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OPEN DIOPHANTINE PROBLEMS

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Dédié à un jeune septuagénaire, Pierre Cartier, qui m'a beaucoup appris

Abstract. Diophantine Analysis is a very active domain of mathematical research where one finds more conjectures than results.

We collect here a number of open questions concerning Diophantine equations (including Pillai's Conjectures), Diophantine approximation (featuring the *abc* Conjecture) and transcendental number theory (with, for instance, Schanuel's Conjecture). Some questions related to Mahler's measure and Weil absolute logarithmic height are then considered (e. g., Lehmer's Problem). We also discuss Mazur's question regarding the density of rational points on a variety, especially in the particular case of algebraic groups, in connexion with transcendence problems in several variables. We say only a few words on metric problems, equidistribution questions, Diophantine approximation on manifolds and Diophantine analysis on function fields.

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1. Diophantine Equations

1.1. Points on Curves. Among the 23 problems posed by Hilbert [Hi], [Gu] the tenth one has the shortest statement.

> *Given a Diophantine equation with any number of unknown quantities and with integral numerical coefficients*: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*

An equation of the form $f(\underline{x}) = 0$, where $f \in \mathbb{Q}[X_1, \ldots, X_n]$ is a given polynomial, while the unknowns $\underline{x} = (x_1, \ldots, x_n)$ take rational integer values, is a *Diophantine equation*. To solve this equation amounts to determining the integer points on the corresponding hypersurface of the affine space. Hilbert's Tenth Problem is to

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find an algorithm which tells us whether or not such a Diophantine equation has a solution.

There are other types of Diophantine equations. First of all one may consider rational solutions instead of integer ones. In this case, one considers rational points on a hypersurface. Next, one may consider integer or rational points over a number field. There is a situation which is intermediate between integer and rational points, where the unknowns take *S-integral point values*. This means that S is a fixed, finite set of prime numbers (rational primes, or prime ideals in the number field), and that the denominators of the solutions are restricted to those belonging to S. Examples are the Thue–Mahler equation

$$F(x, y) = p_1^{z_1} \cdots p_k^{z_k}$$

where F is a homogeneous polynomial with integer coefficients and $p_1, \dots, p_k$ are fixed primes (the unknowns are $x, y, z_1, \dots, z_k$ and take rational integer values with $z_i \geq 0$) and the generalized Ramanujan–Nagell equation $x^2 + D = p^n$, where D is a fixed integer, p a fixed prime, and the unknowns are x, n which take rational integer values with $n \geq 0$ (see for instance [ST], [Ti3], [Sh1], and [BS] for these and other similar questions).

Also, it is interesting to deal with simultaneous Diophantine equations, i. e., to study rational or integer points on algebraic varieties.

The final answer to Hilbert's original Tenth Problem was given in 1970 by Matiyasevich, following the works of Davis, Putnam and Robinson. This was the culminating stage of a rich and beautiful theory (see [DMR], [Ma] and [Mat]). The solution is negative, there is no hope of producing a complete theory of the subject. But one may still hope that there is a positive answer if one restricts Hilbert's initial question to equations in a limited number of variables, say $n = 2$, which amounts to considering integer points on a plane curve. In this case, deep results were achieved during the 20th century and many results are now known, but many more remain to be discovered.

The most basic results are those of Siegel (1929) and Faltings (1983). Siegel's Theorem deals with integer points and produces an algorithm to decide whether the set of solutions forms a finite or an infinite set. Faltings's result, solving Mordell's Conjecture, does the same for rational solutions, i. e., rational points on curves. To these two outstanding achievements of the 20th century, one may add Wiles's contribution, which not only settles the Last Fermat Theorem, but also provides a quantity of similar results for other curves [K].

Some natural questions arise.

(a) To answer Hilbert's tenth Problem for this special case of plane curves, which means to find an algorithm to decide whether a given Diophantine equation $f(x, y) = 0$ has a solution in $\mathbb{Z}$ (and the same problem in $\mathbb{Q}$).

(b) To find an upper bound for the number of either rational or integral points on a curve.

(c) To find an algorithm for solving explicitly a given Diophantine equation in two unknowns.

Further questions may be asked. For instance in question b) one might ask for the exact number of solutions; it may be more relevant to consider more generally the number of points on any number field, or the number of points of bounded degree and to investigate the related generating series... The number of open problems is endless!

Our goal here is not to describe in detail the state of the art regarding these questions (see for instance [La8]). It suffices to say

– that a complete answer to question (a) is not yet available. There is no algorithm (not even a conjectural one) to decide whether a curve has a rational point or not,

– that a number of results are known about question (b), the latest work on this topic being due to G. Rémond [Re] who produces an effective upper bound for the number of rational points on a curve of genus ≥ 2,

– and that question (c) is unanswered even for integer points, and even for the special case of curves of genus 2.

We do not request a practical algorithm, but only (to start with) a theoretical one. So our first open problem will be an effective refinement to Siegel's Theorem.

Problem 1.1. *Let $f \in \mathbb{Z}[X, Y]$ be a polynomial such that the equation $f(x, y) = 0$ has only finitely many solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$. Give an upper bound for $\max\{|x|, |y|\}$ when (x, y) is such a solution, in terms of the degree of f and of the maximal absolute value of the coefficients of f.*

That such a bound exists is part of the hypothesis, but the problem is to state it explicitly (and, if possible, in a closed form).

Further similar questions might also be asked regarding equations involving more variables (rational points on varieties), for instance Schmidt's norm form equations. We refer the reader to [La3] and [La8] for such questions, including the Lang–Vojta Conjectures.

Even the simplest case of quadratic forms suggests open problems. The determination of all positive integers which are represented by a given binary form is far from being solved. It is also expected that infinitely many real quadratic fields have class number one, but it is not even known that there are infinitely many number fields (without restriction on the degree) with class number one. Recall that the first complete solution of Gauss's class number 1 and 2 Problems (for imaginary quadratic fields) has been obtained by transcendence methods (A. Baker and H. M. Stark), so it may be considered to be a Diophantine problem. Nowadays more efficient methods (Goldfeld, Gross–Zagier, ... — see [La8, Chap. V, §5]) are available.

A related open problem is the determination of Euler's *numeri idonei* [Ri2]. Fix a positive integer n. If p is an odd prime for which there exist integers $x \geq 0$ and $y \geq 0$ with $p = x^2 + ny^2$, then

(i) $\gcd(x, ny) = 1$,

(ii) the equation $p = X^2 + nY^2$ in integers $X \geq 0$ and $Y \geq 0$ has the only one solution, $X = x$ and $Y = y$.

Now let p be an odd integer such that there exist integers $x \geq 0$ and $y \geq 0$ with $p = x^2 + ny^2$ and such that the conditions (i) and (ii) above are satisfied. If these properties imply that p is prime, then the number n belongs to the set of so-called *numeri idonei*. Euler found 65 such integers n

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 16, 18, 21, 22, 24, 25, 28, 30, 33, 37, 40, 42, 45, 48, 57, 58, 60, 70, 72, 78, 85, 88, 93, 102, 105, 112, 120, 130, 133, 165, 168, 177, 190, 210, 232, 240, 253, 273, 280, 312, 330, 345, 357, 385, 408, 462, 520, 760, 840, 1320, 1365, 1848.

It is known that there is at most one more number in the list, but one expects there be no other.

Here is just one example ([Sie1, problem 58, p. 112], [Guy, D18]) of an open problem dealing with simultaneous Diophantine quadratic equations. *Is there a perfect integer cuboid*? The existence of a box with integer edges x_1, x_2, x_3, integer face diagonals y_1, y_2, y_3 and integer body diagonal z, amounts to solving the system of four simultaneous Diophantine equations in seven unknowns

$$\begin{cases} x_1^2 + x_2^2 = y_3^2, \\ x_2^2 + x_3^2 = y_1^2, \\ x_3^2 + x_1^2 = y_2^2, \\ x_1^2 + x_2^2 + x_3^2 = z^2 \end{cases}$$

in $\mathbb{Z}$. We don't know whether there is a solution, but it is known that there is no perfect integer cuboid with the smallest edge $\leq 2^{31}$.

1.2. Exponential Diophantine Equations. In a Diophantine equation, the unknowns occur as the variables of polynomials, while in an *exponential Diophantine equation* (see [ST]), some exponents also are variables. One may consider the above-mentioned Ramanujan–Nagell equation $x^2 + D = p^n$ as an exponential Diophantine equation.

A famous problem which was open until 2002 is Catalan's one which dates back to 1844 [Cat], the same year where Liouville constructed the first examples of transcendental numbers (see also [Sie1, problem 77, p. 116], [Sie2, n° 60, p. 42], [ST, Chap. 12], [N1, Chap. 11], [Ri1], [Guy, D9], [Ri2, Chap. 7]). The "Note extraite d'une lettre adressée à l'Éditeur par Monsieur E. Catalan, Répétiteur à l'école polytechnique de Paris", published in Crelle Journal [Cat], reads:

> "*Je vous prie, Monsieur, de bien vouloir énoncer, dans votre recueil, le théorème suivant, que je crois vrai, bien que je n'aie pas encore réussi à le démontrer complètement: d'autres seront peut-être plus heureux.*
>
> *Deux nombres entiers consécutifs, autres que* 8 *et* 9, *ne peuvent être des puissances exactes; autrement dit: l'équation* $x^m - y^n = 1$, *dans laquelle les inconnues sont entières et positives, n'admet qu'une seule solution.*"

This means that the only example of consecutive numbers which are perfect powers x^p with $p \geq 2$ should be 8 and 9. Further information on the history of this question is available in Ribenboim's book [Ri1]. Tijdeman's result [Ti2] in 1976 shows that there are only finitely many solutions. More precisely, for any solution x, y, p, q, the number $\max\{p, q\}$ can be bounded by an effectively computable absolute constant. Once $\max\{p, q\}$ is bounded, only finitely many exponential Diophantine equations remain to be considered, and there are algorithms to complete the solution (based on Baker's method). Such a bound has been computed, but it is somewhat large: M. Mignotte proved that any solution x, y, p, q to Catalan's equation should satisfy

$$\max\{p, q\} < 8 \cdot 10^{16}.$$

Catalan's claim was finally substantiated by P. Mihăilescu [Mi] (see also [Bi] and [Me]).

Theorem 1.2 (Catalan's Conjecture)**.** *The equation*

$$x^p - y^q = 1,$$

where the unknowns x, y, p and q take integer values all ≥ 2, has only one solution, namely $(x, y, p, q) = (3, 2, 2, 3)$.

The final solution by Mihăilescu involves deep results from the theory of cyclotomic fields. Initially sharp measures of linear independence of logarithms of algebraic numbers were required, namely a specific estimate for two logarithms due to M. Laurent, M. Mignotte and Yu. V. Nesterenko, but then a solution using neither results from transcendental number theory nor the help of a computer was derived.

Catalan asked for integral solutions, like in Siegel's Theorem, while Faltings's Theorem deals with rational points. D. Prasad suggested that the set of tuples (x, y, p, q) in $\mathbb{Q}^2 \times \mathbb{N}^2$ satisfying the conditions

$$x^p - y^q = 1, \text{ and the curve } X^p - Y^q = 1 \text{ has genus} \geq 1$$

should be finite — evidence for this is provided by the *abc* Conjecture (see Section 2.1).

The fact that the right hand side in Catalan's equation is 1 is crucial. Nothing is known if one replaces it by another positive integer. The next conjecture was proposed by S. S. Pillai [Pi] at a conference of the Indian Mathematical Society held in Aligarh (see also [Sie1, problem 78, p. 117], [ST], [Ti3], [Sh1]).

Conjecture 1.3 (Pillai)**.** *Let k be a positive integer. The equation*

$$x^p - y^q = k,$$

where the unknowns x, y, p and q take integer values, all ≥ 2, has only finitely many solutions (x, y, p, q).

This means that in the increasing sequence of perfect powers x^p, with $x \geq 2$ and $p \geq 2$:

$$4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, 144, 169, \ldots,$$

the difference between two consecutive terms tends to infinity. It is not even known whether for, say, $k = 2$, Pillai's equation has only finitely many solutions. A related open question is whether the number 6 occurs as a difference between two perfect powers: *Is there a solution to the Diophantine equation* $x^p - y^q = 6$? (see [Sie2, problem 238a, p. 116]).

A conjecture which implies Pillai's has been suggested by T. N. Shorey in [Sh2]. This is the very problem which motivated C. L. Siegel in [Si1]. Let $f \in \mathbb{Z}[X]$ be a polynomial of degree n with at least two distinct roots and $f(0) \neq 0$. Let L be the number of non-zero coefficients of f. Write

$$f(X) = b_1 X^{n_1} + \cdots + b_{L-1} X^{n_{L-1}} + b_L$$

with $n = n_1 > n_2 > \cdots > n_{L-1} > 0$ and $b_i \neq 0$ $(1 \leq i \leq L)$. Set $H = H(f) = \max_{1 \leq i \leq L} |b_i|$.

Conjecture 1.4 (Shorey)**.** *There exists a positive number C which depends only on L and H with the following property. Let m, x and y be rational integers with $m \geq 2$ and $|y| > 1$ satisfying*

$$y^m = f(x).$$

Then either $m \leq C$, or else there is a proper sub-sum in

$$y^m - b_1 x^{n_1} - \cdots - b_{L-1} x^{n_{L-1}} - b_L$$

which vanishes.

An example with a vanishing proper sub-sum is

$$y^m = x^{n_1} + x - 2$$

where $H = 2$, $L = 3$ and a solution $(m, x, y) = (n_1, 2, 2)$.

Consider now the positive integers which are perfect powers y^q, with $q \geq 2$, and such that all digits in some basis $x \geq 2$ are 1's. Examples are 121 in basis 3, 400 in basis 7 and 343 in basis 18. To find all solutions amounts to solving the exponential Diophantine equation

$$\frac{x^n - 1}{x - 1} = y^q,$$

where the unknown x, y, n, q take positive, rational, integer values with $x \geq 2$, $y \geq 1$, $n \geq 3$ and $q \geq 2$. Only 3 solutions are known

$$(x, y, n, q) = (3, 11, 5, 2), \ (7, 20, 4, 2), \ (18, 7, 3, 3),$$

corresponding to the above-mentioned three examples. One does not know whether these are the only solutions (see [ST], [Guy, D10], [Ti3], [Sh1], [BM] and [Sh2]), but it is expected that there are no others.

The next question is to determine all the perfect powers with identical digits in some basis, which amounts to solving the equation

$$z \frac{x^n - 1}{x - 1} = y^q,$$

where the unknown x, y, n, q, z take positive, rational, integer values with $x \geq 2$, $y \geq 1$, $n \geq 3$, $1 \leq z < x$ and $q \geq 2$.

Another type of exponential Diophantine equation has been studied in a joint paper by H. P. Schlickewei and C. Viola [SV] where they state the following conjecture.

Conjecture 1.5. *Let $k \geq 2$ be an integer and $\alpha_1, \ldots, \alpha_n$ be non-zero elements in a field K of zero characteristic, such that, no quotient α_i/α_j with $j \neq i$ is a root of unity. Consider the function*

$$F(X_1, \ldots, X_k) = \det \begin{pmatrix} \alpha_1^{X_1} & \cdots & \alpha_k^{X_1} \\ \ldots & \ldots & \ldots \\ \alpha_1^{X_k} & \cdots & \alpha_k^{X_k} \end{pmatrix}.$$

Then the equation

$$F(0, x_2, \ldots, x_k) = 0$$

has only finitely many solutions $(x_2, \ldots, x_k) \in \mathbb{Z}^{k-1}$ such that, in the corresponding determinant, all $(k-1) \times k$ and all $k \times (k-1)$ submatrices have rank $k-1$.

Other exponential Diophantine equations are worth of study. See for instance [N1, Chap. III], [ST], [Ti3] and [Sh1].

Among the numerous applications of Baker's transcendence method are several questions related to the greatest prime factors of certain numbers. In this connexion we mention Grimm's Conjecture ([Gri], [N1, Chap. III, §3], [Guy, B32]).

Conjecture 1.6 (Grimm)**.** *Given k consecutive, composite integers, $n+1, \ldots, n+k$, there exist k distinct primes $p_1, \ldots, p_k$ such that $n+j$ is divisible by p_j, $1 \leq j \leq k$.*

This conjecture may be rephrased as follows. *Given an increasing sequence of positive integers $n_1 < \cdots < n_k$ for which the product $n_1 \cdots n_k$ has fewer than k distinct prime factors, there is a prime p in the range $n_1 \leq p \leq n_k$.* The equivalence of this with the original formulation follows from the "marriage theorem".

According to P. Erdős and J. L. Selfridge, a consequence of Conjecture 1.6 is that between two consecutive squares there is always a prime number.

A weaker form of Conjecture 1.6, which is also an open problem, is

Conjecture 1.7. *If there is no prime in the interval $[n+1, n+k]$, then the product $(n+1)\cdots(n+k)$ has at least k distinct prime divisors.*

M. Langevin (personal communication) pointed out that Grimm's Conjecture cannot be extended to arithmetical progressions without a proviso. For instance the numbers 12, 25, 38, 51, 64, 77, 90 belong to an arithmetic progression of ratio 13, but the number of distinct prime factors of $12 \cdot 25 \cdot 64 \cdot 90$ is only 3. However in [Lan1] he proposed a stronger conjecture than Conjecture 1.6.

Conjecture 1.8 (Langevin)**.** *Given an increasing sequence $n_1 < n_2 < \cdots < n_k$ of positive integers such that $n_1, n_2, \ldots, n_k$ are multiplicatively dependent, there exists a prime number in the interval $[n_1, n_k]$.*

Even if they may not be classified as Diophantine questions, the following open problems (see [La10]) are related to this topic: the twin prime conjecture, the Goldbach Problem (*is every even integer ≥ 4 the sum of two primes*?), Bouniakovsky's conjecture, Schinzel's hypothesis (H) (see also [Sie1, §29]) and the Bateman–Horn Conjecture.

The Diophantine equation

$$x^p + y^q = z^r$$

has also a long history in relation with Fermat's last Theorem ([K], [Ri2, §9.2.D]). If we look at the solutions in positive integers (x, y, z, p, q, r) for which

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$$

and such that x, y, z are relatively prime, then only 10 solutions[1] are known,

$$1+2^3=3^2, \qquad 2^5+7^2=3^4, \qquad 7^3+13^2=2^9, \qquad 2^7+17^3=71^2,$$
$$3^5+11^4=122^2, \qquad 17^7+76271^3=21063928^2, \qquad 1414^3+2213459^2=65^7,$$
$$9262^3+15312283^2=113^7, \quad 43^8+96222^3=30042907^2, \quad 33^8+1549034^2=15613^3.$$

Since the condition

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1 \quad \text{implies} \quad \frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le\frac{41}{42},$$

the *abc* Conjecture (see Section 2.1) predicts that the set of such solutions is finite (the "Fermat–Catalan" Conjecture formulated by Darmon and Granville — see [Mau]). For all known solutions, one of p, q, r is 2; this led R. Tijdeman and D. Zagier to conjecture[2] that there is no solution with the further restriction that each of p, q and r is ≥ 3.

