Cours de DEA *méthodes algébriques* 2001/2002, UFR 921, Université P. et M. Curie (Paris VI)

Fascicule 3

Valeurs spéciales de polylogarithmes multiples

par

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Mélanges, automates, intégrales de Chen et polylogarithmes

Ce troisième fascicule est consacré au produit de mélange de deux séries, aux d'automates, aux intégrales de Chen et aux relations de mélanges des polylogarithmes multiples en une variable.

1. Mélanges et automates

Les références principales pour l'interprétation du produit de mélange de deux séries en termes d'automates et les applications aux identités syntaxiques sont [Lo 2002] 1.3 et [J 1980].

Dans le cours du 14 mars les exemples suivant sont traités:

1. Mélange de deux éléments de $K < X >$
2. L'identité $x_0^* ш (x_1 \cdots x_n) = x_0^* x_1 x_0^* x_2 \cdots x_0^* x_n x_0^*$
3. L'identité $(x_0 + x_1)^* = x_0^* ш x_1^*$
4. L'identité $(1 + x_0) ш x_0^* = (x_0^*)^2$
5. L'identité $x_0^* ш (x_0 x_1)^* = (x_0 + x_0 x_0^* x_1)^* = (2x_0 + x_0 x_1 - x_0^2)^*(1 - x_0)$
6. L'identité $(x_0 x_1)^* ш (-x_0 x_1)^* = (-4x_0^2 x_1^2)^*$ (cet exemple est détaillé ci-dessous)

There is a description of the shuffle product in terms of automata due to Schutzenberger (see [R 1993]). Here is an example of a so-called "syntaxic" identity (Minh-Petitot):

Lemma 1.1. *The following identity holds:*

$$(x_0 x_1)^\star ш (-x_0 x_1)^\star = (-4x_0^2 x_1^2)^\star.$$

Sketch of proof. Following [Lo 2002], we associate to a finite automaton the series in the algebra $K << X >>$ which is the sum of the labels of its successful paths. Hence the series associated to the automaton

$$\begin{array}{l} \Longleftarrow \\ \Longrightarrow \end{array} \boxed{1} \; \begin{array}{c} \xleftarrow{\;x_1\;} \\ \xrightarrow[\;x_0\;]{} \end{array} \; \boxed{2}$$

is

$$S_1 = e + x_0x_1 + (x_0x_1)^2 + \cdots + (x_0x_1)^n + \cdots = (x_0x_1)^\star,$$

while the series associated to

$$\begin{tikzcd} \Longleftarrow \\[-20pt] \Longrightarrow \end{tikzcd}\begin{tikzcd} \boxed{A} \arrow[r, shift right, "-x_0"'] & \boxed{B} \arrow[l, shift right, "x_1"'] \end{tikzcd}$$

is

$$S_A = e - x_0x_1 + (x_0x_1)^2 + \cdots + (-x_0x_1)^n + \cdots = (-x_0x_1)^\star.$$

The following automaton is the cartesian product of the automata associated with S_1 and S_A:

$$\begin{tikzcd} \Longleftarrow \\[-20pt] \Longrightarrow \end{tikzcd}\begin{tikzcd} \boxed{1A} \arrow[r, shift right, "x_0"'] \arrow[d, shift right, "-x_0"'] & \boxed{2A} \arrow[l, shift right, "x_1"'] \arrow[d, shift right, "-x_0"'] \\ \boxed{1B} \arrow[u, shift right, "x_1"'] \arrow[r, shift right, "x_0"'] & \boxed{2B} \arrow[l, shift right, "x_1"'] \arrow[u, shift right, "x_1"'] \end{tikzcd}$$

hence the associated series S_{1A} is the shuffle product $S_{1A} = S_1 ш S_A$. One computes it by solving a system of linear (noncommutative) equations as follows. Define also S_{1B}, S_{2A} and S_{2B} as the series of labels of the paths starting at the corresponding state and ending at a terminal state. Then

$$\begin{aligned} S_{1A} &= e - x_0S_{1B} + x_0S_{2A}, \\ S_{1B} &= x_1S_{1A} + x_0S_{2B}, \\ S_{2A} &= x_1S_{1A} - x_0S_{2B}, \\ S_{2B} &= x_1S_{1B} + x_1S_{2A}. \end{aligned}$$

