March 6, 2023

The square root of 2, the Golden Ratio and the Fibonacci sequence

Michel Waldschmidt

Professeur Émérite, Sorbonne Université,
Institut de Mathématiques de Jussieu, Paris
http://www.imj-prg.fr/~michel.waldschmidt/

Abstract

The square root of 2,

$$\sqrt{2} = 1.414\,213\,562\,373\,095\ldots,$$

and the Golden ratio

$$\Phi = \frac{1+\sqrt{5}}{2} = 1.618\,033\,988\,749\,894\ldots$$

are two irrational numbers with many remarkable properties. The Fibonacci sequence

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233\ldots$$

occurs in many situations, in mathematics as well as in the real life. We review some of these properties.

Tablet YBC 7289 : 1800 – 1600 BC

Babylonian clay tablet, accurate sexagesimal approximation to $\sqrt{2}$ to the equivalent of six decimal digits.

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = 1.\mathbf{414\,212}\,962\,962\,962\ldots$$

$$\sqrt{2} = 1.\mathbf{414\,213}\,562\,373\,095\,048\ldots$$

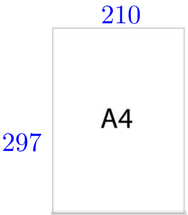
https://en.wikipedia.org/wiki/YBC_7289

A4 format 21×29.7

ISO 216 International standard https://en.wikipedia.org/wiki/ISO_216

$$\frac{297}{210} = \frac{99}{70} = 1.\mathbf{414\,28}5\,714\,285\,714\,285\ldots$$

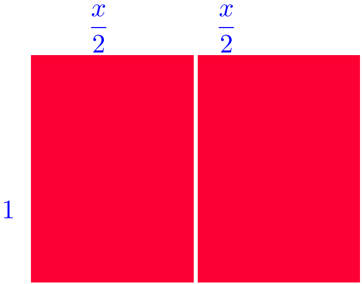
$$\sqrt{2} = 1.\mathbf{414\,21}3\,562\,373\,095\,048\ldots$$



A, B, C formats

Large rectangle : sides x, 1 ; proportion $\frac{x}{1} = x$

Small rectangles : sides 1, $\frac{x}{2}$; proportion $\frac{1}{x/2} = \frac{2}{x}$

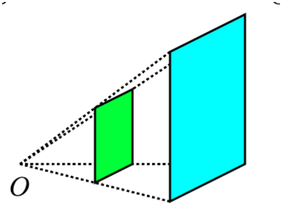


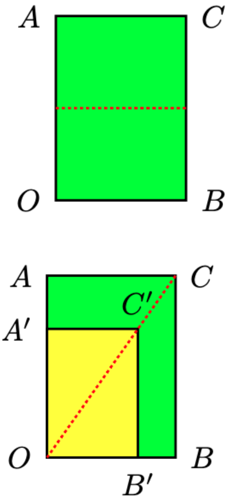
$$x = \frac{2}{x}, \qquad x^2 = 2.$$

https://en.wikipedia.org/wiki/Paper_size

Rectangle format $\sqrt{2}$

The large rectangle and half of it are proportional.



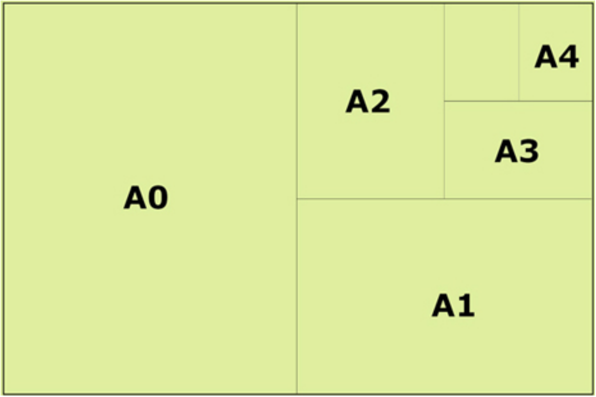


Reference : Paul Gérardin

http://www.lpsm.paris/pageperso/SemaineCollege-Avril08/web/doc/Coll08A4.pdf

A format

The number $\sqrt{2}$ is twice its inverse : $\sqrt{2} = 2/\sqrt{2}$.
Folding a rectangular piece of paper with sides in proportion $\sqrt{2}$ yields a new rectangular piece of paper with sides in proportion $\sqrt{2}$ again.



A0 is 118.9cm $\times 84.1$cm - area 1 m^2.

B and C formats

https://papersizes.io/

B0 is 1m $\times 1.414$m.
B7 (passeport) is 88mm $\times 125$mm.

C0 is 917mm $\times 1297$mm, approximately $\frac{1}{\sqrt[8]{2}} \times \sqrt[8]{8}$.

C6 : 114mm $\times 162$mm

enveloppe for a A6 paper 105mm $\times 148$mm

Xerox machine : enlarging and reducing

141%	119%	84%	71%
1.41	1.19	0.84	0.71
1.4142	1.1892	0.8409	0.7071
$\sqrt{2}$	$\sqrt[4]{2}$	$1/\sqrt[4]{2}$	$1/\sqrt{2}$

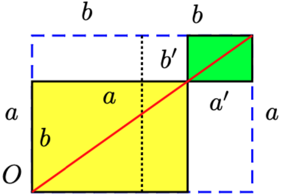
Paper format A0, A1, A2,... in cm

$$x_1 = 100\sqrt[4]{2} = 118.9, \qquad x_2 = \frac{100}{\sqrt[4]{2}} = 84.1.$$

$$\begin{array}{llll}
A0: & x_1 = 118.9 & & x_2 = 84.1 \\
A1: & x_2 = 84.1 & & \frac{x_1}{2} = 59.4 \\
A2: & \frac{x_1}{2} = 59.4 & & \frac{x_2}{2} = 42 \\
A3: & \frac{x_2}{2} = 42 & & \frac{x_1}{4} = 29.7 \\
A4: & \frac{x_1}{4} = 29.7 & & \frac{x_2}{4} = 21 \\
A5: & \frac{x_2}{4} = 21 & & \frac{x_1}{8} = 14.8
\end{array}$$

Irrationality of $\sqrt{2}$

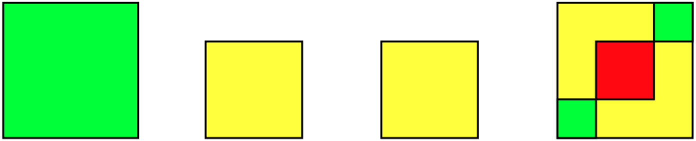
Assume $\frac{a}{b} = \frac{2b}{a}$. Then $a' = 2b - a$ and $b' = a - b$.



