A First Course on Pseudo-Differential Operators

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Chapter 1

Basic Fourier Analysis

1.1 Preliminaries

The Fourier transform of a function $u \in L^1(\mathbb{R}^n)$ can be defined as

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} u(x) e^{-2i\pi x\cdot\xi} dx. \tag{1.1.1}$$

Lemma 1.1.1 (Riemann-Lebesgue Lemma)**.** *Let u be in $L^1(\mathbb{R}^n)$. Then we have*

$$\widehat{u}(\xi) \underset{|\xi|\to\infty}{\longrightarrow} 0.$$

Moreover the function $\hat{u}$ is uniformly continuous on $\mathbb{R}^n$.

Proof. We note first that (1.1.1) is meaningful as the integral of an L^1 function and we have also

$$\sup_{\xi\in\mathbb{R}^n} |\widehat{u}(\xi)| \le \|u\|_{L^1(\mathbb{R}^n)}. \tag{1.1.2}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. With $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we define

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}, \quad D_j = \frac{1}{2i\pi}\frac{\partial}{\partial x_j}, \quad \xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}. \tag{1.1.3}$$

We find the identities

$$\xi_1 \widehat{\varphi}(\xi) = \widehat{D_1\varphi}(\xi), \quad \widehat{D^\alpha\varphi}(\xi) = \xi^\alpha \widehat{\varphi}(\xi), \tag{1.1.4}$$

entailing $\bigl(1 + |\xi|^2\bigr)\widehat{\varphi}(\xi) = \text{Fourier}\Bigl(\varphi + \sum_{1\le j\le n} D_j^2\varphi\Bigr)$. We find thus

$$\bigl(1 + |\xi|^2\bigr)|\widehat{\varphi}(\xi)| \le \|\varphi + \sum_{1\le j\le n} D_j^2\varphi\|_{L^1(\mathbb{R}^n)},$$

which implies $\lim_{|\xi|\to+\infty} \widehat{\varphi}(\xi) = 0$. For $u \in L^1(\mathbb{R}^n)$, we have

$$|\widehat{u}(\xi)| \le |\widehat{(u - \varphi)}(\xi)| + |\widehat{\varphi}(\xi)| \le \|u - \varphi\|_{L^1(\mathbb{R}^n)} + |\widehat{\varphi}(\xi)|,$$

so that for all $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\limsup_{|\xi|\to\infty} |\widehat{u}(\xi)| \le \|u-\varphi\|_{L^1(\mathbb{R}^n)} \Longrightarrow \limsup_{|\xi|\to\infty} |\widehat{u}(\xi)| \le \inf_{\varphi\in C_c^\infty(\mathbb{R}^n)} \|u-\varphi\|_{L^1(\mathbb{R}^n)} = 0.$$

We have also $\widehat{u}(\xi+\eta)-\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi}\big(e^{-2i\pi x\cdot\eta}-1\big)u(x)dx$, so that

$$|\widehat{u}(\xi+\eta)-\widehat{u}(\xi)| \le \int_{\mathbb{R}^n} |u(x)| \underbrace{|e^{-2i\pi x\cdot\eta}-1|}_{\le 2} dx,$$

and Lebesgue's Dominated Convergence Theorem shows that, for all $\xi \in \mathbb{R}^n$,

$$\lim_{\eta\to 0} |\widehat{u}(\xi+\eta)-\widehat{u}(\xi)| = 0,$$

proving continuity. We have also for $R > 1, |\eta| \le 1$,

$$|\widehat{u}(\xi+\eta)-\widehat{u}(\xi)| \le \sup_{|\xi|\le R} |\widehat{u}(\xi+\eta)-\widehat{u}(\xi)| + 2\sup_{|\xi|\ge R-1} |\hat{u}(\xi)|,$$

so that for $0 < \varepsilon < 1$, if ω_ρ is a modulus of continuity[1] of the continuous function $\hat{u}$ on the compact set $\{|x| \le \rho\}$,

$$\sup_{|\eta|\le\varepsilon,\xi\in\mathbb{R}^m} |\widehat{u}(\xi+\eta)-\widehat{u}(\xi)| \le \omega_{R+1}(\varepsilon) + 2\sup_{|\xi|\ge R-1} |\hat{u}(\xi)|,$$

proving that the lim sup of the lhs when ε goes to 0 is smaller than

$$2\sup_{|\xi|\ge R-1} |\hat{u}(\xi)|, \quad \text{for all } R > 1.$$

Since that quantity is already proven to go to 0 when R goes to $+\infty$, we obtain the uniform continuity of $\hat{u}$. □

We need to extend this transformation to various other situations and it turns out that L. Schwartz' point of view to define the Fourier transformation on the very large space of tempered distributions is the simplest. However, the cost of the distribution point of view is that we have to define these objects, which is not a completely elementary matter. We have chosen here to limit our presentation to the tempered distributions, topological dual of the so-called Schwartz space of rapidly decreasing functions; this space is a Fréchet space, so its topology is defined by a countable family of semi-norms and is much less difficult to understand than the space of test functions with compact support on an open set. Proving the Fourier inversion formula on the Schwartz space is a truly elementary matter, which yields almost immediately the most general case for tempered distributions, by a duality abstract nonsense argument. This chapter may also serve to the reader as a motivation to the explore the more difficult local theory of distributions.

[1] For a continuous function v defined on a compact subset K of $\mathbb{R}^m$, the modulus of continuity ω is defined on $\mathbb{R}_+$ by $\omega(\rho) = \sup_{\substack{x,y\in K\\ |x-y|\le\rho}} |v(x)-v(y)|$. We have $\lim_{\rho\to 0_+}\omega(\rho) = 0$.

1.2 Fourier Transform of tempered distributions

The Fourier transformation on $\mathscr{S}(\mathbb{R}^n)$

Definition 1.2.1. Let $n \geq 1$ be an integer. The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is defined as the vector space of C^∞ functions u from $\mathbb{R}^n$ to $\mathbb{C}$ such that, for all multi-indices. $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)| < +\infty.$$

Here we have used the multi-index notation: for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we define

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}, \quad |\alpha| = \sum_{1\leq j\leq n} \alpha_j. \tag{1.2.1}$$

A simple example of such a function is $e^{-|x|^2}$, ($|x|$ is the Euclidean norm of x) and more generally, if A is a symmetric positive definite $n \times n$ matrix, the function

$$v_A(x) = e^{-\pi\langle Ax,x\rangle} \tag{1.2.2}$$

belongs to the Schwartz class. The space $\mathscr{S}(\mathbb{R}^n)$ is a Fréchet space equipped with the countable family of semi-norms $(p_k)_{k\in\mathbb{N}}$

$$p_k(u) = \sup_{\substack{x\in\mathbb{R}^n \\ |\alpha|,|\beta|\leq k}} |x^\alpha \partial_x^\beta u(x)|. \tag{1.2.3}$$

Lemma 1.2.2. *The Fourier transform sends continuously $\mathscr{S}(\mathbb{R}^n)$ into itself.*

Proof. Just notice that

$$\xi^\alpha \partial_\xi^\beta \hat{u}(\xi) = \int e^{-2i\pi x\xi} \partial_x^\alpha (x^\beta u)(x) dx (2i\pi)^{|\beta|-|\alpha|} (-1)^{|\beta|},$$

and since $\sup_{x\in\mathbb{R}^n} (1+|x|)^{n+1} |\partial_x^\alpha (x^\beta u)(x)| < +\infty$, we get the result. □

Lemma 1.2.3. *For a symmetric positive definite $n \times n$ matrix A, we have*

$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi,\xi\rangle}, \tag{1.2.4}$$

where v_A is given by (1.2.2).

Proof. In fact, diagonalizing the symmetric matrix A, it is enough to prove the one-dimensional version of (1.2.4), i.e. to check

$$\int e^{-2i\pi x\xi} e^{-\pi x^2} dx = \int e^{-\pi(x+i\xi)^2} dx e^{-\pi\xi^2} = e^{-\pi\xi^2},$$

where the second equality is obtained by taking the ξ-derivative of $\int e^{-\pi(x+i\xi)^2} dx$: we have indeed

$$\frac{d}{d\xi}\Big(\int e^{-\pi(x+i\xi)^2} dx\Big) = \int e^{-\pi(x+i\xi)^2} (-2i\pi)(x+i\xi) dx$$
$$= i \int \frac{d}{dx}\big(e^{-\pi(x+i\xi)^2}\big) dx = 0.$$

For $a>0$, we obtain

$$\int_{\mathbb{R}} e^{-2i\pi x\xi} e^{-\pi a x^2} dx = a^{-1/2} e^{-\pi a^{-1}\xi^2},$$

which is the sought result in one dimension. If $n\ge 2$, and A is a positive definite symmetric matrix, there exists an orthogonal $n\times n$ matrix P (i.e. ${}^tPP=\mathrm{Id}$) such that

$$D={}^tPAP,\quad D=\mathrm{diag}(\lambda_1,\dots,\lambda_n),\ \text{all } \lambda_j>0.$$

As a consequence, we have, since $|\det P|=1$,

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} e^{-\pi\langle Ax,x\rangle} dx &= \int_{\mathbb{R}^n} e^{-2i\pi (Py)\cdot\xi} e^{-\pi\langle APy,Py\rangle} dy\\ &= \int_{\mathbb{R}^n} e^{-2i\pi y\cdot({}^tP\xi)} e^{-\pi\langle Dy,y\rangle} dy\\ (\text{with } \eta={}^tP\xi)\quad &= \prod_{1\le j\le n}\int_{\mathbb{R}} e^{-2i\pi y_j\eta_j} e^{-\pi\lambda_j y_j^2} dy_j = \prod_{1\le j\le n}\lambda_j^{-1/2} e^{-\pi\lambda_j^{-1}\eta_j^2}\\ &= (\det A)^{-1/2} e^{-\pi\langle D^{-1}\eta,\eta\rangle} = (\det A)^{-1/2} e^{-\pi\langle {}^tPA^{-1}P\ {}^tP\xi,{}^tP\xi\rangle}\\ &= (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi,\xi\rangle}.\end{aligned}$$

□

Proposition 1.2.4. *The Fourier transformation is an isomorphism of the Schwartz class and for $u\in\mathscr{S}(\mathbb{R}^n)$, we have*

$$u(x)=\int e^{2i\pi x\xi}\hat u(\xi)d\xi. \tag{1.2.5}$$

Proof. Using (1.2.4) we calculate for $u\in\mathscr{S}(\mathbb{R}^n)$ and $\epsilon>0$, dealing with absolutely converging integrals,

$$\begin{aligned} u_\epsilon(x) &= \int e^{2i\pi x\xi}\hat u(\xi) e^{-\pi\epsilon^2|\xi|^2} d\xi\\ &= \iint e^{2i\pi x\xi} e^{-\pi\epsilon^2|\xi|^2} u(y) e^{-2i\pi y\xi} dy d\xi\\ &= \int u(y) e^{-\pi\epsilon^{-2}|x-y|^2}\epsilon^{-n} dy\\ &= \int \underbrace{\big(u(x+\epsilon y)-u(x)\big)}_{\text{with absolute value}\le\epsilon|y|\|u'\|_{L^\infty}} e^{-\pi|y|^2} dy + u(x).\end{aligned}$$

Taking the limit when ϵ goes to zero, we get the Fourier inversion formula

$$u(x)=\int e^{2i\pi x\xi}\hat u(\xi)d\xi. \tag{1.2.6}$$

We have also proven for $u\in\mathscr{S}(\mathbb{R}^n)$ and $\check u(x)=u(-x)$

$$u=\check{\hat{\hat u}}. \tag{1.2.7}$$

Since $u\mapsto\hat u$ and $u\mapsto\check u$ are continuous homomorphisms of $\mathscr{S}(\mathbb{R}^n)$, this completes the proof of the proposition. □

Proposition 1.2.5. *Using the notation*

$$D_{x_j} = \frac{1}{2i\pi}\frac{\partial}{\partial x_j}, \qquad D_x^\alpha = \prod_{j=1}^n D_{x_j}^{\alpha_j} \quad \textit{with } \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \tag{1.2.8}$$

we have, for $u \in \mathscr{S}(\mathbb{R}^n)$

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi), \qquad (D_\xi^\alpha \hat{u})(\xi) = (-1)^{|\alpha|}\widehat{x^\alpha u(x)}(\xi) \tag{1.2.9}$$

Proof. We have for $u \in \mathscr{S}(\mathbb{R}^n)$, $\hat{u}(\xi) = \int e^{-2i\pi x\cdot\xi} u(x)dx$ and thus

$$(D_\xi^\alpha \hat{u})(\xi) = (-1)^{|\alpha|}\int e^{-2i\pi x\cdot\xi} x^\alpha u(x)dx,$$
$$\xi^\alpha \hat{u}(\xi) = \int (-2i\pi)^{-|\alpha|}\partial_x^\alpha\big(e^{-2i\pi x\cdot\xi}\big)u(x)dx = \int e^{-2i\pi x\cdot\xi}(2i\pi)^{-|\alpha|}(\partial_x^\alpha u)(x)dx,$$

proving both formulas. □

N.B. The normalization factor $\frac{1}{2i\pi}$ leads to a simplification in Formula (1.2.9), but the most important aspect of these formulas is certainly that the Fourier transformation exchanges the operation of derivation with the operation of multiplication. For instance with

$$P(D) = \sum_{|\alpha|\le m} a_\alpha D_x^\alpha,$$

we have for $u \in \mathscr{S}(\mathbb{R}^n)$, $\widehat{Pu}(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha \hat{u}(\xi) = P(\xi)\hat{u}(\xi)$, and thus

$$(Pu)(x) = \int_{\mathbb{R}^n} e^{2i\pi x\cdot\xi} P(\xi)\hat{u}(\xi)d\xi. \tag{1.2.10}$$

Proposition 1.2.6. *Let* ϕ, ψ *be functions in* $\mathscr{S}(\mathbb{R}^n)$. *Then the convolution* $\phi * \psi$ *belongs to the Schwartz space and the mapping*

$$\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \ni (\phi, \psi) \mapsto \phi * \psi \in \mathscr{S}(\mathbb{R}^n)$$

is continuous. Moreover we have

$$\widehat{\phi * \psi} = \hat{\phi}\hat{\psi}. \tag{1.2.11}$$

Proof. The mapping $(x, y) \mapsto F(x, y) = \phi(x - y)\psi(y)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$ since x, y derivatives of the smooth function F are linear combinations of products

$$(\partial^\alpha \phi)(x - y)(\partial^\beta \psi)(y)$$

and moreover

$$\begin{aligned}(1 + |x| + |y|)^N &|(\partial^\alpha \phi)(x - y)(\partial^\beta \psi)(y)| \\ &\le (1 + |x - y|)^N |(\partial^\alpha \phi)(x - y)|(1 + 2|y|)^N |(\partial^\beta \psi)(y)| \le p(\phi)q(\psi),\end{aligned}$$

where p, q are semi-norms on $\mathscr{S}(\mathbb{R}^n)$. This proves that the bilinear mapping $(\phi,\psi)\mapsto F(\phi,\psi)$ is continuous from $\mathscr{S}(\mathbb{R}^n)\times\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^{2n})$. We have now directly $\partial_x^\alpha(\phi*\psi)=(\partial_x^\alpha\phi)*\psi$ and

$$\begin{aligned}(1+|x|)^N|\partial_x^\alpha(\phi*\psi)| &\le \int |F(\partial^\alpha\phi,\psi)(x,y)|(1+|x|)^N dy\\ &\le \int \underbrace{|F(\partial^\alpha\phi,\psi)(x,y)|(1+|x|)^N(1+|y|)^{n+1}}_{\le p(F)}(1+|y|)^{-n-1}dy,\end{aligned}$$

where p is a semi-norm of F (thus bounded by a product of semi-norms of ϕ and ψ), proving the continuity property. Also we obtain from Fubini's Theorem

$$\widehat{(\phi*\psi)}(\xi)=\iint e^{-2i\pi(x-y)\cdot\xi}e^{-2i\pi y\cdot\xi}\phi(x-y)\psi(y)dydx=\hat\phi(\xi)\hat\psi(\xi),$$

completing the proof of the proposition. □

The Fourier transformation on $\mathscr{S}'(\mathbb{R}^n)$

Definition 1.2.7. Let n be an integer ≥ 1. We define the space $\mathscr{S}'(\mathbb{R}^n)$ as the topological dual of the Fréchet space $\mathscr{S}(\mathbb{R}^n)$: this space is called the space of *tempered distributions* on $\mathbb{R}^n$.

We note that the mapping

$$\mathscr{S}(\mathbb{R}^n)\ni\phi\mapsto\frac{\partial\phi}{\partial x_j}\in\mathscr{S}(\mathbb{R}^n),$$

is continuous since for all $k\in\mathbb{N}$, $p_k(\partial\phi/\partial x_j)\le p_{k+1}(\phi)$, where the semi-norms p_k are defined in (1.2.3). This property allows us to define by duality the derivative of a tempered distribution.

Definition 1.2.8. Let $u\in\mathscr{S}'(\mathbb{R}^n)$. We define $\partial u/\partial x_j$ as an element of $\mathscr{S}'(\mathbb{R}^n)$ by

$$\langle\frac{\partial u}{\partial x_j},\phi\rangle_{\mathscr{S}',\mathscr{S}}=-\langle u,\frac{\partial\phi}{\partial x_j}\rangle_{\mathscr{S}',\mathscr{S}}. \tag{1.2.12}$$

The mapping $u\mapsto\partial u/\partial x_j$ is a well-defined endomorphism of $\mathscr{S}'(\mathbb{R}^n)$ since the estimates

$$\forall\phi\in\mathscr{S}(\mathbb{R}^n),\quad |\langle\frac{\partial u}{\partial x_j},\phi\rangle|\le C_u p_{k_u}(\frac{\partial\phi}{\partial x_j})\le C_u p_{k_u+1}(\phi),$$

ensure the continuity on $\mathscr{S}(\mathbb{R}^n)$ of the linear form $\partial u/\partial x_j$.

Definition 1.2.9. Let $u\in\mathscr{S}'(\mathbb{R}^n)$ and let P be a polynomial in n variables with complex coefficients. We define the product Pu as an element of $\mathscr{S}'(\mathbb{R}^n)$ by

$$\langle Pu,\phi\rangle_{\mathscr{S}',\mathscr{S}}=\langle u,P\phi\rangle_{\mathscr{S}',\mathscr{S}}. \tag{1.2.13}$$

The mapping $u \mapsto Pu$ is a well-defined endomorphism of $\mathscr{S}'(\mathbb{R}^n)$ since the estimates

$$\forall \phi \in \mathscr{S}(\mathbb{R}^n), \quad |\langle Pu, \phi\rangle| \le C_u p_{k_u}(P\phi) \le C_u p_{k_u+D}(\phi),$$

where D is the degree of P, ensure the continuity on $\mathscr{S}(\mathbb{R}^n)$ of the linear form Pu.

Lemma 1.2.10. *Let Ω be an open subset of $\mathbb{R}^n$, $f \in L^1_{loc}(\Omega)$ such that, for all $\varphi \in C_c^\infty(\Omega)$, $\int f(x)\varphi(x)dx = 0$. Then we have $f = 0$.*

Proof. Let K be a compact subset of Ω and let $\chi \in C_c^\infty(\Omega)$ equal to 1 on a neighborhood of K (see e.g. Exercise 2.8.7 in [15]). With $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\int \rho(t)dt = 1$, and for $\epsilon > 0$, $\rho_\epsilon(x) = \rho(x/\epsilon)\epsilon^{-n}$, we get that

$$\lim_{\epsilon \to 0_+} \rho_\epsilon * (\chi f) = \chi f \quad \text{in } L^1(\mathbb{R}^n),$$

since for $w \in L^1(\mathbb{R}^n)$,

$$\begin{aligned}\|\rho_\epsilon * w - w\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \rho_\epsilon(y)\big(w(x-y) - w(x)\big)dy \right| dx \\ &\le \iint |\rho(z)||w(x-\epsilon z) - w(x)|dzdx = \int |\rho(z)| \|\tau_{\epsilon z} w - w\|_{L^1(\mathbb{R}^n)} dz.\end{aligned}$$

We know[2] that $\lim_{h\to 0} \|\tau_h w - w\|_{L^1(\mathbb{R}^n)} = 0$ and $\|\tau_h w - w\|_{L^1(\mathbb{R}^n)} \le 2\|w\|_{L^1(\mathbb{R}^n)}$ so that Lebesgue's dominated convergence theorem provides

$$\lim_{\epsilon \to 0} \|\rho_\epsilon * w - w\|_{L^1(\mathbb{R}^n)} = 0.$$

We have $\big(\rho_\epsilon * (\chi f)\big)(x) = \int f(y) \underbrace{\chi(y)\rho\big((x-y)\epsilon^{-1}\big)\epsilon^{-n}}_{=\varphi_x(y)} dy$, with $\operatorname{supp} \varphi_x \subset \operatorname{supp} \chi$, $\varphi_x \in C_c^\infty(\Omega)$, and from the assumption of the lemma, we obtain $\big(\rho_\epsilon * (\chi f)\big)(x) = 0$ for all x, implying $\chi f = 0$ from the convergence result and thus $f = 0$, a.e. on K; the conclusion of the lemma follows since Ω is a countable union of compact sets (see e.g. Exercise 2.8.10 in [15]). □

Definition 1.2.11 (support of a distribution). For $u \in \mathscr{S}'(\mathbb{R}^n)$, we define the support of u and we note $\operatorname{supp} u$ the closed subset of $\mathbb{R}^n$ defined by

$$(\operatorname{supp} u)^c = \{x \in \mathbb{R}^n, \exists V \text{ open} \in \mathscr{V}_x, \quad u_{|V} = 0\}, \tag{1.2.14}$$

where $\mathscr{V}_x$ stands for the set of neighborhoods of x and $u_{|V} = 0$ means that for all $\phi \in C_c^\infty(V)$, $\langle u, \phi\rangle = 0$.

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[2] For $\phi \in C_c^0(\mathbb{R}^n)$, we have $\|\tau_h w - w\|_{L^1(\mathbb{R}^n)} \le \|\tau_h w - \tau_h\phi\|_{L^1(\mathbb{R}^n)} + \|\tau_h\phi - \phi\|_{L^1(\mathbb{R}^n)} + \|\phi - w\|_{L^1(\mathbb{R}^n)}$, so that for $|h| \le 1$,

$$\begin{aligned}\|\tau_h w - w\|_{L^1(\mathbb{R}^n)} &\le 2\|\phi - w\|_{L^1(\mathbb{R}^n)} + \int |\phi(x-h) - \phi(x)|dx \\ &\le 2\|\phi - w\|_{L^1(\mathbb{R}^n)} + |\operatorname{supp}\phi + \mathbb{B}^n| \sup|\phi(x-h) - \phi(x)|\end{aligned}$$

which implies that $\limsup_{h\to 0} \|\tau_h w - w\| \le 2 \inf_{\phi \in C_c^0(\mathbb{R}^n)} \|\phi - w\|_{L^1(\mathbb{R}^n)} = 0$.

Proposition 1.2.12.
(1) *We have* $\mathscr{S}'(\mathbb{R}^n) \supset \cup_{1\le p\le +\infty} L^p(\mathbb{R}^n)$, *with a continuous injection of each* $L^p(\mathbb{R}^n)$ *into* $\mathscr{S}'(\mathbb{R}^n)$. *As a consequence* $\mathscr{S}'(\mathbb{R}^n)$ *contains as well all the derivatives in the sense* (1.2.12) *of all the functions in some* $L^p(\mathbb{R}^n)$.
(2) *For* $u \in C^1(\mathbb{R}^n)$ *such that*

$$\big(|u(x)| + |du(x)|\big)(1+|x|)^{-N} \in L^1(\mathbb{R}^n), \tag{1.2.15}$$

for some non-negative N, *the derivative in the sense* (1.2.12) *coincides with the ordinary derivative.*

Proof. (1) For $u \in L^p(\mathbb{R}^n)$ and $\phi \in \mathscr{S}(\mathbb{R}^n)$, we can define

$$\langle u, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \int_{\mathbb{R}^n} u(x)\phi(x)dx, \tag{1.2.16}$$

which is a continuous linear form on $\mathscr{S}(\mathbb{R}^n)$:

$$|\langle u, \phi\rangle_{\mathscr{S}',\mathscr{S}}| \le \|u\|_{L^p(\mathbb{R}^n)}\|\phi\|_{L^{p'}(\mathbb{R}^n)},$$

$$\|\phi\|_{L^{p'}(\mathbb{R}^n)} \le \sup_{x\in\mathbb{R}^n}\big((1+|x|)^{\frac{n+1}{p'}}|\phi(x)|\big)C_{n,p} \le C_{n,p}p_k(\phi), \text{ for } k \ge k_{n,p} = \frac{n+1}{p'},$$

with p_k given by (1.2.3) (when $p = 1$, we can take $k = 0$). We indeed have a continuous injection of $L^p(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$: in the first place the mapping described by (1.2.16) is well-defined and continuous from the estimate

$$|\langle u, \phi\rangle| \le \|u\|_{L^p} C_{n,p} p_{k_{n,p}}(\phi).$$

Moreover, this mapping is linear and injective from Lemma 1.2.10.

(2) We have for $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\chi_0 \in C_c^\infty(\mathbb{R}^n)$, $\chi_0 = 1$ near the origin,

$$A = \langle \frac{\partial u}{\partial x_j}, \phi\rangle_{\mathscr{S}',\mathscr{S}} = -\langle u, \frac{\partial \phi}{\partial x_j}\rangle_{\mathscr{S}',\mathscr{S}} = -\int_{\mathbb{R}^n} u(x)\frac{\partial \phi}{\partial x_j}(x)dx$$

so that, using Lebesgue's dominated convergence theorem, we find

$$A = -\lim_{\epsilon\to 0_+}\int_{\mathbb{R}^n} u(x)\frac{\partial \phi}{\partial x_j}(x)\chi_0(\epsilon x)dx.$$

Performing an integration by parts on C^1 functions with compact support, we get

$$A = \lim_{\epsilon\to 0_+}\Big\{\int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)\chi_0(\epsilon x)dx + \epsilon\int_{\mathbb{R}^n} u(x)\phi(x)(\partial_j\chi_0)(\epsilon x)dx\Big\},$$

with $\partial_j u$ standing for the ordinary derivative. We have also

$$\int_{\mathbb{R}^n}|u(x)\phi(x)(\partial_j\chi_0)(\epsilon x)|dx \le \|\partial_j\chi_0)\|_{L^\infty(\mathbb{R}^n)}\int |u(x)|(1+|x|)^{-N}dx\ p_N(\phi) < +\infty,$$

so that

$$\langle \frac{\partial u}{\partial x_j}, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \lim_{\epsilon\to 0_+}\int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)\chi_0(\epsilon x)dx.$$

Since the lhs is a continuous linear form on $\mathscr{S}(\mathbb{R}^n)$ so is the rhs. On the other hand for $\phi \in C_c^\infty(\mathbb{R}^n)$, the rhs is $\int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)dx$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathscr{S}(\mathbb{R}^n)$, we find that

$$\langle \frac{\partial u}{\partial x_j}, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)dx,$$

since the mapping $\phi \mapsto \int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)dx$ belongs to $\mathscr{S}'(\mathbb{R}^n)$, thanks to the assumption on du in (1.2.15). This proves that $\frac{\partial u}{\partial x_j} = \partial_j u$. □

The Fourier transformation can be extended to $\mathscr{S}'(\mathbb{R}^n)$. We start with noticing that for T, ϕ in the Schwartz class we have, using Fubini Theorem,

$$\int \hat{T}(\xi)\phi(\xi)d\xi = \iint T(x)\phi(\xi)e^{-2i\pi x\cdot\xi}dxd\xi = \int T(x)\hat{\phi}(x)dx,$$

and we can use the latter formula as a definition.

Definition 1.2.13. Let T be a tempered distribution ; the Fourier transform $\hat{T}$ of T is the tempered distribution defined by the formula

$$\langle \hat{T}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\varphi}\rangle_{\mathscr{S}',\mathscr{S}}. \tag{1.2.17}$$

The linear form $\hat{T}$ is obviously a tempered distribution since the Fourier transformation is continuous on $\mathscr{S}$. Thanks to Lemma 1.2.10, if $T \in \mathscr{S}$, the present definition of $\hat{T}$ and (1.1.1) coincide.

This definition gives that, with δ_0 standing as the Dirac mass at 0, $\langle \delta_0, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \phi(0)$ (obviously a tempered distribution), we have

$$\widehat{\delta_0} = 1, \tag{1.2.18}$$

since $\langle \widehat{\delta_0}, \varphi\rangle = \langle \delta_0, \widehat{\varphi}\rangle = \widehat{\varphi}(0) = \int \varphi(x)dx = \langle 1, \varphi\rangle$.

Theorem 1.2.14. *The Fourier transformation is an isomorphism of $\mathscr{S}'(\mathbb{R}^n)$. Let T be a tempered distribution. Then we have*[3]

$$T = \check{\hat{\hat{T}}}, \quad \check{\hat{T}} = \hat{\check{T}}. \tag{1.2.19}$$

With obvious notations, we have the following extensions of (1.2.9),

$$\widehat{D_x^\alpha T}(\xi) = \xi^\alpha \hat{T}(\xi), \qquad (D_\xi^\alpha \hat{T})(\xi) = (-1)^{|\alpha|}\widehat{x^\alpha T(x)}(\xi). \tag{1.2.20}$$

Proof. We have for $T \in \mathscr{S}'$

$$\langle \check{\hat{\hat{T}}}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{\hat{T}}, \check{\varphi}\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{T}, \hat{\check{\varphi}}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\hat{\check{\varphi}}}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \varphi\rangle_{\mathscr{S}',\mathscr{S}},$$

[3]We define $\check{T}$ as the distribution given by $\langle \check{T}, \varphi\rangle = \langle T, \check{\varphi}\rangle$ and if $T \in \mathscr{S}'$, $\check{T}$ is also a tempered distribution since $\varphi \mapsto \check{\varphi}$ is an involutive isomorphism of $\mathscr{S}$.

where the last equality is due to the fact that $\varphi \mapsto \check{\varphi}$ commutes[4] with the Fourier transform and (1.2.6) means

$$\check{\hat{\check{\varphi}}} = \varphi,$$

a formula also proven true on $\mathscr{S}'$ by the previous line of equality. Formula (1.2.9) is true as well for $T \in \mathscr{S}'$ since, with $\varphi \in \mathscr{S}$ and $\varphi_\alpha(\xi) = \xi^\alpha \varphi(\xi)$, we have

$$\langle \widehat{D^\alpha T}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, (-1)^{|\alpha|} D^\alpha \hat{\varphi} \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \widehat{\varphi_\alpha} \rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{T}, \varphi_\alpha \rangle_{\mathscr{S}',\mathscr{S}},$$

and the other part is proven the same way. □

The Fourier transformation on $L^1(\mathbb{R}^n)$

Theorem 1.2.15. *The Fourier transformation is linear continuous from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ and for $u \in L^1(\mathbb{R}^n)$, we have*

$$\hat{u}(\xi) = \int e^{-2i\pi x\cdot\xi} u(x)dx, \quad \|\hat{u}\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)}. \tag{1.2.21}$$

Proof. Formula (1.1.1) can be used to define directly the Fourier transform of a function in $L^1(\mathbb{R}^n)$ and this gives a $L^\infty(\mathbb{R}^n)$ function which coincides with the Fourier transform: for a test function $\varphi \in \mathscr{S}(\mathbb{R}^n)$, and $u \in L^1(\mathbb{R}^n)$, we have by the definition (1.2.17) above and Fubini theorem

$$\langle \hat{u}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \int u(x)\hat{\varphi}(x)dx = \iint u(x)\varphi(\xi)e^{-2i\pi x\cdot\xi}dxd\xi = \int \widetilde{u}(\xi)\varphi(\xi)d\xi$$

with $\widetilde{u}(\xi) = \int e^{-2i\pi x\cdot\xi}u(x)dx$ which is thus the Fourier transform of u. □

The Fourier transformation on $L^2(\mathbb{R}^n)$

Theorem 1.2.16 (Plancherel formula)**.**
The Fourier transformation can be extended to a unitary operator of $L^2(\mathbb{R}^n)$, i.e. there exists a unique bounded linear operator $F : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, such that for $u \in \mathscr{S}(\mathbb{R}^n)$, $Fu = \hat{u}$ and we have $F^*F = FF^* = \mathrm{Id}_{L^2(\mathbb{R}^n)}$. Moreover

$$F^* = CF = FC, \quad F^2 C = \mathrm{Id}_{L^2(\mathbb{R}^n)}, \tag{1.2.22}$$

where C is the involutive isomorphism of $L^2(\mathbb{R}^n)$ defined by $(Cu)(x) = u(-x)$. This gives the Plancherel formula: for $u, v \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi = \int u(x)\overline{v(x)}dx. \tag{1.2.23}$$

Proof. For test functions $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$, using Fubini theorem and (1.2.6), we get[5]

$$(\hat{\psi}, \hat{\varphi})_{L^2(\mathbb{R}^n)} = \int \hat{\psi}(\xi)\overline{\hat{\varphi}(\xi)}d\xi = \iint \hat{\psi}(\xi)e^{2i\pi x\cdot\xi}\overline{\varphi(x)}dxd\xi = (\psi, \varphi)_{L^2(\mathbb{R}^n)}.$$

---

[4] If $\varphi \in \mathscr{S}$, we have $\widehat{\check{\varphi}}(\xi) = \int e^{-2i\pi x\cdot\xi}\varphi(-x)dx = \int e^{2i\pi x\cdot\xi}\varphi(x)dx = \hat{\varphi}(-\xi) = \check{\hat{\varphi}}(\xi)$.

[5] We have to pay attention to the fact that the scalar product $(u, v)_{L^2}$ in the complex Hilbert space $L^2(\mathbb{R}^n)$ is linear with respect to u and antilinear with respect to v: for $\lambda, \mu \in \mathbb{C}$, $(\lambda u, \mu v)_{L^2} = \lambda\bar{\mu}(u, v)_{L^2}$.

Next, the density of $\mathscr{S}$ in L^2 shows that there is a unique continuous extension F of the Fourier transform to L^2 and that extension is an isometric operator (i.e. satisfying for all $u \in L^2(\mathbb{R}^n)$, $\|Fu\|_{L^2} = \|u\|_{L^2}$, i.e. $F^*F = \mathrm{Id}_{L^2}$). We note that the operator C defined by $Cu = \check{u}$ is an involutive isomorphism of $L^2(\mathbb{R}^n)$ and that for $u \in \mathscr{S}(\mathbb{R}^n)$,

$$CF^2u = u = FCFu = F^2Cu.$$

By the density of $\mathscr{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, the bounded operators

$$CF^2,\ \mathrm{Id}_{L^2(\mathbb{R}^n)},\ FCF,\ F^2C,$$

are all equal. On the other hand for $u, \varphi \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}(F^*u, \varphi)_{L^2} = (u, F\varphi)_{L^2} = \int u(x)\overline{\hat{\varphi}(x)}dx \\ = \iint u(x)\bar{\varphi}(\xi)e^{2i\pi x\cdot\xi}dxd\xi = (CFu, \varphi)_{L^2},\end{aligned}$$

so that $F^*u = CFu$ for all $u \in \mathscr{S}$ and by continuity $F^* = CF$ as bounded operators on $L^2(\mathbb{R}^n)$, thus $FF^* = FCF = \mathrm{Id}$. The proof is complete. □

Some standard examples of Fourier transform

Let us consider the Heaviside function defined on $\mathbb{R}$ by $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x \leq 0$; as a bounded measurable function, it is a tempered distribution, so that we can compute its Fourier transform. With the notation of this section, we have, with δ_0 the Dirac mass at 0, $\check{H}(x) = H(-x)$,

$$\widehat{H} + \widehat{\check{H}} = \hat{1} = \delta_0, \quad \widehat{H} - \widehat{\check{H}} = \widehat{\mathrm{sign}}, \qquad \frac{1}{i\pi} = \frac{1}{2i\pi}2\widehat{\delta_0}(\xi) = \widehat{D\,\mathrm{sign}}(\xi) = \xi\widehat{\mathrm{sign}}\xi.$$

We note that $\mathbb{R} \mapsto \ln|x|$ belongs to $\mathscr{S}'(\mathbb{R})$ and[6] we define the so-called principal value of $1/x$ on $\mathbb{R}$ by

$$\mathrm{pv}\big(\frac{1}{x}\big) = \frac{d}{dx}(\ln|x|), \tag{1.2.24}$$

$$\begin{aligned}\text{so that, } \langle \mathrm{pv}\,\frac{1}{x}, \phi\rangle &= -\int \phi'(x)\ln|x|dx = -\lim_{\epsilon\to 0_+}\int_{|x|\geq\epsilon}\phi'(x)\ln|x|dx \\ &= \lim_{\epsilon\to 0_+}\Big(\int_{|x|\geq\epsilon}\phi(x)\frac{1}{x}dx + \underbrace{\big(\phi(\epsilon)-\phi(-\epsilon)\big)\ln\epsilon}_{\to 0}\Big) \\ &= \lim_{\epsilon\to 0_+}\int_{|x|\geq\epsilon}\phi(x)\frac{1}{x}dx.\end{aligned} \tag{1.2.25}$$

This entails $\xi\big(\widehat{\mathrm{sign}}\xi - \frac{1}{i\pi}pv(1/\xi)\big) = 0$ and from Remark 1.2.17 below, we get

$$\widehat{\mathrm{sign}}\xi - \frac{1}{i\pi}pv(1/\xi) = c\delta_0,$$

with $c = 0$ since the lhs is odd[7].

[6] For $\phi \in \mathscr{S}(\mathbb{R})$, we have $\langle \ln|x|, \phi(x)\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} = \int_{\mathbb{R}} \phi(x)\ln|x|dx$.

[7] A distribution T on $\mathbb{R}^n$ is said to be odd (resp. even) when $\check{T} = -T$ (resp. T).

Remark 1.2.17. Let $T \in \mathscr{S}'(\mathbb{R})$ such that $xT = 0$. Then we have $T = c\delta_0$. Let $\phi \in \mathscr{S}(\mathbb{R})$ and let $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ such that $\chi_0(0) = 1$. We have

$$\phi(x) = \chi_0(x)\phi(x) + (1 - \chi_0(x))\phi(x).$$

Applying Taylor's formula with integral remainder, we define the smooth function ψ by

$$\psi(x) = \frac{(1 - \chi_0(x))}{x}\phi(x)$$

and, applying Leibniz' formula, we see also that ψ belongs to $\mathscr{S}(\mathbb{R})$. As a result

$$\langle T, \phi\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} = \langle T, \chi_0\phi\rangle = \langle T, \chi_0\big(\phi - \phi(0)\big)\rangle + \phi(0)\langle T, \chi_0\rangle = \phi(0)\langle T, \chi_0\rangle,$$

since the function $x \mapsto \chi_0(x)\big(\phi(x) - \phi(0)\big)/x$ belongs to $C_c^\infty(\mathbb{R})$. As a result $T = \langle T, \chi_0\rangle\delta_0$.

