Some remarks on diophantine equations and diophantine approximation

Claude LEVESQUE and Michel WALDSCHMIDT

Dedicated to professor Hà Huy Khoái.

Abstract. We give many equivalent statements of Mahler's generalization of the fundamental theorem of Thue. In particular, we show that the theorem of Thue–Mahler for degree 3 implies the theorem of Thue for arbitrary degree ≥ 3, and we relate it with a theorem of Siegel on the rational integral points on the projective line $\mathbf{P}^1(K)$ minus 3 points.

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1 Introduction

The fundamental theorem of Thue obtained in 1908–1909 can be stated equivalently (Proposition 2.1) as a result about the finiteness of the set of integral points on an algebraic curve, or as a result of diophantine approximation of algebraic numbers by rational numbers improving Liouville's inequality. Over a number field K, Thue's result on Diophantine equations is equivalent (Proposition 3.1) with finite statements on the number of integral points on Thue curves, on Mordell curves, on elliptic curves, on hyperelliptic curves, on superelliptic curves, and also to the finiteness of the set of solutions of the unit equation $E_1 + E_2 = 1$, where the unknowns E_1, E_2 take their values in the group of units of K.

In Proposition 4.1, we will give many equivalent statements of a generalization of this theorem of Thue by Mahler. In particular, we will show that the theorem of Thue–Mahler for degree 3 implies the theorem of Thue for arbitrary degree ≥ 3, and we will relate it with a theorem of Siegel on the integral points on the projective line $\mathbf{P}^1(K)$ minus 3 points. We remark that Siegel's theorem has been generalized by Vojta for the integral points on the projective space $\mathbf{P}^n(K)$ minus $n+2$ hyperplanes (see Corollary 5.2). Vojta's proof rests on the subspace theorem of Schmidt and comes also into play in the work of Hà Huy Khoái [5, 6].

2 Rational approximation and diophantine equations

The following link, between the rational approximation on the one hand and the finiteness of the set of solutions of some diophantine equations on the other hand, happens to be well known thanks to the work of A. Thue.

Proposition 2.1. *Let $f \in \mathbf{Z}[X]$ be an irreducible polynomial of degree d and let $F(X,Y) = Y^d f(X/Y)$ be the associated homogeneous binary form of degree d. Then the following two assertions are equivalent:*
(i) For any integer $k \neq 0$, the set of $(x,y) \in \mathbf{Z}^2$ verifying

$$F(x,y) = k \tag{1}$$

is finite.
(ii) For any real number $\kappa > 0$ and for any root $\alpha \in \mathbf{C}$ of f, the set of rational numbers p/q verifying

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{\kappa}{q^d} \tag{2}$$

is finite.

Condition (i) can be also phrased by stating that for any positive integer k, the set of $(x,y) \in \mathbf{Z}^2$ verifying

$$0 < |F(x,y)| \leq k$$

is finite.
Before proceeding with the proof, a few remarks are in order. When we consider an element $p/q \in \mathbf{Q}$, it should be understood that p and q are integers with $q > 0$ and that if $p = 0$ then $q = 1$. Moreover, the set defined in the assertion (ii) would be the same if we added the condition $\gcd(p,q) = 1$.

In the case when $d = 1$, the two assertions are false. As a matter of fact, if we write $f(X) = a_0X + a_1$ with $a_0 \neq 0$, for $k = a_0$ the equation $a_0X + a_1Y = k$ has an infinite number of solutions (x,y):

$$x = na_1 + 1, \quad y = -na_0 \qquad (n \in \mathbf{Z}),$$

and for $\kappa = |a_1|/a_0$ the root $\alpha = -a_1/a_0$ of f has an infinite number of approximations p/q satisfying (2) with $\gcd(p,q) = 1$, namely when

$$\frac{p}{q} = \frac{-na_1}{na_0 - 1}$$

for all integers $n > 0$ (with $n > 1$ whenever $a_0 = 1$).

In the case when $d = 2$, the two assertions can be true, take for instance $f(X) = X^2 + a$ with $a \in \mathbf{Z}$, $a > 0$, and both of them can also be false, take for instance $f(X) = X^2 - a$ with $a \in \mathbf{Z}$, $a > 0$ squarefree. For $d \geq 3$, we know, since the work of Thue, that these two assertions are true. The statement in (ii) with $d \geq 3$ is the first improvement of the Liouville inequality, and in particular it gave birth to the works of C.L. Siegel, F. Dyson, Th. Schneider, K.F. Roth and W.M. Schmidt, culminating with the subspace theorem, including a number of variations with a lot of applications.

Proof of Proposition 2.1. Write

$$f(X) = a_0 X^d + a_1 X^{d-1} + \cdots + a_{d-1} X + a_d$$

and

$$F(X,Y) = a_0 X^d + a_1 X^{d-1} Y + \cdots + a_{d-1} X Y^{d-1} + a_d Y^d.$$

Without loss of generality we may assume $a_0 > 0$.

(1) Suppose now that the assertion (i) is true. Consider a root α of f, a number $\kappa > 0$ and a rational number p/q verifying (2). Without loss of generality we can suppose $q^d \geq \kappa$. We have

$$F(X,Y) = a_0 \prod_\sigma (X - \sigma(\alpha) Y),$$

where σ in the product runs through the set of embeddings of the field $K := \mathbf{Q}(\alpha)$ in $\mathbf{C}$. The element α is in $\mathbf{C}$ and we write Id for the inclusion of K into $\mathbf{C}$. Hence

$$|F(p,q)| = a_0 q^d \left| \alpha - \frac{p}{q} \right| \prod_{\sigma \neq \mathrm{Id}} \left| \sigma(\alpha) - \frac{p}{q} \right|.$$

For $\sigma \neq \mathrm{Id}$, we use the upper bound

$$\left| \sigma(\alpha) - \frac{p}{q} \right| \leq |\alpha - \sigma(\alpha)| + \left| \alpha - \frac{p}{q} \right| \leq |\alpha - \sigma(\alpha)| + 1,$$

which comes from (2) and from $q^d \geq k$. Therefore

$$0 < |F(p,q)| \leq a_0 \kappa \prod_{\sigma \neq \mathrm{Id}} \big(|\alpha - \sigma(\alpha)| + 1 \big).$$

The assertion (i) allows to conclude that the set of elements p/q is finite, from which we deduce the assertion (ii).