$a^2 = 2b^2$, $\quad {a'}^2 = 2{b'}^2$, $\qquad a' < a$, $\quad b' < b$.

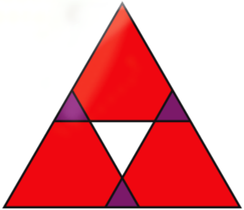
Irrationality of $\sqrt{2}$ (again)

Assume $a^2 = 2b^2$. You have the same amount of green painting and yellow painting. Put two yellow squares of sides b into a green square of side a. They overlap into a red square of side length $a' = 2b - a$. The green squares have a side length $b' = a - b$.



On the right image, you first paint the yellow part. The amount of yellow painting which is left enables you to paint either twice the red square, or once the red square and once both green squares. Hence the red area is the same as the green area : ${a'}^2 = 2{b'}^2$, with $a' < a$, $b' < b$.

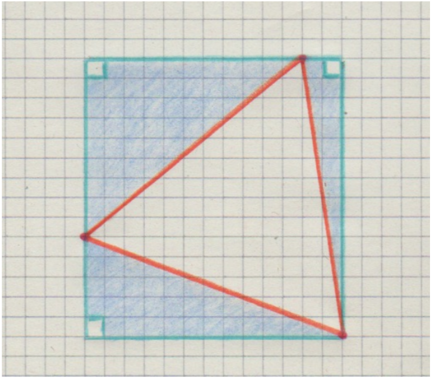
Irrationality of $\sqrt{3}$

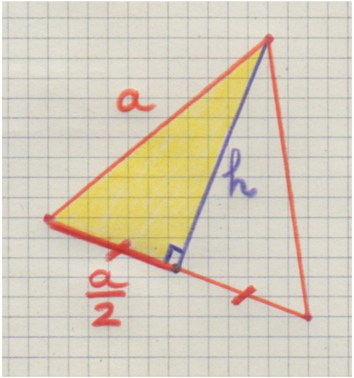


Assume $a^2 = 3b^2$
One large equilateral triangle : side length a
Three red equilateral triangles : side length b
Three purple equilateral triangles : side length $b' = 2b - a$
One white equilateral triangle : side length $a' = 2a - 3b$.

$$0 < a' < a, \qquad 0 < b' < b.$$

There is no equilateral triangle on the screen of a computer



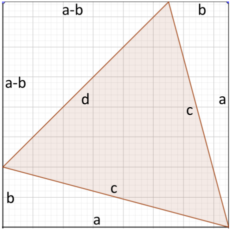


The area is rational

The area is irrational

https://images.math.cnrs.fr/Les-triangles-equilateraux-n-existent-pas.html

How to draw an almost equilateral triangle



Equilateral triangle : $c = d$,
$a^2 + b^2 = 4ab$

$$\frac{a}{b} = 2 + \sqrt{3}.$$

Approximations :
$c^2 = a^2 + b^2$, $d^2 = 2(a-b)^2$

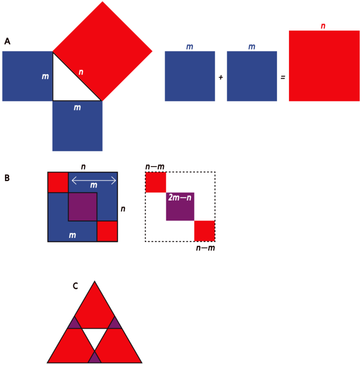
Linear recurrence sequence

$$u_{n+2} = 4u_{n+1} - u_n$$

	a_n	b_n	c_n^2	d_n^2
$n=0$	1	0	1	2
$n=1$	4	1	17	18
$n=2$	15	4	241	242
$n=3$	56	15	3361	3362
$n=4$	209	56	46817	46818

John H. Conway

John H. Conway
(1937 – 2020)



John H. Conway, The power of mathematics.
in : A. Blackwell & D. MacKay (eds.), Power, Cambridge University Press, 2006.
http://matteblogggen.com/wp-content/uploads/2012/11/conway.pdf

Jean-Paul Delahaye, Cinq pépites mathématiques de John Conway.
In : Logique et Calcul, Pour La Science N° 515 / September 2020
https://www.pourlascience.fr/sr/logique-calcul/cinq-pepites-mathematiques-de-john-conway-19963.php

Mathologer, Visualising irrationality with triangular squares.
https://www.youtube.com/watch?v=yk6wbvNPZW0

Irrationality of $\sqrt{d}$

Let d be a positive integer which is not the square of an integer. Then d is not the square of a rational number.

Proof.

Assume $\sqrt{d} = n/m$ with n, m positive integers and $n/m \notin \mathbb{Z}$.
(1) Since $\sqrt{d}$ is not an integer, there is an integer k in the interval

$$\sqrt{d} - 1 < k < \sqrt{d}.$$

Define integers n' and m' by setting

$$n' = dm - kn = n(\sqrt{d} - k) \quad \text{and} \quad m' = n - km = m(\sqrt{d} - k)$$

Then $0 < n' < n$, $0 < m' < m$ and $n/m = n'/m'$.
(2) There is an integer ℓ in the interval

$$\sqrt{d} < \ell < \sqrt{d} + 1.$$

Set

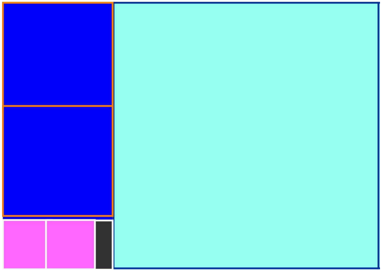
$$n' = \ell n - dm = n(\ell - \sqrt{d}) \quad \text{and} \quad m' = \ell m - n = m(\ell - \sqrt{d})$$