We obtain

$$\widehat{\operatorname{sign}}(\xi) = \frac{1}{i\pi} pv\frac{1}{\xi}, \tag{1.2.26}$$

$$\widehat{pv(\frac{1}{\pi x})} = -i \operatorname{sign} \xi, \tag{1.2.27}$$

$$\hat{H} = \frac{\delta_0}{2} + \frac{1}{2i\pi} pv(\frac{1}{\xi}) = \frac{1}{(x - i0)}\frac{1}{2i\pi}. \tag{1.2.28}$$

Let us consider now for $0 < \alpha < n$ the $L^1_{\text{loc}}(\mathbb{R}^n)$ function $u_\alpha(x) = |x|^{\alpha - n}$ ($|x|$ is the Euclidean norm of x); since u_α is also bounded for $|x| \geq 1$, it is a tempered distribution. Let us calculate its Fourier transform v_α. Since u_α is homogeneous of degree $\alpha - n$, we get that v_α is a homogeneous distribution of degree $-\alpha$. On the other hand, if $S \in \mathrm{O}(\mathbb{R}^n)$ (the orthogonal group), we have in the distribution sense[8] since u_α is a radial function, i.e. such that

$$v_\alpha(S\xi) = v_\alpha(\xi). \tag{1.2.29}$$

The distribution $|\xi|^\alpha v_\alpha(\xi)$ is homogeneous of degree 0 on $\mathbb{R}^n\backslash\{0\}$ and is also "radial", i.e. satisfies (1.2.29). Moreover on $\mathbb{R}^n\backslash\{0\}$, the distribution v_α is a C^1 function which coincides with[9]

$$\int e^{-2i\pi x\cdot\xi}\chi_0(x)|x|^{\alpha-n}dx + |\xi|^{-2N}\int e^{-2i\pi x\cdot\xi}|D_x|^{2N}\big(\chi_1(x)|x|^{\alpha-n}\big)dx,$$

where $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ is 1 near 0 and $\chi_1 = 1 - \chi_0$, $N \in \mathbb{N}, \alpha + 1 < 2N$. As a result $|\xi|^\alpha v_\alpha(\xi) = c_\alpha$ on $\mathbb{R}^n\backslash\{0\}$ and the distribution on $\mathbb{R}^n$ (note that $\alpha < n$)

$$T = v_\alpha(\xi) - c_\alpha|\xi|^{-\alpha}$$

[8] For $M \in Gl(n, \mathbb{R})$, $T \in \mathscr{S}'(\mathbb{R}^n)$, we define $\langle T(Mx), \phi(x)\rangle = \langle T(y), \phi(M^{-1}y)\rangle|\det M|^{-1}$.

[9] We have $\widehat{u_\alpha} = \widehat{\chi_0 u_\alpha} + \widehat{\chi_1 u_\alpha}$ and for ϕ supported in $\mathbb{R}^n\backslash\{0\}$ we get,

$$\langle\widehat{\chi_1 u_\alpha}, \phi\rangle = \langle\widehat{\chi_1 u_\alpha}|\xi|^{2N}, \phi(\xi)|\xi|^{-2N}\rangle = \langle\widehat{|D_x|^{2N}\chi_1 u_\alpha}, \phi(\xi)|\xi|^{-2N}\rangle.$$

is supported in $\{0\}$ and homogeneous (on $\mathbb{R}^n$) with degree $-\alpha$. The condition $0 < \alpha < n$ gives $v_\alpha = c_\alpha |\xi|^{-\alpha}$. To find c_α, we compute

$$\int_{\mathbb{R}^n} |x|^{\alpha-n} e^{-\pi x^2} dx = c_\alpha \int_{\mathbb{R}^n} |\xi|^{-\alpha} e^{-\pi \xi^2} d\xi$$

which yields

$$2^{-1}\Gamma(\frac{\alpha}{2})\pi^{-\frac{\alpha}{2}} = \int_0^{+\infty} r^{\alpha-1} e^{-\pi r^2} dr = c_\alpha \int_0^{+\infty} r^{n-\alpha-1} e^{-\pi r^2} dr$$
$$= c_\alpha 2^{-1}\Gamma(\frac{n-\alpha}{2})\pi^{-(\frac{n-\alpha}{2})}.$$

We have proven the following lemma.

Lemma 1.2.18. *Let $n \in \mathbb{N}^*$ and $\alpha \in (0, n)$. The function $u_\alpha(x) = |x|^{\alpha-n}$ is $L^1_{loc}(\mathbb{R}^n)$ and also a temperate distribution on $\mathbb{R}^n$. Its Fourier transform v_α is also $L^1_{loc}(\mathbb{R}^n)$ and given by*

$$v_\alpha(\xi) = |\xi|^{-\alpha} \pi^{\frac{n}{2}-\alpha} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

Fourier transform of Gaussian functions

Proposition 1.2.19. *Let A be a symmetric nonsingular $n \times n$ matrix with complex entries such that $\operatorname{Re} A \ge 0$. We define the Gaussian function v_A on $\mathbb{R}^n$ by $v_A(x) = e^{-\pi\langle Ax, x\rangle}$. The Fourier transform of v_A is*

$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi, \xi\rangle}. \tag{1.2.30}$$

In particular, when $A = -iB$ with a symmetric real nonsingular matrix B, we get

$$\operatorname{Fourier}(e^{i\pi\langle Bx, x\rangle})(\xi) = \widehat{v_{-iB}}(\xi) = |\det B|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sign} B} e^{-i\pi\langle B^{-1}\xi, \xi\rangle}. \tag{1.2.31}$$

Proof. Let us define Υ_+^* as the set of symmetric $n \times n$ complex matrices with a positive definite real part (naturally these matrices are nonsingular since $Ax = 0$ for $x \in \mathbb{C}^n$ implies $0 = \operatorname{Re}\langle Ax, \bar{x}\rangle = \langle(\operatorname{Re} A)x, \bar{x}\rangle$, so that $\Upsilon_+^* \subset \Upsilon_+$).

Let us assume first that $A \in \Upsilon_+^*$; then the function v_A is in the Schwartz class (and so is its Fourier transform). The set Υ_+^* is an open convex subset of $\mathbb{C}^{n(n+1)/2}$ and the function $\Upsilon_+^* \ni A \mapsto \widehat{v_A}(\xi)$ is holomorphic and given on $\Upsilon_+^* \cap \mathbb{R}^{n(n+1)/2}$ by (1.2.30). On the other hand the function

$$\Upsilon_+^* \ni A \mapsto e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log} A} e^{-\pi\langle A^{-1}\xi, \xi\rangle},$$

is also holomorphic and coincides with previous one on $\mathbb{R}^{n(n+1)/2}$. By analytic continuation this proves (1.2.30) for $A \in \Upsilon_+^*$.

If $A \in \Upsilon_+$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have $\langle \widehat{v_A}, \varphi\rangle_{\mathscr{S}', \mathscr{S}} = \int v_A(x)\hat{\varphi}(x) dx$ so that $\Upsilon_+ \ni A \mapsto \langle \widehat{v_A}, \varphi\rangle$ is continuous and thus (note that the mapping $A \mapsto A^{-1}$ is an homeomorphism of Υ_+), using the previous result on Υ_+^*,

$$\langle \widehat{v_A}, \varphi\rangle = \lim_{\epsilon \to 0_+} \langle \widehat{v_{A+\epsilon I}}, \varphi\rangle = \lim_{\epsilon \to 0_+} \int e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log}(A+\epsilon I)} e^{-\pi\langle (A+\epsilon I)^{-1}\xi, \xi\rangle} \varphi(\xi) d\xi,$$

and by continuity of Log on Υ_+ and dominated convergence,

$$\langle \widehat{v_A}, \varphi \rangle = \int e^{-\frac{1}{2}\operatorname{trace} \operatorname{Log} A} e^{-\pi \langle A^{-1}\xi, \xi \rangle} \varphi(\xi) d\xi,$$

which is the sought result. □

Multipliers of $\mathscr{S}'(\mathbb{R}^n)$

Definition 1.2.20. The space $\mathscr{O}_M(\mathbb{R}^n)$ of multipliers of $\mathscr{S}(\mathbb{R}^n)$ is the subspace of the functions $f \in C^\infty(\mathbb{R}^n)$ such that,

$$\forall \alpha \in \mathbb{N}^n, \exists C_\alpha > 0, \exists N_\alpha \in \mathbb{N}, \quad \forall x \in \mathbb{R}^n, \quad |(\partial_x^\alpha f)(x)| \le C_\alpha (1 + |x|)^{N_\alpha}. \tag{1.2.32}$$

It is easy to check that, for $f \in \mathscr{O}_M(\mathbb{R}^n)$, the operator $u \mapsto fu$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself, and by transposition from $\mathscr{S}'(\mathbb{R}^n)$ into itself: we define for $T \in \mathscr{S}'(\mathbb{R}^n)$, $f \in \mathscr{O}_M(\mathbb{R}^n)$,

$$\langle fT, \varphi \rangle_{\mathscr{S}', \mathscr{S}} = \langle T, f\varphi \rangle_{\mathscr{S}', \mathscr{S}},$$

and if p is a semi-norm of $\mathscr{S}$, the continuity on $\mathscr{S}$ of the multiplication by f implies that there exists a semi-norm q on $\mathscr{S}$ such that for all $\varphi \in \mathscr{S}$, $p(f\varphi) \le q(\varphi)$. A typical example of a function in $\mathscr{O}_M(\mathbb{R}^n)$ is $e^{iP(x)}$ where P is a real-valued polynomial: in fact the derivatives of $e^{iP(x)}$ are of type $Q(x)e^{iP(x)}$ where Q is a polynomial so that (1.2.32) holds.

Definition 1.2.21. Let T, S be tempered distributions on $\mathbb{R}^n$ such that $\hat{T}$ belongs to $\mathscr{O}_M(\mathbb{R}^n)$. We define the convolution $T * S$ by

$$\widehat{T * S} = \hat{T}\hat{S}. \tag{1.2.33}$$

Note that this definition makes sense since $\hat{T}$ is a multiplier so that $\hat{T}\hat{S}$ is indeed a tempered distribution whose inverse Fourier transform is meaningful. We have

$$\langle T * S, \phi \rangle_{\mathscr{S}'(\mathbb{R}^n), \mathscr{S}(\mathbb{R}^n)} = \langle \widehat{T * S}, \hat{\check{\phi}} \rangle_{\mathscr{S}'(\mathbb{R}^n), \mathscr{S}(\mathbb{R}^n)} = \langle \hat{S}, \hat{T}\hat{\check{\phi}} \rangle_{\mathscr{S}'(\mathbb{R}^n), \mathscr{S}(\mathbb{R}^n)}.$$

Proposition 1.2.22. *Let T be a distribution on $\mathbb{R}^n$ such that T is compactly supported. Then $\hat{T}$ is a multiplier which can be extended to an entire function on $\mathbb{C}^n$ such that if* $\operatorname{supp} T \subset \bar{B}(0, R_0)$*,*

$$\exists C_0, N_0 \ge 0, \forall \zeta \in \mathbb{C}^n, \quad |\hat{T}(\zeta)| \le C_0 (1 + |\zeta|)^{N_0} e^{2\pi R_0 |\operatorname{Im} \zeta|}. \tag{1.2.34}$$

In particular, for $S \in \mathscr{S}'(\mathbb{R}^n)$, we may define according to (1.2.33) the convolution $T * S$.

Proof. Let us first check the case $R_0 = 0$: then the distribution T is supported at $\{0\}$ and thus is a linear combination of derivatives of the Dirac mass at 0. Formulas (1.2.18), (1.2.20) imply that $\hat{T}$ is a polynomial, so that the conclusions of Proposition 1.2.22 hold in that case.

Let us assume that $R_0 > 0$ and let us consider a function χ is equal to 1 in neighborhood of $\operatorname{supp} T$ (this implies $\chi T = T$) and

$$\langle \widehat{T}, \phi \rangle_{\mathscr{S}', \mathscr{S}} = \langle \widehat{\chi T}, \phi \rangle_{\mathscr{S}', \mathscr{S}} = \langle T, \chi \hat{\phi} \rangle_{\mathscr{S}', \mathscr{S}}. \tag{1.2.35}$$

On the other hand, defining for $\zeta \in \mathbb{C}^n$ (with $x \cdot \zeta = \sum x_j \zeta_j$ for $x \in \mathbb{R}^n$),

$$F(\zeta) = \langle T(x), \chi(x) e^{-2i\pi x \cdot \zeta} \rangle_{\mathscr{S}', \mathscr{S}}, \tag{1.2.36}$$

we see that F is an entire function (i.e. holomorphic on $\mathbb{C}^n$): calculating

$$\begin{aligned} F(\zeta + h) - F(\zeta) &= \langle T(x), \chi(x) e^{-2i\pi x \cdot \zeta} (e^{-2i\pi x \cdot h} - 1) \rangle \\ &= \langle T(x), \chi(x) e^{-2i\pi x \cdot \zeta} (-2i\pi x \cdot h) \rangle \\ &\quad + \langle T(x), \chi(x) e^{-2i\pi x \cdot \zeta} \int_0^1 (1-\theta) e^{-2i\theta\pi x \cdot h} d\theta (-2i\pi x \cdot h)^2 \rangle, \end{aligned}$$

and applying to the last term the continuity properties of the linear form T, we obtain that the complex differential of F is

$$\sum_{1 \le j \le n} \langle T(x), \chi(x) e^{-2i\pi x \cdot \zeta} (-2i\pi x_j) \rangle d\zeta_j.$$

Moreover the derivatives of (1.2.36) are

$$F^{(k)}(\zeta) = \langle T(x), \chi(x) e^{-2i\pi x \cdot \zeta} (-2i\pi x)^k \rangle_{\mathscr{S}', \mathscr{S}}. \tag{1.2.37}$$

To evaluate the semi-norms of $x \mapsto \chi(x) e^{-2i\pi x \cdot \zeta} (-2i\pi x)^k$ in the Schwartz space, we have to deal with a finite sum of products of type

$$\left| x^\gamma (\partial^\alpha \chi)(x) e^{-2i\pi x \cdot \zeta} (-2i\pi \zeta)^\beta \right| \le (1 + |\zeta|)^{|\beta|} \sup_{x \in \mathbb{R}^n} |x^\gamma (\partial^\alpha \chi)(x) e^{2\pi |x| |\operatorname{Im} \zeta|}|.$$

We may now choose a function χ_0 equal to 1 on $B(0,1)$, supported in $B(0, \frac{R_0 + 2\epsilon}{R_0 + \epsilon})$ such that $\|\partial^\beta \chi_0\|_{L^\infty} \le c(\beta) \epsilon^{-|\beta|}$ with $\epsilon = \frac{R_0}{1 + |\zeta|}$. We find with

$$\chi(x) = \chi_0(x/(R_0 + \epsilon)) \quad \text{(which is 1 on a neighborhood of } B(0, R_0)),$$

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} |x^\gamma (\partial^\alpha \chi)(x) e^{2\pi |x| |\operatorname{Im} \zeta|}| &\le (R_0 + 2\epsilon)^{|\gamma|} \sup_{y \in \mathbb{R}^n} |(\partial^\alpha \chi_0)(y) e^{2\pi (R_0 + 2\epsilon) |\operatorname{Im} \zeta|}| \\ &\le (R_0 + 2\epsilon)^{|\gamma|} e^{2\pi (R_0 + 2\epsilon) |\operatorname{Im} \zeta|} c(\alpha) \epsilon^{-|\alpha|} \\ &= (R_0 + 2 \frac{R_0}{1 + |\zeta|})^{|\gamma|} e^{2\pi (R_0 + 2 \frac{R_0}{1+|\zeta|}) |\operatorname{Im} \zeta|} c(\alpha) (\frac{1 + |\zeta|}{R_0})^{|\alpha|} \\ &\le (3R_0)^{|\gamma|} e^{2\pi R_0 |\operatorname{Im} \zeta|} e^{4\pi R_0} c(\alpha) R_0^{-|\alpha|} (1 + |\zeta|)^{|\alpha|} \end{aligned}$$

yielding

$$|F^{(k)}(\zeta)| \le e^{2\pi R_0 |\operatorname{Im} \zeta|} C_k (1 + |\zeta|)^{N_k},$$

which implies that $\mathbb{R}^n \ni \xi \mapsto F(\xi)$ is indeed a multiplier. We have also

$$\langle T, \chi\hat{\phi}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T(x), \chi(x) \int_{\mathbb{R}^n} \phi(\xi) e^{-2i\pi x\xi} d\xi\rangle_{\mathscr{S}',\mathscr{S}}.$$

Since the function F is entire we have for $\phi \in C_c^\infty(\mathbb{R}^n)$, using (1.2.37) and Fubini Theorem on $\ell^1(\mathbb{N}) \times L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} F(\xi)\phi(\xi)d\xi = \sum_{k\ge 0} \langle T(x), \chi(x)(-2i\pi x)^k\rangle \int_{\operatorname{supp}\phi} \frac{\xi^k}{k!}\phi(\xi)d\xi. \tag{1.2.38}$$

On the other hand, since $\hat{\phi}$ is also entire (from the discussion on F or directly from the integral formula for the Fourier transform of $\phi \in C_c^\infty(\mathbb{R}^n)$), we have

$$\begin{aligned}\langle T, \chi\hat{\phi}\rangle &= \langle T(x), \chi(x) \sum_{k\ge 0} (\hat{\phi})^{(k)}(0) x^k/k!\rangle \\ &= \langle T(x), \underbrace{\chi(x) \lim_{N\to+\infty} \sum_{0\le k\le N} (\hat{\phi})^{(k)}(0) x^k/k!}_{\text{convergence in } C_c^\infty(\mathbb{R}^n)}\rangle \\ &= \lim_{N\to+\infty} \sum_{0\le k\le N} \langle T(x), \chi(x) x^k/k!\rangle \int_{\mathbb{R}^n} \phi(\xi)(-2i\pi\xi)^k d\xi.\end{aligned}$$

Thanks to (1.2.38), that quantity is equal to $\int_{\mathbb{R}^n} F(\xi)\phi(\xi)d\xi$. As a result, the tempered distributions $\hat{T}$ and F coincide on $C_c^\infty(\mathbb{R}^n)$, which is dense in $\mathscr{S}(\mathbb{R}^n)$ and so $\hat{T} = F$, concluding the proof. □

1.3 The Poisson summation formula

Wave packets

We define for $x \in \mathbb{R}^n$, $(y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\varphi_{y,\eta}(x) = 2^{n/4} e^{-\pi(x-y)^2} e^{2i\pi(x-y)\cdot\eta} = 2^{n/4} e^{-\pi(x-y-i\eta)^2} e^{-\pi\eta^2}, \tag{1.3.1}$$

$$\text{where for } \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n, \quad \zeta^2 = \sum_{1\le j\le n} \zeta_j^2. \tag{1.3.2}$$

We note that the function $\varphi_{y,\eta}$ is in $\mathscr{S}(\mathbb{R}^n)$ and with L^2 norm 1. In fact, $\varphi_{y,\eta}$ appears as a *phase translation* of a normalized Gaussian. The following lemma introduces the *wave packets transform* as a Gabor wavelet.

Lemma 1.3.1. *Let u be a function in the Schwartz class $\mathscr{S}(\mathbb{R}^n)$. We define*

$$(Wu)(y,\eta) = (u, \varphi_{y,\eta})_{L^2(\mathbb{R}^n)} = 2^{n/4} \int u(x) e^{-\pi(x-y)^2} e^{-2i\pi(x-y)\cdot\eta} dx \tag{1.3.3}$$

$$= 2^{n/4} \int u(x) e^{-\pi(y-i\eta-x)^2} dx e^{-\pi\eta^2}. \tag{1.3.4}$$

For $u \in L^2(\mathbb{R}^n)$, the function Tu defined by

$$(Tu)(y+i\eta) = e^{\pi\eta^2} Wu(y,-\eta) = 2^{n/4}\int u(x)e^{-\pi(y+i\eta-x)^2}dx \tag{1.3.5}$$

is an entire function. The mapping $u \mapsto Wu$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^{2n})$ and isometric from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. Moreover, we have the reconstruction formula

$$u(x) = \iint_{\mathbb{R}^n\times\mathbb{R}^n}(Wu)(y,\eta)\varphi_{y,\eta}(x)dyd\eta. \tag{1.3.6}$$

Proof. For u in $\mathscr{S}(\mathbb{R}^n)$, we have

$$(Wu)(y,\eta) = e^{2i\pi y\eta}\widehat{\Omega}^1(\eta,y)$$

where $\widehat{\Omega}^1$ is the Fourier transform with respect to the first variable of the $\mathscr{S}(\mathbb{R}^{2n})$ function $\Omega(x,y) = u(x)e^{-\pi(x-y)^2}2^{n/4}$. Thus the function Wu belongs to $\mathscr{S}(\mathbb{R}^{2n})$. It makes sense to compute

$$2^{-n/2}(Wu,Wu)_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+}\int u(x_1)\overline{u}(x_2)e^{-\pi[(x_1-y)^2+(x_2-y)^2+2i(x_1-x_2)\eta+\epsilon^2\eta^2]}dyd\eta dx_1dx_2. \tag{1.3.7}$$

Now the last integral on $\mathbb{R}^{4n}$ converges absolutely and we can use the Fubini theorem. Integrating with respect to η involves the Fourier transform of a Gaussian function and we get $\epsilon^{-n}e^{-\pi\epsilon^{-2}(x_1-x_2)^2}$. Since

$$2(x_1-y)^2+2(x_2-y)^2 = (x_1+x_2-2y)^2+(x_1-x_2)^2,$$

integrating with respect to y yields a factor $2^{-n/2}$. We are left with

$$(Wu,Wu)_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+}\int u(x_1)\ \overline{u}(x_2)e^{-\pi(x_1-x_2)^2/2}\epsilon^{-n}e^{-\pi\epsilon^{-2}(x_1-x_2)^2}dx_1dx_2. \tag{1.3.8}$$

Changing the variables, the integral is

$$\lim_{\epsilon\to 0_+}\int u(s+\epsilon t/2)\ \overline{u}(s-\epsilon t/2)e^{-\pi\epsilon^2t^2/2}e^{-\pi t^2}dtds = \|u\|^2_{L^2(\mathbb{R}^n)}$$

by Lebesgue's dominated convergence theorem: the triangle inequality and the estimate $|u(x)| \le C(1+|x|)^{-n-1}$ imply, with $v = u/C$,

$$\begin{aligned}|v(s+\epsilon t/2)\ \overline{v}(s-\epsilon t/2)| &\le (1+|s+\epsilon t/2|)^{-n-1}(1+|s+\epsilon t/2|)^{-n-1}\\ &\le (1+|s+\epsilon t/2|+|s-\epsilon t/2|)^{-n-1}\\ &\le (1+2|s|)^{-n-1}.\end{aligned}$$

Eventually, this proves that for $u \in \mathscr{S}(\mathbb{R}^n)$,

$$\|Wu\|^2_{L^2(\mathbb{R}^{2n})} = \|u\|^2_{L^2(\mathbb{R}^n)} \tag{1.3.9}$$

so that by density of $\mathscr{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$,

$$W : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}) \quad \text{with} \quad W^*W = \mathrm{id}_{L^2(\mathbb{R}^n)}. \tag{1.3.10}$$

Noticing first that $\iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta$ belongs to $L^2(\mathbb{R}^n)$ (with a norm smaller than $\|Wu\|_{L^1(\mathbb{R}^{2n})}$) and applying Fubini's theorem, we get from the polarization of (1.3.9) for $u, v \in \mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned}(u,v)_{L^2(\mathbb{R}^n)} = (Wu, Wv)_{L^2(\mathbb{R}^{2n})} &= \iint Wu(y,\eta)(\varphi_{y,\eta}, v)_{L^2(\mathbb{R}^n)}dyd\eta \\ &= (\iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta, v)_{L^2(\mathbb{R}^n)},\end{aligned}$$

yielding $u = \iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta$, which is the result of the lemma. □

Poisson's formula

The following lemma is in fact the Poisson summation formula for Gaussian functions in one dimension.

Lemma 1.3.2. *For all complex numbers z, the following series are absolutely converging and*

$$\sum_{m\in\mathbb{Z}} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}} e^{-\pi m^2} e^{2i\pi mz}. \tag{1.3.11}$$

Proof. We set $\omega(z) = \sum_{m\in\mathbb{Z}} e^{-\pi(z+m)^2}$. The function ω is entire and 1-periodic since for all $m \in \mathbb{Z}$, $z \mapsto e^{-\pi(z+m)^2}$ is entire and for $R > 0$,

$$\sup_{|z|\le R} |e^{-\pi(z+m)^2}| \le \sup_{|z|\le R} |e^{-\pi z^2}| e^{-\pi m^2} e^{2\pi|m|R} \in \ell^1(\mathbb{Z}).$$

Consequently, for $z \in \mathbb{R}$, we obtain, expanding ω in Fourier series[10],

$$\omega(z) = \sum_{k\in\mathbb{Z}} e^{2i\pi kz} \int_0^1 \omega(x)e^{-2i\pi kx}dx.$$

We also check, using Fubini's theorem on $L^1(0,1) \times \ell^1(\mathbb{Z})$

$$\begin{aligned}\int_0^1 \omega(x)e^{-2i\pi kx}dx &= \sum_{m\in\mathbb{Z}} \int_0^1 e^{-\pi(x+m)^2} e^{-2i\pi kx}dx \\ &= \sum_{m\in\mathbb{Z}} \int_m^{m+1} e^{-\pi t^2} e^{-2i\pi kt}dt \\ &= \int_{\mathbb{R}} e^{-\pi t^2} e^{-2i\pi kt} = e^{-\pi k^2}.\end{aligned}$$

[10] Note that we use this expansion only for a C^∞ 1-periodic function. The proof is simple and requires only to compute $1 + 2\,\mathrm{Re}\sum_{1\le k\le N} e^{2i\pi kx} = \frac{\sin\pi(2N+1)x}{\sin\pi x}$. Then one has to show that for a smooth 1-periodic function ω such that $\omega(0) = 0$,

$$\lim_{\lambda\to+\infty} \int_0^1 \frac{\sin\lambda x}{\sin\pi x}\omega(x)dx = 0,$$

which is obvious since for a smooth ν (here we take $\nu(x) = \omega(x)/\sin\pi x$), $|\int_0^1 \nu(x)\sin(\lambda x)dx| = O(\lambda^{-1})$ by integration by parts.

So the lemma is proven for real z and since both sides are entire functions, we conclude by analytic continuation. □

It is now straightforward to get the n-th dimensional version of the previous lemma: for all $z \in \mathbb{C}^n$, using the notation (1.3.2), we have

$$\sum_{m\in\mathbb{Z}^n} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}^n} e^{-\pi m^2} e^{2i\pi m\cdot z}. \tag{1.3.12}$$

Theorem 1.3.3 (Poisson summation formula). *Let n be a positive integer and let u be a function in $\mathscr{S}(\mathbb{R}^n)$. Then we have*

$$\sum_{k\in\mathbb{Z}^n} u(k) = \sum_{k\in\mathbb{Z}^n} \hat{u}(k), \tag{1.3.13}$$

where $\hat{u}$ stands for the Fourier transform of u. In other words the tempered distribution $D_0 = \sum_{k\in\mathbb{Z}^n} \delta_k$ is such that $\widehat{D_0} = D_0$.

Proof. We write, according to (1.3.6) and to Fubini's theorem

$$\begin{aligned}\textstyle\sum_{k\in\mathbb{Z}^n} u(k) &= \sum_{k\in\mathbb{Z}^n} \iint Wu(y,\eta)\varphi_{y,\eta}(k)dyd\eta \\ &= \iint Wu(y,\eta) \sum_{k\in\mathbb{Z}^n} \varphi_{y,\eta}(k)dyd\eta.\end{aligned}$$

Now, (1.3.12), (1.3.1) give

$$\sum_{k\in\mathbb{Z}^n} \varphi_{y,\eta}(k) = \sum_{k\in\mathbb{Z}^n} \widehat{\varphi}_{y,\eta}(k),$$

so that (1.3.6) and Fubini's theorem imply the result. □

1.4 Periodic distributions

The Dirichlet kernel

For $N \in \mathbb{N}$, the Dirichlet kernel D_N is defined on $\mathbb{R}$ by

$$\begin{aligned}D_N(x) &= \sum_{-N\le k\le N} e^{2i\pi kx} \\ &= 1 + 2\operatorname{Re} \sum_{1\le k\le N} e^{2i\pi kx} \underbrace{=}_{x\notin\mathbb{Z}} 1 + 2\operatorname{Re}\left(e^{2i\pi x}\frac{e^{2i\pi Nx}-1}{e^{2i\pi x}-1}\right) \\ &= 1 + 2\operatorname{Re}\big(e^{2i\pi x - i\pi x + i\pi Nx}\big)\frac{\sin(\pi Nx)}{\sin(\pi x)} = 1 + 2\cos(\pi(N+1)x)\frac{\sin(\pi Nx)}{\sin(\pi x)} \\ &= 1 + \frac{1}{\sin(\pi x)}\Big(\sin\big(\pi x(2N+1)\big) - \sin(\pi x)\Big) = \frac{\sin\big(\pi x(2N+1)\big)}{\sin(\pi x)},\end{aligned}$$

and extending by continuity at $x \in \mathbb{Z}$ that 1-periodic function, we find that

$$D_N(x) = \frac{\sin\bigl(\pi x(2N+1)\bigr)}{\sin(\pi x)}. \tag{1.4.1}$$

Now, for a 1-periodic $v \in C^1(\mathbb{R})$, with

$$(D_N \star u)(x) = \int_0^1 D_N(x-t)u(t)dt, \tag{1.4.2}$$

we have

$$\lim_{N\to+\infty} \int_0^1 D_N(x-t)v(t)dt = v(x) + \lim_{N\to+\infty} \int_0^1 \sin(\pi t(2N+1)) \frac{\bigl(v(x-t)-v(x)\bigr)}{\sin(\pi t)} dt,$$

and the function θ_x given by $\theta_x(t) = \frac{v(x-t)-v(x)}{\sin(\pi t)}$ is continuous on $[0,1]$, and from the Riemann-Lebesgue Lemma 1.1.1, we obtain

$$\lim_{N\to+\infty} \sum_{-N\le k\le N} e^{2i\pi kx} \int_0^1 e^{-2i\pi kt} v(t)dt = \lim_{N\to+\infty} \int_0^1 D_N(x-t)v(t)dt = v(x).$$

On the other hand if v is 1-periodic and C^{1+l}, the Fourier coefficient

$$c_k(v) = \int_0^1 e^{-2i\pi kt} v(t)dt$$

$$\overbrace{=}^{\text{for } k\neq 0} \frac{1}{2i\pi k}[e^{-2i\pi kt}v(t)]_{t=1}^{t=0} + \int_0^1 \frac{1}{2i\pi k} e^{-2i\pi kt} v'(t)dt,$$

and iterating the integration by parts, we find $c_k(v) = O(k^{-1-l})$ so that for a 1-periodic C^2 function v, we have

$$\sum_{k\in\mathbb{Z}} e^{2i\pi kx} c_k(v) = v(x). \tag{1.4.3}$$

Pointwise convergence of Fourier series

Lemma 1.4.1. *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a 1-periodic $L^1_{loc}(\mathbb{R})$ function and let $x_0 \in [0,1]$. Let us assume that there exists $w_0 \in \mathbb{R}$ such that the Dini condition is satisfied, i.e.*

$$\int_0^{1/2} \frac{|u(x_0+t)+u(x_0-t)-2w_0|}{t} dt < +\infty. \tag{1.4.4}$$

Then, $\lim_{N\to+\infty} \sum_{|k|\le N} c_k(u)e^{2i\pi kx_0} = w_0$ with $c_k(u) = \int_0^1 e^{-2i\pi tk}u(t)dt$.

Proof. Using the above calculations, we find

$$\sum_{|k|\le N} c_k(u)e^{2i\pi kx_0} = (D_N \star u)(x_0) = w_0 + \int_0^1 \frac{\sin\bigl(\pi t(2N+1)\bigr)}{\sin(\pi t)} \bigl(u(x_0-t)-w_0\bigr)dt,$$

so that, using the periodicity of u and the fact that D_N is an even function , we get

$$(D_N \star u)(x_0) - w_0 = \int_0^{1/2} \frac{\sin\bigl(\pi t(2N+1)\bigr)}{\sin(\pi t)} \bigl(u(x_0 - t) + u(x_0 + t) - 2w_0\bigr) dt.$$

Thanks to the hypothesis (1.4.4), the function

$$t \mapsto \mathbf{1}_{[0,\frac{1}{2}]}(t) \frac{u(x_0 - t) + u(x_0 + t) - 2w_0}{\sin(\pi t)}$$

belongs to $L^1(\mathbb{R})$ and Riemann-Lebesgue Lemma 1.1.1 gives the conclusion. □

Theorem 1.4.2. *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a 1-periodic L^1_{loc} function.*
(1) *Let $x_0 \in [0, 1], w_0 \in \mathbb{R}$. We define $\omega_{x_0,w_0}(t) = |u(x_0 + t) + u(x_0 - t) - 2w_0|$ and we assume that*

$$\int_0^{1/2} \omega_{x_0,w_0}(t) \frac{dt}{t} < +\infty. \tag{1.4.5}$$

Then the Fourier series $(D_N \star u)(x_0)$ converges with limit w_0. In particular, if (1.4.5) is satisfied with $w_0 = u(x_0)$, the Fourier series $(D_N \star u)(x_0)$ converges with limit $u(x_0)$. If u has a left and right limit at x_0 and is such that (1.4.5) is satisfied with $w_0 = \frac{1}{2}\bigl(u(x_0 + 0) + u(x_0 - 0)\bigr)$, the Fourier series $(D_N \star u)(x_0)$ converges with limit $\frac{1}{2}\bigl(u(x_0 - 0) + u(x_0 + 0)\bigr)$.
(2) *If the function u is Hölder-continuous*[11]*, the Fourier series $(D_N \star u)(x)$ converges for all $x \in \mathbb{R}$ with limit $u(x)$.*
(3) *If u has a left and right limit at each point and a left and right derivative at each point, the Fourier series $(D_N \star u)(x)$ converges for all $x \in \mathbb{R}$ with limit*

$$\frac{1}{2}\bigl(u(x - 0) + u(x + 0)\bigr).$$

Proof. (1) follows from Lemma 1.4.1; to obtain (2), we note that for a Hölder continuous function of index $\theta \in]0, 1]$, we have for $t \in]0, 1/2]$

$$t^{-1}\omega_{x,u(x)}(t) \le Ct^{\theta-1} \in L^1([0, 1/2]).$$

(3) If u has a right-derivative at x_0, it means that

$$u(x_0 + t) = u(x_0 + 0) + u'_r(x_0)t + t\epsilon_0(t), \qquad \lim_{t\to 0_+} \epsilon_0(t) = 0.$$

As a consequence, for $t \in]0, 1/2]$, $t^{-1}|u(x_0 + t) - u(x_0 + 0)| \le |u'_r(x_0) + \epsilon_0(t)|$. Since $\lim_{t\to 0_+} \epsilon_0(t) = 0$, there exists $T_0 \in]0, 1/2]$ such that $|\epsilon_0(t)| \le 1$ for $t \in [0, T_0]$. As a result, we have

$$\int_0^{1/2} t^{-1}|u(x_0 + t) - u(x_0 + 0)|dt$$
$$\le \int_0^{T_0} (|u'_r(x_0)| + 1)dt + \int_{T_0}^{1/2} |u(x_0 + t) - u(x_0 + 0)|dtT_0^{-1} < +\infty,$$

since u is also L^1_{loc}. The integral $\int_0^{1/2} t^{-1}|u(x_0 - t) - u(x_0 - 0)|dt$ is also finite and the condition (1.4.5) holds with $w_0 = \frac{1}{2}\bigl(u(x_0 - 0) + u(x_0 + 0)\bigr)$. The proof of the lemma is complete. □

[11] Hölder-continuity of index $\theta \in]0, 1]$ means that $\exists C > 0, \forall t, s,\ |u(t) - u(s)| \le C|t - s|^\theta$.

Periodic distributions

We consider now a distribution u on $\mathbb{R}^n$ which is periodic with periods $\mathbb{Z}^n$. Let $\chi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}_+)$ such that $\chi = 1$ on $[0,1]^n$. Then the function χ_1 defined by

$$\chi_1(x) = \sum_{k \in \mathbb{Z}^n} \chi(x-k)$$

is C^∞ periodic[12] with periods $\mathbb{Z}^n$. Moreover since

$$\mathbb{R}^n \ni x \in \prod_{1 \le j \le n} [E(x_j), E(x_j)+1[,$$

the bounded function χ_1 is also bounded from below and such that $1 \le \chi_1(x)$. With $\chi_0 = \chi/\chi_1$, we have

$$\sum_{k \in \mathbb{Z}^n} \chi_0(x-k) = 1, \quad \chi_0 \in C_c^\infty(\mathbb{R}^n).$$

For $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have from the periodicity of u

$$\langle u, \varphi \rangle = \sum_{k \in \mathbb{Z}^n} \langle u(x), \varphi(x)\chi_0(x-k) \rangle = \sum_{k \in \mathbb{Z}^n} \langle u(x), \varphi(x+k)\chi_0(x) \rangle,$$

where the sums are finite. Now if $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have, since χ_0 is compactly supported (say in $|x| \le R_0$),

$$\begin{aligned}
|\langle u(x), \varphi(x+k)\chi_0(x) \rangle| &\le C_0 \sup_{|\alpha| \le N_0, |x| \le R_0} |\varphi^{(\alpha)}(x+k)| \\
&\le C_0 \sup_{|\alpha| \le N_0, |x| \le R_0} |(1+R_0+|x+k|)^{n+1} \varphi^{(\alpha)}(x+k)|(1+|k|)^{-n-1} \\
&\le p_0(\varphi)(1+|k|)^{-n-1},
\end{aligned}$$

where p_0 is a semi-norm of φ (independent of k). As a result u is a tempered distribution and we have for $\varphi \in \mathscr{S}(\mathbb{R}^n)$, using Poisson's summation formula,

$$\langle u, \varphi \rangle = \langle u(x), \sum_{k \in \mathbb{Z}^n} \underbrace{\varphi(x+k)\chi_0(x)}_{\psi_x(k)} \rangle = \langle u(x), \sum_{k \in \mathbb{Z}^n} \widehat{\psi_x}(k) \rangle.$$

Now we see that $\widehat{\psi_x}(k) = \int_{\mathbb{R}^n} \varphi(x+t)\chi_0(x) e^{-2i\pi kt} dt = \chi_0(x) e^{2i\pi kx} \hat{\varphi}(k)$, so that

$$\langle u, \varphi \rangle = \sum_{k \in \mathbb{Z}^n} \langle u(x), \chi_0(x) e^{2i\pi kx} \rangle \hat{\varphi}(k),$$

which means

$$u(x) = \sum_{k \in \mathbb{Z}^n} \langle u(t), \chi_0(t) e^{2i\pi kt} \rangle e^{-2i\pi kx} = \sum_{k \in \mathbb{Z}^n} \langle u(t), \chi_0(t) e^{-2i\pi kt} \rangle e^{2i\pi kx}.$$

[12]Note that the sum is locally finite since for K compact subset of $\mathbb{R}^n$, $(K-k) \cap \operatorname{supp} \chi_0 = \emptyset$ except for a finite subset of $k \in \mathbb{Z}^n$.