(2) Conversely, suppose that the assertion (ii) is true. Let k be a non–zero integer and let $(x,y) \in \mathbf{Z}^2$ satisfy $F(x,y) = k$. We want to show, by assuming (ii), that these couples (x,y) belong to a finite set. Without loss of generality, we may suppose $|y|$ sufficiently large. Let α be a root of f at a minimal distance from x/y. Remark that

$$|k| \;=\; |F(x,y)| \;=\; a_0 |y|^d \left| \alpha - \frac{x}{y} \right| \prod_{\sigma \neq \mathrm{Id}} \left| \sigma(\alpha) - \frac{x}{y} \right| \;\geq\; a_0 |y|^d \left| \alpha - \frac{x}{y} \right|^d,$$

whereupon

$$\left|\alpha - \frac{x}{y}\right|^d \leq \frac{|k|}{a_0|y|^d}\cdot$$

Therefore, for $|y|$ sufficiently large, for instance with

$$|y|^d \geq \frac{2^d|k|}{a_0 \min\limits_{\sigma\neq \mathrm{Id}}(|\alpha-\sigma(\alpha)|^d)},$$

we come up with the inequality

$$\left|\alpha - \frac{x}{y}\right| \;\leq\; \frac{1}{2}\min_{\sigma\neq \mathrm{Id}}(|\alpha-\sigma(\alpha)|,$$

which allows us to deduce that for any $\sigma \neq \mathrm{Id}$, we have

$$\left|\sigma(\alpha) - \frac{x}{y}\right| \;\geq\; \frac{1}{2}|\alpha-\sigma(\alpha)|.$$

Since f is irreducible,

$$f'(\alpha) = a_0 \prod_{\sigma\neq \mathrm{Id}} \bigl(\alpha-\sigma(\alpha)\bigr) \neq 0.$$

Hence we deduce

$$|k| \;=\; |F(x,y)| \;=\; a_0|y|^d\left|\alpha - \frac{x}{y}\right| \prod_{\sigma\neq \mathrm{Id}}\left|\sigma(\alpha) - \frac{x}{y}\right| \;\geq\; 2^{-d+1}|y|^d|f'(\alpha)|\cdot\left|\alpha - \frac{x}{y}\right|,$$

from which we come up with

$$\left|\alpha - \frac{x}{y}\right| \;\leq\; \frac{\kappa}{|y|^d} \quad \text{with} \quad \kappa = \frac{2^{d-1}|k|}{|f'(\alpha)|}\cdot$$

From the the assertion (ii), we can say that the set of rational numbers x/y verifiying this inequality is finite. This allows to conclude that the assertion (i) is true. □

3 Diophantine equations and unit equations

In section 2, we considered the basic situation of rational numbers and points with rational integer coordinates on Thue curves. Here we consider the algebraic numbers while the number field K may vary. We denote by $\mathbf{Z}_K$ the ring of algebraic integers of K and by $\mathbf{Z}_K^\times$ the unit group of K. In the next section, we will consider the S–integers of some fixed number field K. When S is a finite set of places of K containing all the archimedean places, we denote by O_S the ring of S–integers of K and by $O_S^\times$ the group of S–units of K. Let us recall (see for instance [11] §7.1, [12] Chap .7, [15] §3.3.2) that an element α in K is called *an S–integer* if $|\alpha|_v \leq 1$ for all $v \notin S$. The S–units are the invertible elements in O_S, hence they are the elements ε in K which satisfy $|\alpha|_v = 1$ for all $v \notin S$. Let us quote some results whose proofs appear in [14].

Proposition 3.1. *The following statements are equivalent:*

• (M) *For any number field K and for any non–zero element k in K, the Mordell equation*

$$Y^2 = X^3 + k$$

has but a finite number of solutions $(x,y) \in \mathbf{Z}_K \times \mathbf{Z}_K$.

• (E) *For any number field K and for any polynomial f in $K[X]$ of degree* 3 *with three distinct complex roots, the elliptic equation*

$$Y^2 = f(X)$$

has but a finite number of solutions $(x,y) \in \mathbf{Z}_K \times \mathbf{Z}_K$.

• (HE) *For any number field K and for any polynomial f in $K[X]$ with at least three distinct complex roots, the hyperelliptic equation*

$$Y^2 = f(X)$$

has but a finite number of solutions $(x,y) \in \mathbf{Z}_K \times \mathbf{Z}_K$.

• (SE) *For any number field K, for any integer $m \geq 3$ and for any polynomial f in $K[X]$ with at least two distinct complex roots whose orders of multiplicity are prime to m, the superelliptic equation*

$$Y^m = f(X)$$

has but a finite number of solutions $(x,y) \in \mathbf{Z}_K \times \mathbf{Z}_K$.

• (T) *For any number field K, for any non–zero element k in K and for any elements $\alpha_1, \ldots, \alpha_n$ in K with* $\mathrm{Card}\{\alpha_1, \ldots, \alpha_n\} \geq 3$, *the Thue equation*

$$(X - \alpha_1 Y) \cdots (X - \alpha_n Y) = k$$

has but a finite number of solutions $(x,y) \in \mathbf{Z}_K \times \mathbf{Z}_K$.

• (S) *For any number field K and for any elements a_1 and a_2 in K with $a_1 a_2 \neq 0$, the Siegel equation*

$$a_1 E_1 + a_2 E_2 = 1$$

has but a finite number of solutions $(\varepsilon_1, \varepsilon_2) \in \mathbf{Z}_K^\times \times \mathbf{Z}_K^\times$.

Each of these statements is a theorem: the first four ones are due to Siegel who proved that the sets of integral points respectively on a Mordell curve (M), on an elliptic curve (E), on an hyperelliptic curve (HE), on a superelliptic curve (SE) are finite. Statement (T) is due to Thue and (S) deals with the unit equation introduced by Siegel.