Then $0 < n' < n$, $0 < m' < m$ and $n/m = n'/m'$. □

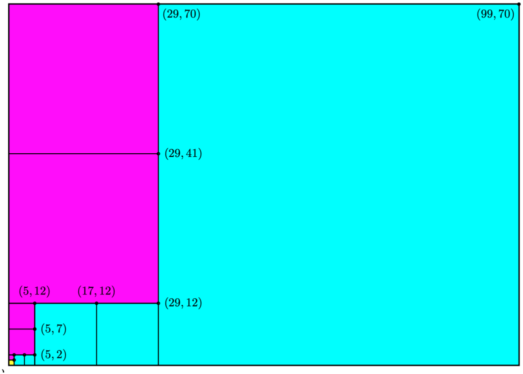
Rectangle with proportion $\sqrt{2}$

One square plus 2 rectangles with proportion $1+\sqrt{2}$:

$$\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}}, \qquad 1+\sqrt{2} = 2 + \frac{1}{1+\sqrt{2}}.$$



Irrationality of $\sqrt{2}$: geometric proof



$$\frac{99}{70} = 1 + \frac{29}{70},$$
$$\frac{70}{29} = 2 + \frac{12}{29},$$
$$\frac{29}{12} = 2 + \frac{5}{12},$$
$$\frac{12}{5} = 2 + \frac{2}{5},$$
$$\frac{5}{2} = 2 + \frac{1}{2}.$$

$$\frac{297}{210} = \frac{99}{70}.$$

Continued fraction of $\sqrt{2}$

The number

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,20\,\ldots$$

satisfies

$$\boxed{\sqrt{2}} = 1 + \cfrac{1}{1 + \boxed{\sqrt{2}}}.$$

Hence

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \boxed{\sqrt{2}}}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}$$

We write the continued fraction expansion of $\sqrt{2}$ using the shorter notation

$$\sqrt{2} = [1,\ 2,\ 2,\ 2,\ 2,\ 2,\ \ldots] = [1,\ \overline{2}].$$

A4 format

$$\frac{297}{210} = 1 + \frac{29}{70},$$
$$\frac{70}{29} = 2 + \frac{12}{29},$$
$$\frac{29}{12} = 2 + \frac{5}{12},$$
$$\frac{12}{5} = 2 + \frac{2}{5},$$
$$\frac{5}{2} = 2 + \frac{1}{2}.$$

Hence

$$\frac{297}{210} = [1, 2, 2, 2, 2, 2].$$

First decimals of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.414213562373095048801688724209698078569671875376948073176679737990732478462107038850387534327641572735013846230912297024924836055850737212644121497099935831413222665927505592755799950501152782060571470109559971605970274534596862014728517418640889198609552329230484308714321450839762603627995251407989687253396546331808829640620615258352395054745750287759961729835575220337531857011354374603408498847160386899970699004815030544027790316454247823068492936918621580578463111596668713013015618568987237235288509264861249497715421833420428568606014682472077143585487415565706967765372022648544701585880162075847492265722600208558446652145839889394437092659180031138824646815708263010059485870400318648034219489727829064104507263688131373985525611732204024509122770022694112757362728049573810896750401836986836845072579936472906076299694138047565482372899718032680247442062926912485905218100445984215059112024944134172853147810580360337107730918286931471017111168391658172688941975871658215212822951848847 . . .

First binary digits of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.0110101000001001111001100110011111110011101111001100100100000
1000101100101111101100010011011001101110101010010101011111010 0
1111100011101011011110110000010111010100010010011101110101000 0
1001100111011010001011110101100100001011000000110011001110011 00
1000101010100101011111100100000110000001000011101010111000101 00
0101100001110101000101100011111111001101111111011100100000111 10
1101100111001000011110111010010101000010111100100001110011100 0
1111011010010100111100000000100100001110011011000111101111110 1
0001001110110100011010010001000000010111010000111010000101010 1
1110001111101001110010100110000010110011100011000000000100011 01
1110000110011011110111100101010110001101111001001000100010110 1
0001000010001011000101001000110000010101011110001110010001011 1
1011111000100111000110011110001101101010110101000101000111000 1
0111011011111101001110111001100101100101010011000110100001100 1
1000111110011110010000100110111110101001011110001001000001111 1
0000011011011100101100000101110111010101010010010100000010001 00
11001000000100000011001010010010101000000010011100101001010 ...

Computation of decimals of $\sqrt{2}$

$1\,542$ decimals computed by hand by Horace Uhler in 1951

$14\,000$ decimals computed in 1967

$1\,000\,000$ decimals in 1971

$137 \cdot 10^9$ decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

- Motivation : computation of π.

Émile Borel (1871–1956)

- *Les probabilités dénombrables et leurs applications arithmétiques,*
Palermo Rend. **27**, 247-271 (1909).
Jahrbuch Database JFM 40.0283.01
http://www.emis.de/MATH/JFM/JFM.html

- *Sur les chiffres décimaux de* $\sqrt{2}$ *et divers problèmes de probabilités en chaînes,*
C. R. Acad. Sci., Paris **230**, 591-593 (1950).
Zbl 0035.08302

Émile Borel : 1950

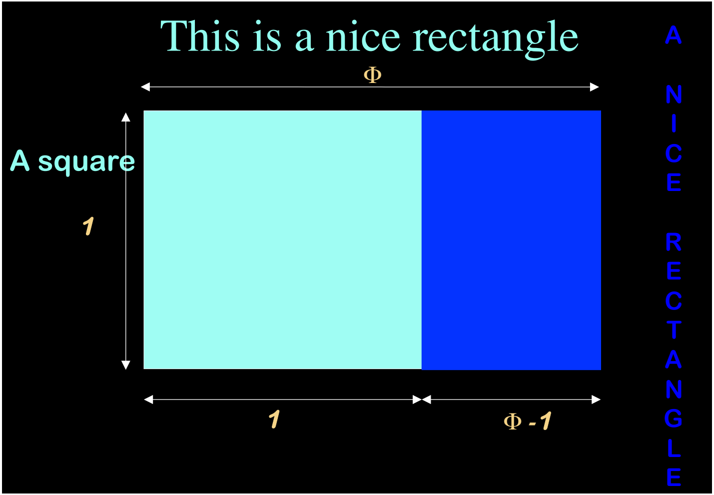
Émile Borel
(1871–1956)

Let $g \geq 2$ be an integer and x a real irrational algebraic number. *The expansion in base g of x should satisfy some of the laws which are valid for almost all real numbers (with respect to Lebesgue's measure).*

Open problem. Select one digit c among $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Choose a real algebraic irrational number α like $\sqrt{2}$. Is it true that the digit c occurs infinitely often in the decimal expansion of α?
It is conjectured that the answer is always yes. There is no example of (c, α) for which we can prove that it is true.