Theorem 1.4.3. *Let u be a periodic distribution on $\mathbb{R}^n$ with periods $\mathbb{Z}^n$. Then u is a tempered distribution and if χ_0 is a $C_c^\infty(\mathbb{R}^n)$ function such that $\sum_{k\in\mathbb{Z}^n}\chi_0(x-k)=1$, we have*

$$u=\sum_{k\in\mathbb{Z}^n} c_k(u)e^{2i\pi kx}, \tag{1.4.6}$$

$$\hat{u}=\sum_{k\in\mathbb{Z}^n} c_k(u)\delta_k, \quad \text{with} \quad c_k(u)=\langle u(t),\chi_0(t)e^{-2i\pi kt}\rangle, \tag{1.4.7}$$

and convergence in $\mathscr{S}'(\mathbb{R}^n)$. If u is in $C^m(\mathbb{R}^n)$ with $m>n$, the previous formulas hold with uniform convergence for (1.4.6) and

$$c_k(u)=\int_{[0,1]^n} u(t)e^{-2i\pi kt}dt. \tag{1.4.8}$$

Proof. The first statements are already proven and the calculation of $\hat{u}$ is immediate. If u belongs to L^1_{loc} we can redo the calculations above choosing $\chi_0=\mathbf{1}_{[0,1]^n}$ and get (1.4.6) with c_k given by (1.4.8). Moreover, if u is in C^m with $m>n$, we get by integration by parts that $c_k(u)$ is $O(|k|^{-m})$ so that the series (1.4.6) is uniformly converging. □

Theorem 1.4.4. *Let u be a periodic distribution on $\mathbb{R}^n$ with periods $\mathbb{Z}^n$. If $u\in L^2_{loc}$ (i.e. $u\in L^2(\mathbb{T}^n)$ with $\mathbb{T}^n=(\mathbb{R}/\mathbb{Z})^n$), then*

$$u(x)=\sum_{k\in\mathbb{Z}^n} c_k(u)e^{2i\pi kx}, \quad \text{with} \quad c_k(u)=\int_{[0,1]^n} u(t)e^{-2i\pi kt}dt, \tag{1.4.9}$$

and convergence in $L^2(\mathbb{T}^n)$. Moreover $\|u\|^2_{L^2(\mathbb{T}^n)}=\sum_{k\in\mathbb{Z}^n}|c_k(u)|^2$. Conversely, if the coefficients $c_k(u)$ defined by (1.4.7) are in $\ell^2(\mathbb{Z}^n)$, the distribution u is $L^2(\mathbb{T}^n)$

Proof. As said above the formula for the $c_k(u)$ follows from changing the choice of χ_0 to $\mathbf{1}_{[0,1]^n}$ in the discussion preceding Theorem 1.4.3. Formula (1.4.6) gives the convergence in $\mathscr{S}'(\mathbb{R}^n)$ to u. Now, since

$$\int_{[0,1]^n} e^{2i\pi(k-l)t}dt=\delta_{k,l}$$

we see from Theorem 1.4.3 that for $u\in C^{n+1}(\mathbb{T}^n)$,

$$\langle u,u\rangle_{L^2(\mathbb{T}^n)}=\sum_{k\in\mathbb{Z}^n}|c_k(u)|^2.$$

As a consequence the mapping $L^2(\mathbb{T}^n)\ni u\mapsto (c_k(u))_{k\in\mathbb{Z}^n}\in\ell^2(\mathbb{Z}^n)$ is isometric with a range containing the dense subset $\ell^1(\mathbb{Z}^n)$ (if $(c_k(u))_{k\in\mathbb{Z}^n}\in\ell^1(\mathbb{Z}^n)$, u is a continuous function); since the range is closed[13], the mapping is onto and is an isometric isomorphism from the open mapping theorem (see e.g. Theorem 2.1.10 in [14]). □

[13]If $A:\mathcal{H}_1\to\mathcal{H}_2$ is an isometric linear mapping between Hilbert spaces and (Au_k) is a converging sequence in $\mathcal{H}_2$, then by linearity and isometry, the sequence (u_k) is a Cauchy sequence in $\mathcal{H}_1$, thus converges. The continuity of A implies that if $u=\lim_k u_k$, we have

$$v=\lim_k Au_k=Au, \quad \text{proving that the range of } A \text{ is closed.}$$

1.5 Convolution of L^2 functions

Let $u, v \in L^2(\mathbb{R}^n)$. We consider $\int u(y)v(x-y)dy = \omega(u,v)(x)$, which makes sense since $\int |u(y)v(x-y)|dy \le \|u\|_{L^2}\|v\|_{L^2} < +\infty$, so that $\omega(u,v) \in L^\infty(\mathbb{R}^n)$. Moreover $\omega(u,v) \in C^0(\mathbb{R}^n)$ since, with $(\tau_h w)(x) = w(x-h)$, we have

$$\omega(u,v)(x+h) - \omega(u,v)(x) = \int u(y)\big((\tau_{-h}v)(x-y) - v(x-y)\big)dy,$$

and thus

$$|\omega(u,v)(x+h) - \omega(u,v)(x)| \le \|u\|_{L^2(\mathbb{R}^n)}\|\tau_{-h}v - v\|_{L^2(\mathbb{R}^n)},$$

and since[14] $\lim_{h\to 0}\|\tau_h v - v\|_{L^2(\mathbb{R}^n)} = 0$, we get the uniform continuity of $\omega(u,v)$. The reader may check the chapter 6 in [15] to see that $\omega(u,v)$ is the convolution of u with v and that $\omega(u,v) = \omega(v,u)$ by a change of variables. However, we have to pay attention to the fact that we have given earlier (Definition 1.2.21) another definition of the convolution when $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$, and we have to verify that these definitions coincide when $u \in L^2_{\text{comp}}(\mathbb{R}^n), v \in L^2(\mathbb{R}^n)$. In fact, for $u, v \in L^2(\mathbb{R}^n), \varphi \in C^0_c(\mathbb{R}^n)$ we have from the Fubini theorem

$$\int \omega(u,v)(x)\varphi(x)dx = \iint u(x)v(y)\varphi(x+y)dxdy, \tag{1.5.1}$$

since with $w(x) = \int |v(y)||\varphi(x+y)|dy = \omega(|\varphi|, |\check{v}|)(x)$, we have[15]

$$\|\omega(|\varphi|, |\check{v}|)\|_{L^2} \le \|v\|_{L^2}\|\varphi\|_{L^1},$$

$$\iint |u(x)||v(y)||\varphi(x+y)|dxdy \le \|u\|_{L^2}\|w\|_{L^2} \le \|u\|_{L^2}\|v\|_{L^2}\|\varphi\|_{L^1} < +\infty,$$

and (1.5.1) gives $\omega(u,v) = u * v$, where the convolution is taken in the distribution sense. We have proven the first part of the following lemma.

Lemma 1.5.1.
(1) *The mapping* $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \ni (u,v) \mapsto u * v \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *as defined above is symmetric and*

$$\|u * v\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)} \tag{1.5.2}$$

[14] For $v \in L^2(\mathbb{R}^n), \varphi \in C^0_c(\mathbb{R}^n)$, $\tau_h v - v = \tau_h(v - \varphi) + \tau_h(\varphi) - \varphi + \varphi - v$, and thus

$$\|\tau_h v - v\|_{L^2} \le 2\|v - \varphi\|_{L^2} + \|\tau_h(\varphi) - \varphi\|_{L^2} \Longrightarrow \limsup_{h\to 0}\|\tau_h v - v\|_{L^2} \le 2\|v - \varphi\|_{L^2},$$

and since $C^0_c(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ this implies $\lim_{h\to 0}\|\tau_h v - v\|_{L^2} = 0$.

[15] This follows from Young's inequality (see e.g. the Théorème 6.2.1 in [15]) but there is a simpler argument: for $w_1 \in L^1, w_2 \in L^2$, then $w_1 * w_2 \in L^2$ with $\|w_1 * w_2\|_{L^2} \le \|w_1\|_{L^1}\|w_2\|_{L^2}$: we have

$$\int \left|\int w_1(y)w_2(x-y)dy\right|^2 dx \le \int \||w_1|^{1/2}\|^2_{L^2} \int |w_1(y)||w_2(x-y)|^2 dydx = \|w_1\|^2_{L^1}\|w_2\|^2_{L^2}.$$

and coincides with the convolution in the distribution sense when u (or v) is compactly supported.
(2) *For $u, v \in L^2(\mathbb{R}^n)$, we have $\widehat{u * v} = \hat{u}\hat{v}$.*

N.B. The formula (2) is already proven for $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$; here, we know that both sides of the equality makes sense, since $u * v \in L^\infty(\mathbb{R}^n)$ and thus is a tempered distribution whose Fourier transform has a meaning. On the other hand, $\hat{u}\hat{v}$ is a product of L^2 functions and thus is a L^1 function.

Proof. We shall see that an approximation argument, the continuity property expressed by the inequality (1.5.2) will imply the result. For $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have with $\chi \in C_c^\infty(\mathbb{R}^n)$, equal to 1 near 0 and $\chi_k(x) = \chi(x/k)$,

$$\langle \widehat{u * v}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle u * v, \hat{\varphi}\rangle_{\mathscr{S}',\mathscr{S}} = \int (u * v)(x)\hat{\varphi}(x)dx = \lim_{k\to+\infty} \int (\chi_k u * v)(x)\hat{\varphi}(x)dx,$$

since $\chi_k u$ tends to u in $L^2(\mathbb{R}^n)$ and thus

$$\int \left|\big((\chi_k u - u) * v\big)(x)\hat{\varphi}(x)\right|dx \le \int |\hat{\varphi}(x)|dx \|\chi_k u - u\|_{L^2}\|v\|_{L^2}.$$

On the other hand, we get, since $\chi_k u, v \in L^2(\mathbb{R}^n)$,

$$\int (\chi_k u * v)(x)\hat{\varphi}(x)dx = \langle \widehat{\chi_k u * v}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle \widehat{\chi_k u}\hat{v}, \varphi\rangle_{\mathscr{S}',\mathscr{S}}$$

$$= \int (F\chi_k u)(x)(Fv)(x)\varphi(x)dx = \langle F(\chi_k u), \overline{\varphi F v}\rangle_{L^2} \underset{k\to+\infty}{\longrightarrow} \langle Fu, \overline{\varphi F v}\rangle_{L^2},$$

a limit which is equal to $\int (Fu)(x)(Fv)(x)\varphi(x)dx$. This completes the proof of (2) in the lemma. □

1.6 Sobolev spaces

Definitions, Injections

For $\xi \in \mathbb{R}^n$, we define

$$\langle \xi \rangle = \sqrt{1 + |\xi|^2}. \tag{1.6.1}$$

It is easy to see that this function as well as all functions $\xi \mapsto \langle \xi \rangle^s$ when $s \in \mathbb{R}$ are elements of the space of multipliers $\mathscr{O}_M$ as given by the definition 1.2.20. In particular, it means that for $u \in \mathscr{S}'(\mathbb{R}^n)$, the product $\langle \xi \rangle^s \hat{u}(\xi)$ makes sense and belongs to $\mathscr{S}'(\mathbb{R}^n)$.

Definition 1.6.1. Let $s \in \mathbb{R}$. We define the Sobolev space $H^s(\mathbb{R}^n)$ as

$$H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n), \langle \xi \rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}. \tag{1.6.2}$$

Proposition 1.6.2. *Let $s \in \mathbb{R}$. The space $H^s(\mathbb{R}^n)$ equipped with the scalar product*

$$\langle u, v\rangle_{H^s(\mathbb{R}^n)} = \int \langle\xi\rangle^{2s}\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi = \langle \hat{u}(\xi)\langle\xi\rangle^s, \hat{v}(\xi)\langle\xi\rangle^s\rangle_{L^2(\mathbb{R}^n)}, \tag{1.6.3}$$

is a Hilbert space. The space $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

Proof. It is obvious that $\langle u, v\rangle_{H^s(\mathbb{R}^n)}$ is a sesquilinear Hermitian and positive-definite form: note in particular that $0 = \langle u, u\rangle_{H^s(\mathbb{R}^n)} = \|\hat{u}(\xi)\langle\xi\rangle^s\|^2_{L^2(\mathbb{R}^n)}$ implies $\hat{u}(\xi)\langle\xi\rangle^s = 0$ in $L^2(\mathbb{R}^n)$ and thus in $\mathscr{S}'(\mathbb{R}^n)$, so that we can muliply that identity by the multiplier $\langle\xi\rangle^{-s}$, get $\hat{u} = 0$ and thus $u = 0$. On the other hand, if $(u_k)_{k\ge1}$ is a Cauchy sequence in $H^s(\mathbb{R}^n)$, the sequence $(v_k)_{k\ge1}$, $v_k(\xi) = \hat{u}_k(\xi)\langle\xi\rangle^s$ converges in $L^2(\mathbb{R}^n)$. Let $v \in L^2$ be its limit; the tempered distribution w defined by the product $w(\xi) = \langle\xi\rangle^{-s}v(\xi)$ is such that $u = \check{\hat{w}} \in H^s(\mathbb{R}^n)$ since $\langle\xi\rangle^s w(\xi) \in L^2$: we have

$$\|u_k - u\|_{H^s} = \|\langle\xi\rangle^s\widehat{u_k}(\xi) - \langle\xi\rangle^s w(\xi)\|_{L^2} = \|v_k - v\|_{L^2} \longrightarrow 0,$$

and the result that H^s is complete. Next we see that, since $\xi \mapsto \langle\xi\rangle^s\hat{u}(\xi)$ is in $\mathscr{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, when $u \in \mathscr{S}(\mathbb{R}^n)$, each $H^s(\mathbb{R}^n)$ contains $\mathscr{S}(\mathbb{R}^n)$. To prove the density of $\mathscr{S}(\mathbb{R}^n)$, we note that if $u \in (\mathscr{S}(\mathbb{R}^n))^{\perp_s}$, i.e.

$$u \in H^s(\mathbb{R}^n), \forall\varphi \in \mathscr{S}(\mathbb{R}^n), \quad \int \langle\xi\rangle^{2s}\hat{u}(\xi)\overline{\hat{\varphi}(\xi)}d\xi = 0,$$

this[16] implies $\forall\psi \in \mathscr{S}(\mathbb{R}^n), \langle\hat{u}, \psi\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = 0$, i.e. $\hat{u} = 0$ as a tempered distribution, thus $u = 0$. □

Theorem 1.6.3. . *Let $s_1 \le s_2$ be real numbers. Then $H^{s_2}(\mathbb{R}^n) \subset H^{s_1}(\mathbb{R}^n)$ with a continuous injection: for $u \in H^{s_2}(\mathbb{R}^n)$ we have*

$$\|u\|_{H^{s_1}(\mathbb{R}^n)} \le \|u\|_{H^{s_2}(\mathbb{R}^n)}. \tag{1.6.4}$$

For a multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$, the operator ∂_x^α is continuous from $H^s(\mathbb{R}^n)$ into $H^{s-m}(\mathbb{R}^n)$.

Proof. The inequality (1.6.4) holds true for $u \in \mathscr{S}(\mathbb{R}^n)$. Now if $u \in H^{s_2}$, $u = \lim_k u_k$ in H^{s_2} with $u_k \in \mathscr{S}(\mathbb{R}^n)$; from (1.6.4) on $\mathscr{S}(\mathbb{R}^n)$, we see that (u_k) is a Cauchy sequence in H^{s_1}, thus converges to $v \in H^{s_1}$. Now the convergence in H^s implies the weak-dual convergence in $\mathscr{S}'(\mathbb{R}^n)$, since for $\varphi \in \mathscr{S}(\mathbb{R}^n)$, $\exists\psi \in \mathscr{S}(\mathbb{R}^n)$ with

$$\langle u_k, \varphi\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle\widehat{u_k}, \check{\hat{\varphi}}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle\langle\xi\rangle^s\widehat{u_k}(\xi), \underbrace{\overline{\langle\xi\rangle^{-s}\check{\hat{\varphi}}(\xi)}}_{\hat{\psi}(\xi)\langle\xi\rangle^s}\rangle_{L^2} = \langle u_k, \psi\rangle_{H^s}.$$

As a result, the sequence (u_k) converges in the weak-dual topology on $\mathscr{S}'(\mathbb{R}^n)$ with limit u (convergence in H^{s_2}) and limit v (convergence in H^{s_1}), thus $u = v$ and the injection property. The inequality (1.6.4) follows from its version with $u \in \mathscr{S}(\mathbb{R}^n)$ and the density, and it implies the continuity. The last property follows from (1.2.9), the density of $\mathscr{S}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$ and the inequality for $m \ge 0$, $|\xi|^m\langle\xi\rangle^{s-m} \le \langle\xi\rangle^s$. □

[16] The mapping $\chi \mapsto \tilde{\chi}$ given by $\tilde{\chi}(\xi) = \langle\xi\rangle^s\chi(\xi)$ is an isomorphism of $\mathscr{S}(\mathbb{R}^n)$.

Identification of $(H^s)^*$ with H^{-s}

Let $s \in \mathbb{R}$. We consider now the following pairing

$$\begin{array}{ccc} H^s(\mathbb{R}^n) \times H^{-s}(\mathbb{R}^n) & \longrightarrow & \mathbb{C} \\ (u,v) & \mapsto & \langle \langle\xi\rangle^s \hat{u}(\xi), \langle\xi\rangle^{-s}\hat{v}(\xi)\rangle_{L^2(\mathbb{R}^n)} = T(u,v) \end{array} \tag{1.6.5}$$

so that

$$|T(u,v)| \le \|u\|_{H^s}\|v\|_{H^{-s}}. \tag{1.6.6}$$

We see that it gives a mapping

$$\Phi : H^{-s}(\mathbb{R}^n) \longrightarrow (H^s(\mathbb{R}^n))^* \tag{1.6.7}$$

defined by

$$\langle \Phi(v), u\rangle_{(H^s)^*, H^s} = T(u,v), \quad \text{with} \quad \|\Phi(v)\|_{(H^s)^*} = \sup_{\|u\|_{H^s}=1} |T(u,v)| = \|v\|_{H^{-s}},$$

since the inequality $\sup_{\|u\|_{H^s}=1} |T(u,v)| \le \|v\|_{H^{-s}}$ follows from (1.6.6) and, for $v \neq 0$, taking u such that $\hat{u}(\xi) = \langle\xi\rangle^{-2s}\hat{v}(\xi)\|v\|_{H^{-s}}^{-1}$, we see that $u \in H^s$ with $\|u\|_{H^s} = 1$ so that $T(u,v) = \|v\|_{H^{-s}}$, providing the equality. The mapping Φ is isometric (thus injective) and to prove that it is an isometric isomorphism, using the open mapping theorem, it is enough to prove that Φ is onto. Let us take $L_0 \in (H^s)^*$: according to the Riesz representation theorem, there exists $u_0 \in H^s$ such that

$$\langle L_0, u\rangle_{(H^s)^*, H^s} = \langle u, u_0\rangle_{H^s} = \langle \langle\xi\rangle^s\hat{u}(\xi), \langle\xi\rangle^s\widehat{u_0}(\xi)\rangle_{L^2} = \langle \langle\xi\rangle^s\hat{u}(\xi), \langle\xi\rangle^{-s}\underbrace{\langle\xi\rangle^{2s}\widehat{u_0}(\xi)}_{\widehat{v_0}(\xi)}\rangle_{L^2},$$

with $v_0 \in H^{-s}$ since $\langle\xi\rangle^{-s}\widehat{v_0}(\xi) = \langle\xi\rangle^s\widehat{u_0}(\xi) \in L^2$, and this gives

$$\langle L_0, u\rangle_{(H^s)^*, H^s} = T(u, v_0) = \Phi(v_0),$$

and the surjectivity of Φ_0. We have proven the following theorem

Theorem 1.6.4. *The pairing (1.6.5) gives a canonical isometric isomorphism Φ (1.6.7) from $H^{-s}(\mathbb{R}^n)$ onto the dual of $H^s(\mathbb{R}^n)$.*

Continuous functions and Sobolev spaces

Theorem 1.6.5. *Let $m \in \mathbb{N}$. Then*

$$H^m(\mathbb{R}^n) = \{u \in \mathscr{D}'(\mathbb{R}^n), \forall \alpha \in \mathbb{N}^n \text{ such that } |\alpha| \le m, \quad \partial_x^\alpha u \in L^2(\mathbb{R}^n)\}. \tag{1.6.8}$$

Moreover, $H^m(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm

$$\Big(\sum_{|\alpha|\le m} \|\partial_x^\alpha u\|_{L^2(\mathbb{R}^n)}^2\Big)^{1/2}. \tag{1.6.9}$$

Proof. Taking $u \in H^m(\mathbb{R}^n)$ in the sense of the definition 1.6.1, we get that $u \in \mathscr{S}'(\mathbb{R}^n)$, $\langle\xi\rangle^m \hat{u}(\xi) \in L^2(\mathbb{R}^n)$ and as a consequence $\hat{u} \in L^2_{\text{loc}}$, $\widehat{D_x^\alpha u} = \xi^\alpha \hat{u}(\xi)$ belongs to $L^2(\mathbb{R}^n)$ if $|\alpha| \le m$ since

$$\int |\xi^\alpha \hat{u}(\xi)|^2 d\xi \le \int \langle\xi\rangle^{2m} |\hat{u}(\xi)|^2 d\xi < +\infty.$$

Conversely, if u satisfies (1.6.8), u belongs to $L^2(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$, and $\xi^\alpha \hat{u}(\xi)$ is in $L^2(\mathbb{R}^n)$ for $|\alpha| \le m$. We have also from Hölder's inequality

$$\langle\xi\rangle^{2m} = (1 + \sum_{1\le j\le n} \xi_j^2)^m \le \big(1 + \sum_{1\le j\le n} \xi_j^{2m}\big)(n+1)^{m-1}, \tag{1.6.10}$$

so that $\int \langle\xi\rangle^{2m} |\hat{u}(\xi)|^2 d\xi \le \big(\|u\|^2_{L^2(\mathbb{R}^n)} + \sum_{1\le j\le n} \|D_j^m u\|^2_{L^2(\mathbb{R}^n)}\big)(n+1)^{m-1} < +\infty$. We have thus proven the first statement of the theorem and also that the Hilbertian norms of $H^m(\mathbb{R}^n)$ and (1.6.9) are equivalent. We have already seen in the proposition 1.6.2 that $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^m(\mathbb{R}^n)$, with a continuous injection since for $\varphi \in \mathscr{S}(\mathbb{R}^n)$,

$$\|\varphi\|^2_{H^s} = \int \langle\xi\rangle^{2s+n+1} |\hat{\varphi}(\xi)|^2 \langle\xi\rangle^{-n-1} d\xi \le C(n) p_s(\varphi), \tag{1.6.11}$$

where p_s is a semi-norm on $\mathscr{S}(\mathbb{R}^n)$.

Lemma 1.6.6. *$C_c^\infty(\mathbb{R}^n)$ is dense in $\mathscr{S}(\mathbb{R}^n)$.*

Proof of the lemma. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and $\chi \in C_c^\infty(\mathbb{R}^n; [0,1])$ equal to 1 on the unit ball of $\mathbb{R}^n$, the sequence of functions $\varphi_k \in C_c^\infty(\mathbb{R}^n)$ defined by $\varphi_k(x) = \chi(x/k)\varphi(x)$ has limit φ in $\mathscr{S}(\mathbb{R}^n)$: we calculate with the standard Leibniz formula

$$\frac{1}{\alpha!}(\partial_x^\alpha \varphi_k)(x) = \sum_{\beta+\gamma=\alpha} \frac{1}{\beta!\gamma!} k^{-|\beta|} (\partial_x^\beta \chi)(x/k)(\partial_x^\gamma \varphi)(x)$$

so that

$$\begin{aligned} &|x^\lambda \big(\partial_x^\alpha(\varphi_k - \varphi)\big)(x)| \\ &\qquad \le |x^\lambda \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\ge 1}} \frac{\alpha!}{\beta!\gamma!} k^{-|\beta|} (\partial_x^\beta \chi)(x/k)(\partial_x^\gamma \varphi)(x)| + |x^\lambda \underbrace{\big(\chi(x/k) - 1\big)}_{\substack{|x|\ge k \\ \text{on its support}}} (\partial_x^\alpha \varphi)(x)| \end{aligned}$$

and

$$\sup_{x\in\mathbb{R}^n} |x^\lambda \big(\partial_x^\alpha(\varphi_k - \varphi)\big)(x)| \le k^{-1} p(\varphi) C(\chi, \alpha) + \frac{2}{k+1} \sup_{|x|\ge k} |(1+|x|) x^\lambda (\partial_x^\alpha \varphi)(x)|,$$

proving that the sequence (φ_k) converges to φ in $\mathscr{S}(\mathbb{R}^n)$ and the lemma. □

The inequality (1.6.11) and the lemma give the density of $C_c^\infty(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$: for $\epsilon > 0$ and $u \in H^s$, there exists $\varphi \in \mathscr{S}(\mathbb{R}^n)$ such that $\|u - \varphi\|_{H^s} < \epsilon/2$ and for that φ there exists $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $p_s(\varphi - \psi) < \frac{\epsilon}{2C(n)+1}$, implying $\|\varphi - \psi\|_{H^s} < \epsilon/2$ and then $\|u - \psi\|_{H^s} < \epsilon$. □

If $f \in \mathscr{O}_M(\mathbb{R}^n)$ (see the definition 1.2.20), we define the operator, called a *Fourier multiplier*, $f(D)$ on $\mathscr{S}'(\mathbb{R}^n)$ by $\widehat{f(D)u} = f(\xi)\hat{u}(\xi)$ and we note that $f(D)$ is an endomorphism of $\mathscr{S}(\mathbb{R}^n)$. The notation is consistent with the fact that for a polynomial P on $\mathbb{R}^n$, the differential operator $P(D)$ is indeed the Fourier multiplier $P(D)$.

Lemma 1.6.7. *Let $s, t \in \mathbb{R}$. Then the Fourier multiplier $\langle D\rangle^s$ is an isomorphism from $H^{s+t}(\mathbb{R}^n)$ onto $H^t(\mathbb{R}^n)$ whose inverse is $\langle D\rangle^{-s}$. If $f \in \mathscr{O}_M$ is bounded, then $f(D)$ is an endomorphism of $H^s(\mathbb{R}^n)$. If $m \in \mathbb{N}$, $H^{-m}(\mathbb{R}^n)$ is the set of linear combinations of derivatives of order $\le m$ of functions of $L^2(\mathbb{R}^n)$.*

Proof. We assume first $t = 0$; we have indeed for $u \in H^s$, $\|u\|_{H^s} = \|\langle D\rangle^s u\|_{L^2}$, and for $u \in L^2$, $\|u\|_{L^2} = \|\langle D\rangle^{-s} u\|_{H^s}$, with $\langle D\rangle^s\langle D\rangle^{-s} = \langle D\rangle^{-s}\langle D\rangle^s = \mathrm{Id}_{\mathscr{S}'(\mathbb{R}^n)}$. If $t \ne 0$, we use the identity $\langle D\rangle^s = \langle D\rangle^{-t}\langle D\rangle^{s+t}$, (valid on $\mathscr{S}'(\mathbb{R}^n)$), so that

$$H^{s+t} \underset{\approx}{\xrightarrow{\langle D\rangle^{s+t}}} H^0 \underset{\approx}{\xrightarrow{\langle D\rangle^{-t}}} H^t.$$

Now if $f \in \mathscr{O}_M$ is bounded, $f(D)$ is bounded on H^0 and the identity $f(D) = \langle D\rangle^{-s} f(D)\langle D\rangle^s$ (valid on $\mathscr{S}'(\mathbb{R}^n)$) proves the boundedness on H^s. For the second part, we consider for a multi-index α with $|\alpha| \le m$, the Fourier multiplier D^α is bounded from L^2 into H^{-m} from the theorem 1.6.3. With $\chi_j(\xi) = \xi_j\langle\xi\rangle^{-1}$, the Fourier multiplier

$$(1 + \sum_{1\le j\le n} \chi_j(D)D_j)^m$$

is an isomorphism from H^0 onto H^{-m}. This implies that for $u \in H^{-m}$,$\exists v \in L^2$ such that

$$u = (1 + \sum_{1\le j\le n} \chi_j(D)D_j)^m v = \sum_{|\alpha|\le m} D^\alpha \psi_\alpha(D) v$$

with each $\psi_\alpha(D)$ bounded on L^2 as a product of $\chi_j(D)$. □

Theorem 1.6.8. *Let $s > n/2$. Then $H^s(\mathbb{R}^n) \subset C^0_{(0)}(\mathbb{R}^n)$ with continuous injection.*

Proof. For $u \in H^s(\mathbb{R}^n)$, we have $\hat{u} \in L^2(\mathbb{R}^n)$ and $\hat{u}(\xi) = \langle\xi\rangle^{-s}\langle\xi\rangle^s\hat{u}(\xi)$ with $\langle\xi\rangle^{-s} \in L^2(\mathbb{R}^n)$, $\langle\xi\rangle^s\hat{u}(\xi) \in L^2(\mathbb{R}^n)$ so that $\hat{u} \in L^1(\mathbb{R}^n)$ and we can apply the Riemann-Lebesgue Lemma . The injection is continuous since (1.2.21) applied to the L^1 function $\hat{u}$ gives

$$\|u\|_{L^\infty} \le \|\hat{u}\|_{L^1} \le \left(\int \langle\xi\rangle^{-2s} d\xi\right)^{1/2} \left(\int \langle\xi\rangle^{2s}|\hat{u}(\xi)|^2 d\xi\right)^{1/2} = c(s,n)\|u\|_{H^s}. \quad (1.6.12)$$

□

Chapter 2

Littlewood-Paley decomposition, Oscillatory integrals

2.1 The Littlewood-Paley decomposition

Let $\varphi_0 \in C_c^\infty(\mathbb{R}^n)$, $1 \geq \varphi_0(\xi) \geq 0$ such that

$$\varphi_0(\xi) = 1 \quad \text{if } |\xi| \leq 1 \text{ and } \varphi_0(\xi) = 0 \text{ if } |\xi| \geq 2,\ \varphi_0 \text{ radial decreasing of } |\xi|.$$

We set

$$\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi).$$

The function φ is supported in the ring $1/2 \leq |\xi| \leq 2$: if $|\xi| \geq 2$, $\varphi(\xi) = 0$ and if $|\xi| \leq 1/2$, $\varphi_0(\xi) = 1 = \varphi_0(2\xi)$ so that $\varphi(\xi) = 0$. We have also $0 \leq \varphi(\xi) \leq 1$. We define, for a positive integer ν, φ_ν to be

$$\varphi_\nu(\xi) = \varphi(\xi/2^\nu)$$

which is supported in the ring $\{2^{\nu-1} \leq |\xi| \leq 2^{\nu+1}\}$. We have then

$$\varphi_\nu(\xi)\varphi_\mu(\xi) = 0 \quad \text{if } |\nu - \mu| \geq 2.$$

We set, for $\nu \in \mathbb{N}$,

$$S_\nu(\xi) = \sum_{0 \leq \mu \leq \nu} \varphi_\mu(\xi).$$

and we have

$$S_\nu(\xi) = \varphi_0(\xi) + \sum_{1 \leq \mu \leq \nu} \varphi_0(\xi/2^\mu) - \varphi_0(\xi/2^{\mu-1}),$$

so that

$$S_\nu(\xi) = \varphi_0(\xi/2^\nu) = 1 \quad \text{if } |\xi| \leq 2^\nu \text{ and } 0 \text{ if } |\xi| \geq 2^{\nu+1}.$$

Consequently, we obtain

$$1 = \sum_{\mu=0}^{+\infty} \varphi_\mu(\xi).$$

Moreover, we get (with $\varphi_{-1} \equiv 0$)

$$1 = \sum_{\mu,\nu} \varphi_\mu(\xi)\varphi_\nu(\xi) = \sum_{\mu\ge 0} \varphi_\mu\varphi_{\mu-1} + \varphi_\mu^2 + \varphi_\mu\varphi_{\mu+1}$$

and thus

$$1/3 \le \sum_{\mu=0}^{+\infty} \varphi_\mu(\xi)^2 \le 1,$$

the last inequality follows from $0 \le \varphi_\mu(\xi) \le 1$. We'll use that $\varphi_\nu(D_x)$ is the convolution with $\hat\varphi(2^\nu x)2^{\nu n}$.

Theorem 2.1.1. *Let $s \in \mathbb{R}$. Then there exists $C_s > c_s > 0$ such that*

$$\forall u \in H^s(\mathbb{R}^n), \quad c_s\|u\|_{H^s}^2 \le \sum_{\mu=0}^{+\infty} \|\varphi_\mu(D_x)u\|_{L^2(\mathbb{R}^n)}^2 2^{2\mu s} \le C_s\|u\|_{H^s}^2.$$

Let $\rho \in (0,1)$. We define the space

$$C^\rho(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n), \sup_{x'\ne x''} \frac{|u(x') - u(x'')|}{|x' - x''|^\rho} < +\infty\}, \tag{2.1.1}$$

$$\|u\|_{C^\rho(\mathbb{R}^n)} = \|u\|_{L^\infty(\mathbb{R}^n)} + \sup_{x'\ne x''} \frac{|u(x') - u(x'')|}{|x' - x''|^\rho}. \tag{2.1.2}$$

For $\rho \in (0,1)$, $C^\rho(\mathbb{R}^n)$ equipped with the above norm is a Banach space; moreover, there exists $C > c > 0$ such that

$$\forall u \in C^\rho(\mathbb{R}^n), \quad c\|u\|_{C^\rho(\mathbb{R}^n)} \le \sup_{\mu\ge 0} \|\varphi_\mu(D_x)u\|_{L^\infty(\mathbb{R}^n)} 2^{\mu\rho} \le C\|u\|_{C^\rho(\mathbb{R}^n)}.$$

Proof. Defining the Besov space $B^s_{p,q}(\mathbb{R}^n)$ for $s \in \mathbb{R}, p, q \ge 1$ by

$$B^s_{p,q}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n), \quad \big(2^{\nu s}\|\varphi_\nu(D)u\|_{L^p(\mathbb{R}^n)}\big)_{\nu\ge 0} \in \ell^q(\mathbb{N})\}, \tag{2.1.3}$$

the theorem is stating that

$$\forall s \in \mathbb{R}, B^s_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n), \quad \forall \rho \in (0,1), B^\rho_{\infty,\infty}(\mathbb{R}^n) = C^\rho(\mathbb{R}^n).$$

The first statement is quite obvious since for $\xi \in \operatorname{supp}\varphi_0$, we have $1 \le \langle\xi\rangle \le 5^{1/2}$, and for

$$\xi \in \operatorname{supp}\varphi_\nu, \quad \nu \ge 1, \quad 2^{-1} \le 2^{-\nu}(1 + 2^{2\nu-2})^{1/2} \le \frac{\langle\xi\rangle}{2^\nu} \le 2^{-\nu}(1 + 2^{2\nu+2})^{1/2} \le 5^{1/2},$$

so that

$$\frac{1}{2^{3|s|}3}\langle\xi\rangle^{2s} \le 2^{-3|s|}\sum_{\nu\ge 0}\langle\xi\rangle^{2s}\varphi_\nu(\xi)^2$$
$$\le \sum_{\nu\ge 0} 2^{2\nu s}\varphi_\nu(\xi)^2 \le 2^{3|s|}\sum_{\nu\ge 0}\langle\xi\rangle^{2s}\varphi_\nu(\xi)^2 \le 2^{3|s|}\langle\xi\rangle^{2s}.$$

Let us now assume that $u \in C^\rho(\mathbb{R}^n)$, i.e. u is a continuous bounded function on $\mathbb{R}^n$ such that $\|u\|_{\Lambda^\rho} < +\infty$. Then, with $\|u\|_{C^\rho} = \|u\|_{L^\infty} + \|u\|_{\Lambda^\rho}$, we have

$$\|\varphi_\nu(D)u\|_{L^\infty(\mathbb{R}^n)} = \|\widehat{\varphi}(2^\nu \cdot)2^{\nu n} * u\|_{L^\infty(\mathbb{R}^n)} \le 2^{-\nu\rho}\|u\|_{C^\rho} C(\varphi_0),$$

since it is obvious for $\nu = 0$ and for $\nu \ge 1$, since $\varphi(0) = 0$ (thus $\int \hat{\varphi} = 0$), we have

$$(\widehat{\varphi}(2^\nu \cdot)2^{\nu n} * u)(x) = \int \widehat{\varphi}(2^\nu y)2^{\nu n}\big(u(x-y) - u(x)\big)dy,$$

which implies $\|\varphi_\nu(D)u\|_{L^\infty(\mathbb{R}^n)} \le \int |\hat{\varphi}(2^\nu y)|2^{\nu n}\|u\|_{\Lambda^\rho}|y|^\rho dy = C(\varphi_0)\|u\|_{\Lambda^\rho}2^{-\nu\rho}$. Conversely if $u \in B^\rho_{\infty,\infty}$, then $u = \sum_{\nu\ge 0}\varphi_\nu(D)u$ and

$$\|u\|_{L^\infty} \le \sum_{\nu\ge 0}\|\varphi_\nu(D)u\|_{L^\infty} \le \sum_{\nu \ge 0} 2^{-\nu\rho}\|u\|_{B^\rho_{\infty,\infty}},$$

so that $u \in L^\infty$. Moreover for $x, h \in \mathbb{R}^n$, we have

$$|u(x+h) - u(x)| \le \underbrace{\sum_{\substack{\nu \\ |h|\le 2^{-\nu}}} |(\varphi_\nu(D)u)(x+h) - (\varphi_\nu(D)u)(x)|}_{=A(h)} + 2\underbrace{\sum_{\substack{\nu \\ |h|>2^{-\nu}}} 2^{-\nu\rho}}_{\le C|h|^\rho}\|u\|_{B^\rho_{\infty,\infty}}.$$

On the other hand, with $\psi \in C^\infty_{\text{c}}(\mathbb{R}^n), \psi = 1$ on the support of φ, $\psi = 0$ near 0, so that with $\nu \ge 1$, $\varphi_\nu(\xi) = \varphi_\nu(\xi)\psi_\nu(\xi)$ with $\psi_\nu(\xi) = \psi(\xi 2^{-\nu})$, $\psi_0 \in C^\infty_{\text{c}}(\mathbb{R}^n)$, $\psi_0 = 1$ on the support of φ_0, we have

$$\begin{aligned}
A(h) &\le \sum_{\substack{\nu \\ |h|\le 2^{-\nu}}} 2\pi|h|\|D\varphi_\nu(D)\psi_\nu(D)u\|_{L^\infty} \\
&\le 2\pi|h| \sum_{\substack{\nu \\ |h|\le 2^{-\nu}}} 2^\nu\|2^{-\nu}D\psi_\nu(D)\varphi_\nu(D)u\|_{L^\infty} \\
&\le 2\pi|h| \sum_{\substack{1\le\nu \\ |h|\le 2^{-\nu}}} 2^\nu\|\varphi_\nu(D)u\|_{L^\infty} \\
&\le 2\pi|h| \sum_{\substack{1\le\nu \\ |h|\le 2^{-\nu}}} 2^{\nu(1-\rho)}\|u\|_{B^\rho_{\infty,\infty}} \\
&\le C\|u\|_{B^\rho_{\infty,\infty}}|h|\big(|h|^{-1}\big)^{1-\rho},
\end{aligned}$$

so that $|u(x+h) - u(x)| \le C'|h|^\rho\|u\|_{B^\rho_{\infty,\infty}}$ and the sought result $u \in C^\rho$. □

Theorem 2.1.2. *The space $B^1_{\infty,\infty}(\mathbb{R}^n)$ given by (2.1.3) has the following characterization: $u \in B^1_{\infty,\infty}(\mathbb{R}^n)$ if and only if $u \in L^\infty(\mathbb{R}^n)$ and*

$$|\!|\!|u|\!|\!|_1 = \sup_{x\in\mathbb{R}^n, 0\ne h\in\mathbb{R}^n} |u(x+h) + u(x-h) - 2u(x)||h|^{-1} < +\infty. \tag{2.1.4}$$

There exists $C > c > 0$ such that, $\forall u \in B^1_{\infty,\infty}(\mathbb{R}^n)$,

$$c\|u\|_{B^1_{\infty,\infty}(\mathbb{R}^n)} \le \|u\|_{L^\infty(\mathbb{R}^n)} + |\!|\!|u|\!|\!|_1 \le C\|u\|_{B^1_{\infty,\infty}(\mathbb{R}^n)}. \tag{2.1.5}$$

Moreover, if $u \in B^1_{\infty,\infty}(\mathbb{R}^n)$*,* $\exists C > 0$ *such that*

$$\forall x \in \mathbb{R}^n, \forall h \in \mathbb{R}^n, \quad |u(x+h) - u(x)| \le C|h|\big(1 + \ln(|h|^{-1})\big). \tag{2.1.6}$$

We define $\operatorname{Lip}(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n), \nabla u \in L^\infty(\mathbb{R}^n)\}$*; this is a Banach space for the norm* $\|u\|_{L^\infty(\mathbb{R}^n)} + \|\nabla u\|_{L^\infty(\mathbb{R}^n)}$*. The inclusion* $\operatorname{Lip}(\mathbb{R}^n) \subset B^1_{\infty,\infty}$ *is continuous and strict.*