In general, if Σ_p is the series associated with a state p, and if the edges with origin p are $x_i : p \to p_i \quad (1 \le i \le m)$, then

$$\Sigma_p = \begin{cases} x_1\Sigma_{p_1} + \cdots + x_m\Sigma_{p_m} + e & \text{if } p \text{ is a terminal state,} \\ x_1\Sigma_{p_1} + \cdots + x_m\Sigma_{p_m} & \text{otherwise.} \end{cases}$$

One could as well for each state p consider the series Σ'_p of labels of the paths starting at an initial state and ending at p, and solve the corresponding system

$$\Sigma'_p = \begin{cases} \Sigma'_{q_1}y_1 + \cdots + \Sigma'_{q_\ell}y_\ell + e & \text{if } p \text{ is an initial state,} \\ \Sigma'_{q_1}y_1 + \cdots + \Sigma'_{q_\ell}y_\ell & \text{otherwise,} \end{cases}$$

where $y_j : q_j \to p \quad (1 \le j \le \ell)$ are the edges with end p.

In the present situation one deduces

$$S_{1A} = e - x_0(S_{1B} - S_{2A}), \quad S_{1B} - S_{2A} = 2x_0S_{2B},$$

$$S_{2B} = x_1(S_{1B} + S_{2A}), \quad S_{1B} + S_{2A} = 2x_1S_{1A}$$

and therefore

$$S_{1A} = e - 4x_0^2x_1^2S_{1A},$$

which completes the proof of Lemma 1.1. $\square$

2. Chen Iterated Integrals

Quelques références sur les intégrales itérées de Chen:

- [Ch 1954], [Ch 1971]
- [K 1995] Chap. XIX, § 11)
- [Lo 2002] exercice 6.3.8
- Le § 2 de [G 1997] et le § 2 de [G 1998]
- et les travaux de Ree (1958) et Fliess (1981) (voir les références dans [Lo 2002])

Chen iterated integrals are defined by induction as follows. Let $\varphi_1, \ldots, \varphi_p$ be holomorphic differential forms on a simply connected open subset D of the complex plane and let x and y be two elements in D. Define, as usual, $\int_x^y \varphi_1$ as the value, at y, of the primitive of φ_1 which vanishes at x. Next, by induction on p, define

$$\int_x^y \varphi_1 \cdots \varphi_p = \int_x^y \varphi_1(t) \int_x^t \varphi_2 \cdots \varphi_p.$$

By means of a change of variables

$$t \longmapsto x + t(y - x)$$

one can assume $x = 0$, $y = 1$ and D contains the real segment $[0, 1]$. In this case the integral is nothing else than

$$\int_{\Delta_p} \varphi_1(t_1)\varphi_2(t_2) \cdots \varphi_p(t_p),$$

where the domain of integration Δ_p is the simplex of $\mathbb{R}^p$ defined by

$$\Delta_p = \bigl\{(t_1, \ldots, t_p) \in \mathbb{R}^p\ ,\ 1 > t_1 > \cdots > t_p > 0\bigr\}.$$

The next statement is due to Kuo-Tsai Chen [Ch 1954] and [Ch 1971]; see also [B^2 2001], § 2, Prop. 1 and [K 1995].

Lemma 2.1. *For complex numbers x_0, x_1 and x, and differential forms $\varphi_1, \ldots, \varphi_p$,*

$$\int_{x_0}^{x_1} \varphi_1 \cdots \varphi_p = \sum_{j=0}^{p} \int_x^{x_1} \varphi_1 \cdots \varphi_j \int_{x_0}^{x} \varphi_{j+1} \cdots \varphi_p.$$