This is a nice rectangle

Golden rectangle



$$\frac{\Phi}{1} = \frac{1}{\Phi - 1}, \qquad \Phi^2 = \Phi + 1.$$

Irrationality of Φ and of $\sqrt{5}$

The number

$$\Phi = \frac{1+\sqrt{5}}{2} = 1.618\,033\,988\,749\,894\ldots$$

satisfies

$$\boxed{\Phi} = 1 + \frac{1}{\boxed{\Phi}}.$$

Hence

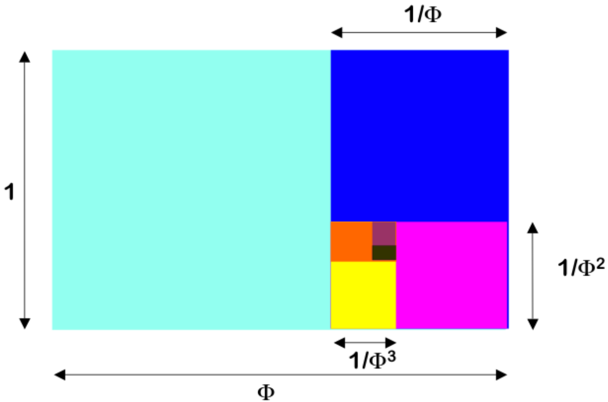
$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{\boxed{\Phi}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$$

If we start from a rectangle with the Golden ratio as proportion of sides lengths, at each step we get a square and a smaller rectangle with the same proportion for the sides lengths.

http://oeis.org/A001622

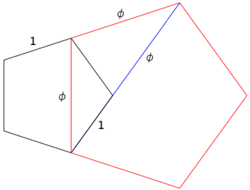
The Golden Ratio $(1+\sqrt{5})/2 = 1.618\,033\,988\,749\,894\ldots$

Golden Rectangle

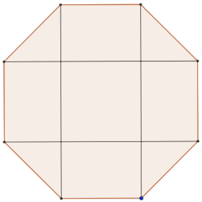


The diagonal of the pentagon and the diagonal of the octogon

The diagonal of the pentagon is Φ



The diagonal of the octogon is $1+\sqrt{2}$



Nested roots

$$\Phi^2 = 1 + \Phi.$$

$$\begin{aligned}
\Phi &= \sqrt{1+\Phi} \\
&= \sqrt{1+\sqrt{1+\Phi}} \\
&= \sqrt{1+\sqrt{1+\sqrt{1+\Phi}}} \\
&= \dots \\
&= \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}
\end{aligned}$$

Nested roots

Journal of the Indian Mathematical Society (1912) – problems solved by Ramanujan

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}} = 3$$

$$\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\cdots}}}} = 4$$

Srinivasa Ramanujan
1887 – 1920

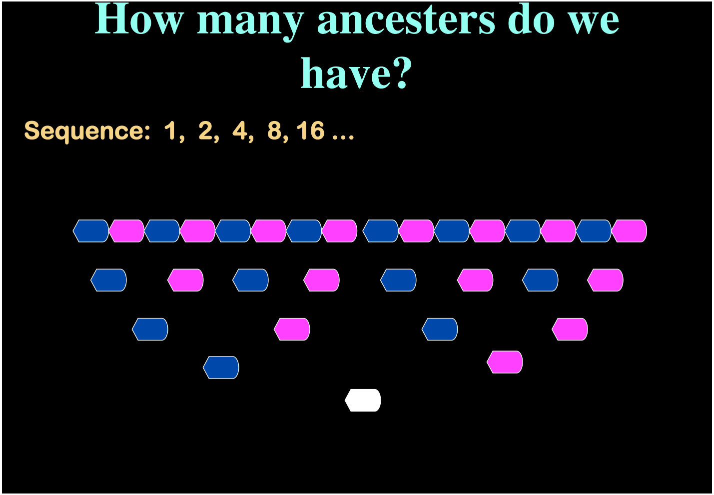
Back to $\sqrt{2}$

$$(1+\sqrt{2})^2 = 1 + 2(1+\sqrt{2}).$$

$$\begin{aligned}
1+\sqrt{2} &= \sqrt{1+2(1+\sqrt{2})} \\
&= \sqrt{1+2\sqrt{1+2(1+\sqrt{2})}} \\
&= \sqrt{1+2\sqrt{1+2\sqrt{1+2(1+\sqrt{2})}}} \\
&= \dots \\
&= \sqrt{1+2\sqrt{1+2\sqrt{1+2\sqrt{1+2\sqrt{1+\cdots}}}}}
\end{aligned}$$

Geometric series

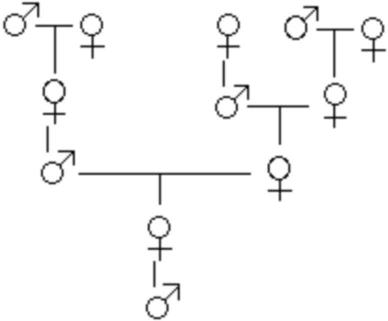
$$u_0 = 1, \quad u_{n+1} = 2u_n$$



$$u_n = 2^n, \quad n \geq 0.$$

Bees genealogy

Male honeybees are born from unfertilized eggs. Female honeybees are born from fertilized eggs. Therefore males have only a mother, but females have both a mother and a father.



Genealogy of a male bee (bottom – up)

Number of bees :