Proof. Let us consider $u \in L^\infty(\mathbb{R}^n)$ such that $\|\!|u|\!\|_1 < +\infty$. Then we have

$$\|\varphi_\nu(D)u\|_{L^\infty(\mathbb{R}^n)} = \|\widehat{\varphi}(2^\nu\cdot)2^{\nu n} * u\|_{L^\infty(\mathbb{R}^n)} \le 2^{-\nu}(\|u\|_{L^\infty} + \|\!|u|\!\|_1)C(\varphi_0),$$

since it is obvious for $\nu = 0$ and for $\nu \ge 1$, since $\varphi(0) = 0$ (thus $\int \hat\varphi = 0$), we have, using that φ is even,

$$2(\widehat{\varphi}(2^\nu\cdot)2^{\nu n} * u)(x) = \int \widehat{\varphi}(2^\nu y)2^{\nu n}\big(u(x-y) + u(x+y) - 2u(x)\big)dy,$$

which implies

$$2\|\varphi_\nu(D)u\|_{L^\infty(\mathbb{R}^n)} \le \int |\hat\varphi(2^\nu y)|2^{\nu n}\|\!|u|\!\|_1|y|dy = 2C(\varphi_0)\|\!|u|\!\|_1 2^{-\nu},$$

and the first inequality in (2.1.5). Conversely if $u \in B^1_{\infty,\infty}$, then $u = \sum_{\nu\ge0}\varphi_\nu(D)u$ and

$$\|u\|_{L^\infty} \le \sum_{\nu\ge0}\|\varphi_\nu(D)u\|_{L^\infty} \le \sum_{\nu\ge0}2^{-\nu}\|u\|_{B^1_{\infty,\infty}} = 2\|u\|_{B^1_{\infty,\infty}},$$

so that $u \in L^\infty$. Moreover for $x, h \in \mathbb{R}^n$, we have

$$|u(x+h) + u(x-h) - 2u(x)| \le \underbrace{\sum_{\substack{\nu \\ |h|\le 2^{-\nu}}} |(\varphi_\nu(D)u)(x+h) + (\varphi_\nu(D)u)(x-h) - 2(\varphi_\nu(D)u)(x)|}_{=A(h)} + 4\underbrace{\sum_{\substack{\nu \\ |h|>2^{-\nu}}} 2^{-\nu}}_{\le C|h|}\|u\|_{B^1_{\infty,\infty}}.$$

We set $v_\nu(x) = (\varphi_\nu(D))u(x)$ and we note that v_ν is a C^∞ function; we have

$$v_\nu(x+h) = v_\nu(x) + v_\nu'(x)h + \int_0^1(1-\theta)v_\nu''(x+\theta h)d\theta h^2$$

and thus $v_\nu(x+h) + v_\nu(x-h) - 2v_\nu(x) = \int_{-1}^1(1-|\theta|)v_\nu''(x+\theta h)d\theta h^2$. As a result, we have

$$A(h) \le |h|^2 4\pi^2 \sum_{\substack{\nu \\ |h|\le 2^{-\nu}}} \|D^2\varphi_\nu(D)u\|_{L^\infty}.$$

We consider $\psi \in C_c^\infty(\mathbb{R}^n), \psi = 1$ on the support of φ, $\psi = 0$ near 0, and ψ even, so that with $\nu \geq 1$, $\varphi_\nu(\xi) = \varphi_\nu(\xi)\psi_\nu(\xi)$ with $\psi_\nu(\xi) = \psi(\xi 2^{-\nu})$ and $\psi_0 \in C_c^\infty(\mathbb{R}^n)$, $\psi_0 = 1$ on the support of φ_0. We have

$$\begin{aligned}
A(h) &\leq |h|^2 4\pi^2 \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} \|D^2\varphi_\nu(D)u\|_{L^\infty} = |h|^2 4\pi^2 \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} \|D^2\varphi_\nu(D)\psi_\nu(D)u\|_{L^\infty} \\
&= |h|^2 4\pi^2 \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} 2^{2\nu}\|2^{-2\nu}D^2\psi_\nu(D)\varphi_\nu(D)u\|_{L^\infty} \\
&\leq C|h|^2 4\pi^2 \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} 2^{2\nu}\|\varphi_\nu(D)u\|_{L^\infty} \\
&\leq C|h|^2 4\pi^2 \|u\|_{B^1_{\infty,\infty}} \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} 2^\nu \leq C_1|h|^2\|u\|_{B^1_{\infty,\infty}}|h|^{-1},
\end{aligned}$$

so that

$$|u(x+h) + u(x-h) - 2u(x)| \leq C'|h|\|u\|_{B^1_{\infty,\infty}},$$

and the second inequality in (2.1.5). Let us consider now $u \in B^1_{\infty,\infty}$. Moreover for $x, h \in \mathbb{R}^n$, with $h \neq 0$, we have

$$|u(x+h) - u(x)| \leq \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} |(\varphi_\nu(D)u)(x+h) - (\varphi_\nu(D)u)(x)| + 2\underbrace{\sum_{\substack{\nu \\ |h|>2^{-\nu}}} 2^{-\nu}}_{\leq C|h|} \|u\|_{B^1_{\infty,\infty}}.$$

With the same ψ as above, we have

$$\begin{aligned}
|u(x+h) - u(x)| &\leq |h|C_1\|u\|_{B^1_{\infty,\infty}} + \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} |h|2\pi\|D\psi_\nu(D)\varphi_\nu(D)u\|_{L^\infty} \\
&\leq |h|C_1\|u\|_{B^1_{\infty,\infty}} + \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} |h|2\pi 2^\nu\|2^{-\nu}D\psi_\nu(D)\varphi_\nu(D)u\|_{L^\infty} \\
&\leq |h|C_1\|u\|_{B^1_{\infty,\infty}} + |h|C_2 \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} 2^\nu\|\varphi_\nu(D)u\|_{L^\infty} \\
&\leq |h|C_1\|u\|_{B^1_{\infty,\infty}} + |h|C_2\|u\|_{B^1_{\infty,\infty}} \underbrace{\operatorname{Card}\{\nu \in \mathbb{N}, 2^\nu \leq |h|^{-1}\}}_{\leq \operatorname{Log}_2(|h|^{-1})},
\end{aligned}$$

which gives (2.1.6). We consider now $u \in \operatorname{Lip}(\mathbb{R}^n)$. We have $\|\varphi_0(D)u\|_{L^\infty} \leq C\|u\|_{L^\infty}$ and for $\nu \geq 1$,

$$(\varphi_\nu(D)u)(x) = (\widehat{\varphi}(2^\nu\cdot)2^{\nu n} * u)(x) = \int \widehat{\varphi}(2^\nu y)2^{\nu n}\big(u(x-y) - u(x)\big)dy.$$

We have also in the distribution sense

$$u(x-y) - u(x) = \int_0^1 u'(x-\theta y)d\theta y \Longrightarrow |u(x-y) - u(x)| \leq \|u'\|_{L^\infty}|y|,$$

so that $\|\varphi_\nu(D)u\|_{L^\infty} \le \int |\varphi(2^\nu y)| 2^{\nu n} |y| dy \|u'\|_{L^\infty} \le C\|u'\|_{L^\infty} 2^{-\nu}$, proving the continuous inclusion $\mathrm{Lip}(\mathbb{R}^n) \subset B^1_{\infty,\infty}(\mathbb{R}^n)$. Let us prove finally that this inclusion is strict: we consider

$$T(x) = \int_1^{+\infty} e^{2i\pi x\xi} \xi^{-2} d\xi.$$

The Fourier transform of T belongs to $L^1(\mathbb{R})$ and thus T is a continuous bounded function. We have also

$$(\varphi_\nu(D)T)(x) = \int_1^{+\infty} e^{2i\pi x\xi} \xi^{-2} \varphi_\nu(\xi) d\xi.$$

and for $\nu \ge 1$,

$$(\varphi_\nu(D)T)(x) = \int_1^{+\infty} e^{2i\pi x\xi} \xi^{-2} \varphi(2^{-\nu}\xi) d\xi = 2^{-2\nu} \int_{2^{-\nu}}^{+\infty} e^{2i\pi x 2^\nu \xi} \xi^{-2} \varphi(\xi) d\xi 2^\nu.$$

Since the function φ is (non-negative and) supported in $1/2 \le |\xi| \le 2$, we get for $\nu \ge 1$ that

$$2^\nu(\varphi_\nu(D)T)(x) = \int_{1/2}^2 e^{2i\pi x 2^\nu \xi} \xi^{-2} \varphi(\xi) d\xi$$

$$\Longrightarrow \|2^\nu \varphi_\nu(D)T\|_{L^\infty(\mathbb{R})} \le \int_{1/2}^2 \xi^{-2} \varphi(\xi) d\xi < +\infty.$$

On the other hand

$$(\varphi_0(D)T)(x) = \int_1^{+\infty} e^{2i\pi x\xi} \xi^{-2} \varphi_0(\xi) d\xi,$$

is a bounded function ; we have proven that $T \in B^1_{\infty,\infty}(\mathbb{R})$. Let us prove that T is not in $\mathrm{Lip}(\mathbb{R}^n)$. We calculate for $\epsilon > 0$,

$$\langle T', \epsilon^{-1} e^{-\pi\epsilon^{-2}x^2} \rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})}$$

$$= 2i\pi \langle \xi\hat{T}, e^{-\pi\epsilon^2\xi^2} \rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} = 2i\pi \int_1^{+\infty} \xi^{-1} e^{-\pi\epsilon^2\xi^2} d\xi \underset{\epsilon\to 0_+}{\longrightarrow} +\infty,$$

say from the Fatou theorem, and if T' were a bounded function, we would have

$$|\langle T', \epsilon^{-1} e^{-\pi\epsilon^{-2}x^2} \rangle| \le \|T'\|_{L^\infty(\mathbb{R})} \|\epsilon^{-1} e^{-\pi\epsilon^{-2}x^2}\|_{L^1(\mathbb{R})} = \|T'\|_{L^\infty(\mathbb{R})} < +\infty.$$

The proof of the theorem is complete. □

2.2 Paley – Wiener's theorem

Lemma 2.2.1. *For $u \in \mathscr{S}'(\mathbb{R}^n)$ the following properties are equivalent.*

(*i*) $u \in C_c^\infty(\mathbb{R}^n)$, $\quad \operatorname{supp} u \subset \{x \in \mathbb{R}^n, |x| \le R\}$.

(*ii*) $\hat{u}$ *can be extended to* $\mathbb{C}^n$ *as an entire function such that*

$$\forall N \in \mathbb{N}, \exists C_N > 0, \quad |\hat{u}(\zeta)| \le C_N(1+|\zeta|)^{-N} e^{2\pi R|\operatorname{Im}\zeta|}. \tag{2.2.1}$$

Proof. Let us assume (i). Using the notation $\mathbb{C}^n \ni \zeta = \xi + i\eta, \xi, \eta \in \mathbb{R}^n$, the Fourier transform of u can be extended to $\mathbb{C}^n$ as an entire function, simply with the formula

$$\hat{u}(\xi + i\eta) = \int e^{-2i\pi x\cdot(\xi+i\eta)} u(x) dx \qquad (\text{note} \quad x\cdot(\xi+i\eta) = x\cdot\xi + ix\cdot\eta).$$

As a result, for a polynomial P on $\mathbb{R}^n$, we have $(\widehat{P(D)u})(\zeta) = P(\zeta)\hat{u}(\zeta)$ and thus

$$|P(\zeta)\hat{u}(\zeta)| \le \|P(D)u\|_{L^1(\mathbb{R}^n)} e^{2\pi R|\operatorname{Im}\zeta|},$$

implying for all multi-indices $\alpha \in \mathbb{N}^n$, $|\zeta^\alpha \hat{u}(\zeta)| \le \|D^\alpha u\|_{L^1(\mathbb{R}^n)} e^{2\pi R|\operatorname{Im}\zeta|}$, i.e.

$$|\zeta_1|^{\alpha_1} \dots |\zeta_n|^{\alpha_n} |\hat{u}(\zeta)| \le \|D^\alpha u\|_{L^1(\mathbb{R}^n)} e^{2\pi R|\operatorname{Im}\zeta|}.$$

As a consequence, for $m \in 2\mathbb{N}$, we have with $\|u\|_{W^{m,1}} = \sum_{|\alpha|\le m} \|D^\alpha u\|_{L^1(\mathbb{R}^n)}$,

$$(1+|\zeta|^2)^{m/2} |\hat{u}(\zeta)| \le C_m \|u\|_{W^{m,1}} e^{2\pi R|\operatorname{Im}\zeta|} \Longrightarrow (ii).$$

Conversely, if (ii) holds, the function $\hat{u}$ is C^∞ on $\mathbb{R}^n$ and for all $N \in \mathbb{N}$, $|\hat{u}(\xi)| \le C_N \langle \xi \rangle^{-N}$. Thus $\hat{u} \in L^1(\mathbb{R}^n)$ and one can apply the theorem 1.2.15, so that $u(x) = \int_{\mathbb{R}^n} e^{2i\pi x\cdot\xi} \hat{u}(\xi) d\xi$. Now we have also for all $\eta \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} e^{2i\pi x\cdot\xi} \hat{u}(\xi) d\xi = \int_{\mathbb{R}^n} e^{2i\pi x\cdot(\xi+i\eta)} \hat{u}(\xi+i\eta) d\xi,$$

where both sides make sense thanks to the estimate (2.2.1), which also allow to shift integration of the entire function $\zeta \mapsto \hat{u}(\zeta) e^{2i\pi x\cdot\zeta}$ from $\mathbb{R}^n$ to $\mathbb{R}^n + i\eta$. Now if $|x| > R$, we obtain for all $\eta \in \mathbb{R}^n$,

$$|u(x)| \le C_N e^{2\pi(R|\eta| - x\cdot\eta)} \int_{\mathbb{R}^n} (1+|\xi|)^{-N} d\xi$$

and in particular choosing $\eta = \lambda x/|x|$, $N = n+1$, we get for all $\lambda > 0$, $|u(x)| \le C'_n e^{2\pi(R\lambda - \lambda|x|)}$, so that for $|x| > R$ we obtain $u(x) = 0$ and (i). □

Lemma 2.2.2. *Let* Ω *be an open set of* $\mathbb{R}^n$, $x_0 \in \Omega$ *and* $u \in \mathscr{D}'(\Omega)$. *The following properties are equivalent.*

(*i*) $x_0 \notin \operatorname{singsupp} u$,

(*ii*) $\exists V_0 \in \mathscr{V}_{x_0}$ *such that for all* $\chi \in C_c^\infty(V_0)$, *for all* $N \in \mathbb{N}$, $\exists C$ *such that*

$$|\widehat{\chi u}(\xi)| \le C(1+|\xi|)^{-N}.$$

(*iii*) $\exists V_0 \in \mathscr{V}_{x_0}$, $\exists \chi_0 \in C_c^\infty(V_0)$, *such that* $\chi_0(x_0) \ne 0$, *for all* $N \in \mathbb{N}$, $\exists C$ *such that*

$$|\widehat{\chi_0 u}(\xi)| \le C(1+|\xi|)^{-N}.$$

Proof. If (i) holds, $\exists V_0 \in \mathscr{V}_{x_0}$ such that for all $\chi \in C_c^\infty(V_0)$, $\chi u \in C_c^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$ and thus $\widehat{\chi u} \in \mathscr{S}(\mathbb{R}^n)$, implying (ii). If (ii) holds, then it is the case of the weaker (iii); we take $\chi_0 \in C_c^\infty(V_0)$, different from 0 on a compact neighborhood V_1 of x_0, and we get $\widehat{\chi_0 u} \in L^1(\mathbb{R}^n)$, so that

$$(\chi_0 u)(x) = \int e^{2i\pi x\cdot\xi} \widehat{\chi_0 u}(\xi) d\xi$$

and the estimate of (iii) gives $\chi_0 u \in C_c^\infty(\mathbb{R}^n)$ and $u_{|V_1} = \frac{1}{\chi_{0|V_1}} (\chi_0 u)_{|V_1} \in C^\infty(V_1)$, implying (i). □

Lemma 2.2.3. *Let Ω be an open set of $\mathbb{R}^n$, $x_0 \in \Omega$ and $u \in \mathscr{D}'(\Omega)$. The following properties are equivalent.*

(i) $x_0 \notin \operatorname{singsupp} u$,

(ii) $\exists V_0 \in \mathscr{V}_{x_0}$ *such that for all* $\chi \in C_c^\infty(V_0)$, *for all* $N \in \mathbb{N}$, $\exists C$ *such that*

$$|\widehat{\chi u}(\xi)| \le C(1 + |\xi|)^{-N}.$$

(iii) $\exists V_0 \in \mathscr{V}_{x_0}$, $\exists \chi_0 \in C_c^\infty(V_0)$, *such that* $\chi_0(x_0) \neq 0$, *for all* $N \in \mathbb{N}$, $\exists C$ *such that*

$$|\widehat{\chi_0 u}(\xi)| \le C(1 + |\xi|)^{-N}.$$

Proof. If (i) holds, $\exists V_0 \in \mathscr{V}_{x_0}$ such that for all $\chi \in C_c^\infty(V_0)$, $\chi u \in C_c^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$ and thus $\widehat{\chi u} \in \mathscr{S}(\mathbb{R}^n)$, implying (ii). If (ii) holds, then it is the case of the weaker (iii); we take $\chi_0 \in C_c^\infty(V_0)$, different from 0 on a compact neighborhood V_1 of x_0, and we get $\widehat{\chi_0 u} \in L^1(\mathbb{R}^n)$, so that

$$(\chi_0 u)(x) = \int e^{2i\pi x\cdot\xi} \widehat{\chi_0 u}(\xi) d\xi$$

and the estimate of (iii) gives $\chi_0 u \in C_c^\infty(\mathbb{R}^n)$ and $u_{|V_1} = \frac{1}{\chi_{0|V_1}} (\chi_0 u)_{|V_1} \in C^\infty(V_1)$, implying (i). □

Lemma 2.2.4. *For $u \in \mathscr{S}'(\mathbb{R}^n)$ the following properties are equivalent.*

(i) $u \in \mathscr{E}'(\mathbb{R}^n)$, $\quad \operatorname{supp} u \subset \{x \in \mathbb{R}^n, |x| \le R_0\}$.

(ii) $\hat{u}$ *can be extended to* $\mathbb{C}^n$ *as an entire function such that*

$$|\hat{u}(\zeta)| \le C_0(1 + |\zeta|)^{N_0} e^{2\pi R_0 |\operatorname{Im}\zeta|}. \tag{2.2.2}$$

Proof. If (i) holds, we get that $\hat{u}$ is the entire function $\hat{u}(\zeta) = \langle u(x), e^{-2i\pi x\cdot\zeta}\rangle_{\mathscr{E}',\mathscr{E}}$. Moreover, since u is compactly supported in $\bar{B}(0, R_0)$, we have for all $\epsilon > 0$ and $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on $B(0,1)$, supported in $B(0, \frac{R_0+2\varepsilon}{R_0+\varepsilon})$, such that $\|\chi_0^{(\beta)}\|_{L^\infty} \le c(\beta)\epsilon^{-|\beta|}$,

$$\hat{u}(\zeta) = \langle u(x), \chi_0\big(\frac{x}{R_0+\epsilon}\big) e^{-2i\pi x\cdot\zeta}\rangle_{\mathscr{E}',\mathscr{E}}.$$

This implies

$$|\hat{u}(\zeta)| \le C_0 \sup_{\substack{|x|\le R_0+2\epsilon \\ |\alpha|+|\beta|\le N_0}} |e^{-2i\pi x\cdot\zeta}\zeta^\alpha(\partial^\beta\chi_0)\big(\frac{x}{R_0+\epsilon}\big)(R_0+\epsilon)^{-|\beta|}|c(\beta)$$

and thus

$$\forall\epsilon>0,\quad |\hat{u}(\zeta)| \le C_0 e^{2\pi(R_0+2\epsilon)|\operatorname{Im}\zeta|}\sup_{|\alpha|+|\beta|\le N_0}|\zeta^\alpha(R_0+\epsilon)^{-|\beta|}|\epsilon^{-|\beta|}c(\beta).$$

We choose now, assuming $R_0>0$ (otherwise the distribution u is supported at the origin and is a linear combination of derivatives of the Dirac mass) $\epsilon=\frac{R_0}{1+|\zeta|}$. We get then

$$|\hat{u}(\zeta)| \le C_0 e^{2\pi R_0|\operatorname{Im}\zeta|}e^{4\pi\frac{R_0|\operatorname{Im}\zeta|}{1+|\zeta|}}(1+|\zeta|)^{N_0}\max_{|\beta|\le N_0} c(\beta)R_0^{-2|\beta|} \Longrightarrow (ii).$$

Conversely, if (ii) holds, we consider a standard mollifier ρ_ϵ given with $\epsilon>0$ by $\rho_\epsilon(x)=\epsilon^{-n}\rho(x/\epsilon)$, $\rho\in C^\infty_{\rm c}(\mathbb{R}^n)$, $\int\rho=1$, ρ supported in the unit ball. We have $\widehat{u*\rho_\epsilon}=\hat{u}\hat{\rho}(\epsilon\cdot)$ and the function $\hat{u}\hat{\rho}(\epsilon\cdot)$ is entire with

$$|\hat{u}(\zeta)\hat{\rho}(\epsilon\zeta)| \le C_{N,\epsilon}(1+|\zeta|)^{-N}e^{2\pi(R_0+\epsilon)|\operatorname{Im}\zeta|}.$$

From the first lemma 2.2.1, we have $\operatorname{supp}(u*\rho_\epsilon)\subset\bar{B}(0,R_0+\epsilon)$. For $\varphi\in C^\infty_{\rm c}(\mathbb{R}^n)$ we have

$$\langle u*\rho_\epsilon,\varphi\rangle=\langle u,\check{\rho}_\epsilon*\varphi\rangle\underset{\epsilon\to0_+}{\longrightarrow}\langle u,\varphi\rangle,$$

and thus if $\operatorname{supp}\varphi\subset\big(\bar{B}(0,R_0+\epsilon)\big)^c$, we get $\langle u*\rho_\epsilon,\varphi\rangle=0=\langle u,\varphi\rangle$, so that $\operatorname{supp}u\subset\bar{B}(0,R_0+\epsilon)$ for all $\epsilon>0$ and eventually

$$\operatorname{supp}u\subset\cap_{\epsilon>0}\bar{B}(0,R_0+\epsilon)=\bar{B}(0,R_0),$$

yielding the conclusion. □

Remark 2.2.5. Let us recall the expression of E_+, fundamental solution of the wave equation:

$$\widehat{E_+}^x(t,\xi)=cH(t)\frac{\sin\big(2\pi ct|\xi|\big)}{2\pi|\xi|}=c^2H(t)\int_0^t\cos(2\pi cs|\xi|)ds. \tag{2.2.3}$$

Since $\cos(2\pi cs|\xi|)=\sum_{k\ge0}\frac{(-1)^k(2\pi cs)^{2k}}{(2k)!}(\sum_{1\le j\le d}\xi_j^2)^k$ the function $\widehat{E}(t,\cdot)$ is entire on $\mathbb{C}^d$ and we have for $\zeta\in\mathbb{C}^d$, using the notation $\zeta^2=\sum_{1\le j\le d}\zeta_j^2$,

$$\widehat{E_+}^x(t,\zeta)=c^2H(t)\int_0^t\sum_{k\ge0}\frac{(-1)^k(2\pi cs)^{2k}}{(2k)!}(\zeta^2)^k ds=c^2H(t)\int_0^t\cos(2\pi cs(\zeta^2)^{1/2})ds.$$

We have also for $z\in\mathbb{C}$

$$2|\cos z|^2=2(\cos z)(\cos\bar{z})=\cos(2\operatorname{Re}z)+\cos(2i\operatorname{Im}z)\le 1+e^{2|\operatorname{Im}z|}\le 2e^{2|\operatorname{Im}z|},$$

and as a consequence

$$\text{for } 0 \le s \le t, \quad |\cos(2\pi cs(\zeta^2)^{1/2})| \le \exp 2\pi ct |\operatorname{Im}\big((\zeta^2)^{1/2}\big)|. \tag{2.2.4}$$

We note that with $\zeta = \xi + i\eta, \xi, \eta \in \mathbb{R}^n$,

$$\zeta^2 = |\xi|^2 - |\eta|^2 + 2i\langle \xi, \eta\rangle = |\xi|^2 - |\eta|^2 + 2i\sigma|\xi||\eta|, \quad \text{with } \sigma \in \mathbb{R}, |\sigma| \le 1.$$

So if $z = a + ib \in \mathbb{C}, a, b \in \mathbb{R}$ is such that $z^2 = \zeta^2$, we have

$$a^2 - b^2 = |\xi|^2 - |\eta|^2, \quad |ab| \le |\xi||\eta|.$$

If we had $|b| > |\eta|$, that would imply from the first equation that $|a| > |\xi|$ and $|ab| > |\xi||\eta|$, which contradicts the second equation; as a result we have $|b| \le |\eta|$ and $|\operatorname{Im}\big((\zeta^2)^{1/2}\big)| \le |\operatorname{Im}\zeta|$, implying

$$|\widehat{E_+}^x(t,\zeta)| \le ctH(t)\exp 2\pi ct|\operatorname{Im}\zeta|,$$

which gives from the Paley-Wiener theorem 2.2.4 that

$$\operatorname{supp} E_+(t,\cdot) \subset \{x \in \mathbb{R}^n, |x| \le ct\}. \tag{2.2.5}$$

2.3 Stationary phase method

Preliminary remarks

It is well-known that

$$\int_{\mathbb{R}} \frac{\sin x}{x} dx = \pi, \quad \text{although} \quad \int_{\mathbb{R}} \left|\frac{\sin x}{x}\right| dx = +\infty. \tag{2.3.1}$$

To get this, we integrate the function e^{iz}/z on the following path: the segment $[\epsilon, R]$, the half-circle $(R, iR, -R)$, the segment $[-R, -\epsilon]$, the half-circle $(-\epsilon, i\epsilon, \epsilon)$. We get

$$0 = 2i\int_\epsilon^R \frac{\sin x}{x}dx + \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta}d\theta - \int_0^\pi \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}d\theta.$$

The third integral has limit $i\pi$ for $\epsilon \to 0$. The absolute value of the second integral is bounded above by $\int_0^\pi e^{-R\sin\theta}d\theta$ which goes to zero when R goes[1] to infinity, yielding the value π in (2.3.1). On the other hand, for $n \in \mathbb{N}^*$, we have

$$\int_{2n\pi}^{(2n+1)\pi} \left|\frac{\sin x}{x}\right| dx \ge \frac{1}{(2n+1)\pi}\int_{2n\pi}^{(2n+1)\pi} \sin x dx = \frac{2}{(2n+1)\pi},$$

[1] One may apply Lebesgue's dominated convergence theorem, but it is way too much: it is enough to note that $0 \le \frac{2\theta}{\pi} \le \sin\theta$ for $\theta \in [0, \pi/2]$ and

$$\int_0^\pi e^{-R\sin\theta}d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}d\theta \le 2\int_0^{\pi/2} e^{-2R\theta/\pi}d\theta \le \pi/R.$$

the general term of a diverging series, so that (2.3.1) is proven. In the integral $\int_{\mathbb{R}} \frac{\sin x}{x} dx$, the *amplitude* $1/x$ is too large at infinity to guarantee the absolute convergence of the integral, although the *oscillations* of the term $\sin x = \operatorname{Im} e^{ix}$ compensate the size of the amplitude and lead to some cancellation phenomena. We want to study this phenomenon more closely and in more geometrical terms. Although the function $\sin x/x$ does not belong to $L^1(\mathbb{R}^n)$, we still[2] have in the sense of weak-dual convergence

$$\lim_{\lambda\to+\infty} \frac{1}{\pi}\frac{\sin(\lambda x)}{x} = \delta_0. \tag{2.3.2}$$

In fact for $\varphi \in C_c^1(\mathbb{R}), \operatorname{supp}\varphi \subset [-M_0, M_0]$, the function ψ defined by

$$\psi(x) = x^{-1}\big(\varphi(x) - \varphi(0)\big) = \int_0^1 \varphi'(\theta x)d\theta$$

is continuous and equal to $-\varphi(0)x^{-1}$ for $|x| \geq M_0(> 0)$. As a consequence, we have

$$\int \frac{\sin(\lambda x)}{x}\varphi(x)dx = \int \underbrace{\psi(x)\mathbf{1}_{[-M_0,M_0]}(x)}_{\in L^1(\mathbb{R})} \sin(\lambda x)dx + \varphi(0)\int_{|x|\leq M_0} x^{-1}\sin(\lambda x)dx.$$

The Riemann-Lebesgue Lemma implies that the first term in the rhs tends to 0 with $1/\lambda$, whereas

$$\int_{|x|\leq M_0} x^{-1}\sin(\lambda x)dx = \int_{|y|\leq \lambda M_0} x^{-1}\sin x dx \underset{\lambda\to+\infty}{\longrightarrow} \pi,$$

proving 2.3.2.

Non-stationary phase

Theorem 2.3.1. *Let $a \in C_c^\infty(\mathbb{R}^n)$ and ϕ be a real-valued C^∞ function defined on $\mathbb{R}^n$ such that $d\phi \neq 0$ on the support of a. We define for $\lambda \in \mathbb{R}$,*

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)}a(x)dx. \tag{2.3.3}$$

Then for all $N \geq 0$, $\sup_{\lambda\in\mathbb{R}} |\lambda^N I(\lambda)| < +\infty$.

Proof. Since the support of a is compact, we know that $\inf_{x\in\operatorname{supp} a} |d\phi(x)| = c_0 > 0$. We define then the differential operator L on the open set $\Omega = \{x \in \mathbb{R}^n, d\phi(x) \neq 0\} \supset \operatorname{supp} a$ by

$$L = \frac{1}{i}\sum_{1\leq j\leq n} |d\phi|^{-2}\frac{\partial\phi}{\partial x_j}\frac{\partial}{\partial x_j}. \tag{2.3.4}$$

[2] If $u \in L^1(\mathbb{R}^n)$, $\varphi \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then with $\lambda > 0$, we have $\int u(\lambda x)\lambda^n\varphi(x)dx = \int u(x)\varphi(\lambda^{-1}x)dx$, and using the Lebesgue dominated convergence theorem, this gives

$$\lim_{\lambda\to+\infty}\int u(\lambda x)\lambda^n\varphi(x)dx = \varphi(0)\int u(x)dx.$$

On Ω, we have $L(e^{i\lambda\phi}) = \lambda e^{i\lambda\phi} \sum_{1\le j\le n} |d\phi|^{-2} \frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_j} = \lambda e^{i\lambda\phi}$, as well as for all $N \in \mathbb{N}$, $e^{i\lambda\phi} = (\lambda^{-N} L^N)(e^{i\lambda\phi})$, implying that, for $\lambda \neq 0$,

$$I(\lambda) = \lambda^{-N}\int_\Omega L^N(e^{i\lambda\phi})a(x)dx = \lambda^{-N}\int_{\operatorname{supp} a} e^{i\lambda\phi(x)}({}^tL^N a)(x)dx.$$

As a result we get for $\lambda \in \mathbb{R}$, $|\lambda^N I(\lambda)| \le \|{}^tL^N a\|_{L^1(\mathbb{R}^n)} < +\infty$, since

$${}^tL = i\sum_{1\le j\le n}\frac{\partial}{\partial x_j}|d\phi|^{-2}\frac{\partial\phi}{\partial x_j}, \qquad {}^tL^N = \sum_{|\alpha|\le N} c_\alpha(x)\partial_x^\alpha, \quad c_\alpha \in C^\infty(\Omega).$$

□

This theorem means that the integral (2.3.3) is rapidly decreasing with respect to the large parameter λ, provided the real phase ϕ does not have stationary points on the support of the amplitude a. We shall now concentrate our attention on the case where the phase does have stationary points ; a first simple model is concerned with (real) quadratic phases.

Quadratic phase

We recall part of the proposition 1.2.19 as a lemma.

Lemma 2.3.2. *Let A be a real symmetric nonsingular $n\times n$ matrix. Then $x \mapsto e^{i\pi\langle Ax,x\rangle}$ is a bounded measurable function, thus a tempered distribution and we have*

$$\operatorname{Fourier}(e^{i\pi\langle Ax,x\rangle})(\xi) = |\det A|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sign} A} e^{-i\pi\langle A^{-1}\xi,\xi\rangle}. \tag{2.3.5}$$

Theorem 2.3.3. *Let $a \in \mathscr{S}(\mathbb{R}^n)$ and A be a real symmetric nonsingular $n\times n$ matrix. Defining $I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\langle Ax,x\rangle}a(x)dx$, we have for $\lambda > 0$,*

$$I(\lambda) = \frac{\pi^{n/2} e^{\frac{i\pi}{4}\operatorname{sign} A}}{\lambda^{\frac{n}{2}}|\det A|^{1/2}}\Big(\sum_{0\le k<N}\lambda^{-k}\frac{\pi^{2k}}{i^k k!}\big(\langle A^{-1}D,D\rangle^k a\big)(0) + r_N(\lambda)\Big), \tag{2.3.6}$$

$$|r_N(\lambda)| \le \lambda^{-N}\frac{\pi^{2N}}{N!}\|\langle A^{-1}D,D\rangle^N a\|_{FL^1}, \tag{2.3.7}$$

where $\|u\|_{FL^1} = \|\hat u\|_{L^1(\mathbb{R}^n)}$, so that $\|\langle A^{-1}D,D\rangle^N a\|_{FL^1} = \|\langle A^{-1}\xi,\xi\rangle^N \hat a\|_{L^1(\mathbb{R}^n)}$ (see also the notation (1.2.8)).

Proof. We write with $\lambda = \pi\mu$ that

$$I(\lambda) = \langle e^{i\pi\langle \mu Ax,x\rangle}, a(x)\rangle_{\mathscr{S}',\mathscr{S}} = \langle \operatorname{Fourier}(e^{i\pi\langle\mu Ax,x\rangle}), \check{\hat a}\rangle_{\mathscr{S}',\mathscr{S}}$$
$$= \mu^{-n/2}|\det A|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sign} A}\int e^{-i\pi\mu^{-1}\langle A^{-1}\xi,\xi\rangle}\hat a(\xi)d\xi,$$

and since

$$\int e^{-i\pi\mu^{-1}\langle A^{-1}\xi,\xi\rangle}\hat{a}(\xi)d\xi = \sum_{0\le k<N}\frac{(-i\pi\mu^{-1})^k}{k!}\int\langle A^{-1}\xi,\xi\rangle^k\hat{a}(\xi)d\xi$$
$$+\int_0^1\int e^{-i\theta\pi\mu^{-1}\langle A^{-1}\xi,\xi\rangle}\langle A^{-1}\xi,\xi\rangle^N\hat{a}(\xi)d\xi\frac{(1-\theta)^{N-1}}{(N-1)!}d\theta\left(\frac{-i\pi}{\mu}\right)^N,$$

we get (2.3.6) with $|r_N(\lambda)| \le \|\langle A^{-1}\xi,\xi\rangle^N\hat{a}(\xi)\|_{L^1}\frac{\pi^{2N}}{N!\lambda^N}$. □

Remark 2.3.4. In particular, under the assumptions of the theorem, we have, if $a(0)\neq 0$,

$$\int_{\mathbb{R}^n} e^{i\lambda\langle Ax,x\rangle}a(x)dx = I(\lambda) \underset{\lambda\to+\infty}{\sim} \frac{\pi^{\frac{n}{2}}e^{\frac{i\pi}{4}\operatorname{sign} A}}{\lambda^{\frac{n}{2}}|\det A|^{1/2}}a(0), \tag{2.3.8}$$

a sharp contrast with the results of the previous subsection 2.3. Naturally, in this case, the phase has a (unique) stationary point at the origin. Note also that in one dimension, we can recover[3] the so-called Fresnel integrals

$$\int_{\mathbb{R}} e^{ix^2}dx = \pi^{1/2}e^{i\pi/4}, \quad \text{i.e.} \quad \int_{\mathbb{R}}\cos(x^2)dx = \int_{\mathbb{R}}\sin(x^2)dx = \sqrt{\frac{\pi}{2}}. \tag{2.3.9}$$

The Morse lemma

The most important step in the proof is the following lemma.

Lemma 2.3.5. *Let U be a neighborhood of 0 in $\mathbb{R}^n$, and $f: U\to\mathbb{R}$ be a C^∞ function such that $df(0)=0$, $\frac{\partial^2 f}{\partial x_1^2}(0)\neq 0$. Then there exists a local diffeomorphism ν of neighborhoods of 0 such that*

$$(f\circ\nu)(y_1,y') = g(y') + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(0)y_1^2.$$

Proof. We may assume that $f(0)=0$. Thanks to the implicit function theorem, we note that the equation $\frac{\partial f}{\partial x_1}(x_1,x')=0$ has a unique solution $x_1=\alpha(x')$ near the origin: there exists $r_0>0$, a neighborhood W of 0 in $\mathbb{R}^{n-1}$ and a C^∞ function $\alpha: W\to\mathbb{R}$ such that $\alpha(0)=0$ and for $|x_1|<r_0, x'\in W$,

$$\frac{\partial f}{\partial x_1}(x_1,x')=0 \Longleftrightarrow x_1=\alpha(x').$$

[3] We have with $\chi\in C_c^\infty(\mathbb{R})$ even, equal to 1 on $[-1,1]$, supported in $[-2,2]$,

$$2\int_0^T e^{ix^2}dx = \int e^{ix^2}\chi(\frac{x}{T})dx - 2\int_{x\ge T}e^{ix^2}\chi(\frac{x}{T})dx$$
$$= \int e^{iT^2x^2}\chi(x)dxT - 2\int_{x\ge T}2ixe^{ix^2}\chi(\frac{x}{T})(2ix)^{-1}dx.$$

From (2.3.8), $\lim_{T\to+\infty}\int e^{iT^2x^2}\chi(x)dxT = \pi^{1/2}e^{i\pi/4}$ and an integration by parts yields that the last term is $O(T^{-1})$.

As a result, we have for $|x_1| < r_0, x' \in W$,

$$f(x_1, x') = f\big(\alpha(x'), x'\big) + \int_0^1 (1-\theta)\frac{\partial^2 f}{\partial x_1^2}\big(\alpha(x') + \theta(x_1 - \alpha(x')), x'\big)d\theta\big(x_1 - \alpha(x')\big)^2,$$

i.e. with a C^∞ function e defined in $]-r_0, r_0[\times W$, a C^∞ function g defined in W,

$$f(x_1, x') = g(x') + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(0)e(x)\big(x_1 - \alpha(x')\big)^2, \quad e(0) = 1.$$

Shrinking if necessary the neighborhoods, we define near 0 the local diffeomorphism κ by

$$\kappa(x_1, x') = \big(e(x)^{1/2}(x_1 - \alpha(x')), x'\big) = (y_1, y')$$

and we have with $\nu = \kappa^{-1}$

$$(f \circ \nu)(y_1, y') = f(x_1, x') = g(y') + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(0)y_1^2,$$

yielding the conclusion. □

Theorem 2.3.6. *Let $x_0 \in \mathbb{R}^n$, $U \in \mathscr{V}_{x_0}$ and $f : U \to \mathbb{R}$ be a C^∞ function such that $df(x_0) = 0, \det f''(x_0) \neq 0$. Then there exists an open neighborhood U_0 of x_0, an open neighborhood V_0 of 0 and a C^∞ diffeomorphism $\nu : V_0 \to U_0$ such that $U_0 \subset U$, $\det \nu'(0) = 1$, and for $y \in V_0$,*

$$(f \circ \nu)(y) - (f \circ \nu)(0) = \frac{1}{2}\sum_{1\le j\le n} \mu_j y_j^2, \tag{2.3.10}$$

where $(\mu_1, \dots, \mu_n)$ are the eigenvalues of the symmetric matrix $f''(x_0)$.