A *Diophantine tuple* is a tuple $(a_1, \dots, a_n)$ of distinct positive integers such that a_ia_j+1 is a square for $1\le i<j\le n$ (see [Guy] and [G]). Fermat gave the example $(1, 3, 8, 120)$, and Euler showed that any Diophantine pair (a_1, a_2) can be extended to a Diophantine quadruple (a_1, a_2, a_3, a_4). It is not known whether there exists a Diophantine quintuple $(a_1, a_2, a_3, a_4, a_5)$, but A. Dujella [Du] proved that each Diophantine quintuple has $\max\{a_1, a_2, a_3, a_4, a_5\}\le 10^{10^{26}}$. He also proved that there is no Diophantine sextuple.

1.3. Markoff Spectrum. The original Markoff[3] equation (1879) is $x^2+y^2+z^2=3xyz$ (see [Ca, Chap. II], [CF, Chap. 2], [Guy, D12] and [Ri2, §10.5.B]). This is an algorithm which produces all solutions in positive integers. Given any solution $(x, y, z)=(m, m_1, m_2)$, we fix two of the three coordinates; then we obtain a quadratic equation in the third coordinate, for which we already know a solution. By the usual process of cutting with a rational line we deduce another solution. From one solution (m, m_1, m_2), this produces three other solutions

$$(m', m_1, m_2), \quad (m, m_1', m_2), \quad (m, m_1, m_2'),$$

where

$$m'=3m_1m_2-m, \quad m_1'=3mm_2-m_1, \quad m_2'=3mm_1-m_2.$$

These three solutions are called *neighbors* of the original one. Apart from the two so-called *singular* solutions $(1, 1, 1)$ and $(2, 1, 1)$, the three components of

[1] Up to obvious symmetries; in particular $1+2^3=3^2$ counts only for one solution.

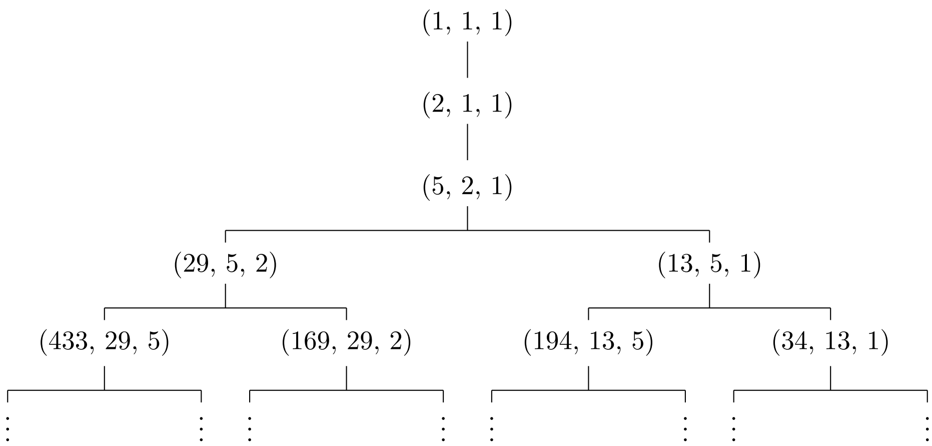
[2] This conjecture is also known as Beal's Conjecture — see [Mau].

[3] His name is spelled *Markov* in probability theory.

(m, m_1, m_2) are pairwise distinct, and the three neighbors of (m, m_1, m_2) are pairwise distinct. Assuming $m > m_1 > m_2$, one deduces

$$m_2' > m_1' > m > m'.$$

Hence there is a neighbor of (m, m_1, m_2) with maximum component less than m, and two neighbors, namely (m_1', m, m_2) and (m_2', m, m_1), with maximum component greater than m. It follows that one produces all solutions, starting from $(1, 1, 1)$, by taking successively the neighbors of the known solutions. Here is the Markoff tree, in the notation of H. Cohn [Coh], where (m_1', m, m_2) is written on the right and (m_2', m, m_1) on the left.



The main open problem in this topic ([Ca, p. 33], [CF, p. 11] and [Guy, D12]) is to prove that each largest component occurs only once in a triple of this tree.

Conjecture 1.9. *Fix a positive integer m for which the equation*

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$$

has a solution in positive integers (m_1, m_2) with $0 < m_1 \le m_2 \le m$. Then such a pair (m_1, m_2) is unique.

This conjecture has been proven for $m \le 10^{105}$.

The sequence

$$1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, \ldots$$

of integers m satisfying the hypotheses of Conjecture 1.9 is closely related to the question of the best rational approximation to quadratic, irrational, real numbers: for each m in this sequence, there is an explicit quadratic form $f_m(x, y)$ such that $f_m(x, 1) = 0$ has a root α_m for which

$$\limsup_{q\in\mathbb{Z},\ q\to\infty} (q\|q\alpha_m\|) = \frac{m}{\sqrt{9m^2-4}}. \tag{1.10}$$

The sequence of $(m, f_m, \alpha_m, \mu_m)$ where $\mu_m = \sqrt{9m^2-4}/m$ starts as follows,

m	1	2	5	13
$f_m(x, 1)$	x^2+x-1	x^2+2x-1	$5x^2+11x-5$	$13x^2+29x-13$
α_m	$\overline{1}$	$\overline{2}$	$\overline{2211}$	$\overline{221111}$
μ_m	$\sqrt{5}$	$\sqrt{8}$	$\sqrt{221}/5$	$\sqrt{1517}/13$

The third row gives the continued fraction expansion for α_m, where $\overline{2211}$, for instance, stands for $[2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 1, 1, \ldots]$. Conjecture 1.9 amounts to claiming that there is no ambiguity in the notation f_m: given m, two quadratic numbers α_m satisfying (1.10) should be roots of equivalent quadratic forms.

Hence the Markoff spectrum is closely related to rational approximation to a single real number. A generalization to simultaneous approximation is considered in Section 2.2 below.

2. Diophantine Approximation

In this section we restrict ourselves to problems in Diophantine approximation which do not require introducing a notion of height for algebraic numbers, those will be discussed in Section 4.

2.1. The *abc* Conjecture. For a positive integer n, we denote by

$$R(n) = \prod_{p|n} p$$

the *radical* or *square free part* of n.

The *abc* Conjecture resulted from a discussion between D. W. Masser and J. Œsterlé ([Œ, p. 169]; see also [Mas], as well as [La7], [La8, Chap. II, §1], [La9, Ch. IV, §7], [Guy, B19], [Bro], [Ri2, §9.4.E], [V], [Maz4] and [Ni]).

Conjecture 2.1 (*abc* Conjecture)**.** *For each $\varepsilon > 0$ there exists a positive number $\kappa(\varepsilon)$ which has the following property: if a, b and c are three positive rational integers which are relatively prime and satisfy $a+b=c$, then*

$$c < \kappa(\varepsilon) R(abc)^{1+\varepsilon}.$$

Conjecture 2.1 implies a previous conjecture by L. Szpiro on the conductor of elliptic curves. *Given any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that, for every elliptic curve with minimal discriminant Δ and conductor N, $|\Delta| < C(\varepsilon) N^{6+\varepsilon}$.*

When a, b and c are three positive relatively prime integers satisfying $a+b=c$, define

$$\lambda(a, b, c) = \frac{\log c}{\log R(abc)}$$

and

$$\varrho(a, b, c) = \frac{\log abc}{\log R(abc)}.$$

Here are the six largest known values for $\lambda(abc)$ (in [Bro] p. 102–105 as well as in [Ni], one can find all the 140 known values of $\lambda(a,\, b,\, c)$ which are ≥ 1.4).

	$a+b=c$	$\lambda(a,\, b,\, c)$	author(s)
1	$2+3^{10}\cdot 109=23^5$	1.629912...	É. Reyssat
2	$11^2+3^2 5^6 7^3=2^{21}\cdot 23$	1.625991...	B. de Weger
3	$19\cdot 1307+7\cdot 29^2\cdot 31^8=2^8\cdot 3^{22}\cdot 5^4$	1.623490...	J. Browkin, J. Brzezinski
4	$283+5^{11}\cdot 13^2=2^8\cdot 3^8\cdot 17^3$	1.580756...	J. Browkin, J. Brzezinski; A. Nitaj
5	$1+2\cdot 3^7=5^4\cdot 7$	1.567887...	B. de Weger
6	$7^3+3^{10}=2^{11}\cdot 29$	1.547075...	B. de Weger

Here are the six largest known values for $\varrho(abc)$, according to [Ni], where one can find the complete list of 46 known triples $(a,\, b,\, c)$ with $0<a<b<c$, $a+b=c$ and $\gcd(a,\, b)=1$ satisfying $\varrho(a,\, b,\, c)>4$.

	$a+b=c$	$\varrho(a,\, b,\, c)$	author(s)
1	$13\cdot 19^6+2^{30}\cdot 5=3^{13}\cdot 11^2\cdot 31$	4.41901...	A. Nitaj
2	$2^5\cdot 11^2\cdot 19^9+5^{15}\cdot 37^2\cdot 47=3^7\cdot 7^{11}\cdot 743$	4.26801...	A. Nitaj
3	$2^{19}\cdot 13\cdot 103+7^{11}=3^{11}\cdot 5^3\cdot 11^2$	4.24789...	B. de Weger
4	$2^{35}\cdot 7^2\cdot 17^2\cdot 19+3^{27}\cdot 107^2=5^{15}\cdot 37^2\cdot 2311$	4.23069...	A. Nitaj
5	$3^{18}\cdot 23\cdot 2269+17^3\cdot 29\cdot 31^8=2^{10}\cdot 5^2\cdot 7^{15}$	4.22979...	A. Nitaj
6	$17^4\cdot 79^3\cdot 211+2^{29}\cdot 23\cdot 29^2=5^{19}$	4.22960...	A. Nitaj

As observed by M. Langevin [Lan2], a consequence of the *abc* Conjecture is the solution of the following open problem [E1].

Conjecture 2.2 (Erdős–Woods)**.** *There exists a positive integer k such that, for m and n positive integers, the conditions*

$$R(m+i)=R(n+i)\quad (i=0,\,\ldots,\,k-1)$$

imply $m=n$.

Conjecture 2.2 is motivated by the following question raised by J. Robinson: *Is first order arithmetic definable using only the successor function $S\colon x\mapsto x+1$ and the coprimarity $x\perp y \Leftrightarrow (x,\, y)=1$?* It would suffice to decide whether the function $x\mapsto 5^x$ can be defined in the language $(S,\,\perp)$; see [Woo], [Guy, B29 and B35], [BLSW].

From the *abc* Conjecture (or even the weaker version with "some $\varepsilon<1$" rather than "all $\varepsilon>0$"), it follows that, apart from possibly finitely many exceptions

$(m,\,n)$, $k=3$ is an admissible value. Indeed, assume $m>n$. Using the *abc* Conjecture with $a=m(m+2)$, $b=1$, $c=(m+1)^2$, we obtain

$$m^2 \le \kappa(\varepsilon) R\bigl(m(m+1)(m+2)\bigr)^{1+\varepsilon}.$$

Now if $R(m+i)=R(n+i)$ for $i=0,\,1,\,2$ then $R(m+i)$ divides $m-n$, hence the number

$$R\bigl(m(m+1)(m+2)\bigr)=\operatorname{lcm}\bigl(R(m),\,R(m+1),\,R(m+2)\bigr)$$

divides $m-n$ and therefore $m^2 \le \kappa(\varepsilon)m^{1+\varepsilon}$. This shows that m is bounded.

One suspects that there is no exception at all with $k=3$. This would mean that if m and n have the same prime divisors, $m+1$, $n+1$ have the same prime divisors and $m+2$, $n+2$ have the same prime divisors, then $m=n$.

That $k=2$ is not an admissible value is easily seen: 75 and 1215 have the same prime divisors, and this is true also for 76 and 1216,

$$R(75)=15=R(1215), \qquad R(76)=2\cdot 19=R(1216).$$

Apart from this sporadic example, there is also a sequence of examples: for $m=2^h-2$ and $n=m(m+2)=2^hm$,

$$R(m)=R(n) \quad \text{and} \quad R(m+1)=R(n+1)$$

because $n+1=(m+1)^2$.

A generalization of the Erdős–Woods Problem to arithmetic progressions has been suggested by T. N. Shorey.

Question. *Does there exist a positive integer k such that, for any non-zero integers m, n, d and d' satisfying* $\gcd(m,\,d)=\gcd(n,\,d')=1$, *the conditions*

$$R(m+id)=R(n+id') \quad (i=0,\,\ldots,\,k-1)$$

imply $m=n$ and $d=d'$?

On the one hand, if the answer is positive, k is at least 4, as shown by several examples of quadruples $(m,\,n,\,d,\,d')$, like $(2,\,2,\,1,\,7)$, $(2,\,8,\,79,\,1)$ or $(4,\,8,\,23,\,1)$:

$$R(2)=R(2), \qquad R(3)=R(2+7), \qquad R(4)=R(2+2\cdot 7),$$
$$R(2)=R(4)=R(8), \qquad R(2+79)=R(4+23)=R(9),$$
$$R(2+2\cdot 79)=R(4+2\cdot 23)=R(10).$$

On the other hand, under the *abc* Conjecture, Shorey's question has a positive answer for $k=5$ (see [Lan3]).

Another related problem of T. S. Motzkin and E. G. Straus ([Guy, B19]) is to determine the pairs of integers m, n such that m and $n+1$ have the same prime divisors, and also n and $m+1$ have the same set of prime divisors. The known examples are

$$m=2^k+1, \qquad n=m^2-1 \quad (k\ge 0)$$

and the sporadic example $m=35=5\cdot 7$, $n=4374=2\cdot 3^7$, which yields $m+1=2^2\cdot 3^2$ and $n+1=5^4\cdot 7$.

We also quote another related conjecture attributed to P. Erdős in [Lan2] and to R. E. Dressler in [Ni].

Conjecture 2.3 (Erdős–Dressler)**.** *If a and b are two positive integers with $a < b$ and $R(a) = R(b)$ then there is a prime p with $a < p < b$.*

The first estimates in the direction of the *abc* Conjecture have been achieved by C. L. Stewart and R. Tijdeman, and then refined by C. L. Stewart and Yu Kunrui (see [SY1], [SY2]), using (p-adic) lower bounds for linear forms in logarithms: if a, b, c are relatively prime positive integers with $a + b = c$, then

$$\log c \le \kappa R^{1/3} (\log R)^3$$

where $R = R(abc)$.

An explicit version was worked out by Wong Chi Ho in 1999 [Wo] following an earlier version of [SY2]. For $c > 2$ the estimate

$$\log c \le R^{(1/3)+(15/\log\log R)}$$

is valid.

Further connexions between the *abc* Conjecture and measures of linear independence of logarithms of algebraic numbers have been pointed out by A. Baker [B2] and P. Philippon [P3] (see also [W6, exercise 1.11]). We reproduce here the main conjecture of the addendum of [P3]. For a rational number a/b with relatively prime integers a, b, we denote by $h(a/b)$ the number $\log \max\{|a|, |b|\}$.

Conjecture 2.4 (Philippon)**.** *There exist real numbers ε, α and β with $0 < \varepsilon < 1/2$, $\alpha \ge 1$ and $\beta \ge 0$, and a positive integer B, such that for any non-zero rational numbers x, y satisfying $xy^B \ne 1$, if S denotes the set of prime numbers for which $|xy^B + 1|_p < 1$, then*

$$-\sum_{p\in S} \log |xy^B + 1|_p \le B\left(\alpha h(x) + \varepsilon h(y) + (\alpha B + \varepsilon)\left(\beta + \sum_{p\in S} \log p\right)\right).$$

The conclusion is a lower bound for the p-adic distance between $-xy^B$ and 1; the main point is that several p's are involved. Conjecture 2.4 is telling us something about the prime decomposition of all numbers xy^B+1 for some fixed but unspecified value of B — and it implies the *abc* Conjecture.

Examples of optimistic Archimedean estimates related to measures of linear independence of logarithms of algebraic numbers are the Lang–Waldschmidt Conjectures in [La5] (introduction to Chap. X and XI, p. 212–217). Here is a simple example.

Conjecture 2.5 (Lang–Waldschmidt)**.** *For any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that, for any non-zero rational integers $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ with $a_1^{b_1} \cdots a_m^{b_m} \ne 1$,*

$$|a_1^{b_1} \cdots a_m^{b_m} - 1| \ge \frac{C(\varepsilon)^m B}{(|b_1| \cdots |b_m| \cdot |a_1| \cdots |a_m|)^{1+\varepsilon}},$$

where $B = \max_{1\le i\le m} |b_i|$.

Similar questions related to Diophantine approximation on tori are discussed in [La8, Chap. IX, §7].

Conjecture 2.5 deals with rational integers; we shall consider algebraic numbers more generally in Section 4.3, once we have defined a notion of height.

A very sharp conjectured lower bound for infinitely many elements in a specific sequence

$$|e^{b_0}a_1^{b_1}\cdots a_m^{b_m}-1|$$

with b_0 arbitrary, and where all the exponents b_i have the same sign (compare with Conjecture 2.14) has been shown by J. Sondow in [So] to yield the irrationality of Euler's constant.

From either the *abc* Conjecture or Conjecture 2.5 one deduces a quantitative refinement of Pillai's Conjecture 1.3.

Conjecture 2.6. *For any $\varepsilon > 0$, there is a constant $C(\varepsilon) > 0$ such that, for any positive integers x, y, p, q satisfying $x^p \neq y^q$,*

$$|x^p - y^q| \geq C(\varepsilon)\max\{x^p,\, y^q\}^{1-(1/p)-(1/q)-\varepsilon}.$$

We consider two special cases of Conjecture 2.6: first $(p,\, q) = (2,\, 3)$, which gives rise to Hall's Conjecture [H] (also [La8, Chap. II, §1]),

Conjecture 2.7 (Hall)**.** *If x and y are positive integers with $y^2 \neq x^3$, then*

$$|y^2 - x^3| \geq C\max\{y^2,\, x^3\}^{1/6}.$$

In this statement there is no ε. On the one hand, Conjecture 2.7 may be true by a sort of accident, but one may also expect that the estimate is too strong to be true. On the other hand, with the exponent $(1/6) - \varepsilon$, the *abc* Conjecture provides a lower bound not only for $|y^2 - x^3|$, but also for its radical [Lan3]: *for x and y relatively prime positive integers with $y^2 \neq x^3$,*

$$R\big(|y^2 - x^3|\big) \geq C(\varepsilon)\max\{y^2,\, x^3\}^{(1/6)-\varepsilon}.$$

The exponent 1/6 in Conjecture 2.7 is optimal, as has been shown by L. V. Danilov and A. Schinzel. Indeed, using the polynomial identity

$$(X^2-6X+4)^3-(X^2+1)(X^2-9X+19)^2 = 27(2X-11)$$

which is related to Klein's identity for the icosahedron (cf. [Lan4, Th. 6]), they show that there exist infinitely many pairs of positive integers $(x,\, y)$ such that

$$0 < |y^2 - x^3| < \frac{54}{25\sqrt{5}}\cdot\sqrt{x}.$$

The smallest known value for $|y^2 - x^3|/\sqrt{x}$ (N. Elkies, 1998) is $0.0214\ldots$, with

$$x = 3\cdot 7\,211\cdot 38\,791\cdot 6\,975\,841, \qquad y = 2\cdot 3^2\cdot 15\,228\,748\,819\cdot 1\,633\,915\,978\,229,$$

$$x^3 - y^2 = 3^3\cdot 7^2\cdot 17\cdot 73.$$

The second special case of Conjecture 2.6 we consider is $(x,\, y) = (3,\, 2)$. The question of how small $3^n - 2^m$ can be in comparison with 2^m has been raised by J. E. Littlewood [Guy, F23]. The example

$$\frac{3^{12}}{2^{19}} = 1 + \frac{7153}{524288} = 1.013\ldots$$

is related to music scales.