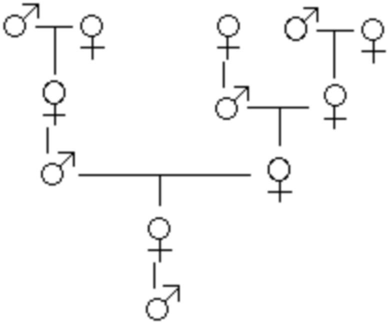
$$1,\ 1,\ 2,\ 3,\ 5\dots$$

Number of females :

$$0,\ 1,\ 1,\ 2,\ 3\dots$$

Rule :

$$u_{n+2} = u_{n+1} + u_n.$$



Bees genealogy $u_1 = 1,\ u_2 = 1,\ u_{n+2} = u_{n+1} + u_n$

Number of females at a given level =
total population at the previous level

Number of males at a given level=
number of females at the previous level

3 + 5 = 8

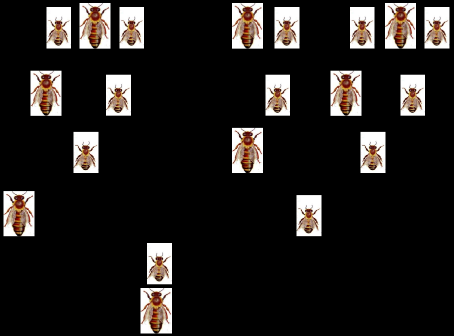
2 + 3 = 5

1 + 2 = 3

1 + 1 = 2

0 + 1 = 1

1 + 0 = 1



The Lamé Series

Gabriel Lamé
1795 – 1870

Edouard Lucas
1842 - 1891

In 1844 the sequence

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\dots$$

was referred to as the Lamé series, because Gabriel Lamé used it to give an upper bound for the number of steps in the Euclidean algorithm for the gcd.
On a trip to Italy in 1876 Edouard Lucas found them in a copy of the Liber Abbaci of Leonardo da Pisa.

Leonardo Pisano (Fibonacci)

The Fibonacci sequence $(F_n)_{n\geq 0}$,

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,$$

$$34,\ 55,\ 89,\ 144,\ 233, \dots$$

is defined by

$$F_0 = 0,\ F_1 = 1,$$

$$F_{n+2} = F_{n+1} + F_n \quad \text{for} \quad n \geq 0.$$

http://oeis.org/A000045

Leonardo Pisano (Fibonacci)
(1170–1250)

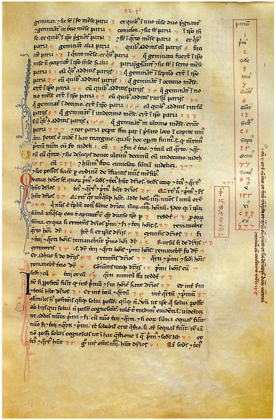
Leonardo Pisano (Fibonacci)

Guglielmo Bonacci : filius Bonacci
or Fibonacci

travels around the mediterranean,

learns the techniques of
Al-Khwarizmi

Liber Abbaci (1202)



https://commons.wikimedia.org/w/index.php?curid=720501

Encyclopedia of integer sequences (again)

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, . . .

The Fibonacci sequence is available online

The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane

Neil J. A. Sloane

http://oeis.org/A000045

Encyclopedia of integer sequences A000045

D. E. Knuth writes: "Before Fibonacci wrote his work, the sequence F_{n} had already been discussed by Indian scholars, who had long been interested in rhythmic patterns that are formed from one-beat and two-beat notes. The number of such rhythms having n beats altogether is F_{n+1}; therefore both Gopāla (before 1135) and Hemachandra (c. 1150) mentioned the numbers 1, 2, 3, 5, 8, 13, 21, ... explicitly." (TAOCP Vol. 1, 2nd ed.) - Peter Luschny, Jan 11 2015

In keeping with historical accounts (see the references by P. Singh and S. Kak), the generalized Fibonacci sequence a, b, a + b, a + 2b, 2a + 3b, 3a + 5b, ... can also be described as the Gopala-Hemachandra numbers H(n) = H(n-1) + H(n-2), with F(n) = H(n) for a = b = 1, and Lucas sequence L(n) = H(n) for a = 2, b = 1. - Lekraj Beedassy, Jan 11 2015

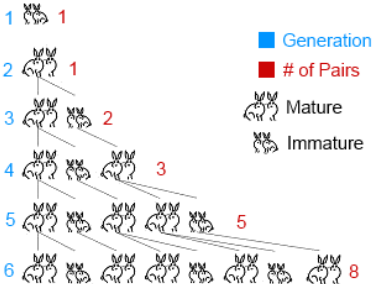
Susantha Goonatilake writes: "[T]his sequence was well known in South Asia and used in the metrical sciences. Its development is attributed in part to Pingala (200 BC), later being associated with Virahanka (circa 700 AD), Gopala (circa 1135), and Hemachandra (circa 1150)-all of whom lived and worked prior to Fibonacci." (Toward a Global Science: Mining Civilizational Knowledge, p. 126) - Russ Cox, Sep 08 2021

Also sometimes called Hemachandra numbers.

Fibonacci rabbits

Fibonacci considered the growth of a rabbit population.

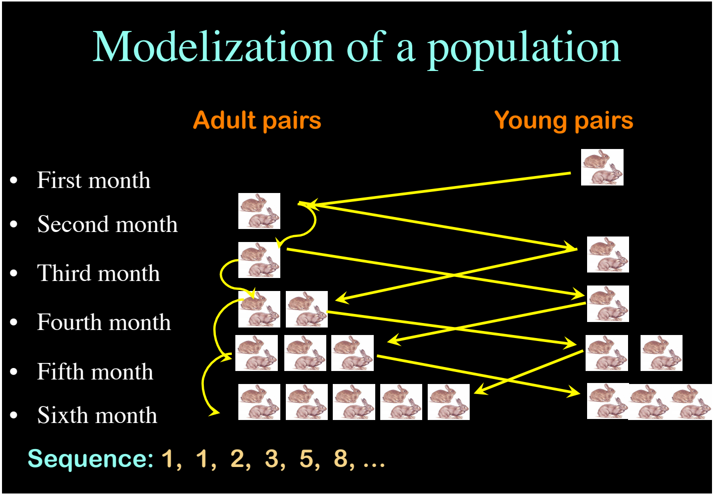
A newly born pair of rabbits, a male and a female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits ; rabbits never die and a mating pair always produces



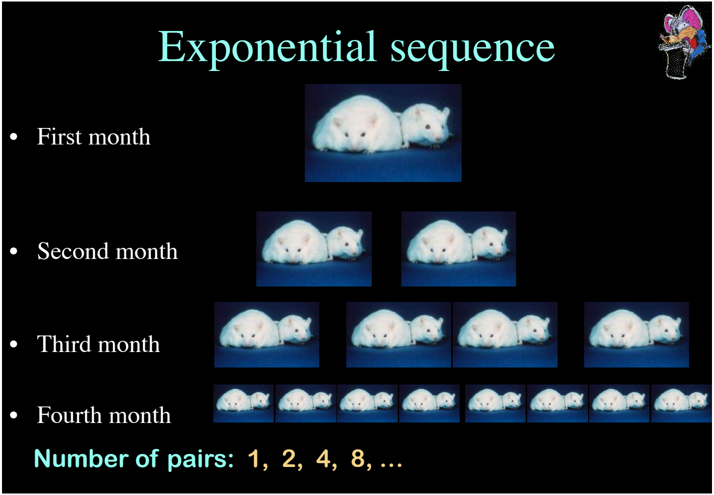
one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was : how many pairs will there be in one year ?