Proof. We may assume for notational simplicity that $x_0 = 0$ and $f(0) = 0$. After composing f with a rotation, we may assume that e_1 is an eigenvector of $f''(0)$, so that in particular, the assumptions of the previous lemma are satisfied. Then we are reduced to tackle a function $g(x') + \frac{1}{2}\mu_1 x_1^2$. We have $dg(0) = 0$, the eigenvalues of $f''(0)$ are $\{\mu_1\} \cup \text{spectrum}(g''(0))$. We get the conclusion by an induction on n. □

Stationary phase formula

We consider now, for $\lambda > 0$ and

$$I(\lambda) = \int e^{i\lambda\phi(x)}a(x)dx, \tag{2.3.11}$$

where the amplitude $a \in C_c^\infty(\mathbb{R}^n)$ and the phase function ϕ is a *Morse function*, i.e. a real-valued smooth function such that

$$\forall x \in \operatorname{supp} a, \quad d\phi(x) = 0 \Longrightarrow \det \phi''(x) \neq 0. \tag{2.3.12}$$

Using the Borel-Lebesgue property, we get that

$$\operatorname{supp} a \subset \underbrace{\{x \in \mathbb{R}^n, d\phi(x) \neq 0\}}_{=\Omega_0} \cup_{1\le j\le N} \Omega_j$$

where Ω_j for $1 \le j \le N$ is an open set such that there exists a C^∞ diffeomorphism $\nu_j : V_j \to \Omega_j$, where V_j is a neighborhood of 0 in $\mathbb{R}^n$ with

$$(\phi \circ \nu_j)(y) = (\phi \circ \nu_j)(0) + \frac{1}{2}\phi''(\nu_j(0))y^2.$$

We are able to find $(\psi_j)_{0\le j\le N}$ with $\psi_j \in C_c^\infty(\Omega_j)$, such that $\sum_{0\le j\le N} \psi_j$ is 1 near $\operatorname{supp} a$. We obtain then that

$$I(\lambda) = \underbrace{\int e^{i\lambda\phi(x)}\psi_0(x)a(x)dx}_{=O(\lambda^{-\infty}) \text{ from Theorem 2.3.1}} + \sum_{1\le j\le N} \int e^{i\lambda\phi(x)}\psi_j(x)a(x)dx,$$

i.e. $I(\lambda) = \sum_{1\le j\le N} \int_{V_j} e^{i\lambda(\phi\circ\nu_j)(y)}(\psi_j a)(\nu_j(y))|\det \nu_j'(y)|dy + O(\lambda^{-\infty})$. We note that, according to the theorem 2.3.3

$$\begin{aligned}\int_{V_j} & e^{i\lambda(\phi\circ\nu_j)(y)}(\psi_j a)(\nu_j(y))|\det \nu_j'(y)|dy \\ &= e^{i\lambda\phi(\nu_j(0))} \int_{V_j} e^{i\lambda\frac{1}{2}\phi''(\nu_j(0))y^2}(\psi_j a)(\nu_j(y))|\det \nu_j'(y)|dy \\ &= \lambda^{-\frac{n}{2}} e^{i\lambda\phi(\nu_j(0))} \frac{(2\pi)^{n/2} e^{i\frac{\pi}{4}\operatorname{sign}\phi''(\nu_j(0))}}{|\det \phi''(\nu_j(0))|^{1/2}}(\psi_j a)(\nu_j(0))|\det \nu_j'(0)| + O(\lambda^{-\frac{n}{2}-1}).\end{aligned}$$

We note also that the stationary points of a Morse function are isolated, since for an invertible symmetric matrix Q, the only singular point of $y \mapsto \langle Qy, y\rangle$ is 0. In particular, there are only finitely many singular points of a Morse function in a compact set.

Theorem 2.3.7. *Let a be a $C_c^\infty(\mathbb{R}^n)$ function and ϕ be a Morse function (see (2.3.12)). We define $I(\lambda)$ by (2.3.11). We have for $\lambda \to +\infty$*

$$I(\lambda) = \lambda^{-\frac{n}{2}}(2\pi)^{n/2} \sum_{\substack{x, d\phi(x)=0 \\ x\in\operatorname{supp} a}} e^{i\lambda\phi(x)} \frac{e^{i\frac{\pi}{4}\operatorname{sign}(\phi''(x))}}{|\det \phi''(x)|^{1/2}} a(x) + O(\lambda^{-\frac{n}{2}-1}). \tag{2.3.13}$$

Proof. We note that the determinant of $\nu'(0)$ is 1 in the theorem 2.3.6 and the formula of Theorem 2.3.3 gives the result if we replace $\psi_j a$ by a; it is indeed harmless to do this since we can assume that $x_1, \dots, x_N$ are the distinct singular points of ϕ in $\operatorname{supp} a$ and write, with $C_c^\infty(\mathbb{R}^n) \ni \widetilde{\psi}_j = 1$ near x_j, $\widetilde{\psi}_j\widetilde{\psi}_k = 0$ if $1 \le j \ne k \le N$

$$a = \sum_{1\le j\le N} \widetilde{\psi}_j a + \underbrace{a - \sum_{1\le j\le N} \widetilde{\psi}_j a}_{\text{supported in } \Omega_0}.$$

□

2.4 The Wave-Front set of a distribution, the H^s wave-front set

Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. Let us recall that the support and the singular support of u are defined by

$$\operatorname{supp} u = \{x \in \Omega, \text{there is no open } V \ni x \text{ with } u_{|V} = 0\}, \tag{2.4.1}$$
$$\operatorname{singsupp} u = \{x \in \Omega, \text{there is no open } V \ni x \text{ with } u_{|V} \in C^\infty(V)\}. \tag{2.4.2}$$

Both sets are closed and we have obviously $\operatorname{singsupp} u \subset \operatorname{supp} u$. The Fourier transform allows a more refined analysis of singularities: first we notice that $x_0 \notin \operatorname{singsupp} u$ iff there exists a neighborhood U of x_0 such that for all $\chi \in C_c^\infty(U)$,

$$\forall N \in \mathbb{N}, \quad \sup_{\xi \in \mathbb{R}^n} |(\widehat{\chi u})(\xi)||\xi|^N < \infty. \tag{†}$$

This is obvious when we assume $x_0 \notin \operatorname{singsupp} u$ since there exists a neighborhood U of x_0 such that $\chi u \in C_c^\infty(\mathbb{R}^n)$ and thus $\widehat{\chi u} \in \mathscr{S}(\mathbb{R}^n)$. Conversely, since $\widehat{\chi u}$ is the Fourier transform of a compactly supported distribution, it is an entire function on $\mathbb{C}^n$, and assuming (†), we see that $(\chi u)(x) = \int e^{2i\pi x\cdot\xi}\widehat{\chi u}(\xi)d\xi$, and the rhs is a C^∞ function, qed.

We use the notation $\Omega \times \mathbb{R}^n\backslash\{0\} = \dot{T}^*(\Omega)$, the cotangent bundle minus the zero section.

Definition 2.4.1. Let Ω be an open set of $\mathbb{R}^n$ and let $u \in \mathscr{D}'(\Omega)$. The *wave-front-set* of u, denoted by WFu, is defined as the complement in $\dot{T}^*(\Omega)$ of the set of points (x_0, ξ_0) such that there exist some neighborhoods U, V respectively of x_0, ξ_0 (with $U \times V \subset \dot{T}^*(\Omega)$) such that for all $\chi \in C_c^\infty(U)$,

$$\forall N \in \mathbb{N}, \quad \sup_{\xi \in \widetilde{V}} |(\widehat{\chi u})(\xi)||\xi|^N < \infty, \quad \text{with } \widetilde{V} = \cup_{\tau > 0}\tau V. \tag{2.4.3}$$

Remark 2.4.2. Note that the wave-front-set is a closed (its complement is open) conic subset of $\dot{T}^*(\Omega)$: conic means here that for all $\tau > 0$, $(x, \xi) \in WFu \Longrightarrow (x, \tau\xi) \in WFu$. On the other hand, with $\operatorname{pr} : \dot{T}^*(\Omega) \to \Omega$ defined by $\operatorname{pr}((x, \xi)) = x$, we get that

$$\operatorname{pr} WFu = \operatorname{singsupp} u. \tag{2.4.4}$$

Let $x_0 \notin \operatorname{singsupp} u$. Then from (†), we see that for all $\xi \in \mathbb{S}^{n-1}$, $(x_0, \xi) \notin WFu$, so that $x_0 \notin \operatorname{pr} WFu$. Conversely, if $x_0 \notin \operatorname{pr} WFu$, for all $\eta \in \mathbb{S}^{n-1}$, there exists some neighborhoods U_η, V_η of x_0, η such that for all $\chi \in C_c^\infty(U_\eta)$,

$$\forall N \in \mathbb{N}, \quad \sup_{\xi \in \widetilde{V}_\eta} |(\widehat{\chi u})(\xi)||\xi|^N < \infty.$$

By compactness, we get $\mathbb{S}^{n-1} \subset \cup_{1\le j\le\nu} V_{\eta_j}$ and defining $U = \cap_{1\le j\le\nu} U_{\eta_j}$, we get that for all $\chi \in C_c^\infty(U)$,

$$\forall j \in \{1, \dots, \nu\}, \forall N \in \mathbb{N}, \quad \sup_{\xi \in \widetilde{V}_{\eta_j}} |(\widehat{\chi u})(\xi)||\xi|^N < \infty,$$

which gives the result (†) since $\cup_{1\le j\le\nu}\widetilde{V}_{\eta_j} = \mathbb{R}^n\backslash\{0\}$ and $\widehat{\chi u}$ is a smooth function.

Examples. It is easy to see that
(1) $WF(\delta_0) = \{0\} \times \mathbb{R}^n \backslash \{0\}$, $\quad \delta_0$ is the Dirac mass at zero in $\mathbb{R}^n$,
(2) $WF(\frac{1}{x+i0}) = \{0\} \times (0, +\infty)$, $\quad \frac{1}{x+i0} = \frac{d}{dx}(\ln|x|) - i\pi\delta_0$, distribution on $\mathbb{R}$,
(3) and with $H = \mathbf{1}_{\mathbb{R}_+}$, considering the distribution on $\mathbb{R}^2$,

$$WF\big(H(x_1)H(x_2)\big) = \{(0, x_2, \xi_1, 0)\}_{x_2>0,\xi_1\neq 0} \cup \{(x_1, 0, 0, \xi_2)\}_{x_1>0,\xi_2\neq 0} \cup \{(0,0)\} \times \mathbb{R}^2 \backslash \{(0,0)\}.$$

(4) If u is a distribution, one can easily define the complex conjugate by duality[4] and we have

$$WF\bar{u} = W\check{F}u = \{(x,\xi) \text{ such that} (x, -\xi) \in WFu\}$$

and in particular, a real-valued distribution (i.e. such that $\bar{u} = u$) has a *projective* wave-front-set, i.e. $(x,\xi) \in WFu \Longleftrightarrow (x, -\xi) \in WFu$, so that, instead of being included in the sphere fiber $S^*(\Omega)$ image of the fiber bundle $\dot{T}^*(\Omega)$ by the mapping $(x,\xi) \mapsto (x, \xi/|\xi|)$, the wave-front-set of a real-valued distribution can be seen as a part of the projective bundle for which the fibers are the quotient of the sphere $\mathbb{S}^{n-1}$ by $\{-1, 1\}$, that is $\mathbb{P}^{n-1}(\mathbb{R})$. In particular for a real-valued distribution u on an open set Ω of the real line, then the wave-front-set does not carry more information than the singular support since $WFu = \operatorname{singsupp} u \times \mathbb{R}^*$.

The following lemma provides a characterization of the wave-front-set which is closer of the pseudo-differential approach.

Lemma 2.4.3. *Let $\theta_0 \in C_c^\infty(\mathbb{R}^n; [0,1]), \operatorname{supp}\theta_0 \subset B(0,1)$, $\theta_0 = 1$ on $B(0,1/2)$. Let Ω be an open set of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. The complement of WFu in $\dot{T}^*(\Omega)$ is the set of (x,ξ) such that there exists $r > 0$ such that*

$$T_r(D)t_r u \quad \textit{belongs to } \mathscr{S}(\mathbb{R}^n),$$

where $T_r(\xi) = \theta_0\left(\frac{\xi}{r|\xi|} - \frac{\xi_0}{r|\xi_0|}\right)(1-\theta_0)(\frac{r\xi}{2}), \quad t_r(x) = \theta_0\left(\frac{x-x_0}{r}\right).$

Proof. Let us assume first that $\dot{T}^*(\Omega) \ni (x_0, \xi_0) \notin WFu$. Using the definition 2.4.1, we get that for some positive r, for all N, $T_r(\xi)\widehat{t_r u}(\xi) = O(\langle\xi\rangle^{-N})$ and since the functions $D_\xi^\alpha\big(\widehat{t_r u}\big) = (-1)^{|\alpha|}\widehat{x^\alpha t_r u}$ are also rapidly decreasing on the support of T_r (from the definition 2.4.1), we get that $\xi \mapsto T_r(\xi)\widehat{t_r u}(\xi)$ is in the Schwartz class as well as its inverse Fourier transform $T_r(D)t_r u$.

Conversely, if for $(x_0, \xi_0) \in \dot{T}^*(\Omega)$ (we may assume $|\xi_0| = 1$) and some positive r, $T_r(D)t_r u \in \mathscr{S}(\mathbb{R}^n)$, we get indeed as in (2.4.3)

$$\forall N \in \mathbb{N}, \quad \sup_{\xi\in\widetilde{V}} |\widehat{t_r u}(\xi)||\xi|^N < \infty, \quad \text{with } V \text{ neighborhood of } \xi_0.$$

[4] We define $\prec \bar{u}, \varphi \succ_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \overline{\prec u, \bar{\varphi} \succ}_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}$.

Now if $\chi \in C_c^\infty(B(x_0, r/2)$, we have $\chi = \chi t_r$ and

$$T_{r/4}(\xi)\widehat{\chi u}(\xi) = T_{r/4}(\xi)\widehat{\chi t_r u}(\xi) = T_{r/4}(\xi)\int \underbrace{\hat{\chi}(\xi-\eta)}_{O(\langle\xi-\eta\rangle^{-N})}\underbrace{T_r(\eta)\widehat{t_r u}(\eta)}_{O(\langle\eta\rangle^{-2N})}d\eta + T_{r/4}(\xi)\int \hat{\chi}(\xi-\eta)\underbrace{(1-T_r(\eta))\widehat{t_r u}(\eta)}_{O(\langle\eta\rangle^{M_0})}d\eta.$$

Using the Peetre inequality[5], we get that the first term is $O(\langle\xi\rangle^{-N})$. To handle the next term we note that, on the support of $T_{r/4}$, we have

$$|\xi| \ge 4/r, \qquad \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| \le r/4$$

and on the integrand we have either $|\eta| \le 1/r$ (harmless term since $\hat{\chi} \in \mathscr{S}$) or

$$|\eta| \ge 1/r \quad \text{and} \quad \left|\frac{\eta}{|\eta|} - \frac{\xi_0}{|\xi_0|}\right| \ge r/2 \quad \Longrightarrow \left|\frac{\eta}{|\eta|} - \frac{\xi}{|\xi|}\right| \ge r/4. \tag{$\star$}$$

Using the inequality[6]

$$\big||\eta|\xi - |\xi|\eta\big|\big(|\xi| + |\eta|\big) \le 4|\xi||\eta||\xi - \eta|, \tag{2.4.6}$$

we obtain here (for the nonzero vectors ξ, η satisfying ($\star$)), $4|\xi - \eta| \ge \frac{r}{4}\big(|\xi| + |\eta|\big)$, so that the rapid decay of $\hat{\chi}(\xi - \eta)$ gives the result of the lemma. □

The wave-front-set of a distribution depends only on the manifold structure of the open set Ω.

Theorem 2.4.4. *let $\kappa : \Omega_2 \longrightarrow \Omega_1$ a C^∞ diffeomorphism of open subsets of $\mathbb{R}^n$ and let $u_1 \in \mathscr{D}'(\Omega_1)$. Then we have*

$$WF\big(\kappa^*(u_1)\big) = \kappa^*\big(WFu_1\big) = \Big\{\Big(\kappa^{-1}(x_1), {}^t\kappa'\big(\kappa^{-1}(x_1)\big)\xi_1\Big)\Big\}_{(x_1,\xi_1)\in WFu_1}.$$

Proof. Let us define $u_2 = \kappa^*(u_1)$, so that for $\chi_2 \in C_c^\infty(\Omega_2)$, we have, for $\varphi_2 \in C_c^\infty(\Omega_2)$, with brackets of duality and $\nu = \kappa^{-1}$, $\chi_1(x_1) = \chi_2(\nu(x_1))|\det \nu'(x_1)|$ (note

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[5] We use $\langle\xi + \eta\rangle \le 2^{1/2}\langle\xi\rangle\langle\eta\rangle$ so that, for all $s \in \mathbb{R}$,

$$\langle\xi + \eta\rangle^s \le 2^{|s|/2}\langle\xi\rangle^s\langle\eta\rangle^{|s|}, \tag{2.4.5}$$

a convenient inequality (to get it for $s \ge 0$, raise the first inequality to the power s, and for $s < 0$, replace ξ by $-\xi - \eta$) a.k.a. Peetre's inequality.

[6] The proof of (2.4.6) is the following: we have $\big||\eta|\xi - |\xi|\eta\big| \le |\eta||\xi - \eta| + |\eta|\big||\xi| - |\eta|\big| \le 2|\eta||\xi - \eta|$ and thus $\big||\eta|\xi - |\xi|\eta\big| \le 2|\xi - \eta|\min(|\xi|, |\eta|)$ which gives

$$\big||\eta|\xi - |\xi|\eta\big|\big(|\xi| + |\eta|\big) \le 2|\xi - \eta|\min(|\xi|, |\eta|)2\max(|\xi|, |\eta|) = 4|\xi||\eta||\xi - \eta|.$$

that χ_1 belongs to $C_c^\infty(\Omega_1)$ and $\chi_1|dx_1|$ is the κ-push-forward of the density $\chi_2|dx_2|$), $\psi_1 \in C_c^\infty(\Omega_1)$ equal to 1 on the support of χ_1,

$$\widehat{\chi_2 u_2}(\xi_2) = \int \chi_1(x_1)u_1(x_1)e^{-2i\pi\nu(x_1)\cdot\xi_2}dx_1$$
$$= \int \widehat{\chi_1 u_1}(\xi_1)\left(\int e^{2i\pi(\xi_1 x_1 - \xi_2\nu(x_1))}\psi_1(x_1)dx_1\right)d\xi_1$$

where the integral with respect to ξ_1 is in fact a bracket of duality. We may thus consider the identity

$$\Big(1 + \big(\xi_1 - {}^t\nu'(x_1)\xi_2\big)\cdot D_{x_1}\Big)\big(e^{2i\pi(\xi_1 x_1 - \xi_2\nu(x_1))}\big) = e^{2i\pi(\xi_1 x_1 - \xi_2\nu(x_1))}\big(1 + \|\xi_1 - {}^t\nu'(x_1)\xi_2\|^2\big)$$

which gives with $L = \big(1 + \|\xi_1 - {}^t\nu'(x_1)\xi_2\|^2\big)^{-1}\Big(1 + \big(\xi_1 - {}^t\nu'(x_1)\xi_2\big)\cdot D_{x_1}\Big)$,

$$\forall N \in \mathbb{N}, \quad L^N\big(e^{2i\pi(\xi_1 x_1 - \xi_2\nu(x_1))}\big) = e^{2i\pi(\xi_1 x_1 - \xi_2\nu(x_1))}$$

so that $\widehat{\chi_2 u_2}(\xi_2) = \int \widehat{\chi_1 u_1}(\xi_1)\left(\int e^{2i\pi(\xi_1 x_1 - \xi_2\nu(x_1))}({}^tL)^N(\psi_1)(x_1)dx_1\right)d\xi_1$ and

$$|\widehat{\chi_2 u_2}(\xi_2)| \le C_N \iint |\widehat{\chi_1 u_1}(\xi_1)|\langle \xi_1 - {}^t\nu'(x_1)\xi_2\rangle^{-N}\mathbf{1}_{\operatorname{supp}\psi}(x_1)dx_1 d\xi_1. \qquad (\star)$$

Let us assume that $\dot T^*(\Omega_1) \ni (x_{01}, \xi_{01}) \notin WFu_1$; the point (x_{02}, ξ_{02}) is defined as $(\nu(x_{01}), {}^t\nu'(x_{01})^{-1}\xi_{01})$. We assume that ξ_2 belongs to a conic neighborhood Γ_2 of ξ_{02}. We consider first for $r > 0$ the conic subset of $\mathbb{R}^n$ defined by

$$\Gamma_1(r) = \{\xi_1 \in \mathbb{R}^n, \forall \xi_2 \in \Gamma_2, \inf_{x_1 \in \operatorname{supp}\psi_1} |\xi_1 - {}^t\nu'(x_1)\xi_2| < r(|\xi_1| + |\xi_2|)\}.$$

The set $\Gamma_1(r)$ is also open and contains ξ_{01}. If r is small enough and the support of χ_2 is included in a small enough ball around x_{02}, we have from our assumption $|\widehat{\chi_1 u_1}(\xi_1)| = O(\langle\xi_1\rangle^{-2N})$ on $\Gamma_1(r)$. When the integration in $(\star)$ takes place in $\Gamma_1(r)$, we estimate that part of the integral, using the footnote on page 52 by

$$C'_N \iint \langle\xi_1\rangle^{-2N+N}\langle {}^t\nu'(x_1)\xi_2\rangle^{-N}\mathbf{1}_{\operatorname{supp}\psi}(x_1)dx_1 d\xi_1 = O(\langle\xi_2\rangle^{-N}).$$

When the integration in $(\star)$ takes place outside $\Gamma_1(r)$, we know that for some $r > 0$ and all $x_1 \in \operatorname{supp}\psi$, $|\xi_1 - {}^t\nu'(x_1)\xi_2| \ge r(|\xi_1| + |\xi_2|)$. We have thus the estimate, with a fixed M_0,

$$C''_N \iint \langle\xi_1\rangle^{M_0}(\langle\xi_1\rangle + \langle\xi_2\rangle)^{-2N}\mathbf{1}_{\operatorname{supp}\psi}(x_1)dx_1 d\xi_1 = O(\langle\xi_2\rangle^{-N}), \text{ for } N > M_0 + n.$$

The proof of the theorem is complete. □

Definition 2.4.5. Let Ω be an open set of $\mathbb{R}^n$, let $u \in \mathscr{D}'(\Omega)$ and $s \in \mathbb{R}$. The *H^s-wave-front-set* of u, denoted by $WF_s u$, is defined as the complement in $\dot T^*(\Omega)$ of the set of points (x_0, ξ_0) such that there exist some neighborhoods U, V respectively of x_0, ξ_0 (with $U \times V \subset \dot T^*(\Omega)$) such that for all $\chi \in C_c^\infty(U)$,

$$\int_{\widetilde V \cap \{|\xi| \ge 1\}} |(\widehat{\chi u})(\xi)|^2 |\xi|^{2s} d\xi < \infty, \quad \text{with } \widetilde V = \cup_{\tau > 0}\tau V.$$

2.5 Oscillatory Integrals

Definition 2.5.1. Let Ω be an open subset of $\mathbb{R}^n$, $m \in \mathbb{R}, N \in \mathbb{N}^*$. The space $S^m(\Omega \times \mathbb{R}^N)$ is defined as the set of functions $a \in C^\infty(\Omega \times \mathbb{R}^N; \mathbb{C})$ such that, for all K compact subset of Ω, for all $\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^N$, there exists $C_{K,\alpha,\beta}$ such that

$$\forall x \in K, \forall \theta \in \mathbb{R}^N, \quad |(\partial_x^\alpha \partial_\theta^\beta a)(x,\theta)| \leq C_{K,\alpha,\beta} \langle \theta \rangle^{m-|\beta|}. \tag{2.5.1}$$

It is a easy exercise left to the reader, consequence of the Leibniz formula, to prove that the space $S^m(\Omega \times \mathbb{R}^N)$ is a Fréchet space and that the mappings

$$S^{m_1}(\Omega \times \mathbb{R}^N) \times S^{m_2}(\Omega \times \mathbb{R}^N) \ni (a_1, a_2) \mapsto a_1 a_2 \in S^{m_1+m_2}(\Omega \times \mathbb{R}^N)$$

are continuous. Moreover for any multi-indices $\alpha, \beta \in \mathbb{N}^n \times \mathbb{N}^N$, the mapping

$$S^m(\Omega \times \mathbb{R}^N) \ni a \mapsto \partial_x^\alpha \partial_\theta^\beta a \in S^{m-|\beta|}(\Omega \times \mathbb{R}^N)$$

is continuous.

Definition 2.5.2. Let Ω be an open subset of $\mathbb{R}^n$, $N \in \mathbb{N}^*$, $\phi \in S^1(\Omega \times \mathbb{R}^N)$. The function ϕ is called a standard phase function on $\Omega \times \mathbb{R}^N$ whenever $\phi \in S^1(\Omega \times \mathbb{R}^N)$ is real-valued and such that, for all K compact subset of Ω, there exists $c_K > 0$ such that

$$\forall x \in K, \forall \theta \in \mathbb{R}^N \text{ with } |\theta| \geq 1, \quad \left|\frac{\partial \phi}{\partial x}(x,\theta)\right|^2 + |\theta|^2 \left|\frac{\partial \phi}{\partial \theta}(x,\theta)\right|^2 \geq c_K |\theta|^2. \tag{2.5.2}$$

For $a \in S^m(\Omega \times \mathbb{R}^N)$ with $m < -N$ and ϕ a standard phase function, we define

$$T_{a,\phi}(x) = \int e^{i\phi(x,\theta)} a(x,\theta) d\theta \tag{2.5.3}$$

which is a continuous function on Ω ; note also that if $m < -N - k$ with $k \in \mathbb{N}$, $T_{a,\phi}$ belongs to $C^k(\Omega)$.

Theorem 2.5.3. *Let Ω be an open subset of $\mathbb{R}^n$, $m \in \mathbb{R}, N \in \mathbb{N}^*, a \in S^m(\Omega \times \mathbb{R}^N)$ and ϕ be a standard phase function on $\Omega \times \mathbb{R}^N$. Then $T_{a,\phi}$ is a distribution on Ω with order $> m + N$ in the following sense. The mapping*

$$\begin{array}{ccc} C_c^\infty(\Omega) \times S^m(\Omega \times \mathbb{R}^N) & \longrightarrow & \mathbb{C} \\ (u,a) & \mapsto & \iint e^{i\phi(x,\theta)} a(x,\theta) u(x) dx d\theta \end{array} \tag{2.5.4}$$

extends the formula (2.5.3) defined for $m < -N$ in a unique way and continuously.

2.6 Singular integrals, examples

The Hilbert transform

A basic object in the classical theory of harmonic analysis is the Hilbert transform, given by the one-dimensional convolution with $pv(1/\pi x) = \frac{d}{\pi dx}(\ln|x|)$, where we consider here the distribution derivative of the $L^1_{\text{loc}}(\mathbb{R})$ function $\ln|x|$. We can also compute the Fourier transform of $pv(1/\pi x)$, which is given by $-i\,\text{sign}\,\xi$. As a result the Hilbert transform $\mathscr{H}$ is a unitary operator on $L^2(\mathbb{R})$ defined by

$$\widehat{\mathscr{H}u}(\xi) = -i\,\text{sign}\,\xi\hat{u}(\xi). \tag{2.6.1}$$

It is also given by the formula

$$(\mathscr{H}u)(x) = \lim_{\epsilon\to 0_+}\frac{1}{\pi}\int_{|x-y|\geq\epsilon}\frac{u(y)}{x-y}dy.$$

The Hilbert transform is certainly the first known example of a *Fourier multiplier* ($\mathscr{H}u = F^{-1}(a\hat{u})$ with a bounded a).

The Riesz operators, the Leray-Hopf projection

The Riesz operators are the natural multidimensional generalization of the Hilbert transform. We define for $u \in L^2(\mathbb{R}^n)$,

$$\widehat{R_j u}(\xi) = \frac{\xi_j}{|\xi|}\hat{u}(\xi), \quad \text{so that } R_j = D_j/|D| = (-\Delta)^{-1/2}\frac{\partial}{i\partial x_j}. \tag{2.6.2}$$

The R_j are selfadjoint bounded operators on $L^2(\mathbb{R}^n)$ with norm 1.

We can also consider the $n\times n$ matrix of operators given by $Q = R\otimes R = (R_jR_k)_{1\leq j,k\leq n}$ sending the vector space of $L^2(\mathbb{R}^n)$ vector fields into itself. The operator Q is selfadjoint and is a projection since $\sum_l R_l^2 = \text{Id}$ so that $Q^2 = (\sum_l R_jR_lR_lR_k)_{j,k} = Q$. As a result the operator

$$\mathbb{P} = \text{Id} - R\otimes R = \text{Id} - |D|^{-2}(D\otimes D) = \text{Id} - \Delta^{-1}(\nabla\otimes\nabla) \tag{2.6.3}$$

is also an orthogonal projection, the Leray-Hopf projector (a.k.a. the Helmholtz-Weyl projector); the operator $\mathbb{P}$ is in fact the orthogonal projection onto the closed subspace of L^2 vector fields with null divergence. We have for a vector field $u = \sum_j u_j\partial_j$, the identities $\text{grad}\,\text{div}\,u = \nabla(\nabla\cdot u)$, $\text{grad}\,\text{div} = \nabla\otimes\nabla = (-\Delta)(iR\otimes iR)$, so that

$$Q = R\otimes R = \Delta^{-1}\,\text{grad}\,\text{div}, \quad \text{div}\,R\otimes R = \text{div},$$

which implies $\text{div}\,\mathbb{P}u = \text{div}\,u - \text{div}(R\otimes R)u = 0$, and if $\text{div}\,u = 0$, $\mathbb{P}u = u$. The Leray-Hopf projector is in fact the $(n\times n)$-matrix-valued Fourier multiplier given by $\text{Id} - |\xi|^{-2}(\xi\otimes\xi)$. This operator plays an important role in fluid mechanics since the Navier-Stokes system for incompressible fluids can be written for a given divergence-free v_0,

$$\begin{cases}\partial_t v - \nu\triangle v = -\mathbb{P}\nabla(v\otimes v),\\ \mathbb{P}v = v,\\ v_{|t=0} = v_0.\end{cases}$$

As already said for the Riesz operators, $\mathbb{P}$ is not a classical pseudo-differential operator, because of the singularity at the origin: however it is indeed a Fourier multiplier with the same functional properties as those of R.

In three dimensions the curl operator is given by the matrix

$$\operatorname{curl} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} = \operatorname{curl}^* \tag{2.6.4}$$

so that $\operatorname{curl}^2 = -\Delta \operatorname{Id} + \operatorname{grad}\operatorname{div}$ and (the Biot-Savard law)

$$\operatorname{Id} = (-\Delta)^{-1}\operatorname{curl}^2 + \Delta^{-1}\operatorname{grad}\operatorname{div}, \quad \text{also equal to } (-\Delta)^{-1}\operatorname{curl}^2 + \operatorname{Id} - \mathbb{P},$$

which gives $\operatorname{curl}^2 = -\Delta\mathbb{P}$, so that

$$[\mathbb{P}, \operatorname{curl}] = \Delta^{-1}\big(\Delta\mathbb{P}\operatorname{curl} - \Delta\operatorname{curl}\mathbb{P}\big) = \Delta^{-1}\big(-\operatorname{curl}^3 + \operatorname{curl}(-\Delta\mathbb{P})\big) = 0,$$
$$\mathbb{P}\operatorname{curl} = \operatorname{curl}\mathbb{P} = \operatorname{curl}(-\Delta)^{-1}\operatorname{curl}^2 = \operatorname{curl}\big(\operatorname{Id} - \Delta^{-1}\operatorname{grad}\operatorname{div}\big) = \operatorname{curl}$$

since $\operatorname{curl}\operatorname{grad} = 0$ (note also that $\operatorname{div}\operatorname{curl} = 0$).

Theorem 2.6.1. *Let Ω be a function in $L^1(\mathbb{S}^{n-1})$ such that $\int_{\mathbb{S}^{n-1}} \Omega(\omega) d\sigma(\omega) = 0$. Then the following formula defines a tempered distribution T:*

$$\langle T, \varphi\rangle = \lim_{\epsilon\to 0_+} \int_{|x|\ge\epsilon} \Omega\big(\frac{x}{|x|}\big)|x|^{-n}\varphi(x)dx = -\int (x\cdot\partial_x\varphi(x))\Omega\big(\frac{x}{|x|}\big)|x|^{-n}\ln|x|dx.$$

The distribution T is homogeneous of degree $-n$ on $\mathbb{R}^n$ and, if Ω is odd, the Fourier transform of T is a bounded function.

N.B. We shall use the principal-value notation

$$T = pv\Big(|x|^{-n}\Omega\big(\frac{x}{|x|}\big)\Big).$$

When $n = 1$ and $\Omega = \operatorname{sign}$, we recover the principal value $pv(1/x) = \frac{d}{dx}(\ln|x|)$ which is odd, homogeneous of degree -1, and whose Fourier transform is $-i\pi\operatorname{sign}\xi$.

Proof. Let φ be in $\mathscr{S}(\mathbb{R}^n)$ and $\epsilon > 0$. Using polar coordinates, we check

$$\int_{\mathbb{S}^{n-1}} \Omega(\omega)\int_\epsilon^{+\infty} \varphi(r\omega)\frac{dr}{r}d\sigma(\omega)$$
$$= \int_{\mathbb{S}^{n-1}} \Omega(\omega)\Big[\varphi(\epsilon\omega)\ln(\epsilon^{-1}) - \int_\epsilon^{+\infty} \omega\cdot d\varphi(r\omega)\ln r dr\Big]d\sigma(\omega).$$

Since the mean value of Ω is 0, we get the first statement of the theorem, noticing that the function $x \mapsto \Omega(x/|x|)|x|^{-n+1}\ln(|x|)(1+|x|)^{-2}$ is in $L^1(\mathbb{R}^n)$. We have

$$\langle x\cdot\partial_x T, \varphi\rangle = -\langle T, x\cdot\partial_x\varphi\rangle - n\langle T, \varphi\rangle \tag{$\circledast$}$$

and we see that

$$\langle T, x\cdot\partial_x\varphi\rangle = \lim_{\epsilon\to 0_+}\int_{\mathbb{S}^{n-1}}\Omega(\omega)\int_\epsilon^{+\infty} r\omega\cdot(d\varphi)(r\omega)\frac{dr}{r}d\sigma(\omega)$$
$$= \int_{\mathbb{S}^{n-1}}\Omega(\omega)\int_0^{+\infty} \omega\cdot(d\varphi)(r\omega)dr d\sigma(\omega)$$
$$= \int_{\mathbb{S}^{n-1}}\Omega(\omega)\int_0^{+\infty} \frac{d}{dr}\big(\varphi(r\omega)\big)dr d\sigma(\omega) = -\varphi(0)\int_{\mathbb{S}^{n-1}}\Omega(\omega)d\sigma(\omega) = 0$$

so that (⊛) implies that $x \cdot \partial_x T = -nT$ which is the homogeneity of degree $-n$ of T. As a result the Fourier transform of T is an homogeneous distribution with degree 0.

N.B. Note that the formula

$$-\int (x \cdot \partial_x \varphi(x)) \Omega\big(\frac{x}{|x|}\big) |x|^{-n} \ln |x| dx$$

makes sense for $\Omega \in L^1(\mathbb{S}^{n-1}), \varphi \in \mathscr{S}(\mathbb{R}^n)$ and defines a tempered distribution. For instance, if $n = 1$ and $\Omega = 1$, we get the distribution derivative $\frac{d}{dx}\big(\operatorname{sign} x \ln |x|\big)$. However, the condition of mean value 0 for Ω on the sphere is necessary to obtain T as a principal value, since in the discussion above, the term factored out by $\ln(1/\epsilon)$ is $\int_{\mathbb{S}^{n-1}} \Omega(\omega)\varphi(\epsilon\omega) d\sigma(\omega)$ which has the limit $\varphi(0) \int_{\mathbb{S}^{n-1}} \Omega(\omega) d\sigma(\omega)$. On the other hand, from the defining formula of T, we get with $\Omega_j(\omega) = \frac{1}{2}(\Omega(\omega) + (-1)^j \Omega(-\omega))$ (Ω_1(resp.Ω_2) is the odd (resp. even) part of Ω)

$$\begin{aligned}\langle T, \varphi \rangle = \int_{\mathbb{S}^{n-1}} \Omega_1(\omega) \langle pv(\frac{1}{2t}), \varphi(t\omega) \rangle_{\mathscr{S}'(\mathbb{R}_t), \mathscr{S}(\mathbb{R}_t)} d\sigma(\omega) \\ + \int_{\mathbb{S}^{n-1}} \Omega_2(\omega) \langle \frac{d}{dt}\big(H(t) \ln t\big), \varphi(t\omega) \rangle_{\mathscr{S}'(\mathbb{R}_t), \mathscr{S}(\mathbb{R}_t)} d\sigma(\omega). \quad (2.6.5)\end{aligned}$$

Let us show that, when Ω is odd, the Fourier transform of T is bounded. We get

$$\begin{aligned}\langle \hat{T}, \psi \rangle = \int_{\mathbb{S}^{n-1}} \Omega(\omega) \langle pv(\frac{1}{2t}), \widehat{\psi}(t\omega) \rangle d\sigma(\omega) \\ = -\frac{i\pi}{2} \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \Omega(\omega) \operatorname{sign}(\omega \cdot \xi) \varphi(\xi) d\xi d\sigma(\omega)\end{aligned}$$

proving that

$$\hat{T}(\xi) = -\frac{i\pi}{2} \int_{\mathbb{S}^{n-1}} \Omega(\omega) \operatorname{sign}(\omega \cdot \xi) d\sigma(\omega) \quad (2.6.6)$$

which is indeed a bounded function. □

Chapter 3

Pseudo-differential operators

3.1 Prolegomena

To illustrate the power of pseudo-differential methods, we begin with a simple and classical regularity result for the Laplace equation.

Theorem 3.1.1. *Let Ω be an open subset of $\mathbb{R}^n$, let $s \in \mathbb{R}$ and let $f \in H^s_{loc}(\Omega)$. If u is a distribution on Ω such that*

$$\Delta u = f, \tag{3.1.1}$$

then u belongs to $H^{s+2}_{loc}(\Omega)$. If f belongs to $C^\infty(\Omega)$ and if u is a distribution solution of (3.1.1) in Ω, then u belongs to $C^\infty(\Omega)$.