Démonstration. As we have seen, using if necessary a change of variables, we may assume x_0 and x_1 are real with $x_0 < x_1$, and the differential forms are holomorphic on an open set containing the real segment $[x_0, x_1]$. The simplex

$$\Delta_p(x_0,x_1) = \{(t_1,\dots,t_p)\ ;\ x_1 > t_1 > \cdots > t_p > x_0\}$$

is the disjoint union of the Cartesian products

$$\Delta_j(x,x_1)\times\Delta_{p-j}(x_0,x) = \{(t_1,\dots,t_p)\ ;\ x_1 > t_1 > \cdots > t_j > x > t_{j+1} > \cdots > t_p > x_0\}$$

for $j = 0, 1, \dots, p$, hence

$$\int_{x_0}^{x_1} \varphi_1\cdots\varphi_p = \sum_{j=0}^{p} \int_{\Delta_j(x,x_1)\times\Delta_{p-j}(x_0,x)} \varphi_1\cdots\varphi_p$$

and

$$\int_{\Delta_j(x,x_1)\times\Delta_{p-j}(x_0,x)} \varphi_1\cdots\varphi_p = \int_x^{x_1} \varphi_1\cdots\varphi_j \int_{x_0}^{x} \varphi_{j+1}\cdots\varphi_p.$$

□

Remark. The result does not hold with

$$\int_x^{x_1} \varphi_1\cdots\varphi_j \int_{x_0}^{x} \varphi_{j+1}\cdots\varphi_p.$$

For instance

$$\int_0^1 t_1 dt_1 dt_2 = \frac{1}{3}, \quad \int_0^{1/2} t_1 dt_1 dt_2 = \frac{1}{24}, \quad \int_{1/2}^1 t_1 dt_1 dt_2 = \frac{5}{48},$$

and

$$\int_{1/2}^1 t_1 dt_1 \int_0^{1/2} dt_2 = \frac{3}{16}, \quad \int_0^{1/2} t_1 dt_1 \int_{1/2}^1 dt_2 = \frac{1}{16}.$$

One should be careful with the definition of Chen integral, where the conventions differ from one author to another (compare our definition with [K 1995]).

The product of two integrals is a Chen integral, and more generally the product of two Chen integrals is a Chen integral. This is where the shuffle comes in.

Lemme 2.2. *Let $\varphi_1, \dots, \varphi_{p+q}$ be differential forms with $p \geq 0$ and $q \geq 0$. Then*

$$\int_0^1 \varphi_1\cdots\varphi_p \int_0^1 \varphi_{p+1}\cdots\varphi_{p+q} = \int_0^1 \varphi_1\cdots\varphi_p ш \varphi_{p+1}\cdots\varphi_{p+q}.$$

Démonstration. We assume, as we may without loss of generality, $x_0 = 0$ and $x_1 = 1$. Define $\Delta'_{p,q}$ as the subset of $\Delta_p \times \Delta_q$ of those elements $(z_1, \ldots, z_{p+q})$ for which we have $z_i \neq z_j$ for $1 \leq i \leq p < z_j \leq p+q$. Hence

$$\int_0^1 \varphi_1 \cdots \varphi_p \int_0^1 \varphi_{p+1} \cdots \varphi_{p+q} = \int_{\Delta_p \times \Delta_q} \varphi_1 \cdots \varphi_{p+q} = \int_{\Delta'_{p,q}} \varphi_1 \cdots \varphi_{p+q}.$$

Now $\Delta'_{p,q}$ is the disjoint union of the subsets $\Delta^\sigma_{p,q}$ defined by

$$\Delta^\sigma_{p,q} = \bigl\{(t_1, \ldots, t_{p+q})\ ;\ 1 > t_{\sigma(1)} > \cdots > t_{\sigma(p+q)} > 0\bigr\},$$

for σ running over $\mathfrak{S}_{p,q}$. Recall (see § 1.4) that $\mathfrak{S}_{p,q}$ is the set of permutations of $\{1, \ldots, p+q\}$ satisfying

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \quad \text{et} \quad \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q).$$

Hence

$$\int_{\Delta'_{p,q}} \varphi_1 \cdots \varphi_{p+q} = \sum_{\sigma \in \mathfrak{S}_{p,q}} \int_{\Delta^\sigma_{p,q}} \varphi_1 \cdots \varphi_{p+q}.$$

Since

$$\int_{\Delta^\sigma_{p,q}} \varphi_1 \cdots \varphi_{p+q} = \int_0^1 \varphi_{\sigma(1)} \cdots \varphi_{\sigma(p+q)}$$

and

$$\sum_{\sigma \in \mathfrak{S}_{p,q}} \varphi_{\sigma(1)} \cdots \varphi_{\sigma(p+q)} = \varphi_1 \cdots \varphi_p ш \varphi_{p+1} \cdots \varphi_{p+q},$$

Lemma 2.2 follows. □

The next easy result will be used to prove a duality theorem (§ 8.2) relating the multiple zeta values.