For further questions dealing with exponential Diophantine equations, we refer to Chap. 12 of the book of T. N. Shorey and R. Tijdeman [ST], as well as to the more recent surveys [Ti3] and [Sh1].

2.2. Thue–Siegel–Roth–Schmidt. One of the main open problems in Diophantine approximation is to produce an effective version of the Thue–Siegel–Roth Theorem. *For any* $\varepsilon > 0$ *and any irrational algebraic number* α, *there is a positive constant* $C(\alpha, \varepsilon) > 0$ *such that, for any rational number* p/q,

$$\left|\alpha - \frac{p}{q}\right| > \frac{C(\alpha, \varepsilon)}{q^{2+\varepsilon}}. \tag{2.8}$$

In connexion with the negative answer to Hilbert's 10th Problem by Yu. Matiyasevich, it has been suggested by M. Mignotte that an effective version of Schmidt's Subspace Theorem (which extends the Thue–Siegel–Roth Theorem to simultaneous Diophantine approximation) may be impossible. If this turns out to be the case also for the special case of the Thue–Siegel–Roth Theorem itself, then, according to E. Bombieri (see [Ni]), an effective version of the *abc* Conjecture would also be out of reach. M. Langevin noticed that the *abc* Conjecture yields a stronger inequality than Roth's,

$$\left|\alpha - \frac{p}{q}\right| > \frac{C(\varepsilon)}{R(pq)q^{\varepsilon}}.$$

So far, effective improvements are known only for Liouville's bound, and to improve them is already a great challenge.

Another goal would be to improve the estimate in Roth's Theorem. In the lower bound (2.8) one would like to replace $q^{-2-\varepsilon}$ by, say, $q^{-2}(\log q)^{-1-\varepsilon}$. It is expected that for any irrational real algebraic number α of degree ≥ 3, the term $q^{-2-\varepsilon}$ cannot be replaced by q^{-2} in inequality (2.8), but the set of α for which the answer is known is empty! This question is often asked for the special case of the number $\sqrt[3]{2}$, but another interesting example (due to Stanislaw Ulam — see for instance [Guy, F22]) is the real algebraic number ξ defined by

$$\xi = \frac{1}{\xi + y} \quad \text{with} \quad y = \frac{1}{1+y}.$$

Essentially nothing is known about the continued fraction expansion of a real algebraic number of degree ≥ 3; one does not know the answer to any of the following two questions.

Question 2.9. *Does there exist a real algebraic number of degree* ≥ 3 *with bounded partial quotients?*

Question 2.10. *Does there exist a real algebraic number of degree* ≥ 3 *with unbounded partial quotients?*

It is usually expected is that the continued fraction expansion of a real algebraic number of degree at least 3 always has unbounded partial quotients. More precisely one expects that real algebraic numbers of degree ≥ 3 behave like "almost all" real numbers (see Section 5.1).

Let $\psi(q)$ be a continuous positive real valued function. Assume that the function $q\psi(q)$ is non-increasing. Consider the inequality

$$\left|\theta - \frac{p}{q}\right| > \frac{\psi(q)}{q}. \tag{2.11}$$

Conjecture 2.12. *Let θ be real algebraic number of degree at least 3. Then inequality (2.11) has infinitely many solutions in integers p and q with $q > 0$ if and only if*

$$\int_1^\infty \psi(x)dx$$

diverges.

A far-reaching generalization of the Thue–Siegel–Roth Theorem to simultaneous approximation is the Schmidt Subspace Theorem. Here are two special cases.

- *Given real algebraic numbers $\alpha_1, \ldots, \alpha_n$ such that $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$, for any $\varepsilon > 0$,*

$$\max_{1\le i\le n} \left|\alpha_i - \frac{p_i}{q}\right| < \frac{1}{q^{1+(1/n)+\varepsilon}}$$

  has only finitely many solutions $(p_1, \ldots, p_n, q)$ in $\mathbb{Z}^{n+1}$ with $q > 0$.

- *Given real algebraic numbers $\alpha_1, \ldots, \alpha_n$ such that $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$, for any $\varepsilon > 0$,*

$$|q_1\alpha_1 + \cdots + q_n\alpha_n - p| < \frac{1}{q^{n+\varepsilon}}$$

  has only finitely many solutions $(q_1, \ldots, q_n, p)$ in $\mathbb{Z}^{n+1}$ with $q = \max\{|q_1|, \ldots, |q_n|\} > 0$.

These two types of Diophantine statements are parallel to the two types of Padé Approximants. It would be interesting to consider the analogue of Schmidt's Subspace Theorem in case of Padé Approximants, and also to study a corresponding analogue of Khinchine's transference principle [Ca].

One of the most important consequences of Schmidt's Subspace Theorem is the finiteness of nondegenerate solutions of the equation

$$x_1 + \cdots + x_n = 1,$$

where the unknowns take integer values (or S-integer values) in a number field. Here, non-degenerate means that no proper sub-sum vanishes. One main open question is to prove an effective version of this result. Schmidt's Theorem, which is a generalization of Roth's Theorem, is not effective. Only for $n = 2$ does one know bounds for the solutions of the S-unit equation $x_1 + x_2 = 1$, thanks to Baker's method (see [B1, Chap. 5], [La5, Chap. VI], [ST, Chap 1], [Se] and [La8]). One would like to extend Baker's method (or any other effective method) to the higher-dimensional case.

A generalization of the Markoff spectrum to simultaneous approximation is not yet available. Even the first step is missing. Given a positive integer n and real numbers $(\xi_1, \ldots, \xi_n)$, not all of which are rational, define $c_n = c_n(\xi_1, \ldots, \xi_n)$ to be the infimum of all c in the range $0 < c \leq 1$ for which

$$q|q\xi_i - p_i|^n < c$$

has infinitely many solutions. Then define the *n-dimensional simultaneous Diophantine approximation constant* γ_n to be the supremum of c_n over tuples $(\xi_1, \ldots \ldots, \xi_n)$ as above. Following [Fi], here is a summary of what is known about the first values of the approximation constants.

$$\gamma_1 = \frac{1}{\sqrt{5}} = 0.447\ldots \qquad \text{(Hurwitz)}$$

$$0.285\ldots = \frac{2}{7} \leq \gamma_2 \leq \frac{64}{169} = 0.378\ldots \qquad \text{(Cassels and Nowak)}$$

$$0.120\ldots = \frac{2}{5\sqrt{11}} \leq \gamma_3 \leq \frac{1}{2(\pi-2)} = 0.437\ldots \qquad \text{(Cusick and Spohn)}$$

The question remains open as to whether there are pairs with an approximation constant larger than $2/7$ (see [Br]).

We now illustrate with Waring's Problem the importance of proving effective Roth-type inequalities for irrational algebraic numbers.

In 1770, a few months before J. L. Lagrange proved that every positive integer is the sum of at most four squares of integers, E. Waring ([Wa, Chap. 5, Theorem 47 (9)]) wrote:

> "*Every integer is a cube or the sum of two, three, ... nine cubes; every integer is also the square of a square, or the sum of up to nineteen such; and so forth. Similar laws may be affirmed for the correspondingly defined numbers of quantities of any like degree.*"

See also Note 15 of the translator in [Wa].

For $k \geq 2$ define $g(k)$ as the smallest positive integer g such that any integer is the sum of g elements of the form x^k with $x \geq 0$. In other terms, for each positive integer n

$$n = x_1^k + \cdots + x_m^k$$

has a solution if $m = g(k)$, while there is a n which is not the sum of $g(k) - 1$ such k-th powers.

Lagrange's Theorem, which solved a conjecture of Bachet and Fermat, is $g(2) = 4$. Following Chap. IV of [N1], here are the values of $g(k)$ for the first integers k, with the name(s) of the author(s) and the date.

$g(2)=4$	$g(3)=9$	$g(4)=19$	$g(5)=37$	$g(6)=73$	$g(7)=143$
J. L. Lagrange	A. Wieferich	R. Balasubramanian J-M. Deshouillers F. Dress	J. Chen	S. S. Pillai	L. E. Dickson
1770	1909	1986	1964	1940	1936

For each integer $k \geq 2$, define

$$I(k) = 2^k + [(3/2)^k] - 2.$$

It is easy to show that $g(k) \geq I(k)$. Indeed, write

$$3^k = 2^k q + r \quad \text{with} \quad 0 < r < 2^k, \quad q = [(3/2)^k],$$

and consider the integer

$$N = 2^k q - 1 = (q-1)2^k + (2^k - 1)1^k.$$

Since $N < 3^k$, writing N as a sum of k-th powers can involve no term 3^k, and since $N < 2^k q$, it involves at most $(q-1)$ terms 2^k, all others being 1^k; hence it requires a total number of at least $(q-1) + (2^k - 1) = I(k)$ terms.

L. E. Dickson and S. S. Pillai (see for instance [HW] or [N1, Chap. IV]) proved independently in 1936 that $g(k) = I(k)$, provided that $r = 3^k - 2^k q$ satisfies

$$r \leq 2^k - q - 2.$$

Otherwise there is another formula for $g(k)$.

It has been shown that the condition $r \leq 2^k - q - 2$ is satisfied for $3 \leq k \leq 471\,600\,000$, and K. Mahler proved that it is also true for any sufficiently large k. Hence $g(k) = I(k)$ for these values of k. The problem is that Mahler's proof relies on a p-adic version of the Thue–Siegel–Roth Theorem, and therefore is not effective. So there is a gap, of which we don't even know the size. The conjecture, dating back to 1853, is $g(k) = I(k)$ for any $k \geq 2$, and this is true as soon as

$$\left\| \left(\frac{3}{2}\right)^k \right\| \geq \left(\frac{3}{4}\right)^k,$$

where $\|\cdot\|$ denote the distance to the nearest integer. As remarked by S. David, such an estimate (for sufficiently large k) follows not only from Mahler's estimate, but also from the *abc* Conjecture!

In [M1] K. Mahler defined a *Z-number* as a non-zero real number α such that the fractional part r_n of $\alpha(3/2)^n$ satisfies $0 \leq r_n < 1/2$ for any positive integer n. It is not known whether Z-numbers exist (see [FLP]). A related remark by J. E. Littlewood ([Guy, E18]) is that we are not yet able to prove that the fractional part of e^n does not tend to 0 as n tends to infinity (see also Conjecture 2.14 below).

A well known conjecture of Littlewood ([B1, Chap. 10, §1] and [PV]) asserts that *for any pair (x, y) of real numbers and any $\varepsilon > 0$, there exists a positive integer q such that*

$$q\|qx\| \cdot \|qy\| < \varepsilon.$$

According to G. Margulis (communication of G. Lachaud), the proofs in a 1988 paper by B. F. Skubenko (see MR **94d:**11047) are not correct and cannot be fixed.

There are several open questions known as "view obstruction Problems". One of them is the following. *Given n positive integers $k_1, \ldots, k_n$, there exists a real number x such that*

$$\|k_i x\| \geq \frac{1}{n+1} \quad \textit{for} \quad 1 \leq i \leq n.$$

It is known that $1/(n+1)$ cannot be replaced by a larger number [CP].

2.3. Irrationality and Linear Independence Measures. Given a real number θ, the first Diophantine question is to decide whether θ is rational or not. This is a qualitative question, and it is remarkable that an answer is provided by a quantitative property of θ. It depends ultimately on the quality of rational Diophantine approximations to θ. Indeed, on the one hand, if θ is rational, then there exists a positive constant $c = c(\theta)$ such that

$$\left|\theta - \frac{p}{q}\right| > \frac{c}{q}$$

for any $p/q \in \mathbb{Q}$. An admissible value for c is $1/b$ when $\theta = a/b$. On the other hand, if θ is irrational, then there are infinitely many rational numbers p/q such that

$$0 < \left|\theta - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Hence, in order to prove that θ is irrational, it suffices to prove that for any $\varepsilon > 0$ there is a rational number p/q such that

$$0 < \left|\theta - \frac{p}{q}\right| < \frac{\varepsilon}{q}.$$

This is a rather weak requirement. There are rational approximations in $1/q^2$, and we need only to produce rational approximations better than the trivial ones in c/q. Accordingly one should expect that it is rather easy to prove the irrationality of a given real number. In spite of that, the class of "interesting" real numbers which are known to be irrational is not as large as one would expect [KZ]. For instance no proof of irrationality has been given so far for Euler's constant

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.577215\ldots,$$

nor for Catalan's constant

$$G = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^2} = 0.915965\ldots, \tag{2.13}$$

nor for

$$\Gamma(1/5) = \int_0^\infty e^{-t}t^{-4/5}dt = 4.590843\ldots$$

or for numbers like

$$e + \pi = 5.859874\ldots, \qquad e^\gamma = 1.781072\ldots,$$
$$\zeta(5) = 1.036927\ldots, \qquad \zeta(3)/\pi^3 = 0.038768\ldots$$

and

$$\sum_{n\geq 1} \frac{\sigma_k(n)}{n!} \quad (k = 1,\, 2) \quad \text{where} \quad \sigma_k(n) = \sum_{d|n} d^k$$

(see [Guy, B14]).

Here is another irrationality question raised by P. Erdős and E. Straus in 1975 (see [E2] and [Guy, E24]). Define an *irrationality sequence* as an increasing sequence

$(n_k)_{k\geq 1}$ of positive integers such that, for any sequence $(t_k)_{k\geq 1}$ of positive integers, the real number

$$\sum_{k\geq 1} \frac{1}{n_k t_k}$$

is irrational. On the one hand, it has been proved by Erdős that $(2^{2^k})_{k\geq 1}$ is an irrationality sequence. On the other hand, the sequence $(k!)_{k\geq 1}$ is not, since

$$\sum_{k\geq 1} \frac{1}{k!(k+2)} = \frac{1}{2}.$$

An open question is whether an irrationality sequence must increase very rapidly. No irrationality sequence $(n_k)_{k\geq 1}$ is known for which $n_k^{1/2^k}$ tends to 1 as k tends to infinity.

Many further open irrationality questions are raised in [E2]. Another related example is Conjecture 5.4 below.

Assume now that the first step has been completed and that we know our number θ is irrational. Then there are (at least) two directions for further investigation.

(1) Considering several real numbers $\theta_1, \ldots, \theta_n$, a fundamental question is to decide whether or not they are linearly independent over $\mathbb{Q}$. One main example is to start with the successive powers of one number, $1, \theta, \theta^2, \ldots, \theta^{n-1}$. The goal is to decide whether θ is algebraic of degree $< n$. If n is not fixed, the question is whether θ is transcendental. This question, which is relevant also for complex numbers, will be considered in the next section. Observe also that the problem of algebraic independence is included here. It amounts to the linear independence of monomials.

(2) Another direction of research is to consider a quantitative refinement of the irrationality statement, namely an *irrationality measure*. We wish to bound from below the non-zero number $|\theta - (p/q)|$ when p/q is any rational number; this lower bound will depend on θ as well as the denominator q of the rational approximation. In case where a statement weaker than an irrationality result is known, namely if one can prove only that at least one of n numbers $\theta_1, \ldots, \theta_n$ is irrational, then a quantitative refinement will be a lower bound (in terms of q) for

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \ldots, \left|\theta_n - \frac{p_n}{q}\right|\right\},$$

when $p_1/q, \ldots, p_n/q$ are n rational numbers and $q > 0$ a common denominator.

On the one hand, the study of rational approximation of real numbers is achieved in a satisfactory way for numbers whose "regular"[4] continued fraction expansion is

[4] A "regular" continued fraction expansion

$$\left[a_0 + \frac{1}{a_1+}\ \frac{1}{a_2+}\ \cdots\right]$$

is written $[a_0, a_1, a_2, \ldots]$. A continued fraction expansion of the form

$$\left[a_0 + \frac{b_1}{a_1+}\ \frac{b_2}{a_2+}\ \cdots\right]$$

is called "irregular".

known. This is the case for rational numbers (!), for quadratic numbers, as well as for a small set of transcendental numbers, like

$$e = [2,\,1,\,2,\,1,\,1,\,4,\,1,\,1,\,6,\,1,\,1,\,\ldots] = [2,\,\{\overline{1,\,2m,\,1}\}_{m\geq 1}]$$

$$e^2 = [7,\,2,\,1,\,1,\,3,\,18,\,5,\,1,\,1,\,6,\,30,\,8,\,1,\,1,\,9,\,42,\,11,\,\ldots] = [7,\,\{\overline{3m-1,\,1,\,1,\,3m,\,12m+6}\}_{m\geq 1}]$$

and

$$e^{1/n} = [1,\,n-1,\,1,\,1,\,3n-1,\,1,\,1,\,5n-1,\,1,\,1,\,\ldots] = [\{\overline{1,\,(2m-1)n-1,\,1}\}_{m\geq 1}]$$

for $n > 1$. On the other hand, even for a real number x for which an "irregular" continued fraction expansion is known, like

$$\log 2 = \left[\frac{1}{1+}\;\frac{1}{1+}\;\frac{4}{1+}\;\frac{9}{1+}\;\cdots\;\frac{n^2}{1+}\cdots\right]$$

or

$$\frac{\pi}{4} = \left[\frac{1}{1+}\;\frac{9}{2+}\;\frac{25}{2+}\;\frac{49}{2+}\;\cdots\;\frac{(2n+1)^2}{2+}\;\cdots\right],$$

one does not know how well x can be approximated by rational numbers. No regular pattern has been observed or can be expected from the regular continued fraction of π,

$$\pi = [3,\,7,\,15,\,1,\,292,\,1,\,1,\,1,\,2,\,1,\,3,\,1,\,14,\,2,\,1,\,1,\,2,\,2,\,2,\,2,\,1,\,84,\,2,\,1,\,1,\,15,\,3,\,13,\,1,\,4,\,2,\,6,\,6,\,6,\,1,\,\ldots],$$

nor from any number "easily" related to π.

One expects that for any $\varepsilon > 0$ there are constants $C(\varepsilon) > 0$ and $C'(\varepsilon) > 0$ such that

$$\left|\log 2 - \frac{p}{q}\right| > \frac{C(\varepsilon)}{q^{2+\varepsilon}} \quad\text{and}\quad \left|\pi - \frac{p}{q}\right| > \frac{C'(\varepsilon)}{q^{2+\varepsilon}}$$

hold for any $p/q \in \mathbb{Q}$, but this is known only with larger exponents, namely $3.8913\ldots$ and $8.0161\ldots$ respectively (Rukhadze and Hata). The sharpest known exponent for an irrationality measure of

$$\zeta(3) = \sum_{n\geq 1}\frac{1}{n^3} = 1.202056\ldots$$

is $5.513891\ldots$, while for π^2 (or for $\zeta(2) = \pi^2/6$) it is $5.441243\ldots$ (both results due to Rhin and Viola). For a number like $\Gamma(1/4)$, the existence of absolute positive constants C and κ for which

$$\left|\Gamma(1/4) - \frac{p}{q}\right| > \frac{C}{q^\kappa}$$

has been proved only recently [P4]. The similar problem for e^π is not yet solved. In other terms there is no proof so far that e^π is not a Liouville number.