The proof of the equivalence given in [14] (see also [2, 7, 9, 10, 12, 15]) is elementary; it goes as follows:

$$\begin{array}{ccccc} \text{(SE)} & \Longrightarrow & \text{(M)} & \Longleftarrow & \text{(E)} \\ \Uparrow & & \Downarrow & & \Uparrow \\ \text{(T)} & \Longleftarrow & \text{(S)} & \Longrightarrow & \text{(HE)} \end{array}$$

The three implications which are not so easy to prove are

$$\text{(T)} \Longrightarrow \text{(SE)}, \quad \text{(S)} \Longrightarrow \text{(T)} \quad \text{and} \quad \text{(S)} \Longrightarrow \text{(HE)}.$$

4 Thue, Mahler, Siegel, Vojta

The aim of this section is to establish an equivalence between many assertions. The first two concern Thue–Mahler equations; we prove the very interesting fact that it suffices to solve the equation for the very special case of the cubic form $XY(X-Y)$ in order to deduce the general case. The next assertion is a theorem of Siegel on the finiteness of the number of solutions of an equation of the form $E_1 + E_2 = 1$ in S–units $\varepsilon_1, \varepsilon_2$ of a number field. The fourth (resp. fifth) assertion is the particular case $n = 1$ (resp. $n = 2$) of the theorem of Vojta stating that any set of S–integral points on $\mathbf{P}^n(K)$ minus $n+2$ hyperplanes is contained in an algebraic hypersurface.

We will consider an algebraic number field K and a finite set S of places of K containing all the archimedean places. Moreover F will denote a binary homogeneous form with coefficients in K. We will consider the Thue–Mahler equations $F(X,Y) = E$ where the two unknowns X, Y take respectively values x, y in a given set of S–integers of K while the unknown E takes its values ε in the set of S–units of K. If (x, y, ε) is a solution and if m denotes the degree of F, then, for all $\eta \in O_S^\times$, the triple $(\eta x, \eta y, \eta^m \varepsilon)$ is also a solution.

Definition. Two solutions (x, y, ε) and (x', y', ε') in $O_S^2 \times O_S^\times$ of the equation $F(X,Y) = E$ are said to be *equivalent modulo* $O_S^\times$ if the points of $\mathbf{P}^1(K)$ with projective coordinates $(x : y)$ and $(x' : y')$ are the same.

If the two solutions (x, y, ε) and (x', y', ε') are equivalent, there exists $\eta \in K^\times$ such that $x' = \eta x$ and $y' = \eta y$. Since (x, y, ε) and (x', y', ε') are solutions of the equation $F(X,Y) = E$, we also have $\varepsilon' = \eta^m \varepsilon$ where m is the degree of the binary homogeneous form $F(X,Y)$. Since ε and ε' are S–units, η^m is also an S–unit, hence $\eta \in O_S^\times$. In other terms, two solutions (x, y, ε) and (x', y', ε') are equivalent if there exists $\eta \in O_S^\times$ such that

$$x' = \eta x, \quad y' = \eta y, \quad \varepsilon' = \eta^m \varepsilon.$$

Definition. We will say that such a Thue–Mahler equation has *but a finite number of classes of solutions* if the set of solutions $(x, y, \varepsilon) \in O_S^2 \times O_S^\times$ can be written as the union of a finite number of equivalence classes modulo $O_S^\times$.

This last definition is equivalent to saying that the set of points $(x : y)$ in $\mathbf{P}^1(K)$ for which there exists $\varepsilon \in O_S^\times$ such that (x, y, ε) is a solution is finite.

Proposition 4.1. *Let K be an algebraic number field.*
(1) *The following four assertions are equivalent:*
(i) *For any finite set S of places of K containing all the archimedean places, for every $k \in K^\times$ and for any binary homogeneous form $F(X,Y)$ with the property that the polynomial $F(X,1) \in K[X]$ has at least three linear factors involving three distinct roots in K, the Thue-Mahler equation*

$$F(X,Y) = kE$$

has but a finite number of classes of solutions $(x, y, \varepsilon) \in O_S^2 \times O_S^\times$.

(ii) *For any finite set* S *of places of* K *containing all the archimedean places, the Thue-Mahler equation*

$$XY(X - Y) = E$$

has but a finite number of classes of solutions $(x, y, \varepsilon) \in O_S^2 \times O_S^\times$.

(iii) *For any finite set* S *of places of* K *containing all the archimedean places, the* S*–unit equation*

$$E_1 + E_2 = 1$$

has but a finite number of solutions $(\varepsilon_1, \varepsilon_2)$ *in* $O_S^\times \times O_S^\times$.

(iv) *For any finite set* S *of places of* K *containing all the archimedean places, every set of* S*–integral points on* $\mathbf{P}^1(K)$ *minus three points is finite.*

(2) *Moreover, these assertions are consequences of the following one:*

(v) *For any finite set* S *of places of* K *containing all the archimedean places, every set* A *of* S*–integral points on the open variety* $\mathbf{V}$*, obtained by removing from* $\mathbf{P}^2(K)$ *four hyperplanes, is* degenerated*, (i.e., there is a non–zero homogeneous polynomial in* $K[X, Y, E]$ *which vanishes on* A*).*

Before proceeding with the proof, a few remarks are in order. These assertions are true: they are theorems essentially going back to the work of K. Mahler [7, 9, 10, 12, 15]. In (iii) the finiteness statement for the number of solutions of the unit equation was singled out by C.L. Siegel, K. Mahler and S. Lang [7, 9, 10, 12, 15]. The assertion (iv) (resp. (v)) is the particular case $n = 1$ (resp. $n = 2$) of a theorem of P. Vojta who stated it for $\mathbf{P}^n(K)$ minus $n + 2$ hyperplanes, and more generally for a variety minus a suitable divisor (see §5). Moreover, the three missing points in (iv) are classically denoted

$$\mathbf{0} = (0 : 1), \quad \mathbf{1} = (1 : 1), \quad \boldsymbol{\infty} = (1 : 0). \tag{3}$$

It should now be clear that the spirit of the last proposition is to state that the truth of each of the four assertions implies the truth of each of the three other ones.

The remarkably powerful subspace theorem of W. Schmidt, which is the source of all results in this paper, generated vast generalisations of these five assertions together with the statements of Proposition 2.1. The methods of C.L. Siegel, F. Dyson, Th. Schneider, K.F. Roth and W.M. Schmidt are not effective. They allow us to give upper bounds for the number of solutions or of classes of solutions, but one had to wait till the major breakthrough of A. Baker [2, 12] to obtain explicit bounds for the solutions themselves, which bounds we cannot avoid when we want to solve completely these equations.