Answer : $F_{12} = 144$.

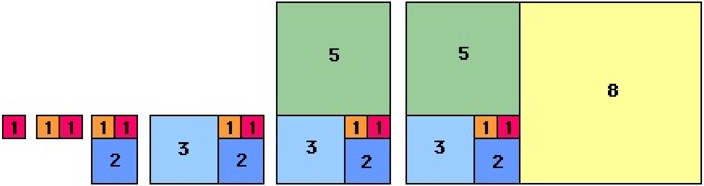
Fibonacci's rabbits



Modelization of a population of mice

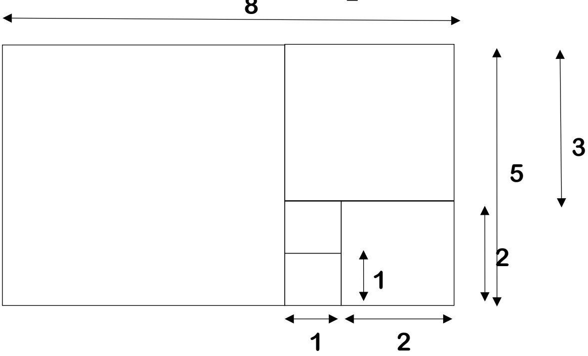


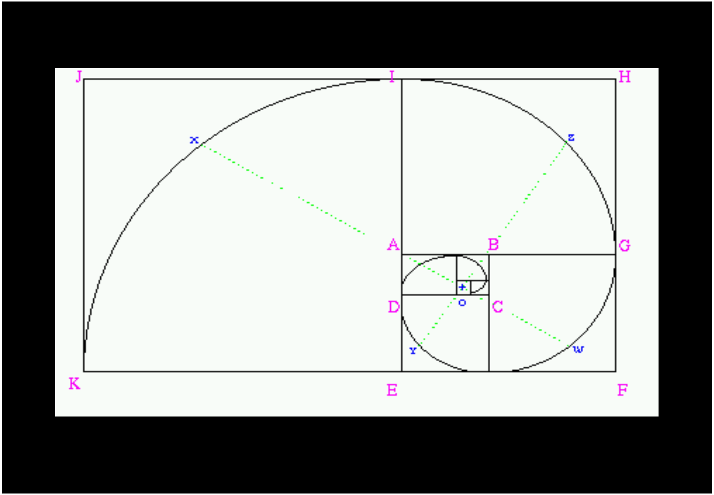
Fibonacci squares



http://mathforum.org/dr.math/faq/faq.golden.ratio.html

Geometric construction of the Fibonacci sequence




J
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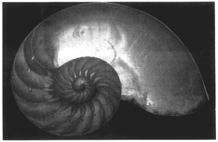
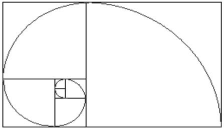
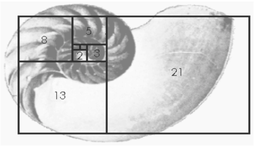
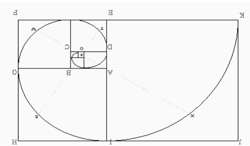
The Fibonacci numbers in nature

Ammonite (Nautilus shape)





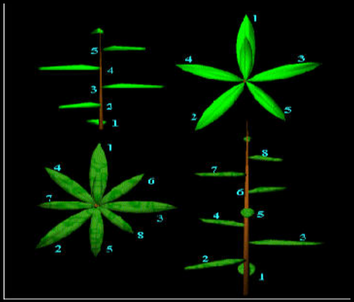




Phyllotaxy

- Study of the position of leaves on a stem and the reason for them
- Number of petals of flowers: daisies, sunflowers, aster, chicory, asteraceae,…
- Spiral patern to permit optimal exposure to sunlight
- Pine-cone, pineapple, Romanesco cawliflower, cactus

Leaf arrangements



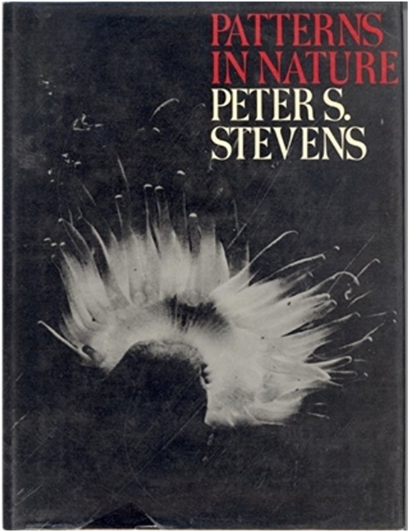
- Université de Nice,
  Laboratoire Environnement Marin Littoral, Equipe d'Accueil "Gestion de la Biodiversité"

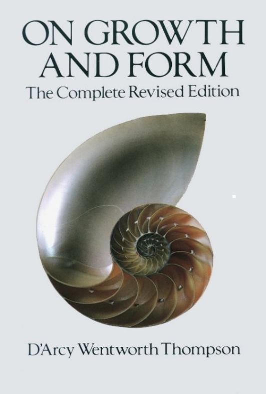
http://www.unice.fr/LEML/coursJDV/tp/tp3.htm

Phyllotaxy

Phyllotaxy

- J. Kepler (1611) uses the Fibonacci sequence in his study of the dodecahedron and the icosaedron, and then of the symmetry of order 5 of the flowers.
- Stéphane Douady and Yves Couder
  Les spirales végétales
  La Recherche *250* (Jan. 1993) vol. **24.**


PATTERNS
IN NATURE
PETER S.
STEVENS


ON GROWTH
AND FORM
The Complete Revised Edition
D'Arcy Wentworth Thompson

Why are there so many occurrences of the Fibonacci numbers and of the Golden ratio in the nature ?

According to Leonid Levin, objects with a small algorithmic Kolmogorov complexity (generated by a short program) occur more often than others.