Lemma 2.3. *Let $\varphi_1, \ldots, \varphi_p$ be differential forms which are holomorphic in a simply connected open set D and let x_0, x_1 two complex numbers in D. Then*

$$\int_{x_0}^{x_1} \varphi_1 \cdots \varphi_p = (-1)^p \int_{x_1}^{x_0} \varphi_p \cdots \varphi_1.$$

Démonstration. Assuming (without loss of generality) $x_0 = 0$ and $x_1 = 1$, the result follows by means of the change of variables

$$t_j \mapsto 1 - t_{p+1-j} \quad (1 \leq j \leq p).$$

□

3. Polylogarithms and Zeta Values

In this section, s is a positive integer and z a complex number. In general we assume $|z|<1$, unless $s\ge 2$ where the condition $|z|\le 1$ will turn out to be sufficient. Define

$$\mathrm{Li}_s(z)=\sum_{n\ge 1}\frac{z^n}{n^s}\cdot$$

For $s\ge 2$ we have $\mathrm{Li}_s(1)=\zeta(s)$. An equivalent definition for these functions Li_s is given by induction on s, starting from

$$\mathrm{Li}_1(z)=\sum_{n\ge 1}\frac{z^n}{n}=-\log(1-z), \tag{3.1}$$

and using the differential equations

$$z\frac{d}{dz}\mathrm{Li}_s(z)=\mathrm{Li}_{s-1}(z)\quad (s\ge 2),$$

together with the initial conditions $\mathrm{Li}_s(0)=0$.

Therefore Li_s is given by integral formulae as follows. For $s=1$ we can write

$$\mathrm{Li}_1(z)=-\log(1-z)=\int_0^z\frac{dt}{1-t},$$

where the complex integral is over any path from 0 to z inside the unit circle. Following [L 1981](*), we shall denote by $\log z$ the logarithm of a nonzero complex number z with argument in $(-\pi,+\pi]$, so that for instance $\log(-1)=i\pi$, and we extend the definition of $\mathrm{Li}_1(z)$ for any $z\in\mathbb{C}\setminus\{1\}$ by setting $\mathrm{Li}_1(z)=-\log(1-z)$.

From the differential equations one deduces (cf. [L 1981], (1.3))

$$\mathrm{Li}_2(z)=\int_0^z\mathrm{Li}_1(t)\frac{dt}{t}=\int_0^z\frac{dt}{t}\int_0^t\frac{du}{1-u},$$

and by induction, for $s\ge 2$, (cf. [L 1981], (7.2))

$$\mathrm{Li}_s(z)=\int_0^z\mathrm{Li}_{s-1}(t)\frac{dt}{t}=\int_0^z\frac{dt_1}{t_1}\int_0^{t_1}\frac{dt_2}{t_2}\cdots\int_0^{t_{s-2}}\frac{dt_{s-1}}{t_{s-1}}\int_0^{t_{s-1}}\frac{dt_s}{1-t_s}\cdot$$

These formulae are valid for $|z|<1$, but for $s\ge 2$ they also yield a definition of $\mathrm{Li}_s(z)$ as an analytic function in any simply connected domain contained in $\mathbb{C}\setminus\{0,1\}$.

(*) In fact one should consider not only one fixed determination of the logarithm, and of each Li_s, but all of them; this gives rise to variation of Hodge structures which yield to deep and quite interesting studies. See for instance § 2 of [G 1997], or the paper *Function Theory of polylogarithms* by S. Bloch, Chap. 12 of [L 1991], pp. 275–285.