Earlier we distinguished two directions for research once we know the irrationality of some given numbers. Either, on the qualitative side, one studies the linear dependence relations, or else, on the quantitative side, one studies the quality of rational approximation. One can combine both. A quantitative version of a result of $\mathbb{Q}$-linear independence of n real numbers $\theta_1, \ldots, \theta_n$, is a lower bound, in terms of $\max\{|p_1|, \ldots, |p_n|\}$, for

$$\left|p_1\theta_1 + \cdots + p_n\theta_n\right|$$

when $(p_1, \ldots, p_n)$ is in $\mathbb{Z}^n \setminus \{0\}$.

For specific classes of transcendental numbers, A. I. Galochkin [G], A. N. Korobov (Th. 1.22 of [FN, Chap. 1, §7]) and more recently P. Ivankov proved extremely sharp measures of linear independence (see [FN, Chap. 2, §6.2 and §6.3]).

A general and important problem is to improve the known measures of linear independence for logarithms of algebraic numbers, as well as elliptic logarithms, Abelian logarithms, and more generally logarithms of algebraic points on commutative algebraic groups. For instance the conjecture that e^π is not a Liouville number should follow from improvements of known linear independence measures for logarithms of algebraic numbers.

The next step, which is to obtain sharp measures of algebraic independence for transcendental numbers, will be considered later (see Sections 4.2 and 4.3).

The so-called Mahler Problem (see [W8, §4.1]) is related to linear combination of logarithms $|b - \log a|$.

Conjecture 2.14 (Mahler)**.** *There exists an absolute constant $c > 0$ such that*

$$\|\log a\| > a^{-c}$$

for all integers $a \geq 2$.

Equivalently,

$$|a - e^b| > a^{-c}$$

for some absolute constant $c > 0$ for all integers a, $b > 1$.

A stronger conjecture is suggested in [W8] (4.1),

$$\|\log a\| > (\log a)^{-c}$$

for some absolute constant $c > 0$ for all integers $a \geq 3$, or equivalently

$$|a - e^b| > b^{-c}$$

for some absolute constant $c > 0$ for all integers a, $b > 1$. So far the best known estimate is

$$|a - e^b| > e^{-c(\log a)(\log b)},$$

so the problem is to replace the product $(\log a)(\log b)$ in the exponent by the sum $\log a + \log b$.

Such explicit lower bounds have interest in theoretical computer science [MT].

Another topic which belongs to Diophantine approximation is the theory of *equidistributed sequences.* For a positive integer $r \geq 2$, a *normal number* in base r is a real number such that the sequence $(xr^n)_{n\geq 1}$ is equidistributed modulo 1. Almost all real numbers for Lebesgue measure are normal (i. e., normal in basis r for any $r > 1$), but it is not known whether any irrational real algebraic number is normal to any integer basis, and it is also not known whether there is an integer r for which any number like e, π, $\zeta(3)$, $\Gamma(1/4)$, γ, G, $e + \pi$, e^γ is normal in basis r (see [Ra]). Further studies by D. H. Bailey and M. E. Crandall have recently been advanced by J. C. Lagarias in [L].

The digits of the expansion (in any basis ≥ 2) of an irrational, real, algebraic number should be equidistributed — in particular any digit should appear infinitely often. But even the following special case is unknown.

Conjecture 2.15 (Mahler). *Let $(\varepsilon_n)_{n\geq 0}$ be a sequence of elements in $\{0, 1\}$. Assume that the real number*

$$\sum_{n\geq 0} \varepsilon_n 3^{-n}$$

is irrational, then it is transcendental.

3. Transcendence

When K is a field and k a subfield, we denote by $\operatorname{trdeg}_k K$ the transcendence degree of the extension K/k. In the case $k = \mathbb{Q}$ we write simply $\operatorname{trdeg} K$ (see [La9, Chap. VIII, §1]).

3.1. Schanuel's Conjecture. We concentrate here on problems related to transcendental number theory. To start with, we consider the classical exponential function $e^z = \exp(z)$. A recent reference on this topic is [W6].

Schanuel's Conjecture is a simple but far-reaching statement — see the historical note to Chap. III of [La1].

Conjecture 3.1 (Schanuel). *Let $x_1, \ldots, x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}\bigl(x_1, \ldots, x_n, e^{x_1}, \ldots \ldots, e^{x_n}\bigr)$ is at least n.*

According to S. Lang ([La1, p. 31]): "From this statement, one would obtain most statements about algebraic independence of values of e^t and $\log t$ which one feels to be true". See also [La2, p. 638–639] and [Ri2, §10.7.G]. For instance the following statements [Ge1] are consequences of Conjecture 3.1.

Question. *Let $\beta_1, \ldots, \beta_n$ be $\mathbb{Q}$-linearly independent algebraic numbers and let $\log\alpha_1, \ldots, \log\alpha_m$ be $\mathbb{Q}$-linearly independent logarithms of algebraic numbers. Then the numbers*

$$e^{\beta_1}, \ldots, e^{\beta_n}, \ \log\alpha_1, \ldots, \log\alpha_m$$

are algebraically independent over $\mathbb{Q}$.

Question. *Let $\beta_1, \ldots, \beta_n$ be algebraic numbers with $\beta_1 \neq 0$ and let $\log\alpha_1, \ldots \ldots, \log\alpha_m$ be logarithms of algebraic numbers with $\log\alpha_1 \neq 0$ and $\log\alpha_2 \neq 0$. Then the numbers*

$$e^{\beta_1 e^{\beta_2 e^{\cdot^{\cdot^{\cdot \beta_{n-1} e^{\beta_n}}}}}} \quad \textit{and} \quad \alpha_1^{\alpha_2^{\cdot^{\cdot^{\cdot \alpha_m}}}}$$

are transcendental, and there is no nontrivial algebraic relation between such numbers.

A quantitative refinement of Conjecture 3.1 is suggested in [W4, Conjecture 1.4].

A quite interesting approach to Schanuel's Conjecture is given in [Ro5] where D. Roy states the next conjecture which he shows to be equivalent to Schanuel's one. Let $\mathcal{D}$ denote the derivation

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

over the ring $\mathbb{C}[X_0, X_1]$. The *height* of a polynomial $P \in \mathbb{C}[X_0, X_1]$ is defined as the maximum of the absolute values of its coefficients.

Conjecture 3.2 (Roy)**.** *Let k be a positive integer, $y_1, \ldots, y_k$ complex numbers which are linearly independent over $\mathbb{Q}$, $\alpha_1, \ldots, \alpha_k$ non-zero complex numbers and s_0, s_1, t_0, t_1, u positive real numbers satisfying*

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\} \quad \text{and} \quad \max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

Assume that, for any sufficiently large positive integer N, there exists a non-zero polynomial $P_N \in \mathbb{Z}[X_0, X_1]$ with partial degree $\leq N^{t_0}$ in X_0, partial degree $\leq N^{t_1}$ in X_1 and height $\leq e^N$ which satisfies

$$\left| (\mathcal{D}^k P_N)\left(\sum_{j=1}^k m_j y_j, \prod_{j=1}^k \alpha_j^{m_j} \right) \right| \leq \exp(-N^u)$$

for any non-negative integers $k, m_1, \ldots, m_k$ with $k \leq N^{s_0}$ and $\max\{m_1, \ldots, m_k\} \leq N^{s_1}$. Then

$$\operatorname{trdeg} \mathbb{Q}(y_1, \ldots, y_k, \alpha_1, \ldots, \alpha_k) \geq k.$$

This work of Roy's also provides an interesting connexion with other open problems related to the Schwarz Lemma for complex functions of several variables (see [Ro8, Conjectures 6.1 and 6.3]).

The most important special case of Schanuel's Conjecture is the *Conjecture of algebraic independence of logarithms of algebraic numbers.*

Conjecture 3.3 (Algebraic Independence of Logarithms of Algebraic Numbers)**.** *Let $\lambda_1, \ldots, \lambda_n$ be $\mathbb{Q}$-linearly independent complex numbers. Assume that the numbers $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are algebraic. Then the numbers $\lambda_1, \ldots, \lambda_n$ are algebraically independent over $\mathbb{Q}$.*

An interesting reformulation of Conjecture 3.3 is due to D. Roy [Ro4]. Denote by $\mathcal{L}$ the set of complex numbers λ for which e^λ is algebraic. Hence $\mathcal{L}$ is a $\mathbb{Q}$-vector subspace of $\mathbb{C}$. Roy's statement is:

Conjecture. *For any algebraic subvariety V of $\mathbb{C}^n$ defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers, the set $V \cap \mathcal{L}^n$ is the union of the sets $E \cap \mathcal{L}^n$, where E ranges over the set of vector subspaces of $\mathbb{C}^n$ which are contained in V.*

Such a statement is reminiscent of several of Lang's conjectures in Diophantine geometry (e. g., [La8, Chap. I, §6, Conjectures 6.1 and 6.3]).

Not much is known about the algebraic independence of logarithms of algebraic numbers, apart from the work of D. Roy on the rank of matrices whose entries are either logarithms of algebraic numbers, or more generally linear combinations of logarithms of algebraic numbers. We refer to [W6] for a detailed study of this question as well as related ones.

Conjecture 3.3 has many consequences. The next three ones are suggested by the work of D. Roy ([Ro1] and [Ro2]) on matrices whose entries are linear combinations of logarithms of algebraic numbers (see also [W6, Conjecture 11.17, §12.4.3 and Exercise 12.12]).

Consider the $\overline{\mathbb{Q}}$-vector space $\widetilde{\mathcal{L}}$ spanned by 1 and $\mathcal{L}$. In other words $\widetilde{\mathcal{L}}$ is the set of complex numbers which can be written

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n,$$

where $\beta_0, \beta_1, \ldots, \beta_n$ are algebraic numbers, $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers, and finally $\log \alpha_1, \ldots, \log \alpha_n$ are logarithms of $\alpha_1, \ldots, \alpha_n$ respectively.

Conjecture 3.4 (Strong Four Exponentials Conjecture)**.** *Let x_1, x_2 be two $\overline{\mathbb{Q}}$-linearly independent complex numbers and y_1, y_2 be also two $\overline{\mathbb{Q}}$-linearly independent complex numbers. Then at least one of the four numbers x_1y_1, x_1y_2, x_2y_1, x_2y_2 does not belong to $\widetilde{\mathcal{L}}$.*

The following special case is also open.

Conjecture 3.5 (Strong Five Exponentials Conjecture)**.** *Let x_1, x_2 be two $\mathbb{Q}$-linearly independent complex numbers, and y_1, y_2 be also two $\mathbb{Q}$-linearly independent complex numbers. Further, let β_{ij} $(i = 1, 2,\ j = 1, 2)$, γ_1 and γ_2 be six algebraic numbers with $\gamma_1 \neq 0$. Assume that the five numbers*

$$e^{x_1y_1-\beta_{11}},\ e^{x_1y_2-\beta_{12}},\ e^{x_2y_1-\beta_{21}},\ e^{x_2y_2-\beta_{22}},\ e^{(\gamma_1x_1/x_2)-\gamma_2}$$

are algebraic. Then all five exponents vanish,

$$x_iy_j = \beta_{ij} \quad (i = 1, 2,\ \ j = 1, 2) \quad \textit{and} \quad \gamma_1x_1 = \gamma_2x_2.$$

A consequence of Conjecture 3.5 is the solution of the open problem of the transcendence of the number e^{π^2}, and more generally of $\alpha^{\log \alpha} = e^{\lambda^2}$ when α is a non-zero algebraic number and $\lambda = \log \alpha$ a non-zero logarithm of α.

The next conjecture is proposed in [Ro4].

Conjecture 3.6 (Roy)**.** *For any 4×4 skew-symmetric matrix $\mathtt{M}$ with entries in $\mathcal{L}$ and rank ≤ 2, either the rows of $\mathtt{M}$ are linearly dependent over $\mathbb{Q}$, or the column space of $\mathtt{M}$ contains a non-zero element of $\mathbb{Q}^4$.*

Finally, a special case of Conjecture 3.6 is the well known Four Exponentials Conjecture due to Schneider ([Schn, Chap. V, end of §4, Problem 1]), S. Lang ([La1, Chap. II, §1], [La2, p. 638]) and K. Ramachandra ([R, II, §4]).

Conjecture 3.7 (Four Exponentials Conjecture)**.** *Let x_1, x_2 be two $\mathbb{Q}$-linearly independent complex numbers and y_1, y_2 also be two $\mathbb{Q}$-linearly independent complex numbers. Then at least one of the four numbers*

$$\exp(x_iy_j) \quad (i = 1, 2,\ \ j = 1, 2)$$

is transcendental.

The four exponentials Conjecture can be stated as follows: *consider a* 2×2 *matrix whose entries are logarithms of algebraic numbers,*

$$M = \begin{pmatrix} \log \alpha_{11} & \log \alpha_{12} \\ \log \alpha_{21} & \log \alpha_{22} \end{pmatrix};$$

assume that the two rows of this matrix are linearly independent over $\mathbb{Q}$ *(in* $\mathbb{C}^2$*), and also that the two columns are linearly independent over* $\mathbb{Q}$*; then the rank of this matrix is* 2.

We refer to [W6] for a detailed discussion of this topic, including the notion of *structural rank of a matrix* and the result, due to D. Roy, that Conjecture 3.3 is equivalent to a conjecture on the rank of matrices whose entries are logarithms of algebraic numbers.

A classical problem on algebraic independence of algebraic powers of algebraic numbers has been raised by A. O. Gelfond [Ge2] and Th. Schneider [Schn, Chap. V, end of §4, Problem 7]. The data are an irrational algebraic number β of degree d, and a non-zero algebraic number α with a non-zero logarithm $\log \alpha$. We write α^z in place of $\exp\{z \log \alpha\}$. Gelfond's problem is

Conjecture 3.8 (Gelfond)**.** *The two numbers*

$$\log \alpha \quad \textit{and} \quad \alpha^\beta$$

are algebraically independent over $\mathbb{Q}$.

Schneider's question is

Conjecture 3.9 (Schneider)**.** *The* $d-1$ *numbers*

$$\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}}$$

are algebraically independent over $\mathbb{Q}$.

The first partial results towards a proof of Conjecture 3.9 are due to A. O. Gelfond [Ge3]. For the more recent ones, see [NP, Chap. 13 and 14].

Combining both questions 3.8 and 3.9 yields a stronger conjecture.

Conjecture 3.10 (Gelfond–Schneider)**.** *The* d *numbers*

$$\log \alpha, \alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}}$$

are algebraically independent over $\mathbb{Q}$.

Partial results are known. They deal, more generally, with the values of the usual exponential function at products $x_i y_j$, when $x_1, \dots, x_d$ and $y_1, \dots, y_\ell$ are $\mathbb{Q}$-linearly independent complex (or p-adic) numbers. The six exponentials Theorem states that, in these circumstances, the $d\ell$ numbers $e^{x_i y_j}$ $(1 \le i \le d, 1 \le j \le \ell)$ cannot all be algebraic if $d\ell > d + \ell$. Assuming stronger conditions on d and ℓ, namely $d\ell \ge 2(d+\ell)$, one deduces that two at least of these $d\ell$ numbers $e^{x_i y_j}$ are algebraically independent over $\mathbb{Q}$. Other results are available involving, in addition to $e^{x_i y_j}$, either the numbers $x_1, \dots, x_d$ themselves, or $y_1, \dots, y_\ell$, or both. But an interesting point is that, if we wish to obtain a higher transcendence degree, say to obtain that three at least of the numbers $e^{x_i y_j}$ are algebraically independent over $\mathbb{Q}$, one needs a further assumption, which is a measure of linear independence over $\mathbb{Q}$

for the tuple $x_1, \ldots, x_d$ as well as for the tuple $y_1, \ldots, y_\ell$. To remove this so-called *technical hypothesis* does not seem to be an easy challenge (see [NP, Chap. 14, §2.2 and §2.3]).

The need for such a technical hypothesis seems to be connected with the fact that the actual transcendence methods produce not only a qualititative statement (lower bound for the transcendence degree), but also quantitative statements (transcendence measures and measures of algebraic independence).

Several complex results have not yet been established in the ultrametric situation. Two noticeable instances are

Conjecture 3.11 (p-adic analogue of Lindemann–Weierstrass's Theorem)**.** *Let $\beta_1, \ldots, \beta_n$ be p-adic algebraic numbers in the domain of convergence of the p-adic exponential function $\exp_p$. Then the n numbers $\exp_p \beta_1, \ldots, \exp_p \beta_n$ are algebraically independent over $\mathbb{Q}$.*

Conjecture 3.12 (p-adic analogue of an algebraic independence result of Gelfond)**.** *Let α be a non-zero algebraic number in the domain of convergence of the p-adic logarithm $\log_p$, and let β be a p-adic cubic algebraic number, such that $\beta \log_p \alpha$ is in the domain of convergence of the p-adic exponential function $\exp_p$. Then*

$$\alpha^\beta = \exp_p(\beta \log_p \alpha) \quad \textit{and} \quad \alpha^{\beta^2} = \exp_p(\beta^2 \log_p \alpha)$$

are algebraically independent over $\mathbb{Q}$.

The p-adic analogue of Conjecture 3.3 would solve Leopoldt's Conjecture on the p-adic rank of the units of an algebraic number field [Le] (see also [N2] and [Gra]), by proving the nonvanishing of the p-adic regulator.

Algebraic independence results for the values of the exponential function (or more generally for analytic subgroups of algebraic groups) in several variables have already been established, but they are not yet satisfactory. The conjectures stated in [W2, p. 292–293] as well as those of [NP, Chap. 14, §2] are not yet proved. One of the main obstacles is the above-mentioned open problem with the technical hypothesis.

The problem of extending the Lindemann–Weierstrass Theorem to commutative algebraic groups is not yet completely solved (see conjectures by P. Philippon in [P1]).

Algebraic independence proofs use elimination theory. Several methods are available; one of them, developed by Masser, Wüstholz and Brownawell, relies on the Hilbert Nulstellensatz. In this context we quote the following conjecture of Blum, Cucker, Shub and Smale (see [Sm] and [NP, Chap. 16, §6.2]), related to the open problem "$P = NP$?" [J].

Conjecture 3.13 (Blum, Cucker, Shub and Smale)**.** *Given an absolute constant c and polynomials $P_1, \ldots, P_m$ with a total of N coefficients and no common complex zeros, there is no program to find, in at most N^c step, the coefficients of polynomials A_i satisfying Bézout's relation,*

$$A_1 P_1 + \cdots + A_m P_m = 1.$$

In connexion with complexity in theoretical computer science, W. D. Brownawell suggests investigating Diophantine approximation from a new point of view in [NP, Chap. 16, §6.3].