Let us also make some remarks which will prove useful when we shall deal with the assertion (iv). There is a general notion of *set of integral points on a projective variety relative to a very ample effective divisor* (see for instance [13], Chap. 1, §4). We will deal with the very special case of this situation where the variety is a projective space $\mathbf{P}^n(K)$ and the divisor is a union of finitely many hyperplanes. For this special case see also [15], Remark 3.14.

We take projective coordinates $(X : Y)$ on $\mathbf{P}^1(K)$. A point on $\mathbf{P}^1(K)$ which is not $(1 : 0)$ has projective coordinates $(\alpha : 1)$ for some $\alpha \in K$. This point is called *an S–integral point on* $\mathbf{P}^1(K) \setminus \{(1:0)\}$ if and only if α is an S–integer. It is clear that if α is an S–integer, then, for each place v of S, it reduces, in the projective line on the residue field, to a point which is not $(1 : 0)$. The converse is true. Indeed, if α is not an S–integer, then there is a place v of K not in S such that $|\alpha|_v > 1$. For this place v the reduction of $(\alpha : 1) = (1 : \alpha^{-1})$, in the projective line on the residue field, is $(1 : 0)$.

Suppose now that the projective coordinates of an S–integral point on $\mathbf{P}^1(K) \setminus \{(1 : 0)\}$ are $(u : 1)$. Then this point is also an S–integral point on $\mathbf{P}^1(K)\setminus\{(0 : 1)\}$ if and only if, for each place v of S, it reduces, in the projective line on the residue field, to a point which is not $(0 : 1)$, hence if and only if u is an S–unit. If these conditions are satisfied, then the same point $(u : 1)$ is an S–integral point on $\mathbf{P}^1(K) \setminus \{(1 : 1)\}$ if and only if $u - 1$ is an S–unit of K.

In the same way, a point on $\mathbf{P}^n(K)$ which is not in the hyperplane H_0 of equation $X_0 = 0$ has coordinates $(1 : \alpha_1 : \cdots : \alpha_n)$. It is an *S–integral point on* $\mathbf{P}^n(K) \setminus H_0$ if and only if $\alpha_1, \dots, \alpha_n$ are in O_S. This is equivalent to the fact that, for each place v of S, it reduces, in the projective space $\mathbf{P}^n$ on the residue field, to a point which is not $(1 : 0 : \cdots : 0)$. Further, for $1 \le i \le n$, denote by H_i the hyperplane of equation $X_i = 0$. Then the point $(1 : \alpha_1 : \cdots : \alpha_n)$ is an *S–integral point on* $\mathbf{P}^n(K) \setminus (H_0 \cup \cdots \cup H_n)$ if and only if $\alpha_1, \dots, \alpha_n$ are in $O_S^\times$. Furthermore, if this conditions are satisfied, then the same point is an S–integral point on the complement of the hyperplane of equation $X_0 + \cdots + X_n = 0$ if and only if $1 + \alpha_1 + \cdots + \alpha_n$ is an S–unit.

Dealing with the standard hyperplanes allows us to avoid denominators. For the more general case of hyperplanes in $\mathbf{P}^n$, one allows a bounded denominator (see e.g. [8] §2 p. 259–260), which is what Serre calls *quasi integral sets on an affine variety* in [11], §7.1 and §8.

The S–unit equation $E_1 + E_2 = 1$ in assertion (iii) is in a non–homogeneous form. The associated homogeneous S–unit equation is $E_1 + E_2 = E_3$, a special case of the generalized Siegel unit equation we will consider in §5.

Definition. Two solutions $(\varepsilon_0, \dots, \varepsilon_n)$ and $(\varepsilon'_0, \dots, \varepsilon'_n)$ in $O_S^\times$ of the equation $E_0 + \cdots + E_n$ are said to be equivalent modulo $O_S^\times$ if the points of $\mathbf{P}^n(K)$ with projective coordinates $(\varepsilon_0 : \cdots : \varepsilon_n)$ and $(\varepsilon_0 : \cdots : \varepsilon_n)$ are the same.

This means that there exists $\eta \in O_S^\times$ such that

$$\varepsilon_j = \eta\varepsilon_j \quad \text{for} \quad 0 \le j \le n.$$

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Proof of Proposition 4.1. If the homogeneous linear form F of degree $n \ge 3$ in assertion (i) is such that $F(X, 1)$ has at least three linear factors involving three distinct roots $\alpha_1, \alpha_2, \alpha_3$ in K, then there exists a homogeneous linear form $H(X, Y) \in K[X, Y]$ of degree $n - 3 \ge 0$ such that

$$F(X, Y) = (X - \alpha_1 Y)(X - \alpha_2 Y)(X - \alpha_3 Y)H(X, Y), \tag{4}$$

where $H(X,1)$ need not be a monic polynomial and may have its roots outside K (though assuming $H(X,1)$ to be monic with roots in K would not restrict the generality). Moreover, we let $d \in \mathbf{Z}$ be a positive integer such that $dH \in \mathbf{Z}[X,Y]$.

We are going to prove the implications

$$(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (i) \quad \text{and} \quad (iii) \Longleftrightarrow (iv) \quad \text{and} \quad (v) \Longrightarrow (iii).$$

This will complete the proof of Proposition 4.1.

$(i) \Longrightarrow (ii)$. We make a change of variables

$$X' = X - Y, \quad Y' = X + Y$$

and we apply (i) to the cubic form $F(X',Y') = X'(X'-Y')(X'+Y')$. □

$(ii) \Longrightarrow (iii)$. Let $(\varepsilon_1, \varepsilon_2) \in O_S^\times$ satisfy $\varepsilon_1 + \varepsilon_2 = 1$. Set $x = 1$ and $y = \varepsilon_1$, so that

$$xy(x-y) = \varepsilon_1\varepsilon_2.$$

Each class modulo $O_S^\times$ of solutions $(x,y,\varepsilon) \in O_S^2 \times O_S^\times$ of $XY(X-Y) = E$ contains a single element with the first component 1, namely $(1, x^{-1}y, x^{-3}\varepsilon)$. Since there is a finite number of classes of solutions, the set of $(1, \varepsilon_1, \varepsilon_1\varepsilon_2)$ with $\varepsilon_1 + \varepsilon_2 = 1$ is finite, hence there is only a finite number of ε_1's in $O_S^\times$ such that $1 - \varepsilon_1 \in O_S^\times$. □

$(iii) \Longrightarrow (i)$. Suppose that the assertion (iii) is true and that $(x,y,\varepsilon) \in O_S^2 \times O_S^\times$ is a solution of the equation $F(X,Y) = kE$. Write, as in (4),

$$F(X,Y) = (X - \alpha_1 Y)(X - \alpha_2 Y)(X - \alpha_3 Y)H(X,Y),$$

where $\alpha_1, \alpha_2, \alpha_3$ are three roots of $F(X,1)$ which are distinct and in K.