4. Multiple Polylogarithms in One Variable, Multiple Zeta Values and Shuffle

Let k, $s_1, \ldots, s_k$ be positive integers. Write $\underline{s}$ in place of $(s_1, \ldots, s_k)$. One defines a complex function of one variable by

$$\mathrm{Li}_{\underline{s}}(z) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_k^{s_k}}.$$

This function is analytic in the open unit disc, and, in the case $s_1 \geq 2$, it is also continuous on the closed unit disc. In the latter case we have

$$\zeta(\underline{s}) = \mathrm{Li}_{\underline{s}}(1).$$

For $k = 1$, we recover the functions studied in § 1. In the same way as in § 1, one can also define in an equivalent way these functions by induction on the number $p = s_1 + \cdots + s_k$ (the *weight* of $\underline{s}$) as follows. If $s_1 \geq 2$, we plainly have

$$z\frac{d}{dz}\mathrm{Li}_{(s_1,\ldots,s_k)}(z) = \mathrm{Li}_{(s_1-1,s_2,\ldots,s_k)}(z). \tag{4.1}$$

If $s_1 = 1$, writing

$$\sum_{n_1=n_2+1}^{\infty} z^{n_1-n_2-1} = \frac{1}{1-z},$$

we find

$$(1-z)\frac{d}{dz}\mathrm{Li}_{(1,s_2,\ldots,s_k)}(z) = \mathrm{Li}_{(s_2,\ldots,s_k)}(z). \tag{4.2}$$

Together with the initial conditions $\mathrm{Li}_{\underline{s}}(0) = 0$, these differential equations (4.1) and (4.2) determine all the $\mathrm{Li}_{\underline{s}}$.

For $\underline{s} = (s_1, \ldots, s_k)$, we set

$$\omega_{\underline{s}} = \omega_0^{s_1-1}\omega_1 \cdots \omega_0^{s_k-1}\omega_1.$$

This is a non-commutative product of differential forms, the total number of factors ω_i is the weight p of $\underline{s}$, and the number of factors ω_1 is the *depth* k of $\underline{s}$.

Using Chen's integrals, we can write

$$\mathrm{Li}_{\underline{s}}(z) = \int_0^z \omega_{\underline{s}}. \tag{4.3}$$

Example 1. For any $n \geq 1$ we have

$$\mathrm{Li}_{\{1\}_n}(z) = \frac{1}{n!}\bigl(-\log(1-z)\bigr)^n. \tag{4.4}$$

For $n=1$ this is (3.1). By induction, (4.4) follows from (4.2):

$$\mathrm{Li}_{\{1\}_n}(z)=\int_0^z \mathrm{Li}_{\{1\}_{n-1}}(t)\frac{dt}{1-t}\cdot$$

An equivalent formulation for (4.4) is given by writing that the generating series is

$$\sum_{n=0}^{\infty}\mathrm{Li}_{\{1\}_n}(z)x^n=(1-z)^{-x}. \tag{4.5}$$

The constant term $\mathrm{Li}_{\{1\}_0}(z)$ is 1.

A direct proof of (4.5) can also be obtained (see Theorem 8.1 of [$\mathrm{B}^3\mathrm{L}$ 2001]) by expanding the polynomial

$$(-1)^m\binom{-x}{m}=\frac{x}{m}\cdot\prod_{i=1}^{m-1}\left(1+\frac{x}{i}\right)$$

using

$$\prod_{i=1}^{m-1}\left(1+\frac{x}{i}\right)=\sum_{n\geq 0}x^n\sum_{\substack{i_1,\dots,i_n\\ m>i_1>\cdots>i_n\geq 1}}\frac{1}{i_1\cdots i_n}\cdot$$

From (4.4) one deduces

$$(-1)^n\mathrm{Li}_{\{1\}_n}(-1)=\mathrm{Li}_{\{1\}_n}(1/2)=\frac{1}{n!}(\log 2)^n,$$

which generalize the relations

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}=\sum_{m=1}^{\infty}\frac{1}{m2^m}=\log 2.$$

`Remark.` Thanks to the Binomial Theorem (see for instance [GR 1990], (1.3.1)), we also have

$$(1-z)^{-x}={}_2F_1\left(\begin{matrix}x\ ,\ \gamma\\ \gamma\end{matrix}\middle|z\right)={}_1F_0\left(\begin{matrix}x\\ -\end{matrix}\middle|z\right).$$