Proof. Let Ω_0 be a relatively compact open subset of Ω. Let $\chi \in C^\infty_c(\Omega_0)$: we have

$$\chi \Delta u = \chi f \in H^s(\mathbb{R}^n).$$

The Fourier multiplier $(1-\Delta)^{-1}$ sends $H^s(\mathbb{R}^n)$ into $H^{s+2}(\mathbb{R}^n)$, and we have

$$(1-\Delta)^{-1}\chi\Delta u = g \in H^{s+2}(\mathbb{R}^n).$$

This implies

$$\begin{aligned} g &= (1-\Delta)^{-1}\chi(\Delta - 1)u + (1-\Delta)^{-1}\chi u \\ &= (1-\Delta)^{-1}[\chi, (\Delta-1)]u - \chi u + (1-\Delta)^{-1}\chi u \\ &= (1-\Delta)^{-1}[\chi, \Delta]u - \big(1-(1-\Delta)^{-1}\big)\chi u. \end{aligned}$$

We note that the operator $R_1 = [\chi, \Delta]$ is a first-order differential operator with smooth coefficients supported in the support of $\nabla\chi$ and thus compactly supported in Ω_0. As a result, we have

$$\big(1-(1-\Delta)^{-1}\big)\chi u = (1-\Delta)^{-1}R_1 u + g. \tag{3.1.2}$$

The Fourier multiplier $1-(1-\Delta)^{-1}$ has the symbol

$$\omega(\xi) = 1 - (1+4\pi^2|\xi|^2)^{-1} = \begin{cases} \in [\frac{1}{2}, 1) & \text{for } |\xi| \geq \frac{1}{2\pi}, \\ \in [0, 1/2) & \text{for } |\xi| < \frac{1}{2\pi}. \end{cases}$$

Let ψ_0 be a function in $C_c^\infty(\{|\xi| < 1/\pi\}; [0, 1])$ equal to 1 on $|\xi| \le \frac{1}{2\pi}$. Then the function

$$\omega_0(\xi) = \omega(\xi) + \psi_0(\xi) \quad \text{is valued in } [1/2, 2],$$

so that the Fourier multiplier $\omega_0(D)$ is an isomorphism of $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$. We have thus from (3.1.2)

$$\chi u = \omega_0(D)^{-1}\big((1 - \Delta)^{-1}R_1 u + g + \psi_0(D)\chi u\big). \tag{3.1.3}$$

From the Paley-Wiener Theorem, since Ω_0 is relatively compact in Ω, we may assume that $u \in H^{s_0}_{loc}(\Omega_0)$, i.e. $\chi u \in H^{s_0}(\mathbb{R}^n)$ for all $\chi \in C_c^\infty(\Omega_0)$. From (3.1.3), we get that $\omega_0(D)^{-1}(1 - \Delta)^{-1}R_1 u$ belongs to $H^{s_0+1}(\mathbb{R}^n)$, $\omega_0(D)^{-1}g \in H^{s+2}(\mathbb{R}^n)$, $\omega_0(D)^{-1}\psi_0(D)\chi u \in H^{+\infty}(\mathbb{R}^n) = \cap_{\sigma\in\mathbb{R}} H^\sigma(\mathbb{R}^n)$. We obtain thus that

$$\forall \chi \in C_c^\infty(\Omega_0), \quad \chi u \in H^{\min(s+2,s_0+1)}(\mathbb{R}^n), \quad \text{i.e. } u \in H^{\min(s+2,s_0+1)}_{loc}(\Omega_0),$$

which gives the sought result whenever $s_0 \ge s + 1$. If $s_0 < s + 1$, we have proven that $u \in H^{s_0+1}_{loc}(\Omega_0)$.
Claim: $u \in H^{s+2}_{loc}(\Omega_0)$. To prove that claim, we consider

$$I = \{\sigma \in \mathbb{R}, u \in H^\sigma_{loc}(\Omega_0)\}.$$

We know that I is not empty (it contains s_0) and also that

$$s_0 \in I \Longrightarrow \min(s + 2, s_0 + 1) \in I, \quad (-\infty, s_0] \in I.$$

Let $s_1 = \sup I$. If $s_1 < +\infty$, then $s_1 - \frac{1}{2} \in I$ and thus since $s_1 + \frac{1}{2} \notin I$,

$$\min(s + 2, s_1 - \frac{1}{2} + 1) = \min(s + 2, s_1 + \frac{1}{2}) \in I \Longrightarrow s + 2 \in I.$$

If $s_1 = +\infty$, then $s + 2 \in I$. We have proven that u belongs to $H^{s+2}_{loc}(\Omega_0)$ for any relatively compact open subset Ω_0 of $\mathbb{R}^n$, which implies that $H^{s+2}_{loc}(\Omega)$, since Ω is locally compact.

If f is C^∞ on Ω, we find that it is $H^s_{loc}(\Omega)$ for any $s \in \mathbb{R}$ and thus from the previous result that u is $H^{s+2}_{loc}(\Omega)$ for any $s \in \mathbb{R}$, so that $u \in C^\infty(\Omega)$, thanks to the next lemma.

Lemma 3.1.2. *Let Ω be an open subset of $\mathbb{R}^n$. Then ,*

$$C^\infty(\Omega) = H^{+\infty}_{loc}(\Omega) = \cap_{s\in\mathbb{R}} H^s_{loc}(\Omega).$$

Proof of the lemma. If u is a smooth function on Ω and χ belongs to $C_c^\infty(\Omega)$, then $\chi u \in C_c^\infty(\mathbb{R}^n)$, so that $\widehat{\chi u}$ belongs to $\mathscr{S}(\mathbb{R}^n)$ and thus for any $s \in \mathbb{R}$,

$$(1 + |\xi|^2)^{s/2}\widehat{\chi u}(\xi) \in L^2(\mathbb{R}^n),$$

implying $u \in H^{+\infty}_{loc}(\Omega)$. Conversely, if $u \in H^{+\infty}_{loc}(\Omega)$ and $\chi \in C_c^\infty(\Omega)$, we find that for all $s \in \mathbb{R}$,

$$(1 + |\xi|^2)^{s/2}\widehat{\chi u}(\xi) \in L^2(\mathbb{R}^n),$$

so that

$$\big(D_x^\alpha(\chi u)\big)(x) = \int e^{2i\pi x\cdot\xi} \overbrace{\underbrace{\xi^\alpha(1+|\xi|^2)^{-s/2}}_{\substack{\in L^2(\mathbb{R}^n)\\ \text{if } |\alpha|-s<-n/2}} \underbrace{(1+|\xi|^2)^{s/2}\widehat{\chi u}(\xi)}_{\in L^2(\mathbb{R}^n)}}^{\in L^1(\mathbb{R}^n)} d\xi,$$

so that $D_x^\alpha(\chi u)$ is a continuous function for any α and thus that u is a smooth function on Ω. □

The proof of Theorem 3.1.1 is complete. □

If we look back at the proof of this theorem, we note that the key point was to invert the symbol of $-\Delta$, which is $4\pi^2|\xi|^2$, away from 0. We introduced the Fourier multiplier $(1-\Delta)^{-1}$ and we got from (3.1.3) a representation of χu in terms of $\chi\Delta u$, up to some unimportant terms. This should serve as a motivation to study more general Fourier multiplier as well as more general operators of this type.

3.2 Introduction

A differential operator of order m on $\mathbb{R}^n$ can be written as

$$a(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha,$$

where we have used the notation (1.2.8) for the multi-indices. Its *symbol* is a polynomial in the variable ξ and is defined as

$$a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha, \qquad \xi^\alpha = \xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}.$$

We have the formula

$$(a(x,D)u)(x) = \int_{\mathbb{R}^n} e^{2i\pi x\cdot\xi} a(x,\xi)\hat u(\xi) d\xi, \tag{3.2.1}$$

where $\hat u$ is the Fourier transform. It is possible to generalize the previous formula to the case where a is a tempered distribution on $\mathbb{R}^{2n}$.

Let u, v be in the Schwartz class $\mathscr{S}(\mathbb{R}^n)$. Then the function

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x,\xi) \mapsto \hat u(\xi)\bar v(x) e^{2i\pi x\cdot\xi} = \Omega_{u,v}(x,\xi) \tag{3.2.2}$$

belongs to $\mathscr{S}(\mathbb{R}^{2n})$ and the mapping $(u,v)\mapsto \Omega_{u,v}$ is sesquilinear continuous. Using these notations, we can provide the following definition.

Definition 3.2.1. Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$ be a tempered distribution. We define the operator $a(x,D) : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}^*(\mathbb{R}^n)$ by the formula

$$\langle a(x,D)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \prec a, \Omega_{u,v} \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

where $\mathscr{S}^*(\mathbb{R}^n)$ is the antidual of $\mathscr{S}(\mathbb{R}^n)$ (continuous antilinear forms). The distribution a is called the symbol of the operator $a(x,D)$.

N.B. The duality product $\langle u, v\rangle_{\mathscr{S}^*(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}$, is linear in the variable u and anti-linear in the variable v. We shall use the same notation for the dot product in the *complex* Hilbert space L^2 with the notation

$$\langle u, v\rangle_{L^2} = \int u(x)\overline{v(x)}dx.$$

The general rule that we shall follow is to always use the sesquilinear duality as above, except if specified otherwise. For the real duality, as in the left-hand-side of the formula in Definition 3.2.1, we shall use the notation $\prec u, v \succ = \int u(x)v(x)dx$, e.g. for $u, v \in \mathscr{S}(\mathbb{R}^n)$.

Although the previous formula is quite general, since it allows us to *quantize*[1] any tempered distribution on $\mathbb{R}^{2n}$, it is not very useful, since we cannot compose this type of operators. We are in fact looking for an algebra of operators and the following theorem is providing a simple example.

In the sequel we shall denote by $C_b^\infty(\mathbb{R}^{2n})$ the (Fréchet) space of C^∞ functions on $\mathbb{R}^{2n}$ which are bounded as well as all their derivatives.

Theorem 3.2.2. *Let $a \in C_b^\infty(\mathbb{R}^{2n})$. Then the operator $a(x, D)$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself.*

Proof. Using Definition 3.2.1, we have for $u, v \in \mathscr{S}(\mathbb{R}^n), a \in C_b^\infty(\mathbb{R}^{2n})$,

$$\langle a(x, D)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \iint e^{2i\pi x\cdot\xi}a(x, \xi)\hat{u}(\xi)\bar{v}(x)dxd\xi.$$

On the other hand the function $U(x) = \int e^{2i\pi x\cdot\xi}a(x, \xi)\hat{u}(\xi)d\xi$ is smooth and such that, for any multi-indices α, β,

$$\begin{aligned} x^\beta D_x^\alpha U(x) &= (-1)^{|\beta|} \sum_{\alpha'+\alpha''=\alpha} \frac{\alpha!}{\alpha'!\alpha''!} \int e^{2i\pi x\cdot\xi} D_\xi^\beta\big(\xi^{\alpha'}(D_x^{\alpha''}a)(x, \xi)\hat{u}(\xi)\big)d\xi \\ &= (-1)^{|\beta|} \sum_{\alpha'+\alpha''=\alpha} \frac{\alpha!}{\alpha'!\alpha''!} \int e^{2i\pi x\cdot\xi} D_\xi^\beta\big((D_x^{\alpha''}a)(x, \xi)\widehat{D^{\alpha'}u}(\xi)\big)d\xi \end{aligned}$$

and thus

$$\sup_{x\in\mathbb{R}^n} |x^\beta D_x^\alpha U(x)| \le \sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} \frac{\alpha!}{\alpha'!\alpha''!}\frac{\beta!}{\beta'!\beta''!} \|D_\xi^{\beta'}D_x^{\alpha''}a\|_{L^\infty(\mathbb{R}^{2n})}\|D^{\beta''}\widehat{D^\alpha u}\|_{L^1(\mathbb{R}^n)}.$$

Since the Fourier transform and ∂_{x_j} are continuous on $\mathscr{S}(\mathbb{R}^n)$, we get that the mapping $u \mapsto U$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself. The above defining formula for $a(x, D)$ ensures that $a(x, D)u = U$. □

[1] We mean simply here that we are able to define a linear mapping from $\mathscr{S}'(\mathbb{R}^{2n})$ to the set of continuous operators from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$.

The Schwartz space $\mathscr{S}(\mathbb{R}^{2n})$ is not dense in the Fréchet space $C_b^\infty(\mathbb{R}^{2n})$ (e.g. $\forall \varphi \in \mathscr{S}(\mathbb{R}^{2n}), \sup_{x\in\mathbb{R}^{2n}} |1-\varphi(x)| \geq 1$) but, in somewhat pedantic terms, one may say that this density is true for the bornology on $C_b^\infty(\mathbb{R}^{2n})$; in simpler terms, let a be a function in $C_b^\infty(\mathbb{R}^{2n})$ and take for instance

$$a_k(x,\xi) = a(x,\xi)e^{-(|x|^2+|\xi|^2)k^{-2}}.$$

It is easy to see that each a_k belongs to $\mathscr{S}(\mathbb{R}^{2n})$, that the sequence (a_k) is bounded in $C_b^\infty(\mathbb{R}^{2n})$ and converges in $C^\infty(\mathbb{R}^{2n})$ to a. This type of density will be enough for the next lemma.

Lemma 3.2.3. *Let (a_k) be a sequence in $\mathscr{S}(\mathbb{R}^{2n})$ such that (a_k) is bounded in the Fréchet space $C_b^\infty(\mathbb{R}^{2n})$ and (a_k) is converging in $C^\infty(\mathbb{R}^{2n})$ to a function a. Then a belongs to $C_b^\infty(\mathbb{R}^{2n})$ and for any $u \in \mathscr{S}(\mathbb{R}^n)$, the sequence $(a_k(x,D)u)$ converges to $a(x,D)u$ in $\mathscr{S}(\mathbb{R}^n)$.*

Proof. The fact that a belongs to $C_b^\infty(\mathbb{R}^{2n})$ is obvious. Using the identities in the proof of Theorem 3.2.2 we see that

$$\begin{aligned}
&x^\beta D_x^\alpha\big(a_k(x,D)u - a(x,D)u\big) = x^\beta D_x^\alpha\big((a_k-a)(x,D)u\big)\\
&= (-1)^{|\beta|} \sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} \frac{\alpha!}{\alpha'!\alpha''!}\frac{\beta!}{\beta'!\beta''!} \int e^{2i\pi x\cdot\xi}\big(D_\xi^{\beta'}D_x^{\alpha''}(a_k-a)\big)(x,\xi)D_\xi^{\beta''}\widehat{D^{\alpha'}u}(\xi)d\xi\\
&= \sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} \frac{\alpha!}{\alpha'!\alpha''!}\frac{\beta!}{\beta'!\beta''!}(1+|x|^2)^{-1}\\
&\qquad\times \int (1+|D_\xi|^2)\big(e^{2i\pi x\cdot\xi}\big)\big(D_\xi^{\beta'}D_x^{\alpha''}(a_k-a)\big)(x,\xi)D_\xi^{\beta''}\widehat{D^{\alpha'}u}(\xi)d\xi,
\end{aligned}$$

that is a (finite) sum of terms of type $V_k(x) = (1+|x|^2)^{-1}\int e^{2i\pi x\cdot\xi}b_k(x,\xi)w_u(\xi)d\xi$ with the sequence (b_k) bounded in $C_b^\infty(\mathbb{R}^{2n})$ and converging to 0 in $C^\infty(\mathbb{R}^{2n})$, $u\mapsto w_u$ linear continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself. As a consequence we get that, with R_1, R_2 positive parameters,

$$\begin{aligned}
|V_k(x)| \leq &\sup_{\substack{|x|\leq R_1\\ |\xi|\leq R_2}} |b_k(x,\xi)| \int_{|\xi|\leq R_2} |w_u(\xi)|d\xi \mathbf{1}_{|x|\leq R_1}\\
&+ \int_{|\xi|\geq R_2} |w_u(\xi)|d\xi \sup_{k\in\mathbb{N}} \|b_k\|_{L^\infty(\mathbb{R}^{2n})}\mathbf{1}_{|x|\leq R_1}\\
&+ R_1^{-2}\mathbf{1}_{|x|\geq R_1}\sup_{k\in\mathbb{N}}\|b_k\|_{L^\infty(\mathbb{R}^{2n})}\int |w_u(\xi)|d\xi,
\end{aligned}$$

implying

$$\begin{aligned}
|V_k(x)| \leq \varepsilon_k(R_1,R_2)\int |w_u(\xi)|d\xi + \eta(R_2)\sup_{k\in\mathbb{N}}\|b_k\|_{L^\infty(\mathbb{R}^{2n})}\\
+ \theta(R_1)\sup_{k\in\mathbb{N}}\|b_k\|_{L^\infty(\mathbb{R}^{2n})}\int |w_u(\xi)|d\xi,
\end{aligned}$$

with $\lim_{k\to+\infty}\varepsilon_k(R_1,R_2)=0, \lim_{R\to+\infty}\eta(R)=\lim_{R\to+\infty}\theta(R)=0$. Thus we have for all positive R_1, R_2,

$$\limsup_{k\to+\infty}\|V_k\|_{L^\infty}\le\eta(R_2)\sup_{k\in\mathbb{N}}\|b_k\|_{L^\infty(\mathbb{R}^{2n})}+\theta(R_1)\sup_{k\in\mathbb{N}}\|b_k\|_{L^\infty(\mathbb{R}^{2n})}\int|w_u(\xi)|d\xi,$$

entailing (by taking the limit when R_1, R_2 go to infinity) that $\lim_{k\to+\infty}\|V_k\|_{L^\infty}=0$ which gives the result of the lemma. □

Theorem 3.2.4. *Let $a\in C_b^\infty(\mathbb{R}^{2n})$: the operator $a(x,D)$ is bounded on $L^2(\mathbb{R}^n)$.*

Proof. Since $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, it is enough to prove that there exists a constant C such that for all $u,v\in\mathscr{S}(\mathbb{R}^n)$,

$$|\langle a(x,D)u,v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)}|\le C\|u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)}.$$

We introduce the polynomial on $\mathbb{R}^n$ defined by $P_k(t)=(1+|t|^2)^{k/2}$, where $k\in 2\mathbb{N}$, and the function

$$W_u(x,\xi)=\int u(y)P_k(x-y)^{-1}e^{-2i\pi y\cdot\xi}dy.$$

The function W_u is the partial Fourier transform of the function $\mathbb{R}^n\times\mathbb{R}^n\ni(x,y)\mapsto u(y)P_k(x-y)^{-1}$ and if $k>n/2$ (we assume this in the sequel), we obtain that $\|W_u\|_{L^2(\mathbb{R}^{2n})}=c_k\|u\|_{L^2(\mathbb{R}^n)}$. Moreover, since $u\in\mathscr{S}(\mathbb{R}^n)$, the function W_u belongs to $C^\infty(\mathbb{R}^{2n})$ and satisfies for all multi-indices α,β,γ

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}}P_k(x)\xi^\gamma|(\partial_x^\alpha\partial_\xi^\beta W_u)(x,\xi)|<\infty.$$

In fact we have

$$\begin{aligned}\xi^\gamma(\partial_x^\alpha\partial_\xi^\beta W_u)(x,\xi)&=\int\overbrace{u(y)(-2i\pi y)^\beta}^{\in\mathscr{S}(\mathbb{R}^n)}\partial^\alpha(1/P_k)(x-y)(-1)^{|\gamma|}D_y^\gamma(e^{-2i\pi y\cdot\xi})dy\\&=\sum_{\gamma'+\gamma''=\gamma}\frac{\gamma!}{\gamma'!\gamma''!}\int D_y^{\gamma'}\big(u(y)(-2i\pi y)^\beta\big)(-2i\pi)^{-|\gamma''|}\\&\qquad\qquad\partial^{\gamma''+\alpha}(1/P_k)(x-y)(e^{-2i\pi y\cdot\xi})dy\end{aligned}$$

and

$$|\partial^\alpha(1/P_k)(x-y)|\le C_{\alpha,k}(1+|x-y|)^{-k}\le C_{\alpha,k}(1+|x|)^{-k}(1+|y|)^k.$$

From Definition 3.2.1, we have

$$\langle a(x,D)u,v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)}=\iint_{\mathbb{R}^n\times\mathbb{R}^n}e^{2i\pi x\cdot\xi}a(x,\xi)\hat u(\xi)\bar v(x)dxd\xi,$$

and we obtain, using an integration by parts justified by the regularity and decay of the functions W above,

$$\begin{aligned}&\langle a(x,D)u,v\rangle\\&=\iint a(x,\xi)P_k(D_\xi)\Big(\int u(y)P_k(x-y)^{-1}e^{2i\pi(x-y)\cdot\xi}dy\Big)\bar v(x)dxd\xi\\&=\iint a(x,\xi)P_k(D_\xi)\big(\underbrace{e^{2i\pi x\cdot\xi}W_u(x,\xi)\bar v(x)}_{\in\mathscr{S}(\mathbb{R}^{2n})}\big)dxd\xi\end{aligned}$$

$$= \iint (P_k(D_\xi)a)(x,\xi)W_u(x,\xi)P_k(D_x)\left(\int e^{2i\pi x\cdot(\xi-\eta)}P_k(\xi-\eta)^{-1}\overline{\hat{v}(\eta)}d\eta\right)dxd\xi$$
$$= \iint (P_k(D_\xi)a)(x,\xi)W_u(x,\xi)P_k(D_x)\left(W_{\bar{\hat{v}}}(\xi,x)e^{2i\pi x\cdot\xi}\right)dxd\xi$$
$$= \sum_{0\le l\le k/2} C^l_{k/2}\iint |D_x|^{2l}\Big((P_k(D_\xi)a)(x,\xi)W_u(x,\xi)\Big)W_{\bar{\hat{v}}}(\xi,x)e^{2i\pi x\cdot\xi}dxd\xi$$
$$= \sum_{\substack{|\alpha|\le k\\|\beta|+|\gamma|\le k}} c_{\alpha\beta\gamma}\iint \underbrace{(D^\alpha_\xi D^\beta_x a)(x,\xi)}_{\text{bounded}}D^\gamma_x(W_u)(x,\xi)\underbrace{W_{\bar{\hat{v}}}(\xi,x)}_{\substack{\in L^2(\mathbb{R}^{2n})\text{ with norm}\\ c_k\|v\|_{L^2(\mathbb{R}^n)}}}e^{2i\pi x\cdot\xi}dxd\xi.$$

Checking now the x-derivatives of W_u, we see that

$$D^\gamma_x(W_u)(x,\xi) = \int u(y)D^\gamma(1/P_k)(x-y)e^{-2i\pi y\cdot\xi}dy,$$

and since $D^\gamma(1/P_k)$ belongs to $L^2(\mathbb{R}^n)$ (since $k > n/2$), we get that the $L^2(\mathbb{R}^{2n})$ norm of $D^\gamma_x(W_u)$ is bounded above by $c_\gamma\|u\|_{L^2(\mathbb{R}^n)}$. Using the Cauchy-Schwarz inequality, we obtain that

$$|\langle a(x,D)u,v\rangle| \le \sum_{\substack{|\alpha|\le k\\|\beta|+|\gamma|\le k}} c_{\alpha\beta\gamma}\|\partial^\alpha_\xi\partial^\beta_x a\|_{L^\infty(\mathbb{R}^{2n})}\|D^\gamma_x W_u\|_{L^2(\mathbb{R}^{2n})}\|W_{\bar{\hat{v}}}\|_{L^2(\mathbb{R}^{2n})}$$
$$\le C_n\|u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)}\sup_{\substack{|\alpha|\le k\\|\beta|\le k}}\|\partial^\alpha_\xi\partial^\beta_x a\|_{L^\infty(\mathbb{R}^{2n})},$$

where C_n depends only on n and $2\mathbb{N}\ni k > n/2$, which is the sought result. □

The next theorem gives us our first algebra of pseudo-differential operators.

Theorem 3.2.5. *Let a, b be in $C^\infty_b(\mathbb{R}^{2n})$. Then the composition $a(x,D)b(x,D)$ makes sense as a bounded operator on $L^2(\mathbb{R}^n)$ (also as a continuous operator from $\mathscr{S}(\mathbb{R}^n)$ into itself), and $a(x,D)b(x,D) = (a\diamond b)(x,D)$ where $a\diamond b$ belongs to $C^\infty_b(\mathbb{R}^{2n})$ and is given by the formula*

$$(a\diamond b)(x,\xi) = (\exp 2i\pi D_y\cdot D_\eta)(a(x,\xi+\eta)b(y+x,\xi))_{|y=0,\eta=0}, \tag{3.2.3}$$
$$(a\diamond b)(x,\xi) = \iint e^{-2i\pi y\cdot\eta}a(x,\xi+\eta)b(y+x,\xi)dyd\eta, \tag{3.2.4}$$

when a and b belong to $\mathscr{S}(\mathbb{R}^{2n})$. The mapping $a, b\mapsto a\diamond b$ is continuous for the topology of Fréchet space of $C^\infty_b(\mathbb{R}^{2n})$. Also if $(a_k), (b_k)$ are sequences of functions in $\mathscr{S}(\mathbb{R}^{2n})$, bounded in $C^\infty_b(\mathbb{R}^{2n})$, converging in $C^\infty(\mathbb{R}^{2n})$ respectively to a, b, then a and b belong to $C^\infty_b(\mathbb{R}^{2n})$, the sequence $(a_k\diamond b_k)$ is bounded in $C^\infty_b(\mathbb{R}^{2n})$ and converges in $C^\infty(\mathbb{R}^{2n})$ to $a\diamond b$.

Remark 3.2.6. From Lemma 4.1.2 in [13], we know that the operator $e^{2i\pi D_y\cdot D_\eta}$ is an isomorphism of $C^\infty_b(\mathbb{R}^{2n})$, which gives a meaning to the formula (3.2.3), since for $a, b\in C^\infty_b(\mathbb{R}^{2n})$, (x,ξ) given in $\mathbb{R}^{2n}$, the function $(y,\eta)\mapsto a(x,\xi+\eta)b(y+x,\xi) = C_{x,\xi}(y,\eta)$ belongs to $C^\infty_b(\mathbb{R}^{2n})$ as well as $JC_{x,\xi}$ and we can take the value of the latter at $(y,\eta) = (0,0)$.

Proof. Let us first assume that $a, b \in \mathscr{S}(\mathbb{R}^{2n})$. The kernels k_a, k_b of the operators $a(x, D), b(x, D)$ belong also to $\mathscr{S}(\mathbb{R}^{2n})$ and the kernel k_c of $a(x, D)b(x, D)$ is given by (we use Fubini's theorem)

$$k(x, y) = \int k_a(x, z)k_b(z, y)dz = \iiint a(x, \xi)e^{2i\pi(x-z)\cdot\xi}b(z, \zeta)e^{2i\pi(z-y)\cdot\zeta}d\zeta d\xi dz.$$

The function k belongs also to $\mathscr{S}(\mathbb{R}^{2n})$ and we get, for $u, v \in \mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned}
&\langle a(x, D)b(x, D)u, v\rangle_{L^2(\mathbb{R}^n)} \\
&= \int\!\!\!\int\!\!\!\int\!\!\!\int\!\!\!\int a(x, \xi)e^{2i\pi(x-z)\cdot\xi}b(z, \zeta)e^{2i\pi(z-y)\cdot\zeta}u(y)\bar{v}(x)d\zeta d\xi dz dy dx. \\
&= \iiiint a(x, \xi)e^{2i\pi(x-z)\cdot\xi}b(z, \zeta)e^{2i\pi z\cdot\zeta}\hat{u}(\zeta)d\zeta d\xi dz\bar{v}(x)dx. \\
&= \iiiint a(x, \xi)e^{2i\pi(x-z)\cdot\xi}b(z, \zeta)e^{2i\pi(z-x)\cdot\zeta}d\xi dz e^{2i\pi x\cdot\zeta}\hat{u}(\zeta)d\zeta\bar{v}(x)dx. \\
&= \iint c(x, \zeta)e^{2i\pi x\cdot\zeta}\hat{u}(\zeta)d\zeta\bar{v}(x)dx,
\end{aligned}$$

with

$$\begin{aligned}
c(x, \zeta) &= \iint a(x, \xi)e^{2i\pi(x-z)\cdot(\xi-\zeta)}b(z, \zeta)d\xi dz \\
&= \iint a(x, \xi+\zeta)e^{-2i\pi z\cdot\xi}b(z+x, \zeta)d\xi dz, \qquad (3.2.5)
\end{aligned}$$

which is indeed (3.2.4). With $c = a \diamond b$ given by (3.2.4), using that $a, b \in \mathscr{S}(\mathbb{R}^{2n})$ we get, using the notation (1.2.8) and $P_k(t) = (1 + |t|^2)^{1/2}, k \in 2\mathbb{N}$,

$$\begin{aligned}
c(x, \xi) &= \iint P_k(D_\eta)\Big(e^{-2i\pi y\cdot\eta}\Big)P_k(y)^{-1}a(x, \xi+\eta)b(y+x, \xi)dyd\eta \\
&= \iint e^{-2i\pi y\cdot\eta}P_k(y)^{-1}(P_k(D_2)a)(x, \xi+\eta)b(y+x, \xi)dyd\eta \\
&= \iint P_k(D_y)\Big(e^{-2i\pi y\cdot\eta}\Big)P_k(\eta)^{-1}P_k(y)^{-1}(P_k(D_2)a)(x, \xi+\eta)b(y+x, \xi)dyd\eta \\
&= \sum_{0\le l\le k/2} C^l_{k/2}\iint e^{-2i\pi y\cdot\eta}|D_y|^{2l}\Big(P_k(y)^{-1}b(y+x, \xi)\Big) \\
&\qquad\qquad P_k(\eta)^{-1}(P_k(D_2)a)(x, \xi+\eta)dyd\eta. \qquad (3.2.6)
\end{aligned}$$

We denote by $a\tilde{\diamond}b$ the right-hand-side of the previous formula and we note that, when $k > n$, it makes sense as well for $a, b \in C^\infty_b(\mathbb{R}^{2n})$, since $|\partial^\alpha_t(1/P_k)(t)| \le C_{\alpha,k}(1+|t|)^{-k}$. We already know that $a \diamond b = a\tilde{\diamond}b$ for a, b in the Schwartz class and we want to prove that it is also true for $a, b \in C^\infty_b(\mathbb{R}^{2n})$. Choosing an even $k > n$ (take $k = n + 1$ or $n + 2$), we also get

$$\|a\tilde{\diamond}b\|_{L^\infty(\mathbb{R}^{2n})} \le C_n \sup_{|\alpha|\le n+2}\|\partial^\alpha_\xi a\|_{L^\infty(\mathbb{R}^{2n})}\sup_{|\beta|\le n+2}\|\partial^\beta_x b\|_{L^\infty(\mathbb{R}^{2n})}.$$

Moreover, we note from (3.2.6) that

$$\partial_{\xi_j}(a\tilde{\diamond}b) = (\partial_{\xi_j}a)\tilde{\diamond}b + a\tilde{\diamond}(\partial_{\xi_j}b), \partial_{x_j}(a\tilde{\diamond}b) = (\partial_{x_j}a)\tilde{\diamond}b + a\tilde{\diamond}(\partial_{x_j}b)$$

and as a result

$$\begin{aligned}&\|\partial_\xi^\alpha\partial_x^\beta(a\tilde{\diamond}b)\|_{L^\infty(\mathbb{R}^{2n})}\\ &\quad\le C_{n,\alpha,\beta}\sup_{\substack{|\alpha'|\le n+2,|\beta'|\le n+2\\ \alpha''+\alpha'''=\alpha,\ \beta''+\beta'''=\beta}}\|\partial_\xi^{\alpha'+\alpha''}\partial_x^{\beta''}a\|_{L^\infty(\mathbb{R}^{2n})}\|\partial_x^{\beta'+\beta'''}\partial_\xi^{\alpha'''}b\|_{L^\infty(\mathbb{R}^{2n})},\end{aligned} \tag{3.2.7}$$

which gives also the continuity of the bilinear mapping $C_b^\infty(\mathbb{R}^{2n}) \times C_b^\infty(\mathbb{R}^{2n}) \ni (a,b) \mapsto a\tilde{\diamond}b \in C_b^\infty(\mathbb{R}^{2n})$. We have for $u, v \in \mathscr{S}(\mathbb{R}^n)$, $a, b \in C_b^\infty(\mathbb{R}^{2n})$,

$$a_k(x,\xi) = e^{-(|x|^2+|\xi|^2)/k^2}a(x,\xi), \quad b_k(x,\xi) = e^{-(|x|^2+|\xi|^2)/k^2}b(x,\xi),$$

from Lemma 3.2.3 and Theorem 3.2.2, with limits in $\mathscr{S}(\mathbb{R}^n)$,

$$a(x,D)b(x,D)u = \lim_k a_k(x,D)b(x,D)u = \lim_k\Big(\lim_l a_k(x,D)b_l(x,D)u\Big),$$

and thus, with $\Omega_{u,v}(x,\xi) = e^{2i\pi x\cdot\xi}\hat{u}(\xi)\bar{v}(x)$ (which belongs to $\mathscr{S}(\mathbb{R}^{2n})$),

$$\begin{aligned}\langle a(x,D)b(x,D)u, v\rangle_{L^2} &= \lim_k\Big(\lim_l\langle(a_k\diamond b_l)(x,D)u, v\rangle\Big)\\ &= \lim_k\Big(\lim_l\iint(a_k\diamond b_l)(x,\xi)\Omega_{u,v}(x,\xi)dxd\xi\Big) = \iint(a\tilde{\diamond}b)(x,\xi)\Omega_{u,v}(x,\xi)dxd\xi,\end{aligned}$$

which gives indeed $a(x,D)b(x,D) = (a\tilde{\diamond}b)(x,D)$. This property gives at once the continuity properties stated at the end of the theorem, since the weak continuity property follows immediately from (3.2.6) and the Lebesgue dominated convergence theorem, whereas the Fréchet continuity follows from (3.2.7). Moreover, with the same notations as above, we have with

$$C_{x,\xi}^{(a,b)}(y,\eta) = a(x,\xi+\eta)b(y+x,\xi)$$

(see Remark 3.2.6) for each $(x,\xi) \in \mathbb{R}^{2n}$,

$$(JC_{x,\xi}^{(a,b)})(0,0) = \lim_k(JC_{x,\xi}^{(a_k,b_k)})(0,0) = \lim_k\big((a_k\diamond b_k)(x,\xi)\big) = (a\tilde{\diamond}b)(x,\xi)$$

which proves (3.2.3). The proof of the theorem is complete. □

Definition 3.2.7. Let $A : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}'(\mathbb{R}^n)$ be a linear operator. The adjoint operator $A^* : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}'(\mathbb{R}^n)$ is defined by

$$\langle A^*u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \overline{\langle Av, u\rangle}_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)},$$

where $\mathscr{S}^*(\mathbb{R}^n)$ is the antidual of $\mathscr{S}(\mathbb{R}^n)$ (continuous antilinear forms).

Lemma 3.2.8. *Let $n \ge 1$ be an integer and $t \in \mathbb{R}^*$. We define the operator*

$$J^t = \exp 2i\pi t D_x \cdot D_\xi \tag{3.2.8}$$

on $\mathscr{S}'(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ by $(FJ^t a)(\xi, x) = e^{2i\pi t \xi\cdot x}\hat{a}(\xi, x)$, where F stands here for the Fourier transform in $2n$ dimensions. The operator J^t sends also $\mathscr{S}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ into itself continuously, satisfies (for $s, t \in \mathbb{R}$) $J^{s+t} = J^s J^t$ and is given by

$$(J^t a)(x, \xi) = |t|^{-n} \iint e^{-2i\pi t^{-1} y\cdot\eta} a(x + y, \xi + \eta) dy d\eta. \tag{3.2.9}$$

We have

$$J^t a = e^{i\pi t\langle BD, D\rangle} a = |t|^{-n} e^{-i\pi t^{-1}\langle B\cdot,\cdot\rangle} * a, \tag{3.2.10}$$

with the $2n \times 2n$ matrix $B = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. The operator J^t sends continuously $C^\infty_b(\mathbb{R}^{2n})$ into itself.

Proof. We have indeed $(FJ^t a)(\xi, x) = e^{2i\pi t\xi\cdot x}\hat{a}(\xi, x) = e^{i\pi t\langle B\Xi,\Xi\rangle}\hat{a}(\Xi)$. Note that B is a $2n \times 2n$ symmetric matrix with null signature, determinant $(-1)^n$ and that $B^{-1} = B$. According to the proposition 1.2.19, the inverse Fourier transform of $e^{i\pi t\langle B\Xi,\Xi\rangle}$ is $|t|^{-n}e^{-i\pi t^{-1}\langle BX,X\rangle}$ so that $J^t a = |t|^{-n}e^{-i\pi t^{-1}\langle B\cdot,\cdot\rangle} * a$. Since the Fourier multiplier $e^{i\pi t\langle B\Xi,\Xi\rangle}$ is smooth bounded with derivatives polynomially bounded, it defines a continuous operator from $\mathscr{S}(\mathbb{R}^{2n})$ into itself.

In the sequel of the proof, we take $t = 1$, which will simplify the notations without corrupting the arguments. Let us consider $a \in \mathscr{S}(\mathbb{R}^{2n})$: we have with $k \in 2\mathbb{N}$ and the polynomial on $\mathbb{R}^n$ defined by $P_k(y) = (1 + |y|^2)^{k/2}$

$$(Ja)(x, \xi) = \iint e^{-2i\pi y\cdot\eta} P_k(y)^{-1} P_k(D_\eta)\Big(P_k(\eta)^{-1}(P_k(D_y)a)(x + y, \xi + \eta)\Big) dy d\eta,$$

so that, with $|T_{\alpha\beta}(\eta)| \le P_k(\eta)^{-1}$ and constants $c_{\alpha\beta}$, we obtain

$$(Ja)(x, \xi) = \sum_{\substack{|\beta|\le k\\ |\alpha|\le k}} c_{\alpha\beta} \iint e^{-2i\pi y\cdot\eta} P_k(y)^{-1} T_{\alpha\beta}(\eta)(D^\alpha_\xi D^\beta_x a)(x+y, \xi+\eta) dy d\eta. \tag{3.2.11}$$

Let us denote by $\tilde{J}a$ the right-hand-side of (3.2.11). We already know that $\tilde{J}a = Ja$ for $a \in \mathscr{S}(\mathbb{R}^{2n})$. We also note that, using an even integer $k > n$, the previous integral converges absolutely whenever $a \in C^\infty_b(\mathbb{R}^{2n})$; moreover we have

$$\|\tilde{J}a\|_{L^\infty} \le C_n \sup_{\substack{|\alpha|\le n+2\\ |\beta|\le n+2}} \|D^\alpha_\xi D^\beta_x a\|_{L^\infty},$$

and since the derivations are commuting with J and $\tilde{J}$, we also get that

$$\|\partial^\gamma \tilde{J}a\|_{L^\infty} \le C_n \sup_{\substack{|\alpha|\le n+2\\ |\beta|\le n+2}} \|D^\alpha_\xi D^\beta_x \partial^\gamma a\|_{L^\infty}. \tag{3.2.12}$$

It implies that $\tilde{J}$ is continuous from $C_b^\infty(\mathbb{R}^{2n})$ to itself. Let us now consider $a \in C_b^\infty(\mathbb{R}^{2n} \times \mathbb{R}^m)$; we define the sequence (a_k) in $\mathscr{S}(\mathbb{R}^{2n})$ by

$$a_k(x,\xi) = e^{-(|x|^2+|\xi|^2)/k^2} a(x,\xi).$$

We have $\langle Ja, \Phi\rangle_{\mathscr{S}^*(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} =$

$$\iint a(x,\xi)\overline{(J^{-1}\Phi)}(x,\xi)dxd\xi = \lim_{k\to+\infty} \iint a_k(x,\xi)\overline{(J^{-1}\Phi)}(x,\xi)dxd\xi$$
$$= \lim_{k\to+\infty} \iint (Ja_k)(x,\xi)\bar{\Phi}(x,\xi)dxd\xi = \iint (\tilde{J}a)(x,\xi)\bar{\Phi}(x,\xi)dxd\xi,$$

so that we indeed have $\tilde{J}a = Ja$ and from (3.2.12) the continuity property of the lemma whose proof is now complete. □

Theorem 3.2.9. *Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$ and $A = a(x,D)$ be given by Definition 3.2.1. Then the operator A^* is equal to $a^*(x,D)$, where $a^* = J\bar{a}$ (J is given in Lemma 3.2.8 above). If a belongs to $C_b^\infty(\mathbb{R}^{2n})$, $a^* = J\bar{a} \in C_b^\infty(\mathbb{R}^{2n})$ and the mapping $a \mapsto a^*$ is continuous from $C_b^\infty(\mathbb{R}^{2n})$ into itself.*

Proof. According to the definitions 3.2.7 and 3.2.1, we have for $u, v \in \mathscr{S}(\mathbb{R}^n)$, with $\Omega_{v,u}(x,\xi) = e^{2i\pi x\cdot\xi}\hat{v}(\xi)\bar{u}(x)$,

$$\langle A^*u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \overline{\langle Av, u\rangle}_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \overline{\prec a, \Omega_{v,u} \succ}_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}$$
$$= \prec \bar{a}, \overline{\Omega}_{v,u} \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} .$$

On the other hand, we have

$$\big(J^{-1}(\overline{\Omega}_{v,u})\big)(x,\xi) = \iint e^{2i\pi(x-y)\cdot(\xi-\eta)}e^{-2i\pi y\cdot\eta}\bar{\hat{v}}(\eta)u(y)dyd\eta$$
$$= \bar{v}(x)e^{2i\pi x\cdot\xi}\hat{u}(\xi) = \Omega_{u,v}(x,\xi),$$

so that, using (3.2.10), we get

$$\langle A^*u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \prec \bar{a}, J\Omega_{u,v} \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} = \prec J\bar{a}, \Omega_{u,v} \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}$$

and finally $A^* = (J\bar{a})(x,D)$. The last statement in the theorem follows from Lemma 3.2.8. □

N.B. In this introductory section, we have seen a very general definition of quantization (Definition 3.2.1), an easy $\mathscr{S}$ continuity theorem (Theorem 3.2.2), a trickier L^2-boundedness result (Theorem 3.2.4), a composition formula (Theorem 3.2.5) and an expression for the adjoint (Theorem 3.2.7). These five steps are somewhat typical of the construction of a pseudo-differential calculus and we shall see many different examples of this situation. The above prolegomena provide a quite explicit and elementary approach to the construction of an algebra of pseudo-differential operators in a rather difficult framework, since we did not use any asymptotic calculus and did not have at our disposal a "small parameter". The proofs and simple methods that we used here will be useful later as well as many of the results.