Define $\beta_i = x - \alpha_i y$ $(i = 1,2,3)$ so that $\beta_1\beta_2\beta_3 H(x,y) = k\varepsilon$. Then we eliminate x and y from these three linear relations defining β_1, β_2 and β_3 to obtain the homogeneous unit equation (already considered by Siegel)

$$(\alpha_1 - \alpha_2)\beta_3 + (\alpha_2 - \alpha_3)\beta_1 + (\alpha_3 - \alpha_1)\beta_2 = 0. \tag{5}$$

Define S to be the set of places given by the assertion (i), and apply (iii) with the set S' obtained by adding to S the places of K dividing numerators and denominators of the fractional principal ideals (k), (d), $(\alpha_i - \alpha_j)$ $(1 \le i < j \le 3)$. Hence the three terms $(\alpha_i - \alpha_j)\beta_k$ of the left member of (5) are S'–units. We deduce from (iii) that the quotients β_i/β_j $(i,j = 1,2,3)$ belong to a fixed finite set, say, $\{\gamma_1, \ldots, \gamma_t\}$ which is independent of the solution (x,y,ε) considered. Suppose that $\beta_2 = \gamma\beta_1$ with $\gamma \in \{\gamma_1, \ldots, \gamma_t\}$. Set $\eta = \beta_1$ (recall that β_1 is an S'–unit),

$$x_0 = \frac{\alpha_1\gamma - \alpha_2}{\alpha_1 - \alpha_2}, \quad y_0 = \frac{\gamma - 1}{\alpha_1 - \alpha_2} \quad \text{and} \quad \varepsilon_0 = k^{-1}F(x_0, y_0).$$

Then from the values of β_1 and of β_2 $(=\gamma\beta_1)$, we obtain

$$x = x_0\eta, \quad y = y_0\eta, \quad \varepsilon = \varepsilon_0\eta^n.$$

We deduce that modulo $O_{S'}^\times$ there is only a finite number of classes of solutions of $F(X,Y) = kE$. This allows to conclude that the assertion (i) is true. □

$(iv) \Longrightarrow (iii)$. Let $\mathcal{E}$ be the set of $(\varepsilon_1, \varepsilon_2) \in O_S^\times \times O_S^\times$ for which $\varepsilon_1 + \varepsilon_2 = 1$. Then the set

$$\{(\varepsilon_1 : 1) \;;\; \text{there exists } \varepsilon_2 \in O_S^\times \text{ such that } (\varepsilon_1, \varepsilon_2) \in \mathcal{E}\}$$

is a set of S–integral points on $\mathbf{P}^1(K) \setminus \{\mathbf{0}, \mathbf{1}, \boldsymbol{\infty}\}$, where $\mathbf{0}, \mathbf{1}, \boldsymbol{\infty}$ are defined in (3), hence it is finite by (iv), and (iii) follows. □

$(iii) \Longrightarrow (iv)$. Let A be a set of S-integral points $(x : y)$ on $\mathbf{P}^1(K)$ minus three points chosen (without loss of generality) to be $\mathbf{0}, \mathbf{1}, \boldsymbol{\infty}$, as defined in (3). Since A is contained in $\mathbf{P}^1(K) \setminus \{(1:0)\}$, each element P in A has projective coordinates $(u : 1)$ with $u \in K$. Since P does not reduce modulo a finite place v to any of the three points $(1 : 0)$, $(0 : 1)$, $(1 : 1)$, it follows that u and $u' := 1 - u$ are S–units. From $u + u' = 1$ we deduce from (iii) that the set of such u's is finite, hence A is finite. □

$(v) \Longrightarrow (iii)$. Take projective coordinates $(E : E_1 : E_2)$ on $\mathbf{P}^2(K)$ and consider the four hyperplanes H_0, H_1, H_2, H_3 defined respectively by the equations

$$E = 0, \;\; E_1 = 0, \;\; E_2 = 0, \;\; E_1 + E_2 = 0.$$

Let $\mathcal{E}$ be the subset of $(O_S^\times)^2$ which consists of the couples $(\varepsilon_1, \varepsilon_2)$ of S–units verifying $\varepsilon_1 + \varepsilon_2 = 1$. For any ε in $O_S^\times$, the point of $P^2(K)$ with projective coordinates $(\varepsilon : \varepsilon_1 : \varepsilon_2)$ is an S–integral point on $\mathbf{P}^2(K) \setminus (H_0 \cup H_1 \cup H_2 \cup H_3)$. Indeed, such a point reduces modulo each place v in S to a point on the projective plane over the residue field which is not on any of the four corresponding hyperplanes. We deduce from (v) the existence of a non–zero homogeneous polynomial $P(E, E_1, E_2)$ of $K[E, E_1, E_2]$ which is annihilated by each of the points in $O_S^\times \times \mathcal{E}$. Assuming (without loss of generality) that $O_S^\times$ is infinite, it follows that for each $(\varepsilon_1, \varepsilon_2) \in \mathcal{E}$ the polynomial $P(E, \varepsilon_1, \varepsilon_2) \in K[E]$ is the zero polynomial, whereupon the assertion (iii) is true. □

This concludes the proof of the fact that indeed the first four assertions of Proposition 4.1 are equivalent to one another and are consequences of the fifth one. □

It may be a fruitful goal to devise further proofs of direct implications between the assertions of Proposition 4.1: taking shortcuts may be useful for further investigations, and we hope that the proofs of these implications are interesting *per se*. In particular, there are at least two points of view for obtaining sharper statements, and for each of them there is a whole variety of methods, involving deep and powerful tools from Diophantine approximation. Firstly, by having an effective statement via an explicit upper bound for the number of solutions or of classes of solutions. Secondly, by giving an upper bound for the height of the solutions, which is the effective way of dealing with the theory. When it comes to establishing such explicit versions of those mentioned implications, using no detours may prove a winning strategy to obtain more precise bounds. This is why we now prove directly the next implication.