Complexity theory may be related to a question raised by M. Kontsevich and D. Zagier in [KZ]. They defined a *period* as a complex number whose real and imaginary part are values of absolutely convergent integrals of rational functions with rational coefficients over domains of $\mathbb{R}^n$ given by polynomial (in)equalities with rational coefficients. Problem 3 in [KZ] is to *produce at least one number which is* not *a period.* This is the analogue for *periods* of Liouville's Theorem for *algebraic numbers.* A more difficult question is to prove that specific numbers like

$$e, \quad 1/\pi, \quad \gamma$$

(where γ is Euler's constant) are not periods. Since every algebraic number is a period, a number which is not a period is transcendental.

Another important tool missing for transcendence proofs in higher dimension is a Schwarz Lemma in several variables. The following conjecture is suggested in [W1, §5]. For a finite subset Σ of $\mathbb{C}^n$ and a positive integer t, denote by $\omega_t(\Sigma)$ the least total degree of a non-zero polynomial P in $\mathbb{C}[z_1, \ldots, z_n]$ which vanishes on Σ with multiplicity at least t,

$$\left(\frac{\partial}{\partial z_1}\right)^{\tau_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\tau_n} P(z) = 0,$$

for any $z \in \Sigma$ and $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{N}^n$ with $\tau_1 + \cdots + \tau_n < t$. Further, when f is an analytic function in an open neighborhood of a closed polydisc $|z_i| \le r$ $(1 \le i \le n)$ in $\mathbb{C}^n$, denote by $\Theta_f(r)$ the average mass of the set of zeroes of f in that polydisc (see [BL]).

Conjecture 3.14. *Let Σ be a finite subset of $\mathbb{C}^n$, and ε be a positive number. There exists a positive number $r_0(\Sigma, \varepsilon)$ such that, for any positive integer t and any entire function f in $\mathbb{C}^n$ which vanishes on Σ with multiplicity $\ge t$,*

$$\Theta_f(r) \ge \omega_t(\Sigma) - t\varepsilon \quad \textit{for} \quad r \ge r_0(\Sigma, \varepsilon).$$

The next question is to compute $r_0(\Sigma, \varepsilon)$. One may expect that for Σ a chunk of a finitely generated subgroup of $\mathbb{C}^n$, say

$$\Sigma = \left\{ s_1 y_1 + \cdots + s_\ell y_\ell \colon (s_1, \ldots, s_\ell) \in \mathbb{Z}^\ell,\ |s_j| \le S\ (1 \le j \le \ell) \right\} \subset \mathbb{C}^n,$$

an admissible value for the number $r_0(\Sigma, \varepsilon)$ will depend only on ε, $y_1, \ldots, y_\ell$, but not on S. This would have interesting applications, especially in the special case $\ell = n + 1$.

Finally we refer to [Chu] for a connexion between the numbers $\omega_t(S)$ and Nagata's work on Hilbert's 14th Problem.

3.2. Multiple Zeta Values. Many recent papers (see for instance [C]) are devoted to the study of algebraic relations among "multiple zeta values",

$$\sum_{n_1 > \cdots > n_k \ge 1} n_1^{-s_1} \cdots n_k^{-s_k},$$

where $(s_1, \ldots, s_k)$ is a k-tuple of positive integers with $s_1 \geq 2$. The main Diophantine conjecture, suggested by the work of D. Zagier, A. B. Goncharov, M. Kontsevich, M. Petitot, Minh Hoang Ngoc, K. Ihara, M. Kaneko and others (see [Z], [C] and [Zu]), is that all such relations can be deduced from the linear and quadratic ones arising in the *shuffle* and *stuffle* products (including the relations arising from the study of divergent series — see [W7] for instance). For $p \geq 2$, let $\mathfrak{Z}_p$ denote the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by the real numbers $\zeta(\underline{s})$ satisfying $\underline{s} = (s_1, \ldots, s_k)$ and $s_1 + \cdots + s_k = p$. Set $\mathfrak{Z}_0 = \mathbb{Q}$ and $\mathfrak{Z}_1 = \{0\}$. Then the $\mathbb{Q}$-subspace $\mathfrak{Z}$ spanned by all $\mathfrak{Z}_p$, $p \geq 0$, is a subalgebra of $\mathbb{R}$, and part of the Diophantine conjecture states

Conjecture 3.15 (Goncharov)**.** *As a $\mathbb{Q}$-algebra, $\mathfrak{Z}$ is the direct sum of $\mathfrak{Z}_p$ for $p \geq 0$.*

In other terms, all algebraic relations should be consequences of homogeneous ones, involving values $\zeta(\underline{s})$ with different $\underline{s}$ but with the same weight $s_1 + \cdots + s_k$.

Assuming conjecture 3.15, the question of *algebraic independence* of the numbers $\zeta(\underline{s})$ is reduced to the question of *linear independence* of the same numbers. The conjectural situation is described by the next conjecture of Zagier [Z] on the dimension d_p of the $\mathbb{Q}$-vector space $\mathfrak{Z}_p$.

Conjecture 3.16 (Zagier)**.** *For $p \geq 3$,*

$$d_p = d_{p-2} + d_{p-3},$$

with $d_0 = 1$, $d_1 = 0$, $d_2 = 1$.

That the actual dimensions of the spaces $\mathfrak{Z}_p$ are bounded above by the integers which are defined inductively in Conjecture 3.16 has been proved by T. Terasoma in [T], who expresses multiple zeta values as periods of relative cohomologies and uses mixed Tate Hodge structures (see also the work of A. G. Goncharov referred to in [T]). Further work on Conjectures 3.15 and 3.16 is due to J. Écalle. In case $k = 1$ (values of the Riemann zeta function) the conjecture is

Conjecture 3.17. *The numbers π, $\zeta(3)$, $\zeta(5)$, $\ldots$, $\zeta(2n+1)$, $\ldots$ are algebraically independent over $\mathbb{Q}$.*

So far the only known results on this topic [Fis] are:

- $\zeta(2n)$ *is transcendental for* $n \geq 1$ (because π is transcendental and $\zeta(2n)\pi^{-2n} \in \mathbb{Q}$),
- $\zeta(3)$ *is irrational* (Apéry, 1978),

  and
- *For any $\varepsilon > 0$ the $\mathbb{Q}$-vector space spanned by the $n+1$ numbers* 1, $\zeta(3)$, $\zeta(5)$, $\ldots$, $\zeta(2n+1)$ *has dimension*

$$\geq \frac{1-\varepsilon}{1+\log 2} \log n$$

  for $n \geq n_0(\varepsilon)$ (see [Riv] and [BR]). For instance infinitely many of these numbers $\zeta(2n+1)$ $(n \geq 1)$ are irrational. W. Zudilin proved that at least one of the four numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.

Further, more recent results are due to T. Rivoal and W. Zudilin. For instance, in a joint paper they have proved that infinitely many numbers among

$$\sum_{n\geq 1}\frac{(-1)^n}{(2n+1)^{2s}}\qquad (s\in\mathbb{Z},\quad s\geq 1)$$

are irrational, but, as pointed out earlier, the irrationality of Catalan's constant G — see (2.13) — is still an open problem.

It may turn out to be more efficient to work with a larger set of numbers, including special values of multiple polylogarithms,

$$\sum_{n_1>\cdots>n_k\geq 1}\frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}.$$

An interesting set of points $\underline{z}=(z_1,\ldots,z_k)$ to consider is the set of k-tuples consisting of roots of unity. The function of a single variable,

$$\mathrm{Li}_{\underline{s}}(z)=\sum_{n_1>\cdots>n_k\geq 1}\frac{z^{n_1}}{n_1^{s_1}\cdots n_k^{s_k}},$$

is worth of study from a Diophantine point of view. For instance, Catalan's constant mentioned above is the imaginary part of $\mathrm{Li}_2(i)$,

$$\mathrm{Li}_2(i)=\sum_{n\geq 1}\frac{i^n}{n^2}=-\frac{1}{8}\zeta(2)+iG.$$

Also no proof for the irrationality of the numbers

$$\zeta(4,2)=\sum_{n>k\geq 1}\frac{1}{n^4k^2}=\zeta(3)^2-\frac{4\pi^6}{2835},$$

$$\mathrm{Li}_2(1/2)=\sum_{n\geq 1}\frac{1}{n^2 2^n}=\frac{\pi^2}{12}-\frac{1}{2}(\log 2)^2$$

and

$$\text{(Ramanujan)}\quad \mathrm{Li}_{2,1}(1/2)=\sum_{n\geq k\geq 1}\frac{1}{2^n n^2 k}=\zeta(3)-\frac{1}{12}\pi^2\log 2,$$

is known so far.

According to P. Bundschuh [Bun], the transcendence of the numbers

$$\sum_{n=2}^{\infty}\frac{1}{n^s-1}$$

for even $s\geq 4$ is a consequence of Schanuel's Conjecture 3.1. For $s=2$ the sum is $3/4$, and for $s=4$ the value is $(7/8)-(\pi/4)\coth\pi$, which is a transcendental number since π and e^π are algebraically independent over $\mathbb{Q}$ (Yu. V. Nesterenko [NP]).

Nothing is known about the arithmetic nature of the values of the Riemann zeta function at rational or algebraic points which are not integers.

3.3. Gamma, Elliptic, Modular, G and E-Functions. On the one hand, the transcendence problem of the values of the Euler beta function at rational points was solved as early as 1940, by Th. Schneider. *For any rational numbers a and b which are not integers and such that $a+b$ is not an integer, the number*

$$\mathrm{B}(a,\, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental. On the other hand, transcendence results for the values of the gamma function itself are not so precise: apart from G. V. Chudnovsky's results, which imply the transcendence of $\Gamma(1/3)$ and $\Gamma(1/4)$ (and Lindemann's result on the transcendence of π which implies that $\Gamma(1/2) = \sqrt{\pi}$ is also transcendental), not much is known. For instance, as we said earlier, there is no proof so far that $\Gamma(1/5)$ is transcendental. This is because the Fermat curve of exponent 5, viz. $x^5 + y^5 = 1$, has genus 2. Its Jacobian is an Abelian surface, and the algebraic independence results known for elliptic curves like $x^3 + y^3 = 1$ and $x^4 + y^4 = 1$ which were sufficient for dealing with $\Gamma(1/3)$ and $\Gamma(1/4)$, are not yet known for Abelian varieties (see [Grin]).

One might expect that Nesterenko's results (see [NP, Chap. 3]) on the algebraic independence of π, $\Gamma(1/4)$, e^{π} and of π, $\Gamma(1/3)$, $e^{\pi\sqrt{3}}$ should be extended as follows.

Conjecture 3.18. *At least three of the four numbers*

$$\pi,\, \Gamma(1/5),\, \Gamma(2/5),\, e^{\pi\sqrt{5}}$$

are algebraically independent over $\mathbb{Q}$.

So the challenge is to extend Nesterenko's results on modular functions in one variable (and elliptic curves) to several variables (and Abelian varieties).

This may be one of the easiest questions to answer on this topic (but it is still open). But one may ask for a general statement which would produce all algebraic relations between gamma values at rational points. Here is a conjecture of Rohrlich [La4]. Define

$$G(z) = \frac{1}{\sqrt{2\pi}}\Gamma(z).$$

According to the multiplication theorem of Gauss and Legendre [WW, §12.15], for each positive integer N and for each complex number x such that $Nx \not\equiv 0 \pmod{\mathbb{Z}}$,

$$\prod_{i=0}^{N-1} G\left(x + \frac{i}{N}\right) = N^{(1/2)-Nx} G(Nx).$$

The gamma function has no zero and defines a map from $\mathbb{C} \setminus \mathbb{Z}$ to $\mathbb{C}^{\times}$. We restrict that function to $\mathbb{Q} \setminus \mathbb{Z}$ and we compose it with the canonical map $\mathbb{C}^{\times} \to \mathbb{C}^{\times}/\overline{\mathbb{Q}}^{\times}$ which amounts to considering its values modulo the algebraic numbers. The composite map has period 1, and the resulting mapping,

$$\overline{G}\colon \frac{\mathbb{Q}}{\mathbb{Z}} \setminus \{0\} \to \frac{\mathbb{C}^{\times}}{\overline{\mathbb{Q}}^{\times}},$$

is an odd *distribution* on $(\mathbb{Q}/\mathbb{Z}) \setminus \{0\}$,

$$\prod_{i=0}^{N-1} \overline{G}\left(x+\frac{i}{N}\right) = \overline{G}(Nx) \quad \text{for} \quad x \in \frac{\mathbb{Q}}{\mathbb{Z}} \setminus \{0\} \quad \text{and} \quad \overline{G}(-x) = \overline{G}(x)^{-1}.$$

Rohrlich's Conjecture ([La4], [La6, Chap. II, Appendix, p. 66]) asserts that

Conjecture 3.19 (Rohrlich)**.** *$\overline{G}$ is a universal odd distribution with values in groups where multiplication by 2 is invertible.*

In other terms, any multiplicative relation between gamma values at rational points

$$\pi^{b/2} \prod_{a \in \mathbb{Q}} \Gamma(a)^{m_a} \in \overline{\mathbb{Q}}$$

with b and m_a in $\mathbb{Z}$ can be derived for the standard relations satisfied by the gamma function. This leads to the question whether the distribution relations, the oddness relation and the functional equations of the gamma function generate the ideal over $\overline{\mathbb{Q}}$ of all algebraic relations among the values of $G(x)$ for $x \in \mathbb{Q}$.

In [NP] (Chap. 3, §1, Conjecture 1.11) Yu. V. Nesterenko proposed another conjectural extension of his algebraic independence result on Eisenstein series of weight 2, 4 and 6:

$$P(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

$$Q(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$R(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n} = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n.$$

Conjecture 3.20 (Nesterenko)**.** *Let $\tau \in \mathbb{C}$ have positive imaginary part. Assume that τ is not quadratic. Set $q = e^{2i\pi\tau}$. Then at least 4 of the 5 numbers*

$$\tau,\ q,\ P(q),\ Q(q),\ R(q)$$

are algebraically independent.

Finally we remark that essentially nothing is known about the arithmetic nature of the values of either the beta or the gamma function at algebraic irrational points.

A wide range of open problems in transcendental number theory, including not only Schanuel's Conjecture 3.1 and Rohrlich's Conjecture 3.19 on the values of the gamma function, but also a conjecture of Grothendieck on the periods of an algebraic variety (see [La1, Chap. IV, Historical Note], [La2, p. 650], [A1, p. 6] and [Ch, §3]), are special cases of very general conjectures due to Y. André [A2], which deal with periods of mixed motives. A discussion of André's conjectures for certain 1-motives related to the products of elliptic curves and their connexions with elliptic and modular functions is given in [Ber]. Here is a special case of the *elliptico-toric Conjecture* in [Ber].

Conjecture 3.21 (Bertolin). *Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be pairwise non isogeneous elliptic curves with modular invariants $j(\mathcal{E}_h)$. For $h = 1, \ldots, n$, let ω_{1h}, ω_{2h} be a pair of fundamental periods of $\wp_h$ where η_{1h}, η_{2h} are the associated quasi-periods, P_{ih} points on $\mathcal{E}_h(\mathbb{C})$ and p_{ih} (resp. d_{ih}) elliptic integrals of the first (resp. second) kind associated to P_{ih}. Define $\kappa_h = [k_h : \mathbb{Q}]$ and let d_h be the dimension of the k_h-subspace of $\mathbb{C}/(k_h\omega_{1h} + k_h\omega_{2h})$ spanned by $p_{1h}, \ldots, p_{r_h h}$. Then the transcendence degree of the field*

$$\mathbb{Q}\Big(\big\{j(\mathcal{E}_h),\, \omega_{1h},\, \omega_{2h},\, \eta_{1h},\, \eta_{2h},\, P_{ih},\, p_{ih},\, d_{ih}\big\}_{\substack{1\le i\le r_h\\ 1\le h\le n}}\Big)$$

is at least

$$2\sum_{h=1}^{n} d_h + 4\sum_{h=1}^{n}\kappa_h^{-1} - n + 1.$$

A new approach to Grothendieck's Conjecture via Siegel's G-functions was introduced in [A1, Chap. IX]. A development of this method led Y. André to his conjecture on the special points on Shimura varieties [A1, Chap. X, §4], which gave rise to the André–Oort Conjecture [O] (for a discussion of this topic, including a precise definition of "Hodge type", together with relevant references, see [Co]).

Conjecture 3.22 (André–Oort). *Let $\mathcal{A}_g(\mathbb{C})$ denote the moduli space of principally polarized complex Abelian varieties of dimension g. Let Z be an irreducible algebraic subvariety of $\mathcal{A}_g(\mathbb{C})$ such that the complex multiplication points on Z are dense for the Zariski topology. Then Z is a subvariety of $\mathcal{A}_g(\mathbb{C})$ of Hodge type.*

Conjecture 3.22 is a far-reaching generalization of Schneider's Theorem on the transcendence of $j(\tau)$, where j is the modular invariant and τ an algebraic point in the Poincaré upper half plane $\mathfrak{H}$, which is not imaginary quadratic ([Schn, Chap. II, §4, Th. 17]). We also mention a related conjecture of D. Bertrand (see [NP, Chap. 1, §4, Conjecture 4.3]) which may be viewed as a nonholomorphic analogue of Schneider's result and which would answer the following question raised by N. Katz.

Question. *Assume that a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in $\mathbb{C}$ has algebraic invariants $g_2(L)$ and $g_3(L)$ and no complex multiplication. Does this implies that the number*

$$G_2^*(L) = \lim_{s\to 0}\sum_{\omega\in L\setminus\{0\}}\omega^{-2}|\omega|^{-s}$$

is transcendental?

Many open transcendence problems dealing with elliptic functions are consequences of André's conjectures (see [Ber]), most of which are likely to be very hard. The next one, which is still open, may be easier, since a number of partial results are already known, as a result of the work of G. V. Chudnovsky and others (see [Grin]).

Conjecture 3.23. *Given an elliptic curve with Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$, a non-zero period ω, the associated quasi-period η of the zeta function and a complex number u which is not a pole of $\wp$,*

$$\operatorname{trdeg}\mathbb{Q}\big(g_2,\, g_3,\, \pi/\omega,\, \wp(u),\, \zeta(u) - (\eta/\omega)u\big) \ge 2.$$

Given a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in $\mathbb{C}$ with invariants $g_2(L)$ and $g_3(L)$, denote by $\eta_i = \zeta_L(z+\omega_i) - \zeta_L(z)$ $(i = 1, 2)$ the corresponding fundamental quasi-periods of the Weierstrass zeta function. Conjecture 3.23 implies that the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}\big(g_2(L), g_3(L), \omega_1, \omega_2, \eta_1, \eta_2\big)$ is at least 2.This would be optimal in the CM case, while in the non CM case, we expect it to be ≥ 4. These lower bounds are given by the period conjecture of Grothendieck applied to an elliptic curve.

According to [Di2, Conjectures 1 and 2, p. 187], the following special case of Conjecture 3.23 can be stated in two equivalent ways: either in terms of values of elliptic functions, or in terms of values of Eisenstein series E_2, E_4 and E_6 (which are P, Q and R in Ramanujan's notation).