`Example 2.` *Catalan constant* is defined as

$$G=\sum_{n\geq 0}\frac{(-1)^n}{(2n+1)^2}\cdot$$

Since

$$i^k-(-i)^k=\begin{cases}0 & \text{if } k\equiv 0 \text{ or } 2 \pmod 4\\ 2i & \text{if } k\equiv 1 \pmod 4\\ -2i & \text{if } k\equiv -1 \pmod 4),\end{cases}$$

we also have

$$\mathrm{Li}_2(i) - \mathrm{Li}_2(-i) = 2iG.$$

From

$$\mathrm{Li}_2(1) - \mathrm{Li}_2(-1) = \frac{3}{2}\zeta(2) = 2\sum_{k\geq 0}\frac{1}{(2k+1)^2}$$

we deduce

$$\mathrm{Li}_2(i) + \mathrm{Li}_2(-i) = -\frac{1}{4}\zeta(2),$$

hence

$$\mathrm{Li}_2(i) = -\frac{1}{8}\zeta(2) + iG.$$

Example 3. Let us check (see [B^{3}L 2001], Th. 10.3)

$$\mathrm{Li}_{(2,1)}(-1) = \frac{1}{8}\zeta(2,1). \tag{4.6}$$

We have

$$\begin{aligned}\mathrm{Li}_{(2,1)}(z) &= \int_0^z \frac{dt_1}{t_1}\int_0^{t_1}\frac{dt_2}{1-t_2}\int_0^{t_2}\frac{dt_3}{1-t_3}\\ &= \int_0^z \frac{dt_1}{t_1}\int_0^{t_1} -\log(1-t_2)\frac{dt_2}{1-t_2}\\ &= \frac{1}{2}\int_0^z \big(\log(1-t)\big)^2\frac{dt}{t}\cdot\end{aligned}$$

Denote by $J(z)$ this function. We claim:

$$J(-z) = -J(z) + \frac{1}{4}J(z^2) + J\left(\frac{2z}{z+1}\right) - \frac{1}{8}J\left(\frac{4z}{(z+1)^2}\right). \tag{4.7}$$

Since the right hand side takes the value $J(1)/8$ at $z=1$, this will complete the proof of (4.6). Now (4.7) follows from the fact that both sides vanish at $z=0$ and have the same derivative. □

We have seen in § 3 (cf. (4.3)) that for $\underline{s}$ of weight p, $\mathrm{Li}_{\underline{s}}(z)$ is the Chen integral from 0 to z of a product of p terms

$$\omega_{\underline{s}} = \omega_0^{s_1-1}\omega_1\cdots\omega_0^{s_k-1}\omega_1.$$

Define $y_s = x_0^{s-1}x_1$ for $s\geq 1$ and

$$y_{\underline{s}} = y_{s_1}\cdots y_{s_k} = x_0^{s_1-1}x_1\cdots x_0^{s_k-1}x_1$$

for $\underline{s} = (s_1,\ldots,s_k)$. Further, introduce the notation:

$$\widehat{\mathrm{Li}}_{y_{\underline{s}}}(z) = \mathrm{Li}_{\underline{s}}(z). \tag{4.8}$$

This defines $\widehat{\mathsf{Li}}_x(z)$ when $x \in X^* x_1$ is a word in x_0 and x_1 which ends with x_1. In other terms

$$\widehat{\mathsf{Li}}_x(z) = \int_0^z \omega_{\epsilon_1} \cdots \omega_{\epsilon_p}$$

when $x = x_{\epsilon_1} \cdots x_{\epsilon_p}$, where each ϵ_i is 0 or 1, and $\epsilon_p = 1$ (otherwise the integral does not converge). If k is the number of x_1, we define positive integers $s_1, \ldots, s_k$ by writing

$$x = x_0^{s_1 - 1} x_1 \cdots x_0^{s_k - 1} x_1,$$

and then

$$\widehat{\mathsf{Li}}_x(z) = \mathsf{Li}_{\underline{s}}(z), \qquad \widehat{\zeta}(x) = \zeta(\underline{s}).$$

By linearity we extend the definition of $\widehat{\mathsf{Li}}_w(z)$ and $\widehat{\zeta}(w)$ to $\mathfrak{H}^1 = \mathbb{Q}e + \mathbb{Q}\langle x_0, x_1 \rangle x_1$: for convergence, we need that each monomial ends with x_1 (however see § 6)

$$\widehat{\mathsf{Li}}_S(z) = \sum_{w \in X^*} (S|w) \widehat{\mathsf{Li}}_w(z) \quad \text{for} \quad S = \sum_{w \in X^*} (S|w) w \in \mathfrak{H}^1.$$

Consider now the product of two $\mathsf{Li}_{\underline{s}}$ with the same argument z:

$$\mathsf{Li}_{\underline{s}}(z) \mathsf{Li}_{\underline{s}'}(z) = \int_0^z y_{\underline{s}} \int_0^z y_{\underline{s}'}.$$

Lemme 2.2 shows that the right hand side can be written as a linear combination of Chen integrals. We repeat the proof of this lemma for our application.