$(ii) \Longrightarrow (i)$. Suppose that the assertion (ii) is true. We want to prove (i) for a homogeneous binary form of degree n that we write as in (4). Change the variables as follows: set

$$X' = (\alpha_2 - \alpha_3)(X - \alpha_1 Y), \quad Y' = (\alpha_1 - \alpha_3)(X - \alpha_2 Y),$$

so that

$$X' - Y' = (\alpha_2 - \alpha_1)(X - \alpha_3 Y).$$

Given the set S of (i), we will use the set S' of (ii) which is the union of S with the set of places of K dividing numerators and denominators of the fractional principal ideals (d), (α_1), (α_2), (α_3), $(\alpha_2 - \alpha_3)$, $(\alpha_1 - \alpha_3)$, $(\alpha_2 - \alpha_1)$ and (k), and also the principal ideals generated by the coefficients of the form dH.

If $(x, y, \varepsilon) \in O_S^2 \times O_S^\times$ satisfies $F(x, y) = k\varepsilon$, then the corresponding elements x', y' obtained by the change of variables are S'–integers with the property that the number $\varepsilon' := x'y'(x' - y')$ is an S'–unit. The assertion (ii) provides the finiteness of the set of classes modulo $O_{S'}^\times$ of solutions (x', y', ε') in $O_{S'}^2 \times O_{S'}^\times$ of the equation $X'Y'(X' - Y') = E'$, and we deduce that the assertion (i) is true. □

From the equivalence between (i) and (ii), we deduce that these two properties are also equivalent to the special case of (i) where one assumes $H = 1$ (hence the form F is a cubic form with $F(X, 1)$ a monic polynomial), so that

$$F(X, Y) = (X - \alpha_1 Y)(X - \alpha_2 Y)(X - \alpha_3 Y) \in K[X, Y]$$

and where one assumes also $k = 1$.

We conclude this section with the remark that it would be very interesting to produce a proof of (v) as a consequence of the previous assertions (we already pointed out that all of them are theorems). Indeed, the assertion (v) has further far reaching consequences, besides assertions (i) to (iv). In particular it can be used to prove that any homogeneous diophantine unit equation

$$E_1 + E_2 + E_3 + E_4 = 0$$

has only finitely many solutions $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ in S–units for which none of the three subsums

$$\varepsilon_1 + \varepsilon_2, \quad \varepsilon_1 + \varepsilon_3, \quad \varepsilon_1 + \varepsilon_4$$

vanishes. So far, no effective proof of this result has been produced in general, while effective versions of assertions (i) to (iv) are known.

5 Generalized Siegel unit equation and Vojta's theorem

This section is motivated by the implication $(v) \Rightarrow (iv)$ of Proposition 4.1. When K is a number field, we show how to deduce Vojta's Theorem on the S–integral points on $\mathbf{P}^n(K)$ minus $n+2$ hyperplanes from the result on the generalized unit equation which happens to be a consequence of Schmidt's subspace Theorem.

Theorem 5.1 on the generalized unit equation (see [13], Theorem 2.3.1, [4], Theorem 7.24 and [15] remark 3.14(v)) has been proved independently by J.H. Evertse on the one hand, by H.P. Schlickewei and A.J. van der Poorten (1982) on the other hand. A special (but significant) case had been obtained earlier by E. Dubois and G. Rhin (see [15]).

Theorem 5.1. *Let $n \geq 1$ be an integer. Then for any number field K, any finite set S of places of K containing all the archimedean places, the equation*

$$E_0 + \cdots + E_n = 0$$

has only finitely many classes modulo $O_S^\times$ of solutions $(\varepsilon_0, \ldots, \varepsilon_n) \in (O_S^\times)^{n+1}$ for which no proper subsum $\sum_{i\in I} \varepsilon_i$ vanishes, with I being a subset of $\{0, \ldots, n\}$, with at least two elements and at most n.

There is a more general version of Theorem 5.1, where the number field is replaced by any field K of zero characteristic, and the group of S–units is replaced by any subgroup of $K^\times$ of finite rank. The first general result in this direction is due to M. Laurent, it has been extended by Schmidt, Evertse, van der Poorten and Schlickewei (see [3], §7.4), and recently refined by Amoroso and Viada (Theorem 6.2 of [1]).

In his thesis on integral points on a variety (1983), P. Vojta started a fertile analogy between Diophantine approximation and Nevanlinna theory. In the case of holomorphic functions, the analog of Theorem 5.1 is a result of E. Borel in 1896 (see [13], Chap. 2, §4) according to which, if $g_1, \ldots, g_n$ are entire functions satisfying $e^{g_1} + \cdots + e^{g_n} = 1$, then some g_i is constant. A connexion between Theorem 5.1 on S–units and integral points on the complement in a projective space of a divisor was found by P. Vojta. In 1991, Min Ru and Pit Man Wong considered the case when the divisor is a union of $2n+1$ hyperplanes in general position and showed that the set of S–integral points is finite. Independently, K. Győry proved the same result in 1994, but formulated it in terms of decomposable form equations (see e.g. [8] p. 261 for the dictionary between decomposable form equations and integral points on the complements of hypersurfaces). Further related results are due to Ta Thi Hoai An, Julie Tzu-Yueh Wang, Zhihua Chen, and more recently Aaron Levin (see [8]).

Here we deduce from Theorem 5.1 the following result, which may be seen as a theorem on integral points which partially extends Siegel's Theorem to higher dimensional varieties [13].

Corollary 5.2. *Let $n \geq 1$ be an integer. Then for any number field K, any finite set S of places of K containing all the archimedean places and any set of*

$n+2$ distinct hyperplanes $H_0, \ldots, H_{n+1}$ in $\mathbf{P}^n(K)$, the set of S–integral points on $\mathbf{P}^n(K) \setminus (H_0 \cup \cdots \cup H_{n+1})$ is contained in a finite union of proper linear subspaces of $\mathbf{P}^n(K)$.