Conjecture. *For any lattice L in $\mathbb{C}$ without complex multiplication and for any non-zero period ω of L,*

$$\operatorname{trdeg} \mathbb{Q}\big(g_2(L), g_3(L), \omega/\sqrt{\pi}, \eta/\sqrt{\pi}\big) \geq 2.$$

Conjecture. *For any $\tau \in \mathfrak{H}$ which is not imaginary quadratic,*

$$\operatorname{trdeg} \mathbb{Q}\big(\pi E_2(\tau), \pi^2 E_4(\tau), \pi^3 E_6(\tau)\big) \geq 2.$$

Moreover, each of these two statements implies the following one, which is stronger than one of Lang's conjectures ([La2, p. 652]).

Conjecture. *For any $\tau \in \mathfrak{H}$ which is not imaginary quadratic,*

$$\operatorname{trdeg} \mathbb{Q}\big(j(\tau), j'(\tau), j''(\tau)\big) \geq 2.$$

Further related open problems are proposed by G. Diaz in [Di1] and [Di2], in connexion with conjectures due to D. Bertrand on the values of the modular function $J(q)$, where $j(\tau) = J(e^{2i\pi\tau})$ (see [Bert2] as well as [NP, Chap. 1, §4 and Chap. 2, §4]).

Conjecture 3.24 (Bertrand)**.** *Let $q_1, \ldots, q_n$ be non-zero algebraic numbers in the unit open disc such that the $3n$ numbers*

$$J(q_i),\ DJ(q_i),\ D^2J(q_i) \quad (i = 1, \ldots, n)$$

are algebraically dependent over $\mathbb{Q}$. Then there exist two indices $i \neq j$ $(1 \leq i \leq n$, $1 \leq j \leq n)$ such that q_i and q_j are multiplicatively dependent.

Conjecture 3.25 (Bertrand)**.** *Let q_1 and q_2 be two non-zero algebraic numbers in the unit open disc. Suppose that there is an irreducible element $P \in \mathbb{Q}[X, Y]$ such that*

$$P\big(J(q_1), J(q_2)\big) = 0.$$

Then there exist a constant c and a positive integer s such that $P = c\Phi_s$, where Φ_s is the modular polynomial of level s. Moreover q_1 and q_2 are multiplicatively dependent.

Among Siegel's G-functions are the algebraic functions. Transcendence methods produce some information, in particular in connexion with Hilbert's Irreducibility Theorem. Let $f \in \mathbb{Z}[X, Y]$ be a polynomial which is irreducible in $\mathbb{Q}(X)[Y]$. According to Hilbert's Irreducibility Theorem, the set of positive integers n such that

$P(n, Y)$ is irreducible in $\mathbb{Q}[Y]$ is infinite. Effective upper bounds for an admissible value for n have been studied (especially by M. Fried, P. Dèbes and U. Zannier), but do not yet answer the next question.

Question 3.26. *Is there such a bound depending polynomially on the degree and height of P?*

Such questions are also related to the *Galois inverse Problem* [Se].

Also the polylogarithms

$$\mathrm{Li}_s(z) = \sum_{n\geq 1} \frac{z^m}{n^s},$$

where s is a positive integer, are G-functions; unfortunately no way has yet been found to use the Siegel–Shidlovskii method to prove the irrationality of the values of the Riemann zeta function ([FN, Chap. 5, §7, p. 247]).

With G-functions, the other class of analytic functions introduced by C. L. Siegel in 1929 is the class of E-functions, which includes the hypergeometric ones. One main open question is the arithmetic nature of the values at algebraic points of hypergeometric functions with *algebraic* parameters,

$${}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix}\middle| z\right) = \sum_{n\geq 0} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \cdot \frac{z^n}{n!},$$

defined for $|z| < 1$ and $\gamma \notin \{0, -1, -2, \dots\}$.

In 1949, C. L. Siegel ([Si2, Chap. 2, §9, p. 54 and 58]; see also [FS, p. 62] and [FN, Chap. 5, §1.2]) asked whether *any E-function satisfying a linear differential equation with coefficients in $\mathbb{C}(z)$ can be expressed as a polynomial in z and a finite number of hypergeometric E-functions or functions obtained from them by a change of variables of the form $z \mapsto \gamma z$ with algebraic γ's?*

Finally, we quote from [W4]: a folklore conjecture is that the zeroes of the Riemann zeta function (say their imaginary parts, assuming it > 0) are algebraically independent. As suggested by J-P. Serre, one might also be tempted to consider

- The eigenvalues of the zeroes of the hyperbolic Laplacian in the upper half plane modulo $\mathrm{SL}_2(\mathbb{Z})$ (i. e., to study the algebraic independence of the zeroes of the Selberg zeta function).
- The eigenvalues of the Hecke operators acting on the corresponding eigenfunctions (Maass forms).

3.4. Fibonacci and Miscellanea. Many further open problems arise in transcendental number theory. An intriguing question is to study the arithmetic nature of real numbers given in terms of power series involving the Fibonacci sequence

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \quad F_1 = 1.$$

Several results are due to P. Erdős, R. André-Jeannin, C. Badea, J. Sándor, P. Bundschuh, A. Pethő, P. G. Becker, T. Töpfer, D. Duverney, Ku. et Ke. Nishioka, I. Shiokawa and T. Tanaka. It is known that the number

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1$$

is rational, while

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1-\sqrt{5}}{2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}+1} = \frac{\sqrt{5}}{2}$$

are irrational algebraic numbers. Each of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n + F_{n+2}} \quad \text{and} \quad \sum_{n\geq 1} \frac{1}{F_1 F_2 \cdots F_n}$$

is irrational, but it is not known whether they are algebraic or transcendental. The numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n^2},$$

$$\sum_{n=1}^{\infty} \frac{n}{F_{2n}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2^n-1}+F_{2^n+1}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{F_{2^n+1}}$$

are all transcendental (further results of algebraic independence are known). The first challenge here is to formulate a conjectural statement which would give a satisfactory description of the situation.

There is a similar situation for infinite sums $\sum_n f(n)$ where f is a rational function [Ti4]. While

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

and

$$\sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0$$

are rational numbers, the sums

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2, \qquad \sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3},$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{2\pi}{e^{\pi}-e^{-\pi}}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(6n+6)} = \frac{1}{4320}(192 \log 2 - 81 \log 3 - 7\pi\sqrt{3})$$

are transcendental. The simplest example of the Euler sums $\sum_n n^{-s}$ (see Section 3.2) illustrates the difficulty of the question. Here again, even a sufficiently general conjecture is missing. One may remark that there is no known algebraic irrational number of the form

$$\sum_{\substack{n\geq 0\\ Q(n)\neq 0}} \frac{P(n)}{Q(n)},$$

where P and Q are non-zero polynomials having rational coefficients and $\deg Q \geq 2 + \deg P$.

The arithmetic study of the values of power series suggests many open problems. We shall only mention a few of them.

The next question is due to K. Mahler [M3].

Question 3.27 (Mahler)**.** *Are there entire transcendental functions $f(z)$ such that if x is a Liouville number then so is $f(x)$?*

The study of integral valued entire functions gives rise to several open problems; we quote only one of them which arose in the work of D. W. Masser and F. Gramain on entire functions f of one complex variable which map the ring of Gaussian integers $\mathbb{Z}[i]$ into itself. The initial question (namely to derive an analogue of Pólya's Theorem in this setting) has been solved by F. Gramain in [Gr] (following previous work of Fukasawa, Gelfond, Gruman and Masser). *If f is not a polynomial, then*

$$\limsup_{r\to\infty} \frac{1}{r^2} \log |f|_r \geq \frac{\pi}{2e}\cdot$$

Here,

$$|f|_r = \max_{|z|=r} |f(z)|.$$

Preliminary works on this estimate gave rise to the following problem, which is still unsolved. For each integer $k \geq 2$, let A_k be the minimal area of a closed disk in $\mathbb{R}^2$ containing at least k points of $\mathbb{Z}^2$, and for $n \geq 2$ define

$$\delta_n = -\log n + \sum_{k=2}^{n} \frac{1}{A_k}.$$

The limit $\delta = \lim_{n\to\infty} \delta_n$ exists (it is an analogue in dimension 2 of the Euler constant), and the best known estimates for it are [GW]

$$1.811\ldots < \delta < 1.897\ldots$$

(see also [Fi]). F. Gramain conjectures that

$$\delta = 1 + \frac{4}{\pi}\bigl(\gamma L(1) + L'(1)\bigr),$$

where γ is Euler's constant and

$$L(s) = \sum_{n\geq 0} (-1)^n (2n+1)^{-s}$$

is the L function of the quadratic field $\mathbb{Q}(i)$ (Dirichlet beta function). Since

$$L(1) = \frac{\pi}{4} \quad \text{and}$$

$$L'(1) = \sum_{n\geq 0} (-1)^{n+1} \cdot \frac{\log(2n+1)}{2n+1} = \frac{\pi}{4}\bigl(3\log\pi + 2\log 2 + \gamma - 4\log\Gamma(1/4)\bigr),$$

Gramain's conjecture is equivalent to

$$\delta = 1 + 3\log\pi + 2\log 2 + 2\gamma - 4\log\Gamma(1/4) = 1.822825\dots$$

Other problems related to the lattice $\mathbb{Z}[i]$ are described in the section "On the borders of geometry and arithmetic" of [Sie1].

4. Heights

For a non-zero polynomial $f \in \mathbb{C}[X]$ of degree d,

$$f(X) = a_0X^d + a_1X^{d-1} + \cdots + a_{d-1}X + a_d = a_0\prod_{i=1}^{d}(X - \alpha_i),$$

define its *usual height* by

$$H(f) = \max\{|a_0|, \dots, |a_d|\}$$

and its *Mahler's measure* by

$$M(f) = |a_0|\prod_{i=1}^{d}\max\{1, |\alpha_i|\} = \exp\left(\int_0^1 \log|f(e^{2i\pi t})|\,dt\right).$$

The equality between these two formulae follows from Jensen's formula (see [M2, Chap. I, §7], as well as [W6, Chap. 3] and [S]; the latter includes an extension to several variables).

When α is an algebraic number with minimal polynomial $f \in \mathbb{Z}[X]$, define its *Mahler's measure* by $M(\alpha) = M(f)$ and its *usual height* by $H(\alpha) = H(f)$. Further, if α has degree d, define its *logarithmic height* as

$$h(\alpha) = \frac{1}{d}\log M(\alpha).$$

Furthermore, if $\alpha_1, \dots, \alpha_d$ are the complex roots of f (also called the *complex conjugates* of α), then the *house* of α is

$$\overline{|\alpha|} = \max\{|\alpha_1|, \dots, |\alpha_d|\}.$$

The height of an algebraic number is the prototype of a whole collection of height, like the height of projective point $(1\colon \alpha_1\colon \cdots\colon \alpha_n) \in \mathbb{P}_n$ which is denoted by $h(1\colon \alpha_1\colon \cdots\colon \alpha_n)$ (see for instance [W6, §3.2]) and the height of a subvariety $\hat{h}(V)$ (see for instance [D1] and [D2]).

Further, if $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ is a n-tuple of multiplicatively independent algebraic numbers, $\omega(\underline{\alpha})$ denotes the minimum degree of a non-zero polynomial in $\mathbb{Q}[X_1, \dots, X_n]$ which vanishes at $\underline{\alpha}$.

A side remark is that Mahler's measure of a polynomial in a single variable with algebraic coefficients is an algebraic number. The situation is much more intricate for polynomials in several variables and suggests to further very interesting open problems [Boy1], [Boy2].

4.1. Lehmer's Problem. The smallest known value for $dh(\alpha)$, which was found in 1933 by D. H. Lehmer, is $\log \alpha_0 = 0.162357\ldots$, where $\alpha_0 = 1.176280\ldots$ is the real root[5] of the degree 10 polynomial

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1.$$

D. H. Lehmer asked whether it is true that for every positive ε there exists an algebraic integer α for which $1 < M(\alpha) < 1 + \varepsilon$?

Conjecture 4.1 (Lehmer's Problem)**.** *There exists a positive absolute constant c such that, for any non-zero algebraic number α which is not a root of unity,*

$$M(\alpha) \geq 1 + c.$$

Equivalently, there exists a positive absolute constant c such that, for any non-zero algebraic number α of degree at most d which is not a root of unity,

$$h(\alpha) \geq \frac{c}{d}.$$

Since $h(\alpha) \leq \log \lceil\overline{\alpha}\rceil$, the following statement [SZ] is a weaker assertion than Conjecture 4.1.

Conjecture 4.2 (Schinzel–Zassenhaus)**.** *There exists an absolute constant $c > 0$ such that, for any non-zero algebraic integer of degree d which is not a root of unity,*

$$\lceil\overline{\alpha}\rceil \geq 1 + \frac{c}{d}.$$

Lehmer's Problem is related to the multiplicative group $\mathbb{G}_{\mathrm{m}}$. Generalizations to $\mathbb{G}_{\mathrm{m}}^n$ have been considered by many authors (see for instance [Bert1] and [Sch2]). In [AD1, Conjecture 1.4], F. Amoroso and S. David extend Lehmer's Problem 4.1 to simultaneous approximation.

Conjecture 4.3 (Amoroso–David)**.** *For each positive integer $n \geq 1$ there exists a positive number $c(n)$ having the following property. Let $\alpha_1, \ldots, \alpha_n$ be multiplicatively independent algebraic numbers. Define $D = [\mathbb{Q}(\alpha_1, \ldots, \alpha_n) : \mathbb{Q}]$. Then*

$$\prod_{i=1}^{n} h(\alpha_i) \geq \frac{c(n)}{D}.$$

The next statement ([AD1, Conjecture 1.3] and [AD2, Conjecture 1.3]) is stronger.

Conjecture 4.4 (Amoroso–David)**.** *For each positive integer $n \geq 1$ there exists a positive number $c(n)$ such that, if $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is a n-tuple of multiplicatively independent algebraic numbers, then*

$$h(1\colon \alpha_1 \colon \cdots : \alpha_n) \geq \frac{c(n)}{\omega(\underline{\alpha})}.$$

[5] Further properties of this smallest known Salem number are described by D. Zagier in his paper *Special values and functional equations of polylogarithms*, Appendix A of "Structural properties of polylogarithms", ed. L. Lewin, Mathematical Surveys and Monographs, vol. 37, Amer. Math. Soc. 1991, pp. 377–400.

Many open questions are related to the height of subvarieties [D1], [D2]. The next one, dealing with the height of subvarieties of $\mathbb{G}_{\mathrm{m}}^n$ and proposed by F. Amoroso and S. David in [AD2, Conjecture 1.4] (see also Conjecture 1.5 of [AD2], which is due to S. David and P. Philippon [DP]), is more general than Conjecture 4.4.

Conjecture 4.5 (Amoroso–David)**.** *For each integer $n \geq 1$ there exists a positive constant $c(n)$ such that, for any algebraic subvariety V of $\mathbb{G}_{\mathrm{m}}^n$ which is defined over $\mathbb{Q}$, which is $\mathbb{Q}$-irreducible, and which is not a union of translates of algebraic subgroups by torsion points,*

$$\hat{h}(V) \geq c(n) \deg(V)^{(s-\dim V-1)/(s-\dim V)},$$

where s is the dimension of the smallest algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^n$ containing V.

Let V be an open subset of $\mathbb{C}$. The *Lehmer–Langevin constant* of V is defined as

$$L(V) = \inf M(\alpha)^{1/[\mathbb{Q}(\alpha):\mathbb{Q}]},$$

where α ranges over the set of non-zero and non-cyclotomic algebraic numbers, α, lying with all their conjugates outside of V. It was proved by M. Langevin in 1985 that $L(V) > 1$ as soon as V contains a point on the unit circle $|z| = 1$.

Problem 4.6. *For $\theta \in (0, \pi)$, define*

$$V_\theta = \{re^{it} \colon r > 0,\ |t| > \theta\}.$$

Compute $L(V_\theta)$ in terms of θ.

The solution is only known for a very few values of θ. In 1995 G. Rhin and C. Smyth [RS] computed $L(V_\theta)$ for nine values of θ, including

$$L(V_{\pi/2}) = 1.12\ldots$$

In a different direction, an analogue of Lehmer's Problem has been proposed for elliptic curves, and more generally for Abelian varieties. Here is Conjecture 1.4 of [DH]. Let A be an Abelian variety defined over a number field K and equipped with a symmetric ample line bundle $\mathcal{L}$. For any $P \in A(\overline{\mathbb{Q}})$, define

$$\delta(Q) = \min \deg(V)^{1/\operatorname{codim}(V)},$$

where V ranges over the proper subvarieties of A, defined over K, K-irreducible and containing Q, while $\deg(V)$ is the degree of V with respect to $\mathcal{L}$. Also denote by $\hat{h}_{\mathcal{L}}$ the Néron–Tate canonical height on $A(\overline{\mathbb{Q}})$ associated to $\mathcal{L}$.

Conjecture 4.7 (David–Hindry)**.** *There exists a positive constant c depending only on A and $\mathcal{L}$, such that for any $P \in A(\overline{\mathbb{Q}})$ which has infinite order modulo any Abelian subvariety,*

$$\hat{h}_{\mathcal{L}}(P) \geq c\delta(P)^{-1}.$$

An extension of Conjecture 4.7 to linearly independent tuples is also stated in [DH, Conjecture 1.6].

The dependence on A of these "constants" also suggests interesting questions. Take an elliptic curve E and consider the Néron–Tate height $\hat{h}(P)$ of a nontorsion rational point on a number field K. Several invariants are related to E: the modular invariant j_E, the discriminant Δ_E and Faltings height $h(E)$. S. Lang conjectured that

$$\hat{h}(P) \geq c(K) \max\{1,\, h(E)\},$$

while S. Lang ([La5, p. 92]) and J. Silverman ([Sil, Chap. VIII, §10, Conjecture 9.9]) conjecture that

$$\hat{h}(P) \geq c(K) \max\{\log |N_{K/\mathbb{Q}}(\Delta_E)|,\, h(j_E)\}.$$

Partial results are known (J. Silverman, M. Hindry and J. Silverman, S. David), but the conjecture is not yet proved.

There is another Abelian question related to Mahler's measure. According to D. A. Lind, Lehmer's Problem is known to be equivalent to the existence of a continuous endomorphism of the infinite torus $(\mathbb{R}/\mathbb{Z})^{\mathbb{Z}}$ with finite entropy. A similar question has been asked by P. D'Ambros, G. Everest, R. Miles and T. Ward [AEMW] for elliptic curves, and it can be extended to Abelian varieties, and more generally to commutative algebraic groups.

4.2. Wirsing–Schmidt Conjecture. According to Dirichlet's box principle, for any irrational, real number θ there is an infinite set of rational numbers p/q with $q > 0$ such that

$$\left|\theta - \frac{p}{q}\right| \leq \frac{1}{q^2}. \tag{4.8}$$

There are several extensions of this result. For the first one, we write (4.8) as $|q\theta - p| < 1/q$ and we replace $q\theta - p$ by $P(\theta)$ for some polynomial P.

Lemma 4.9. *Let θ be a real number, d and H be positive integers. There exists a non-zero polynomial $P \in \mathbb{Z}[X]$, of degree less than or equal to d and usual height less than or equal to H such that*

$$|P(\theta)| \leq cH^{-d},$$

where $c = 1 + |\theta| + \cdots + |\theta|^d$.