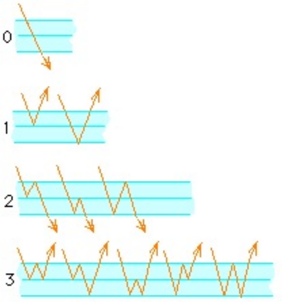
Another example is given by Sierpinski triangles.

Reference : J-P. Delahaye. http://cristal.univ-lille.fr/~jdelahay/pls/

Reflections of a ray of light

Consider three parallel sheets of glass and a ray of light which crosses the first sheet. Each time it touches one of the sheets, it can cross it or reflect on it.

Denote by p_n the number of different paths with the ray going out of the system after n reflections.



$p_0 = 1$,

$p_1 = 2$,

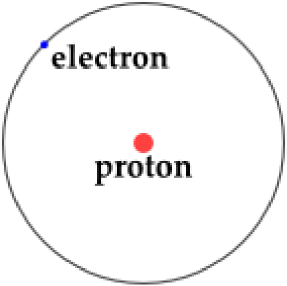
$p_2 = 3$,

$p_3 = 5$.

In general, $p_n = F_{n+2}$.

Levels of energy of an electron of an atom of hydrogen

An atom of hydrogen can have three levels of energy, 0 at the ground level when it does not move, 1 or 2. At each step, it **alternatively** gains and looses some level of energy, either 1 or 2, without going sub 0 nor above 2. Let ℓ_n be the number of different possible scenarios for this electron after n steps.



In general, $\ell_n = F_{n+2}$.

We have $\ell_0 = 1$ (initial state level 0)

$\ell_1 = 2$: state 1 or 2, scenarios (ending with gain) 01 or 02.

$\ell_2 = 3$: scenarios (ending with loss) 010, 021 or 020.

$\ell_3 = 5$: scenarios (ending with gain) 0101, 0102, 0212, 0201 or 0202.

Rhythmic patterns

The Fibonacci sequence appears in Indian mathematics, in connection with Sanskrit prosody. Several Indian scholars, Pingala (200 BC), Virahanka (c. 700 AD), Gopāla (c. 1135), and the Jain scholar Hemachandra (c. 1150). studied rhythmic patterns that are formed by concatenating one beat notes • and double beat notes ■■.

one-beat note • : short syllabe (ti in Morse Alphabet)
double beat note ■■ : long syllabe (ta ta in Morse)

1 beat, 1 pattern : •

2 beats, 2 patterns : • • and ■■

3 beats, 3 patterns : • • •, • ■■ and ■■ •

4 beats, 5 patterns :

• • • •, ■■ • •, • ■■ •, • • ■■, ■■ ■■

n beats, F_{n+1} patterns.

Fibonacci sequence and Golden Ratio

The developments

$$[1], \quad [1,1], \quad [1,1,1], \; [1,1,1,1], \; [1,1,1,1,1], \; [1,1,1,1,1,1], \dots$$

are the quotients

$$\begin{array}{cccccccc} \frac{F_2}{F_1} & \frac{F_3}{F_2} & \frac{F_4}{F_3} & \frac{F_5}{F_4} & \frac{F_6}{F_5} & \frac{F_7}{F_6} & \\ \| & \| & \| & \| & \| & \| & \cdots \\ \frac{1}{1} & \frac{2}{1} & \frac{3}{2} & \frac{5}{3} & \frac{8}{5} & \frac{13}{8} & \end{array}$$

of consecutive Fibonacci numbers.
The development $[1,1,1,1,1,\dots]$ is the continued fraction expansion of the *Golden Ratio*

$$\Phi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1.618\,033\,988\,749\,894\dots$$

which satisfies

$$\Phi = 1 + \frac{1}{\Phi}.$$

The Fibonacci sequence and the Golden ratio

For $n \geq 0$, the Fibonacci number F_n is the nearest integer to

$$\frac{1}{\sqrt{5}}\Phi^n,$$

where Φ is the *Golden Ratio* :

$$\Phi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}.$$

Binet's formula

For $n \geq 0$,

$$F_n = \frac{\Phi^n - (-\Phi)^{-n}}{\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}},$$

Jacques Philippe Marie Binet
(1786 – 1856)

$$\Phi = \frac{1+\sqrt{5}}{2}, \quad -\Phi^{-1} = \frac{1-\sqrt{5}}{2},$$

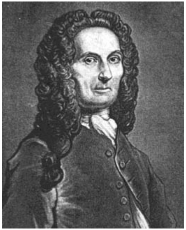
$$X^2 - X - 1 = (X - \Phi)(X + \Phi^{-1}).$$

The so–called Binet Formula

Formula of A. De Moivre (1718, 1730), Daniel Bernoulli (1726), L. Euler (1728, 1765), J.P.M. Binet (1843) : for $n \geq 0$,

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

Abraham de Moivre
(1667–1754)



Daniel Bernoulli
(1700–1782)

Leonhard Euler
(1707–1783)

Jacques P.M. Binet
(1786–1856)

Generating series

A single series encodes all the Fibonacci sequence :

$$\sum_{n\geq 0} F_n X^n = X + X^2 + 2X^3 + 3X^4 + 5X^5 + \cdots + F_n X^n + \cdots$$

Fact : this series is the Taylor expansion of a rational fraction :

$$\sum_{n\geq 0} F_n X^n = \frac{X}{1 - X - X^2}.$$

`Proof` : the product

$$(X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \cdots)(1 - X - X^2)$$

is a telescoping series

$$\begin{aligned} & X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \cdots \\ & \quad -X^2 - X^3 - 2X^4 - 3X^5 - 5X^6 - \cdots \\ & \qquad -X^3 - X^4 - 2X^5 - 3X^6 - \cdots \\ = {} & X. \end{aligned}$$

□

Generating series of the Fibonacci sequence

`Remark.` The denominator $1 - X - X^2$ in the right hand side of

$$X + X^2 + 2X^3 + 3X^4 + \cdots + F_n X^n + \cdots = \frac{X}{1 - X - X^2}$$

is $X^2 f(X^{-1})$, where $f(X) = X^2 - X - 1$ is the irreducible polynomial of the Golden ratio Φ.