For hyperplanes in a general position, Theorem 5.1 and Corollary 5.2 are equivalent. We show how to deduce 5.2 from Theorem 5.1 without restriction.

Proof. Let $X_0, \ldots, X_n$ be projective coordinates on $\mathbf{P}^n(K)$ and let $L_0, \ldots, L_{n+1}$ be homogeneous linear forms in $X_0, \ldots, X_n$ such that, for $i = 0, \ldots, n+1$, the hyperplane H_i is given by the equation $L_i = 0$. Let $r+1$ be the rank of the system of linear forms $L_0, \ldots, L_{n+1}$. Reorder the forms so that $L_0, \ldots, L_r$ are linearly independent, and such that L_{r+1} can be written as $a_0L_0 + \cdots + a_mL_m$ with $m \le r$ and $a_0, \ldots, a_m$ non–zero elements in K. Let $Y_j = a_jL_j$ for $0 \le j \le m$. Complete $Y_0, \ldots, Y_m$ in order to get a new system of projective coordinates $Y_0, \ldots, Y_n$ on $\mathbf{P}^n(K)$. We apply Theorem 5.1 to the projective subspace $\mathbf{P}^m(K)$ of $\mathbf{P}^n(K)$ given by the equation $Y_{m+1} = \cdots = Y_n = 0$ and to the $m+2$ hyperplanes

$$Y_0 = 0, \; \ldots, \; Y_m = 0, \; Y_0 + \cdots + Y_m = 0.$$

The map

$$(y_0 : \cdots : y_n) \longmapsto (y_0 : \cdots : y_m) \in \mathbf{P}^m(K)$$

is well defined on $\mathbf{P}^n(K) \setminus H_0$ (recall that H_0 is the hyperplane of equation $Y_0 = 0$), hence also on the set of S–integral points on $\mathbf{P}^n(K) \setminus (H_0 \cup \cdots \cup H_{n+1})$. An S–integral point on $\mathbf{P}^n(K) \setminus (H_0 \cup \cdots \cup H_{n+1})$ has projective coordinates $(y_0 : \cdots : y_n)$ such that $y_0, \ldots, y_m$ and $y_0 + \cdots + y_m$ are S–units, hence by Theorem 5.1, for all but a finite number of these projective points $(y_0 : \cdots : y_n)$, the tuple $(y_0, \ldots, y_m)$ has a non–trivial vanishing subsum. Therefore the set of S–integral points on $\mathbf{P}^n(K) \setminus (H_0 \cup \cdots \cup H_{n+1})$ is contained in the union of finitely many linear subspaces. □

Remark. Corollary 2.4.3 of [13] of Vojta is different from Corollary 5.2: our hyperplanes H_i are replaced by hypersurfaces, and Vojta's conclusion is that the set of S–integral points on the complement is degenerate (contained in an hypersurface). Vojta deduces his result from a more general result (Theorem 2.4.1 of [13]), according to which the set of D–integral points on a variety V is degenerate, when D is a divisor which is a sum of at least $\dim V + \varrho + r + 1$ distinct prime divisors D_i. Here, r is the rank of the group of rational points on the variety $\mathrm{Pic}^0(V)$ and ϱ is the Picard number of V. For $\mathbf{P}^n(K)$ we have $r = 0$ and $\varrho = 1$. The proof of that result reduces to the unit equation considered in Corollary 5.2, so again everything boils down to Schmidt's subspace Theorem.

6 Potpourri

We received a number of comments on a preliminary version of this paper. We are grateful to Francesco Amoroso, Pietro Corvaja, Jan-Hendrik Evertse, Gaël Rémond, Paul Vojta and Umberto Zannier for their valuable suggestions they

kindly sent us. Some of these arguments were of geometrical nature and we reproduce here the proofs which are elementary and closer to the arguments which we use in the present paper.

Fact 6.1. *The assertion* (v) *of Proposition* 4.1 *implies the assertion* (i).

Proof (after P. Corvaja). Let $(X : Y : E)$ be projective coordinates on $\mathbf{P}^2(K)$. Denote by $\mathcal{E}$ the set of solutions (x, y, ε) in $O_S^2 \times O_S^\times$ of the equation $F(X, Y) = kE$, where F is given by 4. Consider the four hyperplanes H_0, H_1, H_2, H_3 of $\mathbf{P}^2(K)$ of equations respectively

$$E = 0, \quad X - \alpha_1 Y = 0, \quad X - \alpha_2 Y = 0, \quad X - \alpha_3 Y = 0.$$

Let S' be the set obtained by adding to S the places of K dividing numerators and denominators of the fractional principal ideals (k), (α_1), (α_2) and (α_3). Then for each $(x, y, \varepsilon) \in \mathcal{E}$, the point in $\mathbf{P}^2(K)$ with projective coordinates $(x : y : \varepsilon)$ is an S'–integral point on $\mathbf{P}^2(K) \setminus (H_0 \cup H_1 \cup H_2 \cup H_3)$. From (v) we deduce that there exists a non–zero homogeneous polynomial $P \in K[X, Y, E]$ such that $P(x, y, \varepsilon) = 0$ for all $(x, y, \varepsilon) \in \mathcal{E}$. For any $(x, y, \varepsilon) \in \mathcal{E}$ and any $\eta \in O_S^\times$, we have $(\eta x, \eta y, \eta^m \varepsilon) \in \mathcal{E}$, hence $P(\eta x, \eta y, \eta^m \varepsilon) = 0$. Since P is homogeneous, assuming (without loss of generality) that $O_S^\times$ is infinite, we deduce $P(x, y, E) = 0$. Therefore the set of $(x : y) \in \mathbf{P}^1(K)$ is finite. □

Fact 6.2. *The assertion* (v) *of Proposition* 4.1 *implies the assertion* (iii).