There is no assumption on θ, but if θ is algebraic of degree $\leq d$, then there is a trivial solution!

A similar result applies to complex numbers, and more generally when θ is replaced by a m-tuple $(\theta_1, \ldots, \theta_m) \in \mathbb{C}^m$ (see for instance [W6] Lemma 15.11). For simplicity, we only deal here with the easiest case.

Another extension of (4.8) that is interesting to consider is where p/q is replaced by an algebraic number of degree $\leq d$. If the polynomial P given by (4.9) has a single simple root γ close to θ, then

$$|\theta - \gamma| \leq c' H^{-d}$$

where c' depends only on θ and d. However, the root of P which is nearest γ may be a multiple root, and may be not unique. This occurs precisely when the first derivative P' of P has a small absolute value at θ. Dirichlet's box principle does not allow us to construct a polynomial P as in (4.8) with a *lower* bound for $|P'(\theta)|$.

However E. Wirsing [Wi] succeeded in proving the following theorem.

Theorem 4.10. *There exist two positive absolute constants, c and κ, such that, for any transcendental real number θ and any positive integer n, there are infinitely many algebraic numbers γ of degree $\le n$ for which*

$$|\theta-\gamma|\le cH(\gamma)^{-\kappa n}.$$

Wirsing himself obtained his estimate in 1960 with c replaced by $c(n,\,\varepsilon_n,\,\theta)$ and κn by $(n/2)+2-\varepsilon_n$, where $\varepsilon_n\to 0$ as $n\to\infty$. He conjectured that the exponent κn can be replaced by $n+1-\varepsilon$ [Wi]. For $n=2$, H. Davenport and W. M. Schmidt in 1967 reached the exponent 3 without ε. *For any transcendental real number θ, there exists a positive real number $c(\theta)$ such that the inequality*

$$|\theta-\gamma|\le c(\theta)H(\gamma)^{-3}$$

has infinitely many solutions γ with $[\mathbb{Q}(\gamma):\mathbb{Q}]\le 2$. A conjecture of Schmidt ([Sch1, Chap. VIII, §3]; see also [Bu1] and [Bu2]) asserts that (4.10) is valid when κn is replaced by $n+1$.

Conjecture 4.11 (Wirsing and Schmidt)**.** *For any positive integer n and any real number θ which is either transcendental or else is algebraic of degree greater than n, there exists a positive constant $c=c(n,\,\theta)$ with the following property: there exist infinitely many algebraic numbers γ of degree $\le n$ for which*

$$0<|\theta-\gamma|<cH(\gamma)^{-n-1}.$$

A third extension of (4.8) is the study of the simultaneous rational approximation of successive powers of a real number. Let $n\ge 2$ be an integer; denote by $\mathcal{E}_n$ the set of real numbers which are not algebraic of degree $\le n$. For $\xi\in\mathcal{E}_n$, let $\alpha_n(\xi)$ be the infimum of the set of real numbers α such that, for any sufficiently large real number X, there exists $(x_0,\,x_1,\,\dots,\,x_n)\in\mathbb{Z}^n$ satisfying

$$0<x_0\le X\quad\text{and}\quad\max_{1\le j\le n}|x_0\xi^j-x_j|\le X^{-1/\alpha}.$$

From Dirichlet's box principle one deduces $\alpha_n(\xi)\le n$ for any $\xi\in\mathcal{E}_n$ and any $n\ge 2$. Moreover, for any $n\ge 2$, the set of $\xi\in\mathcal{E}_n$ for which $\alpha_n(\xi)<n$ has Lebesgue measure zero. H. Davenport and W. M. Schmidt proved in [DS] that $\alpha_2(\xi)\ge\gamma$ for any $\xi\in\mathcal{E}_2$, where $\gamma=(1+\sqrt{5})/2=1.618\dots$ It was expected that $\alpha_2(\xi)$ would be equal to 2 for any $\xi\in\mathcal{E}_2$, but D. Roy [Ro9] has produced a $\xi\in\mathcal{E}_2$ for which $\alpha_2(\xi)=\gamma$, showing that the result of Davenport and Schmidt is optimal. This raises a number of open problems and suggests that we study the set

$$\mathcal{A}_n=\{\alpha_n(\xi)\colon \xi\in\mathcal{E}_n\}.$$

Recent results concerning the set $\mathcal{A}_2$, by Y. Bugeaud and M. Laurent, S. Fischler, indicate a structure like the Markoff spectrum. For further references on this topic, see [Bu3].

In Section 3 we considered problems of algebraic independence. In Section 2 we discussed questions related to measures of linear independence of logarithms of algebraic numbers. In Section 4 we introduced a notion of height. Connexions between these three topics arise from the study of simultaneous approximation of complex numbers by algebraic numbers (see for instance [W6, Chap. 15]). For a m-tuple $\underline{\gamma} = (\gamma_1, \ldots, \gamma_m)$ of algebraic numbers, we define

$$\mu(\underline{\gamma}) = [\mathbb{Q}(\underline{\gamma}) : \mathbb{Q}] \max_{1 \leq j \leq m} h(\gamma_j),$$

so that for $m = 1$ and $\gamma \in \overline{\mathbb{Q}}$, $\mu(\gamma) = \log M(\gamma)$.

So far, relations between simultaneous approximation and algebraic independence have only been established for small transcendence degrees. The missing link for large transcendence degrees is given by the next statement (see [W6, Conjecture 15.31], [Lau1, §4.2, Conjecture 5], [Lau2, Conjecture 1], [W5, Conjecture 2], as well as [Ro6, Conjectures 1 and 2]).

Conjecture 4.12. *Let $\underline{\theta} = (\theta_1, \ldots, \theta_m)$ be a m-tuple of complex numbers. Define*

$$t = \operatorname{trdeg} \mathbb{Q}(\underline{\theta})$$

and assume $t \geq 1$. There exist positive constants c_1 and c_2 with the following property. Let $(D_\nu)_{\nu \geq 0}$ and $(\mu_\nu)_{\nu \geq 0}$ be sequences of real numbers satisfying

$$c_1 \leq D_\nu \leq \mu_\nu, \quad D_\nu \leq D_{\nu+1} \leq 2D_\nu, \quad \mu_\nu \leq \mu_{\nu+1} \leq 2\mu_\nu \qquad (\nu \geq 0).$$

Assume also that the sequence $(\mu_\nu)_{\nu \geq 0}$ is unbounded. Then for infinitely many ν there exists a m-tuple $(\gamma_1, \ldots, \gamma_m)$ of algebraic numbers satisfying

$$[\mathbb{Q}(\underline{\gamma}) : \mathbb{Q}] \leq D_\nu, \quad \mu(\underline{\gamma}) \leq \mu_\nu$$

and

$$\max_{1 \leq i \leq m} |\theta_i - \gamma_i| \leq \exp\{-c_2 D_\nu^{1/t} \mu_\nu\}.$$

There are two different, but related quantitative refinements to a transcendence result: for a transcendental number θ, either one proves *a transcendence measure*, which is a lower bound for $|P(\theta)|$ when P is a non-zero polynomial with integer coefficients, or else one proves a *measure of algebraic approximation for θ*, which is a lower bound for $|\theta - \gamma|$ when γ is an algebraic number. In both cases such a lower bound will usually depend on the degree (of the polynomial P, or on the algebraic number γ), and on the height of the same.

Next, given several transcendental numbers $\theta_1, \ldots, \theta_n$, one may consider either a measure of simultaneous approximation by algebraic numbers, namely a lower bound for

$$\max\{|\theta_i - \gamma_i|\}$$

when $\gamma_1, \ldots, \gamma_n$ are algebraic numbers, or a measure of algebraic independence, which is a lower bound for

$$|P(\theta_1, \ldots, \theta_n)|$$

when P is a non-zero polynomial with integer coefficients. The first estimate deals with algebraic points (algebraic sets of zero dimension), the second with hypersurfaces (algebraic sets of codimension 1). There is a set of intermediate possibilities which have been studied by Yu. V. Nesterenko and P. Philippon, and are closely connected.

For instance, Conjecture 4.12 deals with simultaneous approximation by algebraic points; M. Laurent and D. Roy asked general questions about the approximation by algebraic subsets of $\mathbb{C}^m$, defined over $\mathbb{Q}$. For instance Conjecture 2 in

[Lau2] as well as the conjecture in §9 of [Ro7] deal with the more general problem of approximation of points in $\mathbb{C}^n$ by points located on $\mathbb{Q}$-varieties of a given dimension.

For an algebraic subset Z of $\mathbb{C}^m$, defined over $\mathbb{Q}$, denote by $t(Z)$ the size of a Chow form of Z.

Conjecture 4.13 (Laurent–Roy). *Let $\theta \in \mathbb{C}^m$. There is a positive constant c, depending only on θ and m, with the following property. Let k be an integer with $0 \le k \le m$. For infinitely many integers $T \ge 1$, there exists an algebraic set $Z \subset \mathbb{C}^m$, defined over $\mathbb{Q}$, of dimension k, and a point $\alpha \in Z$, such that*

$$t(Z) \le T^{m-k} \quad \text{and} \quad |\theta - \alpha| \le \exp\{-cT^{m+1}\}.$$

Further far-reaching, open problems in this direction have been proposed by P. Philippon as Problèmes 7, 8 and 10 in [P2, §5].

4.3. Logarithms of Algebraic Numbers. We have already suggested several questions related to linear independence measures over the field of rational numbers for logarithms of rational numbers (see Conjectures 2.4, 2.5 and 2.14). Now that we have a notion of height for algebraic numbers at our disposal, we can extend our study to linear independence measures over the field of algebraic numbers for the logarithms of algebraic numbers.

The next statement is Conjecture 14.25 of [W6].

Conjecture 4.14. *There exist two positive absolute constants c_1 and c_2 with the following property. Let $\lambda_1, \dots, \lambda_m$ be logarithms of algebraic numbers with $\alpha_i = e^{\lambda_i}$ $(1 \le i \le m)$, let $\beta_0, \dots, \beta_m$ be algebraic numbers, D the degree of the number field*

$$\mathbb{Q}(\alpha_1, \dots, \alpha_m, \beta_0, \dots, \beta_m),$$

and, finally, let $h \ge 1/D$ satisfy

$$h \ge \max_{1\le i\le m} h(\alpha_i), \quad h \ge \frac{1}{D}\max_{1\le i\le m} |\lambda_i| \quad \text{and} \quad h \ge \max_{0\le j\le m} h(\beta_j).$$

(1) *Assume that the number*

$$\Lambda = \beta_0 + \beta_1\lambda_1 + \cdots + \beta_m\lambda_m$$

is non-zero. Then

$$|\Lambda| \ge \exp\{-c_1 m D^2 h\}.$$

(2) *Assume that $\lambda_1, \dots, \lambda_m$ are linearly independent over $\mathbb{Q}$. Then*

$$\sum_{i=1}^m |\lambda_i - \beta_i| \ge \exp\{-c_2 m D^{1+(1/m)} h\}.$$

Assuming both Conjecture 4.12 and part 2 of Conjecture 4.14, one deduces not only Conjecture 3.3, but also further special cases of Conjecture 3.1 (these connexions are described in [W5] as well as [W6, Chap. 15]).

As far as part 1 of Conjecture 4.14 is concerned, weaker estimates are available (see [W6, §10.4]). Here is a much weaker (but still open) statement than either Conjecture 2.5 or part 1 of Conjecture 4.14.

Conjecture 4.15. *There exists a positive absolute constant C with the following property. Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers and $\log\alpha_1, \ldots, \log\alpha_n$ logarithms of $\alpha_1, \ldots, \alpha_n$ respectively. Assume that the numbers $\log\alpha_1, \ldots, \log\alpha_n$ are $\mathbb{Q}$-linearly independent. Let $\beta_0, \beta_1, \ldots, \beta_n$ be algebraic numbers, not all of which are zero. Denote by D the degree of the number field*

$$\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n)$$

over $\mathbb{Q}$. Further, let $A_1, \ldots, A_n$ and B be positive real numbers, each $\geq e$, such that

$$\log A_j \geq \max\left\{h(\alpha_j),\ \frac{|\log\alpha_j|}{D},\ \frac{1}{D}\right\} \quad (1 \leq j \leq n),$$
$$B \geq \max_{1\leq j\leq n-1} h(\beta_j).$$

Then the number

$$\Lambda = \beta_0 + \beta_1\log\alpha_1 + \cdots + \beta_n\log\alpha_n$$

satisfies

$$|\Lambda| > \exp\{-C^n D^{n+2}(\log A_1)\cdots(\log A_n)(\log B + \log D)(\log D)\}.$$

One is rather close to such an estimate (see [W8, §5 and §6], as well as [Matv]). The result is proved now in the so-called rational case, where

$$\beta_0 = 0 \quad \text{and} \quad \beta_i \in \mathbb{Q} \quad \text{for} \quad 1 \leq i \leq n.$$

In the general case, one needs a further condition, namely

$$B \geq \max_{1\leq i\leq n} \log A_i.$$

Removing this extra condition would enable one to prove that numbers like e^π or $2^{\sqrt{2}}$ are not Liouville numbers.

These questions are the first and simplest ones concerning transcendence measures,measures of Diophantine approximation, measures of linear independence and measures of algebraic independence. One may ask many further questions on this topic, including an effective version of Schanuel's conjecture. It is interesting to notice that in this case a "technical condition" cannot be omitted ([W4, Conjecture 1.4]).

Recall that the rank of a prime ideal $\mathfrak{P} \subset \mathbb{Q}[T_1, \ldots, T_m]$ is the largest integer $r \geq 0$ such that there exists an increasing chain of prime ideals

$$(0) = \mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_r = \mathfrak{P}.$$

The rank of an ideal $\mathfrak{I} \subset \mathbb{Q}[T_1, \ldots, T_m]$ is the minimum rank of a prime ideal containing $\mathfrak{I}$.

Conjecture 4.16 (Quantitative Refinement of Schanuel's Conjecture)**.** *Let $x_1, \ldots$ $\ldots, x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Assume that for any $\varepsilon > 0$, there exists a positive number H_0 such that, for any $H \geq H_0$ and n-tuple $(h_1, \ldots, h_n)$ of rational integers satisfying $0 < \max\{|h_1|, \ldots, |h_n|\} \leq H$, the inequality*

$$|h_1x_1 + \cdots + h_nx_n| \geq \exp\{-H^\varepsilon\}$$

is valid. Let d be a positive integer. Then there exists a positive number $C = C(x_1, \ldots, x_n, d)$ with the following property: for any integer $H \geq 2$ and any $n+1$ tuple $P_1, \ldots, P_{n+1}$ of polynomials in $\mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ with degrees $\leq d$ and usual heights $\leq H$, which generate an ideal of $\mathbb{Q}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ of rank $n+1$,

$$\sum_{j=1}^{n+1} \bigl|P_j(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})\bigr| \geq H^{-C}.$$

A consequence of Conjecture 4.16 is a quantitative refinement to Conjecture 3.3 on the algebraic independence of logarithms of algebraic numbers [W4].

Conjecture. *If $\log\alpha_1, \ldots, \log\alpha_n$ are $\mathbb{Q}$-linearly independent logarithms of algebraic numbers and d a positive integer, there exists a constant $C > 0$ such that, for any non-zero polynomial $P \in \mathbb{Z}[X_1, \ldots, X_n]$ of degree $\leq d$ and height $\leq H$, with $H \geq 2$,*

$$|P(\log\alpha_1, \ldots, \log\alpha_n)| \geq H^{-C}.$$

4.4. Density: Mazur's Problem. Let K be a number field with a given real embedding. Let V be a smooth variety over K. Denote by Z the closure, for the real topology, of $V(K)$ in $V(\mathbb{R})$. In his paper [Maz1] on the topology of rational points, Mazur asks,

Question 4.17 (Mazur)**.** *Assume that $K = \mathbb{Q}$ and that $V(\mathbb{Q})$ is Zariski dense; is Z a union of connected components of $V(\mathbb{R})$?*

An interesting fact is that Mazur asks this question in connexion with the rational version of Hilbert's tenth Problem (see [Maz2] and [Maz3]).

The answer to question 4.17 is negative. An example is given in [CSS] by J.-L. Colliot-Thélène, A. N. Skorobogatov and P. Swinnerton-Dyer of a smooth surface V over $\mathbb{Q}$, whose $\mathbb{Q}$-rational points are Zariski-dense, but such that the closure Z in $V(\mathbb{R})$ of the set of $\mathbb{Q}$-points is not a union of connected components.

However for the special case of Abelian varieties, there are good reasons to believe that the answer to question 4.17 is positive. Indeed for this special case a reformulation of question 4.17 is the following.

Conjecture. *Let A be a simple Abelian variety over $\mathbb{Q}$. Assume that the Mordell–Weil group $A(\mathbb{Q})$ has rank ≥ 1. Then $A(\mathbb{Q}) \cap A(\mathbb{R})^0$ is dense in the neutral component, $A(\mathbb{R})^0$ of $A(\mathbb{R})$.*

This statement is equivalent to the next one.

Conjecture 4.18. *Let A be a simple Abelian variety over $\mathbb{Q}$, $\exp_A\colon \mathbb{R}^g \to A(\mathbb{R})^0$ the exponential map of the Lie group $A(\mathbb{R})^0$, and $\Omega = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_g$ its kernel. Let $u = u_1\omega_1 + \cdots + u_g\omega_g \in \mathbb{R}^g$ satisfy $\exp_A(u) \in A(\mathbb{Q})$. Then $1, u_1, \ldots, u_g$ are linearly independent over $\mathbb{Q}$.*

The following quantitative refinement of Conjecture 4.18 is suggested in [W3, Conjecture 1.1].

For $\zeta = (\zeta_0 : \cdots : \zeta_N)$ and $\xi = (\xi_0 : \cdots : \xi_N)$ in $\mathbb{P}_N(\mathbb{R})$, write

$$\operatorname{dist}(\zeta, \xi) = \frac{\max_{0 \le i,j \le N} |\zeta_i \xi_j - \zeta_j \xi_i|}{\max_{0 \le i \le N} |\zeta_i| \cdot \max_{0 \le j \le N} |\xi_j|}.$$

Conjecture 4.19. *Let A be a simple Abelian variety of dimension g over a number field K embedded in $\mathbb{R}$. Denote by ℓ the rank over $\mathbb{Z}$ of the Mordell–Weil group $A(K)$. For any $\varepsilon > 0$, there exists $h_0 > 0$ (which depends only on the Abelian variety A, the real number field K and ε) such that, for any $h \ge h_0$ and any $\zeta \in A(\mathbb{R})^0$, there is a point $\gamma \in A(K)$ with Néron–Tate height $\le h$ such that*

$$\operatorname{dist}(\zeta, \gamma) \le h^{-(\ell/2g)+\varepsilon}.$$

Similar problems arise for commutative algebraic groups. Let us consider the easiest case, a torus $\mathbb{G}_{\mathrm{m}}^n$ over the field of real algebraic numbers. We replace the simple Abelian variety A of dimension g by the torus $\mathbb{G}_{\mathrm{m}}^n$ of dimension n, the Mordell–Weil group $A(K)$ by a finitely generated multiplicative subgroup of $(\overline{\mathbb{Q}}^\times)^n$, and the connected component $A(\mathbb{R})^0$ of the origin in $A(\mathbb{R})$ by $(\mathbb{R}_+^\times)^n$. The corresponding problem is then, given positive algebraic numbers γ_{ij} $(1 \le i \le n,\ 1 \le j \le m)$, to consider the approximation of a tuple $(\zeta_1, \ldots, \zeta_n) \in (\mathbb{R}_+^\times)^n$ by tuples of algebraic numbers of the form

$$\left(\gamma_{11}^{s_1} \cdots \gamma_{1m}^{s_m}, \ldots, \gamma_{n1}^{s_1} \cdots \gamma_{nm}^{s_m}\right)$$

with $\underline{s} = (s_1, \ldots, s_m) \in \mathbb{Z}^m$.