3.3 Quantization formulas

We have already seen in Definition 3.2.1 and in the formula (3.2.1) a way to associate to a tempered distribution $a \in \mathscr{S}'(\mathbb{R}^{2n})$ an operator from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$. This question of quantization has of course many links with quantum mechanics and we want here to study some properties of various quantizations formulas, such as the Weyl quantization and the Feynman formula along with several variations around these examples. We are given a function a defined on the phase space $\mathbb{R}^n \times \mathbb{R}^n$ (a is a "Hamiltonian") and we wish to associate to this function an operator. For instance, we may introduce the one-parameter formulas, for $t \in \mathbb{R}$,

$$(\operatorname{op}_t a)u(x) = \iint e^{2i\pi(x-y)\cdot\xi} a\big((1-t)x + ty, \xi\big)u(y)dyd\xi. \tag{3.3.1}$$

When $t = 0$, we recognize the standard quantization introduced in Definition 3.2.1, quantizing $a(x)\xi_j$ in $a(x)D_{x_j}$ (see (1.2.8)). However, one may wish to multiply first and take the derivatives afterwards: this is what the choice $t = 1$ does, quantizing $a(x)\xi_j$ in $D_{x_j}a(x)$. The more symmetrical choice $t = 1/2$ was done by Hermann Weyl [27]: we have

$$(\operatorname{op}_{\frac{1}{2}} a)u(x) = \iint e^{2i\pi(x-y)\cdot\xi} a\big(\frac{x+y}{2}, \xi\big)u(y)dyd\xi, \tag{3.3.2}$$

and thus $\operatorname{op}_{\frac{1}{2}}(a(x)\xi_j) = \frac{1}{2}\big(a(x)D_{x_j} + D_{x_j}a(x)\big)$. This quantization is widely used in quantum mechanics, because a real-valued Hamiltonian gets quantized by a (formally) selfadjoint operator. We shall see that the most important property of that quantization remains its symplectic invariance, which will be studied in details in Chapter 2; a different symmetrical choice was made by Richard Feynman who used the formula

$$\iint e^{2i\pi(x-y)\cdot\xi}\ \big(a(x,\xi) + a(y,\xi)\big)\frac{1}{2}u(y)dyd\xi, \tag{3.3.3}$$

keeping the selfadjointness of real Hamiltonians, but loosing the symplectic invariance. The reader may be embarrassed by the fact that we did not bother about the convergence of the integrals above. Before providing a definition, we may assume that $a \in \mathscr{S}(\mathbb{R}^{2n}), u, v \in \mathscr{S}(\mathbb{R}^n), t \in \mathbb{R}$ and compute

$$\begin{aligned}
\langle(\operatorname{op}_t a)u, v\rangle &= \iiint a\big((1-t)x + ty, \xi\big)e^{2i\pi(x-y)\cdot\xi}u(y)\bar{v}(x)dyd\xi dx \\
&= \iiint a(z,\xi)e^{-2i\pi s\cdot\xi}u(z + (1-t)s)\bar{v}(z - ts)dzd\xi ds \\
&= \iiint a(x,\xi)e^{-2i\pi z\cdot\xi}u(x + (1-t)z)\bar{v}(x - tz)dxd\xi dz,
\end{aligned}$$

so that with

$$\Omega_{u,v}(t)(x,\xi) = \int e^{-2i\pi z\cdot\xi}u(x + (1-t)z)\bar{v}(x - tz)dz, \tag{3.3.4}$$

which is easily seen[2] to be in $\mathscr{S}(\mathbb{R}^{2n})$ when $u, v \in \mathscr{S}(\mathbb{R}^n)$, we can give the following definition.[3]

Definition 3.3.1. Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$ be a tempered distribution and $t \in \mathbb{R}$. We define the operator $\operatorname{op}_t a : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}^*(\mathbb{R}^n)$ by the formula

$$\langle (\operatorname{op}_t a)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \prec a, \Omega_{u,v}(t) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

where $\mathscr{S}^*(\mathbb{R}^n)$ is the antidual of $\mathscr{S}(\mathbb{R}^n)$ (continuous antilinear forms).

Proposition 3.3.2. *Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$ be a tempered distribution and $t \in \mathbb{R}$. We have*

$$\operatorname{op}_t a = \operatorname{op}_0(J^t a) = (J^t a)(x, D),$$

with J^t defined in Lemma 3.2.8.

Proof. Let $u, v \in \mathscr{S}(\mathbb{R}^n)$. With the $\mathscr{S}(\mathbb{R}^{2n})$ function $\Omega_{u,v}(t)$ given above, we have for $t \neq 0$,

$$\begin{aligned}
\big(J^t\Omega_{u,v}(0)\big)(x,\xi) &= |t|^{-n} \iint e^{-2i\pi t^{-1}(x-y)\cdot(\xi-\eta)}\Omega_{u,v}(0)(y,\eta)dyd\eta \\
&= |t|^{-n} \iint e^{-2i\pi t^{-1}(x-y)\cdot(\xi-\eta)}\hat{u}(\eta)\bar{v}(y)e^{2i\pi y\cdot\eta}dyd\eta \\
&= \iint e^{-2i\pi z\cdot(\xi-\eta)}\hat{u}(\eta)\bar{v}(x-tz)e^{2i\pi(x-tz)\cdot\eta}dzd\eta \\
&= \int e^{-2i\pi z\cdot\xi}u(x+(1-t)z)\bar{v}(x-tz)dz = \Omega_{u,v}(t)(x,\xi),
\end{aligned} \tag{3.3.5}$$

so that

$$\begin{aligned}
\langle (\operatorname{op}_t a)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} &= \prec a, \Omega_{u,v}(t) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} && \text{(definition 3.3.1)}\\
&= \prec a, J^t\Omega_{u,v}(0) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} && \text{(property (3.3.5))}\\
&= \prec J^t a, \Omega_{u,v}(0) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} && \text{(easy identity for } J^t)\\
&= \langle (J^t a)(x,D)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} && \text{(definition 3.2.1)},
\end{aligned}$$

completing the proof. □

Remark 3.3.3. The theorem 3.2.9 and the previous proposition give in particular that $a(x, D)^* = \operatorname{op}_1(\bar{a}) = (J\bar{a})(x, D)$, a formula which in fact motivates the study of the group J^t. On the other hand, using the Weyl quantization simplifies somewhat the matter of taking adjoints since we have

$$\big(\operatorname{op}_{1/2}(a)\big)^* = \big(\operatorname{op}_0(J^{1/2}a)\big)^* = \operatorname{op}_0(J(\overline{J^{1/2}a})) = \operatorname{op}_0(J^{1/2}\bar{a}) = \operatorname{op}_{1/2}(\bar{a})$$

[2] In fact the linear mapping $\mathbb{R}^n \times \mathbb{R}^n \ni (x, z) \mapsto (x - tz, x + (1-t)z)$ has determinant 1 and $\Omega_{u,v}(t)$ appears as the partial Fourier transform of the function $\mathbb{R}^n \times \mathbb{R}^n \ni (x, z) \mapsto \bar{v}(x - tz)u(x + (1-t)z)$, which is in the Schwartz class.

[3] The reader can check that this is consistent with Definition 3.2.1.

and in particular if a is real-valued, $\mathrm{op}_{1/2}(a)$ is formally selfadjoint. The Feynman formula as displayed in (3.3.3) amounts to quantize the Hamiltonian a by

$$\frac{1}{2}\mathrm{op}_0(a + Ja)$$

and we see that $\big(\mathrm{op}_0(a+Ja)\big)^* = \mathrm{op}_0(J\bar{a}+J(\overline{Ja})) = \mathrm{op}_0(J\bar{a}+\bar{a})$, which also provides selfadjointness for real-valued Hamiltonians.

Lemma 3.3.4. *Let $a \in \mathscr{S}(\mathbb{R}^{2n})$. Then for all $t \in \mathbb{R}$, $op_t(a)$ is a continuous mapping from $\mathscr{S}'(\mathbb{R}^n)$ in $\mathscr{S}(\mathbb{R}^n)$.*

Proof. Let $a \in \mathscr{S}(\mathbb{R}^{2n})$: we have for $u \in \mathscr{S}'(\mathbb{R}^n)$, $A = a(x, D)$,

$$x^\beta(D_x^\alpha A u)(x) = \sum_{\alpha'+\alpha''=\alpha} \frac{1}{\alpha'!\alpha''!}\langle \hat{u}(\xi), e^{2i\pi x\cdot\xi}\xi^{\alpha'} x^\beta (D_x^{\alpha''} a)(x,\xi)\rangle_{\mathscr{S}'(\mathbb{R}^n_\xi),\mathscr{S}(\mathbb{R}^n_\xi)},$$

so that $Au \in \mathscr{S}(\mathbb{R}^n)$ and the same property holds for $\mathrm{op}_t(a)$ since J^t is an isomorphism of $\mathscr{S}'(\mathbb{R}^{2n})$. □

3.4 The $S^m_{1,0}$class of symbols

Differential operators on $\mathbb{R}^n$ with smooth coefficients are given by a formula (see (3.2.1))

$$a(x, D)u = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha$$

where the a_α are smooth functions. Assuming some behaviour at infinity for the a_α, we may require that they are $C_b^\infty(\mathbb{R}^n)$ (see page 62) and a natural generalization is to consider operators $a(x, D)$ with a symbol a of type $S^m_{1,0}$, i.e. smooth functions on $\mathbb{R}^{2n}$ satisfying

$$|(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|}, \quad \langle\xi\rangle = (1+|\xi|^2)^{1/2}. \tag{3.4.1}$$

The best constants $C_{\alpha\beta}$ in (3.4.1) are the semi-norms of a in the Fréchet space $S^m_{1,0}$. We can define, for $a \in S^m_{1,0}$, $k \in \mathbb{N}$,

$$\gamma_{k,m}(a) = \sup_{(x,\xi)\in\mathbb{R}^{2n}, |\alpha|+|\beta|\le k} |(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi)|\langle\xi\rangle^{-m+|\alpha|}. \tag{3.4.2}$$

Example. The function $\langle\xi\rangle^m$ belongs to $S^m_{1,0}$: the function

$$\mathbb{R}\times\mathbb{R}^n \ni (\tau,\xi) \mapsto (\tau^2 + |\xi|^2)^{m/2}$$

is (positively) homogeneous of degree m on $\mathbb{R}^{n+1}\backslash\{0\}$, and thus $\partial_\xi^\alpha\big((\tau^2+|\xi|^2)^{m/2}\big)$ is homogeneous of degree $m - |\alpha|$ and bounded above by

$$C_\alpha(\tau^2 + |\xi|^2)^{\frac{m-|\alpha|}{2}}.$$

Since the restriction to $\tau = 1$ and the derivation with respect to ξ commute, it gives the answer.

We shall see that the class of operators $\mathrm{Op}(S^m_{1,0})$ is suitable ($\mathrm{Op}(b)$ is $\mathrm{op}_0 b$, see Proposition 3.3.2) to invert elliptic operators, and useful for the study of singularities of solutions of PDE. We see that the elements of $S^m_{1,0}$ are temperate distributions, so that the operator $a(x, D)$ makes sense, according to Definition 3.2.1. We have also the following result.

Theorem 3.4.1. *Let $m \in \mathbb{R}$ and $a \in S^m_{1,0}$. Then the operator $a(x, D)$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself.*

Proof. With $\langle D\rangle = \mathrm{Op}(\langle\xi\rangle)$, we have $a(x, D) = \mathrm{Op}(a(x, \xi)\langle\xi\rangle^{-m})\langle D\rangle^m$. The function $a(x, \xi)\langle\xi\rangle^{-m}$ belongs to $C^\infty_b(\mathbb{R}^{2n})$ so that we can use Theorem 3.2.2 and the fact that $\langle D\rangle^m$ is continuous on $\mathscr{S}(\mathbb{R}^n)$ to get the result. □

Theorem 3.4.2. *Let $a \in S^0_{1,0}$. Then the operator $a(x, D)$ is bounded on $L^2(\mathbb{R}^n)$.*

Proof. Since $S^0_{1,0} \subset C^\infty_b(\mathbb{R}^{2n})$, it follows from Theorem 3.2.4. □

Theorem 3.4.3. *Let m_1, m_2 be real numbers and $a_1 \in S^{m_1}_{1,0}, a_2 \in S^{m_2}_{1,0}$. Then the composition $a_1(x, D)a_2(x, D)$ makes sense as a continuous operator from $\mathscr{S}(\mathbb{R}^n)$ into itself and $a_1(x, D)a_2(x, D) = (a_1 \diamond a_2)(x, D)$ where $a_1 \diamond a_2$ belongs to $S^{m_1+m_2}_{1,0}$ and is given by the formula*

$$(a_1 \diamond a_2)(x, \xi) = (\exp 2i\pi D_y \cdot D_\eta)\Big(a_1(x, \xi + \eta)a_2(y + x, \xi)\Big)_{|y=0,\eta=0}. \tag{3.4.3}$$

N.B. From Lemma 4.1.5 in [13], we know that the operator $e^{2i\pi D_y\cdot D_\eta}$ is an isomorphism of $S^m_{1,0}(\mathbb{R}^{2n})$, which gives a meaning to the formula (3.4.3), since for $a_j \in S^{m_j}_{1,0}(\mathbb{R}^{2n})$, (x, ξ) given in $\mathbb{R}^{2n}$, the function $(y, \eta) \mapsto a_1(x, \xi + \eta)a_2(y + x, \xi) = C_{x,\xi}(y, \eta)$ belongs to $S^{m_1}_{1,0}(\mathbb{R}^{2n})$ as well as $JC_{x,\xi}$ and we can take the value of the latter at $(y, \eta) = (0, 0)$.

Proof. We assume first that both a_j belong to $\mathscr{S}(\mathbb{R}^{2n})$. The formula (3.2.4) provides the answer. Now, rewriting the formula (3.2.6) for an even integer k, we get

$$(a_1 \diamond a_2)(x, \xi) = \sum_{0\le l\le k/2} C^l_{k/2} \iint e^{-2i\pi y\cdot\eta}|D_y|^{2l}\Big(\langle y\rangle^{-k}a_2(y + x, \xi)\Big) \langle\eta\rangle^{-k}(\langle D_\eta\rangle^k a_1)(x, \xi + \eta)dyd\eta. \tag{3.4.4}$$

We denote by $a_1\tilde\diamond a_2$ the right-hand-side of (3.4.4) and we note that, when $k > n + |m_1|$, it makes sense (and it does not depend on k) as well for $a_j \in S^{m_j}_{1,0}$, since

$$|\partial^\alpha_y\langle y\rangle^{-k}| \le C_{\alpha,k}\langle y\rangle^{-k},\ |\partial^\beta_y a_2(y + x, \xi)| \le C_\beta\langle\xi\rangle^{m_2},\ |\partial^\gamma_\eta a_1(x, \xi + \eta)| \le C_\gamma\langle\xi + \eta\rangle^{m_1}$$

so that the absolute value of the integrand above is[4] [5]

$$\lesssim \langle y\rangle^{-k}\langle\eta\rangle^{-k}\langle\xi\rangle^{m_2}\langle\xi + \eta\rangle^{m_1} \lesssim \langle y\rangle^{-k}\langle\eta\rangle^{-k+|m_1|}\langle\xi\rangle^{m_1+m_2}.$$

[4] We use $\langle\xi + \eta\rangle \le 2^{1/2}\langle\xi\rangle\langle\eta\rangle$ so that,

$$\forall s \in \mathbb{R}, \forall\xi, \eta \in \mathbb{R}^n, \quad \langle\xi + \eta\rangle^s \le 2^{|s|/2}\langle\xi\rangle^s\langle\eta\rangle^{|s|}, \tag{3.4.5}$$

a convenient inequality (to get it for $s \ge 0$, raise the first inequality to the power s, and for $s < 0$, replace ξ by $-\xi - \eta$) a.k.a. Peetre's inequality.

[5] We use here the notation $a \lesssim b$ for the inequality $a \le Cb$, where C is a "controlled" constant (here C depends only on k, m_1, m_2).

Remark 3.4.4. Note that this proves that the mapping

$$S^{m_1}_{1,0} \times S^{m_2}_{1,0} \ni (a_1, a_2) \mapsto a_1 \tilde{\diamond} a_2 \in S^{m_1+m_2}_{1,0}$$

is bilinear continuous. In fact, we have already proven that

$$|(a_1 \tilde{\diamond} a_2)(x, \xi)| \leq C\langle\xi\rangle^{m_1+m_2},$$

and we can check directly that $a_1 \tilde{\diamond} a_2$ is smooth and satisfies

$$\partial_{\xi_j}\big(a_1 \tilde{\diamond} a_2\big) = (\partial_{\xi_j} a_1) \tilde{\diamond} a_2 + a_1 \tilde{\diamond} (\partial_{\xi_j} a_2)$$

so that $|\partial_{\xi_j}(a_1 \tilde{\diamond} a_2)(x, \xi)| \leq C\langle\xi\rangle^{m_1+m_2-1}$, and similar formulas for higher order derivatives.

Remark 3.4.5. Let (c_k) be a bounded sequence in the Fréchet space $S^m_{1,0}$ converging in $C^\infty(\mathbb{R}^{2n})$ to c. Then c belongs to $S^m_{1,0}$ and for all $u \in \mathscr{S}(\mathbb{R}^n)$, the sequence $(c_k(x, D)u)$ converges to $c(x, D)u$ in $\mathscr{S}(\mathbb{R}^n)$. In fact, the sequence of functions $(c_k(x, \xi)\langle\xi\rangle^{-m})$ is bounded in $C^\infty_b(\mathbb{R}^{2n})$ and we can apply Lemma 3.2.3 to get that $\lim_k \operatorname{Op}(c_k(x, \xi)\langle\xi\rangle^{-m})\langle D\rangle^m u = \operatorname{Op}(c(x, \xi)\langle\xi\rangle^{-m})\langle D\rangle^m u = \operatorname{Op}(c)u$ in $\mathscr{S}(\mathbb{R}^n)$.

The remaining part of the argument is the same than in the proof of Theorem 3.2.5, after (3.2.7). □

Theorem 3.4.6. *Let s, m be real numbers and $a \in S^m_{1,0}$. Then the operator $a(x, D)$ is bounded from $H^{s+m}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$.*

Proof. Let us recall that $H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n), \langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}$. From the theorem 3.4.3, the operator $\langle D\rangle^s a(x, D)\langle D\rangle^{-m-s}$ can be written as $b(x, D)$ with $b \in S^0_{1,0}$ and so from the theorem 3.4.2, it is a bounded operator on $L^2(\mathbb{R}^n)$. Since $\langle D\rangle^\sigma$ is an isomorphism of $H^\sigma(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$ with inverse $\langle D\rangle^{-\sigma}$, it gives the result. □

Corollary 3.4.7. *Let r be a symbol in $S^{-\infty}_{1,0} = \cap_m S^m_{1,0}$. Then $r(x, D)$ sends $\mathscr{E}'(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$.*

Proof. We have for $v \in \mathscr{E}'$ and $\psi \in C^\infty_c(\mathbb{R}^n)$ equal to 1 on a neighborhood of the support of v, iterating

$$x_j D^\beta r(x, D)v = [x_j, D^\beta r(x, D)]\psi v + D^\beta r(x, D)\psi x_j v = r_j(x, D)v, \quad r_j \in S^{-\infty}_{1,0},$$

that $x^\alpha D^\beta r(x, D)v = r_{\alpha\beta}(x, D)v, r_{\alpha\beta} \in S^{-\infty}_{1,0}$, and thus

$$x^\alpha D^\beta r(x, D)v \in \cap_s H^s(\mathbb{R}^n) \subset C^\infty_b(\mathbb{R}^n),$$

completing the proof. □

Theorem 3.4.8. *Let m_1, m_2 be real numbers and $a_1 \in S^{m_1}_{1,0}, a_2 \in S^{m_2}_{1,0}$. Then $a_1(x,D)a_2(x,D) = (a_1 \diamond a_2)(x,D)$, the symbol $a_1 \diamond a_2$ belongs to $S^{m_1+m_2}_{1,0}$ and we have the asymptotic expansion, for all $N \in \mathbb{N}$,*

$$a_1 \diamond a_2 = \sum_{|\alpha|<N} \frac{1}{\alpha!} D^\alpha_\xi a_1 \partial^\alpha_x a_2 + r_N(a_1, a_2), \tag{3.4.6}$$

with $r_N(a_1, a_2) \in S^{m_1+m_2-N}_{1,0}$. Note that $D^\alpha_\xi a_1 \partial^\alpha_x a_2$ belong to $S^{m_1+m_2-|\alpha|}_{1,0}$.

Proof. We can use the formula (3.4.3) and apply that lemma to get the desired formula with

$$\begin{aligned} &r_N(a_1, a_2)(x,\xi) \\ &\quad = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} e^{2i\pi\theta D_z\cdot D_\zeta} (2i\pi D_z \cdot D_\zeta)^N \big(a_1(x,\zeta)a_2(z,\xi)\big) d\theta_{|z=x,\zeta=\xi}. \end{aligned} \tag{3.4.7}$$

The function $(z,\zeta) \mapsto b_{x,\xi}(z,\zeta) = \langle\xi\rangle^{-m_2}(2i\pi D_z\cdot D_\zeta)^N a_1(x,\zeta)a_2(z,\xi)$ belongs to $S^{m_1-N}_{1,0}(\mathbb{R}^{2n}_{z,\zeta})$ uniformly with respect to the parameters $(x,\xi) \in \mathbb{R}^{2n}$: it satisfies, using the notation (3.4.2), for $\max(|\alpha|,|\beta|) \le k$,

$$|\partial^\alpha_\zeta \partial^\beta_z b_{x,\xi}(z,\zeta)| \le \gamma_{k,m_1}(a_1)\gamma_{k,m_2}(a_2)\langle\zeta\rangle^{m_1-N-|\alpha|}.$$

Lemma 3.4.9. *Let $n \ge 1$ be an integer and $m, t \in \mathbb{R}$. The operator J^t sends continuously $S^m_{1,0}(\mathbb{R}^{2n})$ into itself and for all integers $N \ge 0$,*

$$\begin{aligned} (J^t a)(x,\xi) &= \sum_{|\alpha|<N} \frac{t^{|\alpha|}}{\alpha!}(D^\alpha_\xi \partial^\alpha_x a)(x,\xi) + r_N(t)(x,\xi), \quad r_N(t) \in S^{m-N}_{1,0}, \\ r_N(t)(x,\xi) &= t^N \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} \big(J^{\theta t}(D_\xi\cdot\partial_x)^N a\big)(x,\xi)d\theta. \end{aligned}$$

Proof. We apply Taylor's formula on $J^t = \exp 2i\pi t D_x \cdot D_\xi$ to get for operators on $\mathscr{S}'(\mathbb{R}^{2n})$,

$$J^t = \sum_{0\le k<N} \frac{t^k}{k!}(D_\xi\cdot\partial_x)^k + \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} J^{\theta t}(tD_\xi\cdot\partial_x)^N d\theta, \tag{3.4.8}$$

and since
$$\frac{1}{k!}(D_\xi\cdot\partial_x)^k = \sum_{\substack{\alpha_1+\dots+\alpha_n=k\\ \alpha_j\in\mathbb{N}}} \frac{(D_{\xi_1}\partial_{x_1})^{\alpha_1}}{\alpha_1!}\dots\frac{(D_{\xi_n}\partial_{x_n})^{\alpha_n}}{\alpha_n!},$$

we obtain the above formulas for $a \in \mathscr{S}'(\mathbb{R}^{2n})$. On the other hand, we get from (3.4.1) that the term $D^\alpha_\xi \partial^\alpha_x a$ belongs to $S^{m-|\alpha|}_{1,0}$. It is thus enough that we show that J^t sends continuously $S^m_{1,0}$ into itself. For that purpose, we can use the formula (3.2.11) (and assume that $t = 1$) in the proof of the lemma 3.2.8; also the same reasoning as in the proof of this lemma shows that the right-hand-side of (3.2.11) is meaningful for $a \in S^m_{1,0}$ if $k > n + |m|$ and is indeed the expression of Ja. We get, for all $k \in \mathbb{N}$,

$$|Ja(x, \xi)| \le C_{k,n} \iint \langle y\rangle^{-k} \langle \eta\rangle^{-k} \langle \xi + \eta\rangle^m d\xi d\eta$$

so that Peetre's inequality (3.4.5) yields, for $k > n + |m|$, $|Ja(x, \xi)| \le C'_{k,n} \langle\xi\rangle^m$. The estimates for the derivatives are obtained similarly since they commute with J. The terms involving integrals of J^t can be handled via Remark 4.1.4 in [13], which provides a polynomial control with respect to t. □

Applying Lemma 3.4.9, we obtain that the function

$$\rho_{x,\xi}(z, \zeta) = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} (J^\theta b_{x,\xi})(z, \zeta) d\theta$$

belongs to $S^{m_1-N}_{1,0}(\mathbb{R}^{2n}_{z,\zeta})$ uniformly with respect to x, ξ, so that in particular

$$\sup_{(x,\xi,z,\zeta)\in\mathbb{R}^{4n}} |\rho_{x,\xi}(z, \zeta)\langle\zeta\rangle^{-m_1+N}| = C_0 < +\infty.$$

Since $r_N(a_1, a_2)(x, \xi)\langle\xi\rangle^{-m_2} = \rho_{x,\xi}(x, \xi)$, we obtain

$$|r_N(a_1, a_2)(x, \xi)| \le C_0 \langle\xi\rangle^{m_1+m_2-N}. \tag{3.4.9}$$

Using the formula (3.4.7) above gives as well the smoothness of $r_N(a_1, a_2)$ and with the identities (consequences of $\partial_{x_j}(a_1 \diamond a_2) = (\partial_{x_j} a_1) \diamond a_2 + a_1 \diamond (\partial_{x_j} a_2)$)

$$\begin{aligned}\partial_{x_j}\big(r_N(a_1, a_2)\big) &= r_N(\partial_{x_j} a_1, a_2) + r_N(a_1, \partial_{x_j} a_2)\\ \partial_{\xi_j}\big(r_N(a_1, a_2)\big) &= r_N(\partial_{\xi_j} a_1, a_2) + r_N(a_1, \partial_{\xi_j} a_2),\end{aligned}$$

it is enough to reapply (3.4.9) to get the result $r_N \in S^{m_1+m_2-N}_{1,0}$. □

We have already seen in Theorem 3.2.9 that the adjoint (in the sense of Definition 3.2.7) of the operator $a(x, D)$ is equal to $a^*(x, D)$, where $a^* = J\bar{a}$ (J is given in Lemma 3.2.8). Lemma 3.4.9 gives the following result.

Theorem 3.4.10. *Let $a \in S^m_{1,0}$. Then $a^* = J\bar{a}$ and the mapping $a \mapsto a^*$ is continuous from $S^m_{1,0}$ into itself. Moreover, for all integers N, we have*

$$a^* = \sum_{|\alpha|<N} \frac{1}{\alpha!} D^\alpha_\xi \partial^\alpha_x \bar{a} + r_N(a), \quad r_N(a) \in S^{m-N}_{1,0}.$$

A consequence of the above results is the following.

Corollary 3.4.11. *Let $a_j \in S^{m_j}_{1,0}, j = 1, 2$. Then we have*

$$a_1 \diamond a_2 \equiv a_1 a_2 \mod S^{m_1+m_2-1}_{1,0}, \tag{3.4.10}$$

$$a_1 \diamond a_2 - a_2 \diamond a_1 \equiv \frac{1}{2i\pi}\{a_1, a_2\} \mod S^{m_1+m_2-2}_{1,0}, \tag{3.4.11}$$

where the **Poisson bracket** $$\{a_1, a_2\} = \sum_{1\le j\le n} \frac{\partial a_1}{\partial \xi_j}\frac{\partial a_2}{\partial x_j} - \frac{\partial a_1}{\partial x_j}\frac{\partial a_2}{\partial \xi_j}. \tag{3.4.12}$$

For $a \in S^m_{1,0}$, $$a^* \equiv \bar{a} \mod S^{m-1}_{1,0}. \tag{3.4.13}$$

Theorem 3.4.12. *Let a be a symbol in $S^m_{1,0}$ such that $\inf_{(x,\xi)\in\mathbb{R}^{2n}} |a(x,\xi)|\langle\xi\rangle^{-m} > 0$. Then there exists $b \in S^{-m}_{1,0}$ such that*

$$\begin{array}{rcl} b(x,D)a(x,D) & = & \mathrm{Id} + l(x,D), \\ a(x,D)b(x,D) & = & \mathrm{Id} + r(x,D), \end{array} \quad r, l \in S^{-\infty}_{1,0} = \cap_\nu S^\nu_{1,0}.$$

Proof. We remark first that the smooth function $1/a$ belongs to $S^{-m}_{1,0}$: it follows from the Faà de Bruno formula or more elementarily, from the fact that, for $|\alpha|+|\beta| \ge 1$, $\partial^\alpha_\xi \partial^\beta_x(\frac{1}{a}a) = 0$, entailing with the Leibniz formula

$$a\partial^\alpha_\xi\partial^\beta_x(1/a) = \sum_{\substack{\alpha'+\alpha''=\alpha,\beta'+\beta''=\beta \\ |\alpha'|+|\beta'|<|\alpha|+|\beta|}} \partial^{\alpha'}_\xi\partial^{\beta'}_x(1/a)\partial^{\alpha''}_\xi\partial^{\beta''}_x(a)c(\alpha',\beta'),$$

with constants $c(\alpha',\beta')$. Arguing by induction on $|\alpha|+|\beta|$, we get

$$|a\partial^\alpha_\xi\partial^\beta_x(1/a)| \lesssim \sum_{\alpha'+\alpha''=\alpha} \langle\xi\rangle^{-m-|\alpha'|}\langle\xi\rangle^{m-|\alpha''|} \lesssim \langle\xi\rangle^{-|\alpha|}$$

and from $|a| \gtrsim \langle\xi\rangle^m$, we get $1/a \in S^{-m}_{1,0}$. Now, we can compute, using Theorem 3.4.8,

$$\frac{1}{a} \diamond a = 1 + l_1, \quad l_1 \in S^{-1}_{1,0}.$$

Inductively, we can assume that there exist $(b_0,\cdots,b_N)$ with $b_j \in S^{-m-j}$ such that

$$(b_0+\cdots+b_N)\diamond a = 1 + l_{N+1}, \quad l_{N+1} \in S^{-N-1}_{1,0}. \tag{3.4.14}$$

We can now take $b_{N+1} = -l_{N+1}/a$ which belongs to S^{-m-N-1} and this gives

$$(b_0+\cdots+b_N+b_{N+1})\diamond a = 1 + l_{N+1} - l_{N+1} + l_{N+2}, \quad l_{N+2} \in S^{-N-2}_{1,0}.$$

Lemma 3.4.13. *Let $\mu \in \mathbb{R}$ and $(c_j)_{j\in\mathbb{N}}$ be a sequence of symbols such that $c_j \in S^{\mu-j}_{1,0}$. Then there exists $c \in S^\mu_{1,0}$ such that*

$$c \sim \sum_j c_j, \qquad i.e. \quad \forall N \in \mathbb{N}, \quad c - \sum_{0\le j<N} c_j \in S^{\mu-N}_{1,0}.$$

Proof. The proof is based on a Borel-type argument similar to the one used to construct a C^∞ function with an arbitrary Taylor expansion. Let $\omega \in C^\infty_b(\mathbb{R}^n)$ such that $\omega(\xi) = 0$ for $|\xi| \le 1$ and $\omega(\xi) = 1$ for $|\xi| \ge 2$. Let $(\lambda_j)_{j\in\mathbb{N}}$ be a sequence of numbers ≥ 1. We want to define

$$c(x,\xi) = \sum_{j\ge 0} c_j(x,\xi)\omega(\xi\lambda_j^{-1}), \tag{3.4.15}$$

and we shall show that a suitable choice of λ_j will provide the answer. We note that, since $\lambda_j \ge 1$, the functions $\xi \mapsto \omega(\xi\lambda_j^{-1})$ make a bounded set in the Fréchet space $S^0_{1,0}$. Multiplying the c_j by $\langle\xi\rangle^{-\mu}$, we may assume that $\mu = 0$. We have then, using the notation (3.4.2) (in which we drop the second index),

$$|c_j(x,\xi)|\omega(\xi\lambda_j^{-1}) \le \gamma_0(c_j)\langle\xi\rangle^{-j}\mathbf{1}_{|\xi|\ge\lambda_j} \le \gamma_0(c_j)\lambda_j^{-j/2}\langle\xi\rangle^{-j/2},$$

so that,

$$\forall j \geq 1,\ \lambda_j \geq 2^2 \gamma_0(c_j)^{\frac{2}{j}} = \mu_j^{(0)} \Longrightarrow \forall j \geq 1,\ |c_j(x,\xi)| \omega(\xi \lambda_j^{-1}) \leq 2^{-j} \langle \xi \rangle^{-j/2},$$

showing that the function c can be defined as above in (3.4.15) and is a continuous bounded function. Let $1 \leq k \in \mathbb{N}$ be given. Calculating (with $\omega_j(\xi) = \omega(\xi \lambda_j^{-1})$) the derivatives $\partial_\xi^\alpha \partial_x^\beta (c_j \omega_j)$ for $|\alpha| + |\beta| = k$, we get

$$|\partial_\xi^\alpha \partial_x^\beta (c_j \omega_j)| \leq \gamma_k(c_j \omega_j) \langle \xi \rangle^{-j-|\alpha|} \mathbf{1}_{|\xi| \geq \lambda_j} \leq \tilde{\gamma}_k(c_j) \lambda_j^{-j/2} \langle \xi \rangle^{-|\alpha| - \frac{j}{2}},$$

so that

$$\forall j \geq k,\ \lambda_j \geq 2^2 \big(\tilde{\gamma}_k(c_j)\big)^{\frac{2}{j}} = \mu_j^{(k)} \Longrightarrow \forall j \geq k,\ |\partial_\xi^\alpha \partial_x^\beta (c_j \omega_j)| \leq 2^{-j} \langle \xi \rangle^{-|\alpha| - \frac{j}{2}}, \quad (3.4.16)$$

showing that the function c can be defined as above in (3.4.15) and is a C^k function such that

$$|(\partial_\xi^\alpha \partial_x^\beta c)(x,\xi)| \leq \sum_{0 \leq j < k} \tilde{\gamma}_k(c_j) \langle \xi \rangle^{-j-|\alpha|} + \sum_{j \geq k} 2^{-j} \langle \xi \rangle^{-|\alpha|} \leq C_k \langle \xi \rangle^{-|\alpha|}.$$

It is possible to fulfill the conditions on the λ_j above for all $k \in \mathbb{N}$: just take

$$\lambda_j \geq \sup_{0 \leq k \leq j} \mu_j^{(k)}.$$

The function c belongs to $S^0_{1,0}$ and

$$r_N = c - \sum_{0 \leq j < N} c_j = \sum_{0 \leq j < N} \underbrace{(\omega_j - 1) c_j}_{\in S^{-\infty}_{1,0}} + \sum_{j \geq N} c_j \omega_j,$$

and for $|\alpha| + |\beta| = k$, using the estimates (3.4.16), we obtain

$$\sum_{j \geq N} |\partial_\xi^\alpha \partial_x^\beta (c_j \omega_j)(x,\xi)| \leq \sum_{N \leq j < \max(2N,k)} \overbrace{|\partial_\xi^\alpha \partial_x^\beta (c_j \omega_j)(x,\xi)|}^{\lesssim \langle \xi \rangle^{-|\alpha| - j} \lesssim \langle \xi \rangle^{-|\alpha| - N}} + \sum_{j \geq \max(2N,k)} \underbrace{|\partial_\xi^\alpha \partial_x^\beta (c_j \omega_j)(x,\xi)|}_{\lesssim 2^{-j} \langle \xi \rangle^{-|\alpha| - \frac{j}{2}} \lesssim 2^{-j} \langle \xi \rangle^{-|\alpha| - N}},$$

proving that $r_N \in S^{-N}_{1,0}$. The proof of the lemma is complete. □

Going back to the proof of the theorem, we can take, using Lemma 3.4.13, $S^{-m}_{1,0} \ni b \sim \sum_{j \geq 0} b_j$, and for all $N \in \mathbb{N}$,

$$b \diamond a \in \sum_{0 \leq j < N} b_j \diamond a + S^{-N-m}_{1,0} \diamond a = 1 + S^{-N}_{1,0},$$

providing the first equality in Theorem 3.4.12. To construct a right approximate inverse, i.e. to obtain the second equality in this theorem with an a priori different b

follows the same lines (or can be seen as a direct consequence of the previous identity by applying it to the adjoint a^*); however we are left with the proof that the right and the left approximate inverse could be taken as the same. We have proven that there exists $b^{(1)}, b^{(2)} \in S^{-m}_{1,0}$ such that

$$b^{(1)} \diamond a \in 1 + S^{-\infty}_{1,0}, \quad a \diamond b^{(2)} \in 1 + S^{-\infty}_{1,0}.$$

Now we calculate, using[6] the theorem 3.2.5, $(b^{(1)} \diamond a) \diamond b^{(2)} = b^{(2)} \mod S^{-\infty}_{1,0}$ which is also $b^{(1)} \diamond (a \diamond b^{(2)}) = b^{(1)} \mod S^{-\infty}_{1,0}$ so that $b^{(1)} - b^{(2)} \in S^{-\infty}_{1,0}$, providing the result and completing the proof of the theorem. □

Remark 3.4.14. The mapping $\mathscr{S}'(\mathbb{R}^{2n}) \ni a \mapsto a(x, D)$ is (obviously) linear and one-to-one: if $a(x, D) = 0$, choosing $v(x) = e^{-\pi|x-x_0|^2}, \hat{u}(\xi) = e^{-\pi|\xi-\xi_0|^2}$, we get that the convolution of the distribution $\tilde{a}(x, \xi) = a(x, \xi)e^{2i\pi x\cdot\xi}$ with the Gaussian function $e^{-\pi(|x|^2+|\xi|^2)}$ is zero, so that, taking the Fourier transform shows that the product of the same Gaussian function with $\widehat{\tilde{a}}$ is zero, implying that $\tilde{a}$ and thus a is zero. It is a consequence of a version of the Schwartz kernel theorem that the same mapping $\mathscr{S}'(\mathbb{R}^{2n}) \ni a \mapsto a(x, D) \in$ continuous linear operators from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$ is indeed onto. However the "onto" part of our statement is highly non trivial and a version of this theorem can be found in the theorem 5.2.1 of [5].

An important consequence of the proof of the previous theorem is the possible microlocalization of this result.