Proof (after U. Zannier). Assume that the set $\mathcal{E}$ of $(\varepsilon_1, \varepsilon_2) \in (O_S^\times)^2$ such that $\varepsilon_1 + \varepsilon_2 = 1$ is infinite. Let $(\varepsilon_1, \varepsilon_2)$ and (η_1, η_2) be two elements in $\mathcal{E}$. From $\varepsilon_1 + \varepsilon_2 = 1$ and $\eta_1 + \eta_2 = 1$ one deduces

$$1 - \varepsilon_2 - \varepsilon_1 \eta_2 = \varepsilon_1 \eta_1.$$

Hence the point with projective coordinates $(1 : -\varepsilon_2 : -\varepsilon_1 \eta_2)$ is an S–integral point on $\mathbf{P}^2(K) \setminus (H_0 \cup H_1 \cup H_2 \cup H_3)$, where H_0, H_1, H_2, H_3 are the hyperplanes of equations respectively

$$X_0 = 0, \quad X_1 = 0, \quad X_2 = 0, \quad X_0 + X_1 + X_2 = 0.$$

Assume now the truth of assertion (v) in Proposition 4.1: there exists a non–zero polynomial $P \in K[X, Y]$ such that $P(\varepsilon_2, \varepsilon_1 \eta_2) = 0$ for all $((\varepsilon_1, \varepsilon_2), (\eta_1, \eta_2)) \in \mathcal{E}^2$. Since $\mathcal{E}$ is infinite, there are infinitely many η_2, hence the polynomial $P(\varepsilon_2, \varepsilon_1 T)$ is the zero polynomial, which implies $P(\varepsilon_2, Y) = 0$, and since there are infinitely many ε_2, we obtain the contradiction $P = 0$. □

Proposition 6.3. *Let* n *and* t *be integers with* $1 \le t < n$. *The truth of the statement of Theorem* 5.1 *for* n *implies the truth of the result for* $n - t$.

Proof (after U. Zannier). Denote by $\mathcal{E}$ the set of $(\varepsilon_0, \ldots, \varepsilon_{n-t})$ in $(O_S^\times)^{n-t+1}$ satisfying $\varepsilon_0 + \cdots + \varepsilon_{n-t} = 0$ with the non–vanishing of any non–trivial subsum of the left hand side. Let γ be an element in $O_S \setminus O_S^\times$, so that $r/\gamma \notin O_S$, for

$r = 1, \dots, t$. Let S' be the set obtained by adding to S the places of K dividing $\gamma(\gamma-1)$. Write

$$\gamma\varepsilon_0 + \cdots + \gamma\varepsilon_{n-t-1} + (\gamma - t)\varepsilon_{n-t} + \varepsilon_{n-t} + \cdots + \varepsilon_{n-t} = 0.$$

Consider the left hand side as a sum of $n-t+1$ elements which are S'–units of K. Since $1/\gamma$ is not a sum of S–units of K, no non–trivial subsum on the left hand side vanishes. Assuming that Theorem 5.1 is true for n, it follows that $\mathcal{E}$ is a union of finitely many equivalent classes modulo $O_{S'}^\times$, hence modulo $O_S^\times$. $\square$

Proposition 6.4. *Let n and t be integers with $1 \le t < n$. The truth of the statement of Corollary* 5.2 *for n implies the truth of the result for $n-t$.*

Proof (after G. Rémond). Using the same argument as in the proof of Corollary 5.2, we deduce that there are projective coordinates $(X_0, \dots, X_n)$ on $\mathbf{P}^n(K)$ and there is an integer r in the range $1 \le r \le t$ such that $(X_0, \dots, X_{n-t})$ are projective coordinates on $\mathbf{P}^{n-t}(K)$ and $n-r+2$ of the given hyperplanes in in $\mathbf{P}^{n-t}(K)$ are defined by the equations $X_0 = 0$, $X_1 = 0$, $\dots$, $X_{n-r} = 0$ and $X_0 + \cdots + X_{n-r} = 0$. Let $\mathcal{E}$ be the set of S–integral points on the complements in $\mathbf{P}^{n-r}(K)$ of these hyperplanes. Consider the hyperplanes $X_0 = 0$, $X_1 = 0$, $\dots$, $X_n = 0$ and $X_0 + \cdots + X_{n-r} = 0$ of $\mathbf{P}^n(K)$. Assuming that Corollary 5.2 holds for n, we deduce that there exists a homogeneous polynomial Q in $n+1$ variables which vanishes at $(\varepsilon_0, \dots, \varepsilon_{n-r}, \eta_1, \dots, \eta_r)$ for all $(\varepsilon_0, \dots, \varepsilon_{n-r})$ in $\mathcal{E}$ and all $(\eta_1, \dots, \eta_r)$ in $(O_S^\times)^r$. If $O_S^\times$ is infinite, then the polynomial $Q(\varepsilon_0, \dots, \varepsilon_{n-r}, X_1, \dots, X_r)$ does not depend on $X_1, \dots, X_r$ and we deduce that Corollary 5.2 holds for $n-t$. $\square$

Proposition 6.5. *The truth of the statement of Theorem* 5.1 *for a fixed $n \ge 3$ implies the truth of the result for $n = 2$.*

Proof (after U. Zannier). Let $n \ge 3$. Set $m = n-1$. Let $\varepsilon \in O_S^\times$ satisfy $1 - \varepsilon \in O_S^\times$. Write

$$(1-\varepsilon)^m - m\varepsilon + \cdots + (-1)^j \binom{m}{j} \varepsilon^j + \cdots + (-1)^m \varepsilon^m = 1.$$

Let S' denote the set obtained by adding to S the places of K dividing the binomial coefficients $\binom{m}{j}$ for $1 \le j \le m-1$. The left hand side is a sum of $m+1$ terms which are S'–units. The set of S–units ε for which there is a vanishing non–trivial subsum is finite, namely it is the set of roots of finitely many polynomials of the form

$$(1-E)^m - u_1 mE + \cdots + (-1)^j u_j \binom{m}{j} E^j + \cdots + (-1)^m u_m E^m,$$

where $u_t \in \{0, 1\}$ for $t = 1, \dots, m$. From the assumption that the statement of Theorem 5.1] holds for n, we deduce that the set of these S–units ε is finite. $\square$

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Claude Levesque
Département de mathématiques et de statistique,
Université Laval,
Québec (Québec),
CANADA G1V 0A6
`Claude.Levesque@mat.ulaval.ca`

Michel Waldschmidt
Institut de Mathématiques de Jussieu,
Université Pierre et Marie Curie (Paris 6),
4 Place Jussieu,
F – 75252 PARIS Cedex 05, FRANCE
`miw@math.jussieu.fr`
`http://www.math.jussieu.fr/∼miw/`