Recently D. Prasad [Pr] studied this question in terms of toric varieties.

The qualitative density question is solved by the following statement, which is a consequence of Conjecture 3.3.

Conjecture 4.20. *Let m, n, k be positive integers and $a_{ij\kappa}$ rational integers $(1 \le i \le n,\ 1 \le j \le m,\ 1 \le \kappa \le k)$. For $\underline{x} = (x_1, \ldots, x_k) \in (\mathbb{R}_+^\times)^k$ denote by $\Gamma(\underline{x})$ the following finitely generated subgroup of $(\mathbb{R}_+^\times)^n$,*

$$\Gamma(\underline{x}) = \left\{ \left(\prod_{j=1}^{m} \prod_{\kappa=1}^{k} x_k^{a_{ij\kappa} s_j} \right)_{1 \le i \le n} : \underline{s} = (s_1, \ldots, s_m) \in \mathbb{Z}^m \right\}.$$

Assume that there exists $\underline{x} \in (\mathbb{R}_+^\times)^k$ such that $\Gamma(\underline{x})$ is dense in $(\mathbb{R}_+^\times)^n$. Then for any $\underline{\gamma} = (\gamma_1, \ldots, \gamma_k)$ in $(\mathbb{R}_+^\times)^k$ with $\gamma_1, \ldots, \gamma_k$ algebraic and multiplicatively independent, the subgroup $\Gamma(\underline{\gamma})$ is dense in $(\mathbb{R}_+^\times)^n$.

If there is a $\underline{x}$ in $(\mathbb{R}_+^\times)^k$ such that $\Gamma(\underline{x})$ is dense in $(\mathbb{R}_+^\times)^n$, then the set of such $\underline{x}$ is dense in $(\mathbb{R}_+^\times)^k$. Hence again, loosely speaking, Conjecture 4.20 means that logarithms of algebraic numbers should behave like almost all numbers (see also [La8, Chap. IX, §7, p. 235]).

Conjecture 4.20 would provide an effective solution to the question raised by J.-L. Colliot-Thélène and J.-J. Sansuc and solved by D. Roy (see [Ro3]).

Theorem. *Let k be a number field of degree $d = r_1 + 2r_2$, where r_1 is the number of real embeddings and r_2 the number of pairwise non-conjugate embeddings of k. Then there exists a finitely generated subgroup Γ of $k^\times$, with rank $r_1 + r_2 + 1$, whose image in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ is dense.*

The existence of Γ is known, but the proof by D. Roy does not yield an explicit example.

Density questions are closely related to transcendence questions. For instance the multiplicative subgroup of $\mathbb{R}_+^\times$ generated by e and π is dense if and only if $\log \pi$ is irrational (which is an open question).

The simplest case of Conjecture 4.20 is obtained with $n = 2$ and $m = 3$. It reads as follows.

Conjecture. *Let $\alpha_1,\, \alpha_2,\, \alpha_3,\, \beta_1,\, \beta_2,\, \beta_3$ be non-zero positive algebraic numbers. Assume that for any $(a,\, b) \in \mathbb{Z}^2 \setminus \{(0,\, 0)\}$, two at least of the three numbers*

$$\alpha_1^a\beta_1^b,\; \alpha_2^a\beta_2^b,\; \alpha_3^a\beta_3^b$$

are multiplicatively independent. Then the subgroup

$$\Gamma = \left\{ \left(\alpha_1^{s_1}\alpha_2^{s_2}\alpha_3^{s_3},\, \alpha_1^{s_1}\alpha_2^{s_2}\alpha_3^{s_3} \right) : (s_1,\, s_2,\, s_3) \in \mathbb{Z}^3 \right\}$$

of $(\mathbb{R}_+^\times)^2$ is dense.

It is easy to deduce this statement from the four exponentials Conjecture 3.7.

The next question is to consider a quantitative refinement. Let Γ be a finitely generated subgroup of $(\overline{\mathbb{Q}} \cap \mathbb{R}_+^\times)^n$ which is dense in $(\mathbb{R}_+^\times)^n$. Fix a set of generators $\underline{\gamma}_1,\, \ldots,\, \underline{\gamma}_m$ of Γ. For $\underline{s} = (s_1,\, \ldots,\, s_m) \in \mathbb{Z}^m$ and $1 \le i \le n$ define

$$\gamma_i(\underline{s}) = \prod_{j=1}^m \gamma_{ij}^{s_j} \in \overline{\mathbb{Q}}^\times.$$

The density assumption means that for any $\underline{\zeta} = (\zeta_1,\, \ldots,\, \zeta_n) \in (\mathbb{R}_+^\times)^n$ and any $\varepsilon > 0$, there exists $\underline{s} \in \mathbb{Z}^m$ such that

$$\max_{1\le i\le n} |\gamma_i(\underline{s}) - \zeta_i| \le \varepsilon.$$

We wish to bound $|\underline{s}| = \max_{1\le j\le m} |s_j|$ in terms of ε.

We fix a compact neighborhood $\mathcal{K}$ of the origin $(1,\, \ldots,\, 1)$ in $(\mathbb{R}_+^\times)^n$. For instance

$$\mathcal{K} = \{\underline{\zeta} \in (\mathbb{R}_+^\times)^n : 1/2 \le |\zeta_i| \le 2 \; (1 \le i \le n)\}$$

would do.

Conjecture 4.21. *For any $\varepsilon > 0$ there exists $S_0 > 0$ (depending on $\varepsilon,\, \gamma_1,\, \ldots,\, \gamma_m$ and $\mathcal{K}$) such that, for any $S \ge S_0$ and any $\underline{\zeta} \in \mathcal{K}$, there exists $\underline{s} \in \mathbb{Z}^m$ with $|\underline{s}| \le S$ and*

$$\max_{1\le i\le n} |\gamma_i(\underline{s}) - \zeta_i| \le S^{-1-(1/n)+\varepsilon}.$$

These questions suggest a new kind of Diophantine approximation problem.

5. Further Topics

5.1. Metric Problems. Among the motivations for studying metric problems in Diophantine analysis (not to mention secular perturbations in astronomy and the statistical mechanics of a gas — see [Ha]), one would like to be able to guess the behavior of certain classes of numbers (such as algebraic numbers, logarithms of algebraic numbers, and numbers given as values of classical functions, suitably normalized [La2, p. 658 and 664]).

A first example is related to the Wirsing–Schmidt Conjecture. V. G. Sprindzuk showed in 1965 that the conjecture 4.11 is true for almost all θ (for Lebesgue measure).

A second example is the question of refining Roth's Theorem. Conjecture 2.12 is motivated by Khinchine's Theorem ([Sp, Chap. I, §1, Th. 1, p. 1]) which answers the question of rational Diophantine approximation for almost all real numbers. In 1926 A. Khinchine himself extended his result to the simultaneous Diophantine rational approximation

$$\max_{1\le i\le n} |q\alpha_i - p_i|$$

([Sp, Chap. I, §4, Th. 8, p. 28]), and in 1938 A. V. Groshev proved the first very general theorem of Khinchine type for systems of linear forms,

$$\max_{1\le i\le n} |q_1\alpha_{i1} + \cdots + q_m\alpha_{im} - p_i|$$

([Sp, Chap. I, §5, Th. 12, p. 33]). Using the same heuristic arguments, one may extend Conjecture 2.12 to the context of simultaneous linear combinations of algebraic numbers.

In Conjecture 2.12 (as well as in Khinchine's result for almost all real numbers) the function $q\psi(q)$ is assumed to be non-increasing. A conjecture of Duffin and Schaeffer (see [Sp, Chap. 1, §2, p. 17] and [Ha]) would enable one to work without such a restriction. Denote by $\varphi(n)$ Euler's function

$$\varphi(n) = \sum_{\substack{1\le k\le n\\ \gcd(k,n)=1}} 1.$$

Conjecture 5.1 (Duffin and Schaeffer)**.** *Let ψ be a positive real valued function. Then, for almost all $\theta\in\mathbb{R}$, inequality (2.11) has an infinite number of solutions in integers p and q with $q>0$ and $\gcd(p,q)=1$ if and only if the series*

$$\sum_{q=1}^{\infty} \frac{1}{q}\psi(q)\varphi(q)$$

diverges.

The Khinchine–Goshev Theorem has been extended to certain manifolds (see [Sp], [BD], as well as more recent papers by V. Bernick, M. Dodson, D Kleinbock and G. Margulis). Further, connexions between the metrical theory of Diophantine approximation on one hand, hyperbolic geometry, ergodic theory and dynamics of flows on homogeneous spaces of Lie groups on the other, have been studied by several mathematicians, including D. Sullivan, S. J. Dani, G. Margulis and D. Kleinbock. Also S. Hersonsky and F. Paulin [HP] have recently studied the Diophantine approximation properties of geodesic lines on the Heisenberg group, which suggests new, open questions, for instance to study

$$\max_{1\le i\le n} |q\alpha_i - p_i|^{\kappa_i}$$

when $\kappa_1, \ldots, \kappa_n$ are positive real numbers.

The set of real numbers with bounded partial quotients is countable. This is the set of real numbers which are badly approximable by rational numbers. Y. Bugeaud asks a similar question for numbers which are badly approximable by algebraic numbers of bounded degree.

Question 5.2 (Bugeaud)**.** *Let $n \geq 2$. Denote by $Ж_n$ the set of real numbers ξ with the following property: there exists $c_1(\xi) > 0$ and $c_2(\xi) > 0$ such that for algebraic number α of degree $\leq n$,*

$$|\xi - \alpha| \geq c_2(\xi) H(\alpha)^{-n-1},$$

and such that there are infinitely many algebraic numbers α of degree $\leq n$ with

$$|\xi - \alpha| \leq c_1(\xi) H(\alpha)^{-n-1},$$

Does the set $Ж_n$ strictly contain the set of algebraic numbers of degree $n+1$?

In connexion with the algebraic independence problems of Section 3.1, one would like to understand better the behavior of real (or complex) numbers with respect to Diophantine approximation by algebraic numbers of large degree (see Conjecture 4.12). A natural question is to consider this question from a metrical point of view. Roughly speaking, what is expected is that for almost all real numbers ξ, the quality of approximation by algebraic numbers of degree $\leq d$ and measure $\leq t$ be e^{-dt}. This is the precise suggestion of Y. Bugeaud [Bu2].

For a real number $\kappa > 0$, denote by $\mathcal{F}_\kappa$ the set of real numbers ξ with the following property: for any κ' with $0 < \kappa' < \kappa$ and any $d_0 \geq 1$, there exists a real number $h_0 \geq 1$ such that, for any $d \geq d_0$ and any $t \geq h_0 d$, the inequality

$$|\xi - \gamma| \leq e^{-\kappa' dt}$$

has a solution $\gamma \in \overline{\mathbb{Q}}$ where $[\mathbb{Q}(\gamma) : \mathbb{Q}] \leq d$ and $\mu(\gamma) \leq t$.

Also, denote by $\mathcal{F}'_\kappa$ the set of real numbers ξ with the following property: for any $\kappa' > \kappa$ there exist $d_0 \geq 1$ and $h_0 \geq 1$ such that, for any $d \geq d_0$ and any $t \geq h_0 d$, the inequality

$$|\xi - \gamma| \leq e^{-\kappa' dt}$$

has no solution $\gamma \in \overline{\mathbb{Q}}$ where $[\mathbb{Q}(\gamma) : \mathbb{Q}] \leq d$ and $\mu(\gamma) \leq t$.

These definition are given more concisely in [Bu2]: for $t \geq d \geq 1$ denote by $\overline{\mathbb{Q}}(d,\, t)$ the set of real algebraic numbers γ of degree $\leq d$ and measure $\leq t$. Then

$$\mathcal{F}_\kappa = \bigcap_{\kappa' < \kappa} \bigcap_{d_0 \geq 1} \bigcup_{h_0 \geq 1} \bigcap_{d \geq d_0} \bigcap_{t \geq h_0 d} \bigcup_{\gamma \in \overline{\mathbb{Q}}(d,t) \cap \mathbb{R}}]\gamma - e^{-\kappa' dt},\, \gamma + e^{-\kappa' dt}[,$$

$$\mathcal{F}'_\kappa = \bigcap_{\kappa' > \kappa} \bigcup_{d_0 \geq 1} \bigcup_{h_0 \geq 1} \bigcap_{d \geq d_0} \bigcap_{t \geq h_0 d} \bigcap_{\gamma \in \overline{\mathbb{Q}}(d,t) \cap \mathbb{R}}]\gamma - e^{-\kappa' dt},\, \gamma + e^{-\kappa' dt}[^c$$

where $]a,\, b[^c$ denotes the complement of the intervall $]a,\, b[$. According to Theorem 4 of [Bu2], there exist two positive constants $\widetilde{\kappa}$ and $\widetilde{\kappa}'$ such that, for almost all $\xi \in \mathbb{R}$,

$$\max\{\kappa > 0 \colon \xi \in \mathcal{F}_\kappa\} = \widetilde{\kappa} \quad \text{and} \quad \min\{\kappa > 0 \colon \xi \in \mathcal{F}'_\kappa\} = \widetilde{\kappa}'.$$

Further,

$$\frac{1}{850} \leq \widetilde{\kappa} \leq \widetilde{\kappa}' \leq 1.$$

Bugeaud's conjecture is $\widetilde{\kappa} = \widetilde{\kappa}' = 1$.

It is an important open question to study the simultaneous approximation of almost all tuples in $\mathbb{R}^n$ by algebraic tuples $\underline{\gamma} = (\gamma_1, \ldots, \gamma_n)$ in terms of the degree $[\mathbb{Q}(\underline{\gamma}) : \mathbb{Q}]$ and the measure $\mu(\underline{\gamma})$. Most authors have devoted much attention to the dependence on the height, but now it is necessary to study more thoroughly the behavior of the approximation for large degree.

Further problems which we considered in the previous sections deserve to be studied from the metrical point of view. Our next example is a strong quantitative form of Schanuel's Conjecture for almost all tuples ([W5, Conjecture 4]).

Conjecture 5.3. *Let n be a positive integer. For almost all n-tuples $(x_1, \ldots, x_n)$, there are positive constants c and D_0 (depending on n, $x_1, \ldots, x_n$ and ε), with the following property. For any integer $D \geq D_0$, any real number $\mu \geq D$ and any $2n$-tuple $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ of algebraic numbers satisfying*

$$[\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) : \mathbb{Q}] \leq D$$

and

$$[\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) : \mathbb{Q}] \max\{h(\alpha_i), h(\beta_i) \colon 1 \leq i \leq n\} \leq \mu,$$

$$\max\{|x_i - \beta_i|, |e^{x_i} - \alpha_i| \colon 1 \leq i \leq n\} \geq \exp\{-cD^{1/(2n)}\mu\}.$$

One may also expect that c does not depend on $x_1, \ldots, x_n$.

An open metrical problem of uniform distribution was suggested by P. Erdős to R. C. Baker in 1973 (see [Ha] Chap. 5, p. 163). It is a counterpart to the conjecture of Khinchine which was disproved by J. M. Marstrand in 1970.

Question. *Let f be a bounded measurable function with period 1. Is it true that*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} f(n\alpha) = \int_0^1 f(x)\, dx$$

for almost all $\alpha \in \mathbb{R}$?

5.2. Function Fields. Let K be a field and $\mathcal{C} = K((T^{-1}))$ be the field of Laurent series on K. The field $\mathcal{C}$ has similar properties to the real number field, when $\mathbb{Z}$ is replaced by $K[T]$ and $\mathbb{Q}$ by $K(T)$. An absolute value on $\mathcal{C}$ is defined by selecting $|T| > 1$. We set $|\alpha| = |T|^k$ if $\alpha = \sum_{n\in\mathbb{Z}} a_n T^{-n}$ is a non-zero element of $\mathcal{C}$, where $k = \deg(\alpha)$ denotes the least index such that $a_k \neq 0$. Hence $\mathcal{C}$ is the completion of $K(T)$ for this absolute value.

A theory of Diophantine approximation has been developed on $\mathcal{C}$ in analogy to the classical one. If K has zero characteristic, the results are very similar to the classical ones. But if K has finite characteristic, the situation is completely different (see [dML] and [Sch3]). It is not yet even clear how to describe the situation from a conjectural point of view. A conjectural description of the set of algebraic numbers for which a Roth type inequality is valid is still missing. Some algebraic elements satisfy a Roth type inequality, while for some others, Liouville's estimate is optimal. However, from a certain point of view, much more is known about the function field case, since the exact approximation exponent is known for several classes of algebraic numbers.

There is also a transcendence theory over function fields. The starting point is a paper by Carlitz in the 40's. He defines functions on $\mathcal{C}$ which behave like analogues of the exponential function (*Carlitz module*). A generalization is due to V. G. Drinfeld (*Drinfeld modules*), and a number of results on the transcendence of numbers related to these objects are known, going much further than their classical (complex) counterpart. For example, the *number*

$$\prod_p (1-p^{-1})^{-1}$$

(in a suitable extension of a finite field), where p runs over the monic irreducible polynomials over the given finite field, is known to be transcendental (over the field of rational functions on the finite field) [AT]; it may be considered to be an analogue of Euler's constant γ since

$$\gamma = \lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right).$$

However the theory is far from being complete. An analogue of Schanuel's Conjecture for Drinfeld modules was proposed by W. D. Brownawell in [Brow], together with many further related problems, including large transcendence degree, Diophantine geometry, values of Carlitz–Bessel functions and values of gamma functions.

For the study of Diophantine approximation, an important tool (which is not available in the classical number theoretic case) is the derivation d/dT. In the transcendence theory this suggests new questions which started to be studied by L. Denis. Also in the function field case, interesting new questions are suggested by considering several characteristics. So Diophantine analysis for function fields involve different aspects, some which are reminiscent of the classical theory, and some which have no counterpart.

For the related transcendence theory involving automata theory, we refer to the paper by D. Thakur [Th] (especially on p. 389–390) and to [AS] for the state of the art concerning the following open problem,

Conjecture 5.4 (Loxton and van der Poorten)**.** *Let $(n_i)_{i\ge 0}$ be an increasing sequence of positive integers. Assume that there is a prime number p such that the power series*

$$\sum_{i\ge 0} z^{n_i} \in \mathbb{F}_p[[z]]$$

is algebraic over $\mathbb{F}_p(z)$ and irrational (i. e., does not belong to $\mathbb{F}_p(z)$). Then the real number

$$\sum_{i\ge 0} 10^{-n_i}$$

is transcendental.

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