For simplicity consider the special case where z is real, $0 < z < 1$. One may deduce the general case of a complex z either by modifying suitably the argument, or else by using analytic continuation.

The product $y_{\underline{s}} y_{\underline{s}'}$ is a word of weight $p + p'$, when p is the weight of $\underline{s}$ and p' the weight of $\underline{s}'$. For $0 < z < 1$ we integrate over the Cartesian product

$$\begin{aligned} &\Delta_p(z) \times \Delta_{p'}(z) = \\ &\qquad \big\{ (t_1, \ldots, t_p, u_1, \ldots, u_{p'}) \; ; \; z > t_1 > \cdots > t_p > 0, \; z > u_1 > \cdots > u_{p'} > 0 \big\}. \end{aligned}$$

Clearly, this product is a disjoint union of simplices (we may ignore the tuples for which one t_i is equal to one u_j, since they do not contribute to the integral). A few special cases were already given in § 0. For instance, when $k = k' = 1$ and $s_1 = s'_1 = 1$, we get

$$\big(\mathsf{Li}_1(z)\big)^2 = \int_0^z \frac{dt}{1-t} \int_0^z \frac{du}{1-u} = \int_{z>t>u>0} \omega_1^2 + \int_{z>u>t>0} \omega_1^2 = 2\mathsf{Li}_{(1,1)}(z).$$

By induction, for $n \geq 1$, we infer

$$\mathsf{Li}_{\{1\}_{n-1}}(z) \mathsf{Li}_1(z) = n \mathsf{Li}_{\{1\}_n}(z),$$

hence

$$\mathsf{Li}_{\{1\}_n}(z) = \frac{1}{n!}\big(\mathsf{Li}_1(z)\big)^n$$

(cf. (4.4)).

In the next example we keep $k = k' = 1$ and $s_1 = 1$, but with $s_2 = 2$; we get

$$\mathsf{Li}_1(z)\mathsf{Li}_2(z) = \mathsf{Li}_{(1,2)}(z) + 2\mathsf{Li}_{(2,1)}(z).$$

For our last example we take $k = k' = 1$, $s_1 = s'_1 = 2$ and we get

$$\begin{aligned}\big(\mathsf{Li}_2(z)\big)^2 &= \int_{z>t_1>t_2>0} \omega_0(t_1)\omega_1(t_2) \int_{z>u_1>u_2>0} \omega_0(u_1)\omega_1(u_2)\\ &= 2\mathsf{Li}_{(2,2)}(z) + 4\mathsf{Li}_{(3,1)}(z).\end{aligned}$$

This shows that the product $\mathsf{Li}_{\underline{s}}(z)\mathsf{Li}_{\underline{s}'}(z)$ is a linear combination of $\mathsf{Li}_{\underline{\sigma}}(z)$, with positive coefficients, the sum of the coefficients being the binomial coefficient $(p+p')!/p!p'!$.According to the definition of the shuffle product (§ 1), if both w and w' end with x_1, then

$$\int_0^z w \cdot \int_0^z w' = \int_0^z w ш w'.$$

From Lemma 2.2 one readily deduces:

Proposition 4.9. *For any w and w' in $\mathbb{Q}\langle x_0, x_1\rangle x_1$,*

$$\widehat{\mathsf{Li}}_w(z)\widehat{\mathsf{Li}}_{w'}(z) = \widehat{\mathsf{Li}}_{w ш w'}(z). \tag{4.10}$$

In particular, for $z = 1$, we find

$$\widehat{\zeta}(y_{\underline{s}})\widehat{\zeta}(y_{\underline{s}'}) = \widehat{\zeta}(y_{\underline{s}} ш y_{\underline{s}'}) \tag{4.11}$$

whenever $s_1 \geq 2$ and $s'_1 \geq 2$.

These are the *first standard relations* between multiple zeta values.

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