf{F}_q[\theta]$ and provide complete algebraic independence results for them.

Chieh-Yu Chang

Title : *Periods of third kind for rank 2 Drinfeld modules and algebraic independence of logarithms* arXiv :0909.0101

Abstract : In analogy with the periods of abelian integrals of differentials of third kind for an elliptic curve defined over a number field, we introduce a notion of periods of third kind for a rank 2 Drinfeld $\mathbf{F}_q[t]$–module rho defined over an algebraic function field and derive explicit formulae for them. When rho has complex multiplication by a separable extension, we prove the algebraic independence of rho-logarithms of algebraic points that are linearly independent over the CM field of rho. Together with the main result in [CP08], we completely determine all the algebraic relations among the periods of first, second and third kinds for rank 2 Drinfeld $\mathbf{F}_q[t]$–modules in odd characteristic.