Homogeneous linear differential equation

Consider the homogeneous linear differential equation

$$y'' - y' - y = 0.$$

If $y = \mathrm{e}^{\lambda x}$ is a solution, from $y' = \lambda y$ and $y'' = \lambda^2 y$ we deduce

$$\lambda^2 - \lambda - 1 = 0.$$

The two roots of the polynomial $X^2 - X - 1$ are Φ (the Golden ratio) and Φ' with

$$\Phi' = 1 - \Phi = -\frac{1}{\Phi}\cdot$$

A basis of the space of solutions is given by the two functions $\mathrm{e}^{\Phi x}$ and $\mathrm{e}^{\Phi' x}$. Since (Binet's formula)

$$\sum_{n\geq 0} F_n \frac{x^n}{n!} = \frac{1}{\sqrt{5}}\left(\mathrm{e}^{\Phi x} - \mathrm{e}^{\Phi' x}\right),$$

this exponential generating series of the Fibonacci sequence is a solution of the differential equation.

Fibonacci and powers of matrices

The Fibonacci linear recurrence relation $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ can be written

$$\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}.$$

By induction one deduces, for $n \geq 0$,

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

An equivalent formula is, for $n \geq 1$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

Characteristic polynomial

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is

$$\det(XI - A) = \det \begin{pmatrix} X & -1 \\ -1 & X - 1 \end{pmatrix} = X^2 - X - 1,$$

which is the irreducible polynomial of the Golden ratio Φ.

The Fibonacci sequence and the Golden ratio (continued)

For $n \geq 1$, $\Phi^n \in \mathbb{Z}[\Phi] = \mathbb{Z} + \mathbb{Z}\Phi$ is a linear combination of 1 and Φ with integer coefficients, namely

$$\Phi^n = F_{n-1} + F_n\Phi.$$

$$\begin{aligned}
\Phi &= 0 + \Phi \\
\Phi^2 &= 1 + \Phi \\
\Phi^3 &= 1 + 2\Phi \\
\Phi^4 &= 2 + 3\Phi \\
\Phi^5 &= 3 + 5\Phi \\
\Phi^6 &= 5 + 8\Phi \\
\Phi^7 &= 8 + 13\Phi \\
&\vdots
\end{aligned}$$

The Fibonacci Quarterly

The Fibonacci sequence satisfies a lot of very interesting properties. Four times a year, the *Fibonacci Quarterly* publishes an issue with new properties which have been discovered.

The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

ESZTERHÁZY KÁROLY FŐISKOLA
Tittel Pál Könyvtár
& Médiacentrum, Eger

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Narayana's cows

Narayana was an Indian mathematician in the 14th century who proposed the following problem :
A cow produces one calf every year. Beginning in its fourth year each calf produces one calf at the beginning of each year. How many calves are there altogether after, for example, 17 *years ?*

Narayana sequence

https://oeis.org/A000930

Narayana sequence is defined by the recurrence relation

$$C_{n+3} = C_{n+2} + C_n$$

with the initial values $C_0 = 2$, $C_1 = 3$, $C_2 = 4$.

It starts with

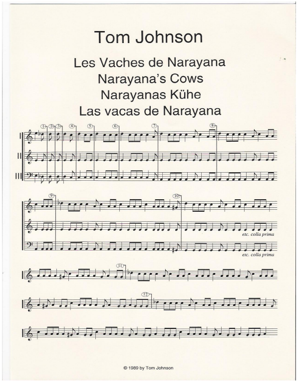
$$2,\ 3,\ 4,\ 6,\ 9,\ 13,\ 19,\ 28,\ 41,\ 60,\ 88,\ 129,\ 189,\ 277, \ldots$$

Real root of $x^3 - x^2 - 1$

$$\frac{\sqrt[3]{\dfrac{29 + 3\sqrt{93}}{2}} + \sqrt[3]{\dfrac{29 - 3\sqrt{93}}{2}} + 1}{3} = 1.465571231876768\ldots$$

Music : http://www.pogus.com/21033.html

In working this out, Tom Johnson found a way to translate this into a composition called *Narayana's Cows.*
Music : Tom Johnson
Saxophones : Daniel Kientzy



Jean-Paul Allouche and Tom Johnson

http://www.math.jussieu.fr/~jean-paul.allouche/bibliorecente.html
http://www.math.jussieu.fr/~allouche/johnson1.pdf

Cows, music and morphisms

Jean-Paul Allouche and Tom Johnson

• Narayana's Cows and Delayed Morphisms
In 3èmes Journées d'Informatique Musicale (JIM '96), Ile de Tatihou, Les Cahiers du GREYC (1996 no. 4), pages 2-7, May 1996.
http://kalvos.org/johness1.html

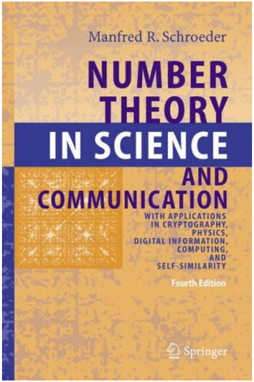
• Finite automata and morphisms in assisted musical composition, Journal of New Music Research, no. 24 (1995), 97 – 108.
http://www.tandfonline.com/doi/abs/10.1080/09298219508570676
http://web.archive.org/web/19990128092059/www.swets.nl/jnmr/vol24_2.html

Music and the Fibonacci sequence

- Dufay, XV$^{\text{ème}}$ siècle
- Roland de Lassus
- Debussy, Bartok, Ravel, Webern
- Stockhausen
- Xenakis
- **Tom Johnson** *Automatic Music for six percussionists*

Number Theory in Science and communication

M.R. Schroeder.
Number theory in science and communication :
with applications in cryptography, physics, digital information, computing and self similarity
Springer series in information sciences **7** 1986.
4th ed. (2006) 367 p.



Applications of Diophantine Approximation

HUA LOO KENG & WANG YUAN – *Application of number theory to numerical analysis*, Springer Verlag (1981).

Hua Loo Keng
(1910 – 1985)

Wang Yuan
(1930 – 2021)

Further applications of Diophantine Approximation include equidistribution modulo 1, discrepancy, numerical integration, interpolation, approximate solutions to integral and differential equations.

http://www-history.mcs.st-and.ac.uk/Biographies/Hua.html

http://www-history.mcs.st-and.ac.uk/PictDisplay/Wang_Yuan.html

The square root of 2, the Golden Ratio and the Fibonacci sequence

Michel Waldschmidt

Professeur Émérite, Sorbonne Université,
Institut de Mathématiques de Jussieu, Paris
http://www.imj-prg.fr/~michel.waldschmidt/