Theorem 3.4.15. *Let χ be a symbol in $S^0_{1,0}$ and let a be a symbol in $S^m_{1,0}$ such that $\inf_{(x,\xi)\in \operatorname{supp}\chi} |a(x, \xi)|\langle\xi\rangle^{-m} > 0$. Let ψ be a symbol in $S^0_{1,0}$ such that $\operatorname{supp}\psi \subset \{\chi = 1\}$. Then there exists $b \in S^{-m}_{1,0}$ such that*

$$b(x, D)a(x, D) = \psi(x, D) + l(x, D), \quad l \in S^{-\infty}_{1,0}.$$

Proof. We consider the symbol $b_0 = \chi/a$, which belongs obviously to $S^{-m}_{1,0}$. We have

$$b_0 \diamond a = \chi + l_1, \quad l_1 \in S^{-1}_{1,0}, \quad \Big(-\frac{\chi l_1}{a} + \frac{\chi}{a}\Big) \diamond a = \chi + l_1(1 - \chi) + l_2,\ l_2 \in S^{-2}_{1,0}.$$

Inductively, we may assume that there exists $(b_0, \dots, b_N)$ with $b_j \in S^{-m-j}$ such that

$$(b_0 + b_1 + \dots + b_N) \diamond a = \chi + \sum_{1\le j\le N} l_j(1 - \chi) + l_{N+1}, \quad l_{N+1} \in S^{-1-N}_{1,0}.$$

Choosing $b_{N+1} = -\chi l_{N+1}/a$, we get

$$(b_0 + b_1 + \dots + b_N + b_{N+1}) \diamond a = \chi + \sum_{1\le j\le N+1} l_j(1 - \chi) + l_{N+2}, \quad l_{N+2} \in S^{-2-N}_{1,0}.$$

[6] A consequence of Theorem 3.2.5 is the associativity of the "law" $\diamond$ since

$$\operatorname{Op}(a \diamond (b \diamond c)) = \operatorname{Op}(a)\big(\operatorname{Op}(b)\operatorname{Op}(c)\big) = \big(\operatorname{Op}(a)\operatorname{Op}(b)\big)\operatorname{Op}(c) = \operatorname{Op}((a \diamond b) \diamond c)$$

so that the injectivity property of Remark 3.4.14 gives the answer.

Taking now a symbol $\psi \in S^0_{1,0}$ such that $\operatorname{supp}\psi \subset \chi^{-1}(\{1\})$, we obtain for all $N \in \mathbb{N}$, the existence of symbols $b_0, \cdots, b_N$ with $b_j \in S^{-m-j}$ such that

$$\psi \diamond (b_0 + b_1 + \cdots + b_N) \diamond a = \psi \diamond \chi + \psi \diamond \sum_{1 \le j \le N} l_j(1-\chi) + \psi \diamond l_{N+1} \qquad (l_{N+1} \in S^{-1-N}_{1,0})$$
$$= \psi + r_{N+1}, \qquad r_{N+1} \in S^{-1-N}_{1,0}.$$

Using now Lemma 3.4.13, we find a symbol $b \in S^{-m}_{1,0}$ such that, for all $N \in \mathbb{N}$, $\psi \diamond b \diamond a \in \psi + S^{-1-N}_{1,0}$, i.e. we find $\tilde b \in S^{-m}_{1,0}$ such that $\tilde b \diamond a \equiv \psi \pmod{S^{-\infty}_{1,0}}$. □

3.5 Gårding's inequality

We end this introduction with the so-called *Sharp Gårding inequality*, a result proven in 1966 by L. Hörmander [3] and extended to systems the same year by P. Lax and L. Nirenberg [9].

Theorem 3.5.1. *Let a be a nonnegative symbol in $S^m_{1,0}$. Then there exists a constant C such that, for all $u \in \mathscr{S}(\mathbb{R}^n)$,*

$$\operatorname{Re}\langle a(x,D)u, u\rangle + C\|u\|^2_{H^{\frac{m-1}{2}}(\mathbb{R}^n)} \ge 0. \tag{3.5.1}$$

Proof. First reductions. We may assume that $m = 1$: in fact, the statement for $m = 1$ implies the result by considering, for a nonnegative $a \in S^m_{1,0}$, the operator $\langle D\rangle^{\frac{1-m}{2}} a(x,D)\langle D\rangle^{\frac{1-m}{2}}$ which, according to Theorem 3.4.8 has a symbol in $S^1_{1,0}$, which belongs to $\langle\xi\rangle^{1-m} a(x,\xi) + S^0_{1,0}$. Applying the result for $m = 1$, and the L^2-boundedeness of operators with symbols in $S^0_{1,0}$, we get for all $u \in \mathscr{S}(\mathbb{R}^n)$,

$$\operatorname{Re}\langle \langle D\rangle^{\frac{1-m}{2}} a(x,D)\langle D\rangle^{\frac{1-m}{2}} u, u\rangle + C\|u\|^2_{L^2(\mathbb{R}^n)} \ge 0,$$

which gives the sought result when applied to $u = \langle D\rangle^{\frac{m-1}{2}} v$. We may also replace $a(x,D)$ by a^w, where a^w is the operator with Weyl symbol a . In fact, according to Lemma 3.2.8, $J^{1/2}a - a \in S^0_{1,0}$ and $\operatorname{Op}(S^0_{1,0})$ is L^2-bounded.
Main step: a result with a small parameter. We consider a nonnegative $a \in S^1_{1,0}$ and

$$\varphi \in C^\infty_c((0,+\infty);\mathbb{R}_+) \text{ such that } \int_0^{+\infty} \varphi(h)\frac{dh}{h} = 1. \tag{3.5.2}$$

This implies

$$a(x,\xi) = \int_0^{+\infty} \underbrace{\varphi(\langle\xi\rangle h) a(x,\xi)}_{=a_h(x,\xi)} \frac{dh}{h}. \tag{3.5.3}$$

We have, with $\Gamma_h(x,\xi) = 2^n \exp -2\pi(h^{-1}|x|^2 + h|\xi|^2)$ and $X = (x,\xi)$,

$$(a_h * \Gamma_h)(X) = a_h(X) + \int_0^1 (1-\theta) a_h''(X+\theta Y)Y^2\Gamma_h(Y)dYd\theta$$
$$= a_h(X) + r_h(X). \tag{3.5.4}$$

The main step of the proof is that $(a_h * \Gamma_h)^w \ge 0$, a result following from the next calculation (for $u \in \mathscr{S}(\mathbb{R}^n)$), due to Definition 3.3.1. We have, with $\Omega_{u,u}$ defined in (3.3.4),

$$\begin{aligned}\langle (a_h * \Gamma_h)^w u, u\rangle &= \iint (a_h * \Gamma_h)(x,\xi)\left(\int e^{-2i\pi z\cdot\xi} u(x+\frac{z}{2})\bar u(x-\frac{z}{2})dz\right)dxd\xi\\ &= \iint a(y,\eta)(\Omega_{u,u}(1/2) * \Gamma_h)(y,\eta)dyd\eta,\end{aligned}$$

and since $(\Omega_{u,u}(1/2) * \Gamma_h)(x,\xi) =$

$$\begin{aligned}&\iiint e^{-2i\pi z\cdot(\xi-\eta)} u(x-y+\frac{z}{2})\bar u(x-y-\frac{z}{2})2^n \exp -2\pi(h^{-1}|y|^2+h|\eta|^2)dzdyd\eta\\ &= \iint e^{-2i\pi z\cdot\xi} u(x-y+\frac{z}{2})\bar u(x-y-\frac{z}{2})2^{n/2} e^{-2\pi h^{-1}|y|^2} h^{-n/2} e^{-\frac{\pi}{2h}|z|^2} dzdy\\ &= \iint u(x-y_1)\bar u(x-y_2) e^{-2i\pi(y_2-y_1)\cdot\xi} 2^{n/2} h^{-n/2} e^{-\frac{\pi}{2h}|y_1+y_2|^2} e^{-\frac{\pi}{2h}|y_1-y_2|^2} dy_1dy_2\\ &= 2^{n/2}h^{-n/2}\left|\int u(x-y_1)e^{2i\pi y_1\cdot\xi}e^{-\pi h^{-1}|y_1|^2}dy_1\right|^2 \ge 0,\end{aligned}$$

we get indeed $(a_h * \Gamma_h)^w \ge 0$. From (3.5.3) and (3.5.4), we get

$$a^w = \int_0^{+\infty} a_h^w h^{-1}dh = \int_0^{+\infty} (a_h * \Gamma_h)^w h^{-1}dh - \int_0^{+\infty} r_h^w h^{-1}dh \ge - \int_0^{+\infty} r_h^w h^{-1}dh.$$

Last step: $\int_0^{+\infty} r_h^w h^{-1}dh$ is L^2-bounded. This is a technical point, where the main difficulty is coming from the integration in h. We have from (3.5.4) and the fact that Γ_h is an even function,

$$r_h(X) = \frac{1}{8\pi}\operatorname{trace}_h a_h''(X) + \frac{1}{3!}\iint_0^1 (1-\theta)^3 a_h^{(4)}(X+\theta Y)Y^4\Gamma_h(Y)dYd\theta,$$

with $\operatorname{trace}_h a_h''(X) = h \operatorname{trace}\partial_x^2 a_h + h^{-1}\operatorname{trace}\partial_\xi^2 a_h$. Since $\varphi \in C_c^\infty((0,+\infty))$, we have

$$\int_0^{+\infty} h \operatorname{trace}\partial_x^2 a_h h^{-1}dh = \operatorname{trace}\partial_x^2 a(x,\xi)\int_0^{+\infty}\varphi(\langle\xi\rangle h)dh = c\operatorname{trace}\partial_x^2 a\langle\xi\rangle^{-1},$$

with $c = \int_0^{+\infty}\varphi(t)dt$. The symbol $c\operatorname{trace}\partial_x^2 a\langle\xi\rangle^{-1}$ belongs to $S^0_{1,0}$ as well as the other term $\int_0^{+\infty} h^{-1}\operatorname{trace}\partial_\xi^2 a_h(x,\xi)h^{-1}dh$: we have

$$\begin{aligned}(\partial_\xi a_h)(x,\xi) &= (\partial_\xi a)(x,\xi)\varphi(h\langle\xi\rangle) + a(x,\xi)\varphi'(h\langle\xi\rangle)h\langle\xi\rangle^{-1}\xi\\ (\partial_\xi^2 a_h)(x,\xi) &= (\partial_\xi^2 a)(x,\xi)\varphi(h\langle\xi\rangle) + 2\partial_\xi a(x,\xi)\varphi'(h\langle\xi\rangle)h\\ &\quad + a(x,\xi)\varphi''(h\langle\xi\rangle)h^2\langle\xi\rangle^{-2}\xi^2 + a(x,\xi)\varphi'(h\langle\xi\rangle)h\partial_\xi\big(\xi\langle\xi\rangle^{-1}\big),\end{aligned}$$

and checking for instance the term $\int_0^{+\infty} h^{-1}(\partial_\xi^2 a)(x,\xi)\varphi(h\langle\xi\rangle)\frac{dh}{h}$, we see that it is equal to

$$(\partial_\xi^2 a)(x,\xi)\int_0^{+\infty} h^{-1}\varphi(h\langle\xi\rangle)\frac{dh}{h} = (\partial_\xi^2 a)(x,\xi)\langle\xi\rangle\int_0^{+\infty} h^{-1}\varphi(h)\frac{dh}{h}$$
$$= c_1(\partial_\xi^2 a)(x,\xi)\langle\xi\rangle \in S^0_{1,0},$$

whereas the other terms are analogous. We are finally left with the term

$$\rho(X) = \frac{1}{3!}\iiint_0^1 (1-\theta)^3 a_h^{(4)}(X+\theta Y)Y^4\Gamma_h(Y)dYh^{-1}dhd\theta,$$

and we note that on the integrand of (3.5.3), the product $h\langle\xi\rangle$ is bounded above and below by fixed constants and that integral can in fact be written as

$$a(x,\xi) = \int_{\kappa_0\langle\xi\rangle^{-1}}^{\kappa_1\langle\xi\rangle^{-1}} \varphi(\langle\xi\rangle h)a(x,\xi)dh/h$$

with $0<\kappa_0 = \min\operatorname{supp}\varphi < \kappa_1 = \max\operatorname{supp}\varphi$. Consequently the symbol a_h satisfies the following estimates:

$$|\partial_\xi^\alpha\partial_x^\beta a_h| \le C_{\alpha\beta}h^{-1+|\alpha|}$$

where the $C_{\alpha\beta}$ are some semi-norms of a (and thus independent of h). As a result, the above estimates can be written in a more concise and convenient way, using the multilinear forms defined by the derivatives. We have, with $T=(t,\tau)\in\mathbb{R}^n\times\mathbb{R}^n$,

$$|a_h^{(l)}(X)T^l| \le C_l h^{-1}g_h(T)^{l/2}, \quad \text{with } g_h(t,\tau) = |t|^2 + h^2|\tau|^2.$$

We calculate

$$\rho^{(k)}(X)T^k = \frac{1}{3!}\iiint_0^1 (1-\theta)^3 a_h^{(4+k)}(X+\theta Y)Y^4T^k\Gamma_h(Y)dYh^{-1}dhd\theta,$$

which satisfies with $\omega_h(t,\tau) = h^{-1}g_h(t,\tau)$,

$$\begin{aligned}&|\rho^{(k)}(X)T^k|\\ &\quad\le \frac{C_{4+k}}{4!}\iint \mathbf{1}\{h\le\kappa_1\}h^{-1}g_h(T)^{k/2}\underbrace{g_h(Y)^2}_{=h^2\omega_h(Y)^2} 2^n e^{-2\pi\omega_h(Y)}dYh^{-1}dh\\ &\quad\le \frac{C_{4+k}}{4!}g_h(T)^{k/2}\iint \omega_h(Y)^2\mathbf{1}\{h\le\kappa_1\}2^ne^{-2\pi\omega_h(Y)}dYdh \le \tilde C_k(|t|+|\tau|)^k\end{aligned}$$

and this proves that the function ρ belongs to $C_b^\infty(\mathbb{R}^{2n})$, as well as $J^{1/2}\rho$ (Lemma 3.2.8) and thus $\rho^w = (J^{1/2}\rho)(x,D)$ is bounded on L^2 (Theorem 3.2.4). The proof is complete. □

Remark 3.5.2. Theorem 3.5.1 remains valid for systems, even in infinite dimension. For definiteness, let us assume simply that $a(x,\xi)$ is a $N\times N$ Hermitian non-negative matrix of symbols in $S^1_{1,0}$. Then for all $u\in\mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$, the inequality (3.5.1) holds. The vector space $\mathbb{C}^N$ can be replaced in the above statement by an infinite-dimensional complex Hilbert space H with a valued in $\mathscr{L}(H)$ and the proof above requires essentially no change.

3.6 The semi-classical calculus

A semiclassical symbol of order m is defined as a family of smooth functions $a(\cdot,\cdot,h)$ defined on the phase space $\mathbb{R}^{2n}$, depending on a parameter $h \in (0,1]$, such that, for all multi-indices α, β

$$\sup_{(x,\xi,h)\in\mathbb{R}^n\times\mathbb{R}^n\times(0,1]} |(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi,h)| h^{m-|\alpha|} < +\infty. \tag{3.6.1}$$

The set of semi-classical symbols of order m will be denoted by S^m_{scl}. A typical example of such a symbol of order 0 is a function $a_1(x, h\xi)$ where a_1 belongs to $C_b^\infty(\mathbb{R}^{2n})$: we have indeed $\partial_\xi^\alpha \partial_x^\beta \big(a_1(x,h\xi)\big) = (\partial_\xi^\alpha \partial_x^\beta a_1)(x,h\xi)h^{|\alpha|}$. It turns out that this version of the semi-classical calculus is certainly the easiest to understand and that Theorem 3.2.4 is implying the main continuity result for these symbols. The reader has also to keep in mind that we are not dealing here with a single function defined on the phase space, but with a family of symbols depending on a (small) parameter h, a way to express that the constants occurring in (3.6.1) are "independent of h". We shall review the results of the section on the $S^m_{1,0}$ class of symbols and show how they can be transferred to the semi-classical framework, *mutatis mutandis* and almost without any new argument. To understand the correspondence between symbols in $S^m_{1,0}$ and semi-classical symbols, it is essentially enough to think of the $S_{1,0}$ calculus as a semi-classical calculus with small parameter $\langle\xi\rangle^{-1}$.

We can define, for $a \in S^m_{scl}$, $k \in \mathbb{N}$,

$$\gamma_{k,m}(a) = \sup_{(x,\xi,h)\in\mathbb{R}^{2n}\times(0,1],|\alpha|+|\beta|\le k} |(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi,h)| h^{m-|\alpha|}. \tag{3.6.2}$$

Theorem 3.6.1. *Let $a \in S^m_{scl}$. Then the operator $a(x,D,h)h^m$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself with constants independent of $h \in (0,1]$.*

Proof. We have $a(x,D,h) = \operatorname{Op}(a(x,\xi,h))$. The set $\big\{a(x,\xi,h)h^m\big\}_{h\in(0,1]}$ is bounded in $C_b^\infty(\mathbb{R}^{2n})$, so that we can use Theorem 3.2.2 to get the result. □

Theorem 3.6.2. *Let $a \in S^m_{scl}$. Then the operator $a(x,D,h)h^m$ is bounded on $L^2(\mathbb{R}^n)$ with a norm bounded above independently of $h \in (0,1]$.*

Proof. The set $\big\{a(x,\xi,h)h^m\big\}_{h\in(0,1]}$ being bounded in $C_b^\infty(\mathbb{R}^{2n})$, it follows from Theorem 3.2.4. □

Theorem 3.6.3. *Let m_1, m_2 be real numbers and $a_1 \in S^{m_1}_{scl}, a_2 \in S^{m_2}_{scl}$. Then the composition $a_1(x,D,h)a_2(x,D,h)$ makes sense as a continuous operator from $\mathscr{S}(\mathbb{R}^n)$ into itself, as well as a bounded operator on $L^2(\mathbb{R}^n)$ and*

$$a_1(x,D,h)a_2(x,D,h) = (a_1 \diamond a_2)(x,D,h)$$

where $a_1 \diamond a_2$ belongs to $S^{m_1+m_2}_{scl}$ and is given by the formula

$$(a_1 \diamond a_2)(x,\xi,h) = (\exp 2i\pi D_y\cdot D_\eta)\Big(a_1(x,\xi+\eta,h)a_2(y+x,\xi,h)\Big)_{|y=0,\eta=0}. \tag{3.6.3}$$

Proof. This is a direct consequence of Theorem 3.2.5 since

$$\cup_{j=1,2}\{h^{m_j}a_j(x,\xi,h)\}_{h\in(0,1]} \quad \text{is bounded in } C_b^\infty(\mathbb{R}^{2n}).$$

□

Theorem 3.6.4. *Let m_1, m_2 be real numbers and $a_1 \in S_{scl}^{m_1}, a_2 \in S_{scl}^{m_2}$. Then $a_1(x,D,h)a_2(x,D,h) = (a_1 \diamond a_2)(x,D,h)$, the symbol $a_1 \diamond a_2$ belongs to $S_{scl}^{m_1+m_2}$ and we have the asymptotic expansion, for all $N \in \mathbb{N}$,*

$$a_1 \diamond a_2 = \sum_{|\alpha|<N} \frac{1}{\alpha!} D_\xi^\alpha a_1 \partial_x^\alpha a_2 + r_N(a_1,a_2), \tag{3.6.4}$$

with $r_N(a_1,a_2) \in S_{scl}^{m_1+m_2-N}$. Note that $D_\xi^\alpha a_1 \partial_x^\alpha a_2$ belongs to $S_{scl}^{m_1+m_2-|\alpha|}$.

Proof. Since $h^{m_j}a_j(x,\xi,h), j=1,2$, belongs to S_{scl}^0, we may assume that $m_1 = m_2 = 0$. We can use the formula (3.4.3) and apply the formula (3.4.8) to get the desired formula with

$$r_N(a_1,a_2)(x,\xi,h) = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} e^{2i\pi\theta D_z\cdot D_\zeta} (2i\pi D_z\cdot D_\zeta)^N \big(a_1(x,\zeta,h)a_2(z,\xi,h)\big) d\theta_{|z=x,\zeta=\xi}. \tag{3.6.5}$$

The function $(z,\zeta) \mapsto b_{x,\xi,h}(z,\zeta) = (2i\pi D_z\cdot D_\zeta)^N a_1(x,\zeta,h)a_2(z,\xi,h)$ belongs to $S_{scl}^{-N}(\mathbb{R}^{2n}_{z,\zeta})$ uniformly with respect to the parameters $(x,\xi)\in\mathbb{R}^{2n}$: it satisfies, using the notation (3.6.2), for $\max(|\alpha|,|\beta|)\le k$,

$$|\partial_\zeta^\alpha \partial_z^\beta b_{x,\xi,h}(z,\zeta)| \le \gamma_{k,m_1}(a_1)\gamma_{k,m_2}(a_2)h^{N+|\alpha|}.$$

Applying Lemma 3.2.8, we obtain that the function

$$\rho_{x,\xi,h}(z,\zeta) = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!}(J^\theta b_{x,\xi,h})(z,\zeta)d\theta$$

belongs to $S_{scl}^{-N}(\mathbb{R}^{2n}_{z,\zeta})$ uniformly with respect to x,ξ,h, so that in particular

$$\sup_{(x,\xi,z,\zeta)\in\mathbb{R}^{4n}, h\in(0,1]} |\rho_{x,\xi,h}(z,\zeta)h^{-N}| = C_0 < +\infty.$$

Since $r_N(a_1,a_2)(x,\xi) = \rho_{x,\xi,h}(x,\xi)$, we obtain

$$|r_N(a_1,a_2)(x,\xi)| \le C_0 h^N. \tag{3.6.6}$$

Using the formula (3.6.5) above gives as well the smoothness of $r_N(a_1,a_2)$ and with the identities (consequences of $\partial_{x_j}(a_1\diamond a_2) = (\partial_{x_j}a_1)\diamond a_2 + a_1 \diamond(\partial_{x_j}a_2)$)

$$\partial_{x_j}\big(r_N(a_1,a_2)\big) = r_N(\partial_{x_j}a_1,a_2) + r_N(a_1,\partial_{x_j}a_2)$$
$$\partial_{\xi_j}\big(r_N(a_1,a_2)\big) = r_N(\partial_{\xi_j}a_1,a_2) + r_N(a_1,\partial_{\xi_j}a_2),$$

it is enough to reapply (3.6.6) to get the result $r_N \in S_{scl}^{-N}$. □

Lemma 3.2.8 and Taylor's expansion (3.6.5) give the following result.

Theorem 3.6.5. *Let $a \in S^m_{scl}$. Then $a^* = J\bar{a}$ and the mapping $a \mapsto a^*$ is continuous from S^m_{scl} into itself. Moreover, for all integers N, we have*

$$a^* = \sum_{|\alpha|<N} \frac{1}{\alpha!} D_\xi^\alpha \partial_x^\alpha \bar{a} + r_N(a), \quad r_N(a) \in S^{m-N}_{scl}.$$

Corollary 3.6.6. *Let $a_j \in S^{m_j}_{scl}, j = 1,2$. Then we have*

$$a_1 \diamond a_2 \equiv a_1 a_2 \mod S^{m_1+m_2-1}_{scl}, \tag{3.6.7}$$

$$a_1 \diamond a_2 - a_2 \diamond a_1 \equiv \frac{1}{2i\pi}\{a_1, a_2\} \mod S^{m_1+m_2-2}_{scl}, \tag{3.6.8}$$

$$\text{For } a \in S^m_{scl}, \quad a^* \equiv \bar{a} \mod S^{m-1}_{scl}. \tag{3.6.9}$$

Lemma 3.6.7. *Let $\mu \in \mathbb{R}$ and $(c_j)_{j\in\mathbb{N}}$ be a sequence of symbols such that $c_j \in S^{\mu-j}_{scl}$. Then there exists $c \in S^\mu_{scl}$ such that*

$$c \sim \sum_j c_j, \qquad i.e. \quad \forall N \in \mathbb{N}, \quad c - \sum_{0\le j<N} c_j \in S^{\mu-N}_{scl}.$$

Proof. The proof is almost identical to the proof of Lemma 3.4.13.

Let $\omega \in C_b^\infty(\mathbb{R};\mathbb{R}_+)$ such that $\omega(t) = 0$ for $t \le 1$ and $\omega(t) = 1$ for $t \ge 2$. Let $(\lambda_j)_{j\in\mathbb{N}}$ be a sequence of numbers ≥ 1. We want to define

$$c(x,\xi,h) = \sum_{j\ge 0} c_j(x,\xi,h)\omega(h^{-1}\lambda_j^{-1}), \tag{3.6.10}$$

and we shall show that a suitable choice of λ_j will provide the answer. Multiplying the c_j by h^μ, we may assume that $\mu = 0$. We have then

$$|c_j(x,\xi,h)|\omega(h^{-1}\lambda_j^{-1}) \le \gamma_0(c_j) h^j \mathbf{1}_{1\ge h\lambda_j} \le \gamma_0(c_j)\lambda_j^{-j},$$

so that,

$$\forall j \ge 1,\ \lambda_j \ge 2\gamma_0(c_j)^{\frac{1}{j}} = \mu_j^{(0)} \Longrightarrow \forall j \ge 1,\ |c_j(x,\xi,h)|\omega(h^{-1}\lambda_j^{-1}) \le 2^{-j},$$

showing that the function c can be defined as above in (3.6.10) and is a continuous bounded function. Let $1 \le k \in \mathbb{N}$ be given. Calculating (with $\omega_j = \omega(h^{-1}\lambda_j^{-1})$) the derivatives $\omega_j \partial_\xi^\alpha \partial_x^\beta(c_j)$ for $|\alpha| + |\beta| = k$, we get

$$\omega_j |\partial_\xi^\alpha \partial_x^\beta(c_j)| \le \gamma_k(c_j) h^{j+|\alpha|}\mathbf{1}_{1\ge h\lambda_j} \le \gamma_k(c_j)\lambda_j^{-j/2} h^{|\alpha|+\frac{j}{2}},$$

so that

$$\forall j \ge k,\ \lambda_j \ge 2^2\big(\gamma_k(c_j)\big)^{\frac{2}{j}} = \mu_j^{(k)} \Longrightarrow \forall j \ge k,\ |\partial_\xi^\alpha \partial_x^\beta(c_j\omega_j)| \le 2^{-j} h^{|\alpha|+\frac{j}{2}}, \tag{3.6.11}$$

showing that the function c can be defined as above in (3.6.10) and is a C^k function such that

$$|(\partial_\xi^\alpha \partial_x^\beta c)(x,\xi,h)| \le \sum_{0\le j<k} \gamma_k(c_j) h^{j+|\alpha|} + \sum_{j\ge k} 2^{-j} h^{|\alpha|} \le C_k h^{|\alpha|}.$$

It is possible to fulfill the conditions on the λ_j above for all $k \in \mathbb{N}$: just take $\lambda_j \ge \sup_{0\le k\le j} \mu_j^{(k)}$. The function c belongs to S^0_{scl} and, with $S^{-\infty}_{scl} = \cap_{m\in\mathbb{R}} S^m_{scl}$,

$$r_N = c - \sum_{0\le j<N} c_j = \sum_{0\le j<N} \underbrace{(\omega_j - 1)c_j}_{\in S^{-\infty}_{scl}} + \sum_{j\ge N} c_j\omega_j,$$

and for $|\alpha| + |\beta| = k$, using the estimates (3.6.11), we obtain

$$\sum_{j\ge N} |\partial_\xi^\alpha \partial_x^\beta (c_j\omega_j)(x,\xi,h)| \le \sum_{N\le j<\max(2N,k)} \overbrace{|\partial_\xi^\alpha \partial_x^\beta (c_j\omega_j)(x,\xi,h)|}^{\lesssim h^{|\alpha|+j} \lesssim h^{|\alpha|+N}}$$
$$+ \sum_{j\ge \max(2N,k)} \underbrace{|\partial_\xi^\alpha \partial_x^\beta (c_j\omega_j)(x,\xi,h)|}_{\lesssim 2^{-j}h^{|\alpha|+\frac{j}{2}} \lesssim 2^{-j}h^{|\alpha|+N}},$$

proving that $r_N \in S^{-N}_{scl}$. The proof of the lemma is complete. □

Remark 3.6.8. These asymptotic results (as well as the example $a_1(x, h\xi)$ with $a_1 \in C_b^\infty(\mathbb{R}^{2n})$ see page 83) led many authors to set a slightly different framework for the semiclassical calculus; instead of dealing with a family of symbols $a(x,\xi,h)$ satisfying the estimates (3.6.1), one deals with a function $a \in C_b^\infty(\mathbb{R}^{2n})$ and consider the operator $a(x, hD_x)$ or the operator $a(x, h\xi)^w$; another way to express this is to modify the quantization formula and to define for instance

$$(a^{w_h}u)(x) = \iint e^{\frac{2i\pi}{h}\langle x-y,\xi\rangle} a(\frac{x+y}{2},\xi)u(y)dyd\xi h^{-n}, \text{ i.e. } a^{w_h} = a(x,h\xi)^w. \tag{3.6.12}$$

Then, using Lemma 3.6.7, given a sequence $(a_j)_{j\ge0}$ in $C_b^\infty(\mathbb{R}^{2n})$, it is possible to consider $a(x,\xi,h) \in S^0_{scl}$ with

$$a(x,\xi,h) \sim \sum_{j\ge0} h^j a_j(x,h\xi), \quad \text{i.e.} \quad \forall N,\ a(x,\xi,h) - \sum_{0\le j<N} h^j a_j(x,h\xi) \in S^{-N}_{scl}.$$

The symbol a_0 is the *principal symbol* and

$$a(x,\xi,h)^w \sim \sum_{j\ge0} h^j a_j^{w_h}, \quad \text{i.e.} \quad \forall N,\ a(x,\xi,h)^w - \sum_{0\le j<N} h^j a_j^{w_h} = h^N r_{N,h}^{w_h},$$

where $\{r_{N,h}\}_{0<h\le1}$ is bounded in $C_b^\infty(\mathbb{R}^{2n})$: in fact we have from Theorem 3.6.4,

$$h^N r_{N,h}(x,h\xi) = s_N(x,\xi,h), \quad s_N \in S^{-N}_{scl}, \quad \text{i.e. } r_{N,h}(x,\xi) = h^{-N}s_N(x,h^{-1}\xi,h),$$

and thus

$$|(\partial_\xi^\alpha\partial_x^\beta r_{N,h})(x,\xi) = h^{-N-|\alpha|}(\partial_\xi^\alpha\partial_x^\beta s_N)(x,h^{-1}\xi,h)| \le h^{-N-|\alpha|}\gamma_{\alpha,\beta,N}h^{N+|\alpha|}.$$

If $a, b \in S^0_{scl}$ and $a \sim \sum_{j\ge0} h^j a_j(x,h\xi)$, $b \sim \sum_{j\ge0} h^j b_j(x,h\xi)$ as above, then one can prove, using Corollary 3.6.6 and Lemma 3.2.8

$$a^w b^w \equiv (a_0 b_0)^{w_h} \mod h(S^0_{scl})^w, \tag{3.6.13}$$

$$[a^w, b^w] \equiv \frac{h}{2i\pi}\{a_0, b_0\}^{w_h} \mod h^2(S^0_{scl})^w. \tag{3.6.14}$$

There are many variations on this theme, and in particular, one can replace the space $C_b^\infty(\mathbb{R}^{2n})$ by a more general one, involving some weight functions, for instance with polynomial growth at infinity. At this point, we are leaving an introduction to the pseudo-differential calculus and can use our more general approach of Chapter 2, involving metrics on the phase space, which incorporate all these variations. Expecting these generalizations, we shall not use the w_h quantization in this book, except for the present remark.

Theorem 3.6.9. *Let a be a symbol in S^0_{scl} such that*

$$\inf_{(x,\xi)\in\mathbb{R}^{2n}, h\in(0,1]} |a(x,\xi,h)| > 0.$$

Then there exists $b \in S^0_{scl}$ such that

$$\begin{aligned} b(x,D,h)a(x,D,h) &= \operatorname{Id} + l(x,D,h), \\ a(x,D,h)b(x,D,h) &= \operatorname{Id} + r(x,D,h), \end{aligned} \qquad r, l \in S^{-\infty}_{scl} = \cap_\nu S^\nu_{scl}.$$

Proof. The only change to perform in the proof of Theorem 3.4.12 to get this result is to replace everywhere $S_{1,0}$ by S_{scl}. □

Theorem 3.6.10. *Let χ be a symbol in S^0_{scl} and let a be a symbol in S^0_{scl} such that $\inf_{h\in(0,1],(x,\xi)\in\operatorname{supp}\chi(\cdot,\cdot,h)} |a(x,\xi,h)| > 0$. Let ψ be a symbol in S^0_{scl} such that $\operatorname{supp}\psi(\cdot,\cdot,h) \subset \{(x,\xi), \chi(x,\xi,h) = 1\}$. Then there exists $b \in S^0_{scl}$ such that*

$$b(x,D,h)a(x,D,h) = \psi(x,D,h) + l(x,D,h), \qquad l \in S^{-\infty}_{scl}.$$

Proof. Here also we have only to follow the proof of Theorem 3.4.15 and use Lemma 3.6.7 instead of Lemma 3.4.13 in the course of the proof. □

Theorem 3.6.11. *Let a be a nonnegative symbol in S^0_{scl}. Then there exists a constant C such that, for all $u \in \mathscr{S}(\mathbb{R}^n)$,*

$$\operatorname{Re}\langle a(x,D,h)u, u\rangle + hC\|u\|^2_{L^2(\mathbb{R}^n)} \geq 0. \tag{3.6.15}$$

Equivalently, there exists $C \geq 0$ such that $a^w + Ch \geq 0$.

Proof. The proof of Theorem 3.5.1 is containing a proof of this result: noticing that it is harmless to replace the standard quantization by the Weyl quantization for this result, since $J^{1/2}a - a$ belongs to S^{-1}_{scl} (see the formula (3.4.8) and Lemma 3.4.9), we use the formula (3.5.4) to obtain than $(a * \Gamma_h)^w \geq 0$. The difference $a * \Gamma_h - a$ is $\int_0^1 (1-\theta) \int_{\mathbb{R}^{2n}} a''(X + \theta Y, h) Y^2 \Gamma_h(Y) dY d\theta$, which belongs to S^0_{scl}. □

Chapter 4

Local versions of pseudo-differential operators

4.1 Pseudo-differential operators on an open subset of $\mathbb{R}^n$

Introduction

The main reason for studying the class $S^m_{1,0}$ of pseudo-differential operators as introduced in the second subsection of the section 3.4 is that the parametrix of an elliptic differential operator of order m has a symbol in the class $S^{-m}_{1,0}$. More specifically, we have the following result.

Proposition 4.1.1. *Let m be a nonnegative integer, Ω an open set of $\mathbb{R}^n$ and let $A = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha_x$ be a differential [1] operator with C^∞ coefficients on Ω (i.e. $a_\alpha \in C^\infty(\Omega)$). We assume that A is elliptic, i.e.*

$$\forall (x,\xi) \in \Omega \times (\mathbb{R}^n \backslash \{0\}), \qquad \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \neq 0.$$

Then, if u is a distribution on Ω such that Au belongs to $H^s_{loc}(\Omega)$, we obtain that u belongs to $H^{s+m}_{loc}(\Omega)$, implying that $\operatorname{singsupp} u = \operatorname{singsupp} Au$ *(for the C^∞ singular supports[2]).*

This result will be proven in the next subsection in a far greater generality; first we shall use the notion of wave-front-set which microlocalizes the notion of singular support and also we shall prove this result for a microelliptic pseudo-differential operator. In fact, the proof relies essentially on Theorem 3.4.12 which allows the invertibility of an operator of the same type as A above. Nevertheless, one should note that the function $(x,\xi) \mapsto \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ does not belongs to $S^m_{1,0}$ since in the first place it is not defined on $\mathbb{R}^{2n}$ when $\Omega \neq \mathbb{R}^n$, and even if Ω were equal to $\mathbb{R}^n$, we do not have any control on the growth of the a_α at infinity. Also we see

[1] We use the notation (1.2.8) for the D^α_x.

[2] For $v \in \mathscr{D}'(\Omega)$, $(\operatorname{singsupp} v)^c$ is the union of the open subsets ω of Ω such that $v_{|\omega} \in C^\infty(\omega)$.

that the ellipticity condition concerns only the *principal symbol*, i.e. the function $\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$. To get a good understanding (and a simple proof) of the previous result, we have to introduce the notion of pseudo-differential operator on an open set of $\mathbb{R}^n$, as well as the proper notion of ellipticity. The elliptic regularity theorem will be a simple consequence of the calculus of pseudo-differential operators on an open set of $\mathbb{R}^n$. One of the most important result of this theory is that pseudo-differential operators are geometrical objects that can be defined on a smooth manifold without reference to a coordinate chart; this invariance by change of coordinates has had a tremendous influence on the success of microlocal methods in geometrical problems such as the index theorem.

Definition 4.1.2. Let Ω be an open subset of $\mathbb{R}^n$ and $m \in \mathbb{R}$. $S^m_{\text{loc}}(\Omega\times\mathbb{R}^n)$ is defined as the set of $a \in C^\infty(\Omega \times \mathbb{R}^{2n})$ such that for any compact subset K of Ω, for all multi-indices $\alpha, \beta \in \mathbb{N}^n$, there exists $C_{K\alpha\beta}$ such that, for $(x, \xi) \in K \times \mathbb{R}^n$,

$$|(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi)| \le C_{K\alpha\beta}\langle\xi\rangle^{m-|\alpha|}. \tag{4.1.1}$$

We note in particular that the differential operators of order m with C^∞ coefficients in Ω have a symbol in $S^m_{\text{loc}}(\Omega \times \mathbb{R}^n)$, i.e. can be written as

$$(Au)(x) = \int e^{2i\pi x\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi, \quad \text{for } u \in C^\infty_{\text{c}}(\Omega), \tag{4.1.2}$$

with $a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$, $a_\alpha \in C^\infty(\Omega)$.

Theorem 4.1.3. *Let Ω be an open set of $\mathbb{R}^n$ and let a be a symbol in $S^m_{loc}(\Omega \times \mathbb{R}^n)$. Then the formula (4.1.2) defines a continuous linear operator (denoted also by $a(x, D)$) from $C^\infty_c(\Omega)$ into $C^\infty(\Omega)$, from $\mathscr{E}'(\Omega)$ into $\mathscr{D}'(\Omega)$, and from $H^{s+m}_{comp}(\Omega)$ to $H^s_{loc}(\Omega)$ for all $s \in \mathbb{R}$.*

Proof. To obtain the last result, we note that for $\chi \in C^\infty_{\text{c}}(\Omega)$, the operator

$$\chi(x)a(x, D),$$

has the symbol $\chi(x)a(x, \xi)$ which belongs to $S^m_{1,0}$ and thus, from Theorem 3.4.6, $\chi(x)a(x, D)$ sends continuously $H^{s+m}(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$, which gives that $a(x, D)$ sends continuously $H^{s+m}_{\text{comp}}(\Omega)$ into $H^s_{\text{loc}}(\Omega)$. This implies also that the formula (4.1.2) defines an operator from $\mathscr{S}(\mathbb{R}^n)$ into $C^\infty(\Omega)$. Moreover the formula (4.1.2) defines a mapping from $\mathscr{E}'(\Omega)$ into $\mathscr{D}'(\Omega)$, via the identity[3]

$$\prec a(x, D)u, \varphi \succ_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \prec \hat{u}(\xi), \int \varphi(x)a(x,\xi)e^{2i\pi x\cdot\xi}dx \succ_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} .$$

[3] Using Theorem 3.2.2, we see that for $\varphi \in C^\infty_{\text{c}}(\Omega)$, the function

$$\xi \mapsto V_\varphi(\xi) = \int_{\mathbb{R}^n} a(x,\xi)\varphi(x)e^{2i\pi x\cdot\xi}dx$$

belongs to $\mathscr{S}(\mathbb{R}^n)$: for $\chi \in C^\infty_{\text{c}}(\Omega)$ equal to 1 on the support of φ, we consider the symbol $b(x, \xi) = \chi(\xi)a(\xi, x)\langle x\rangle^{-m}$ which belongs to $C^\infty_b(\mathbb{R}^{2n})$ and we have

$$V_\varphi(\xi) = \langle\xi\rangle^m \int_{\mathbb{R}^n} \chi(x)a(x,\xi)\langle\xi\rangle^{-m}\varphi(x)e^{2i\pi x\cdot\xi}dx = \langle\xi\rangle^m(\text{Op}(b)\hat{\check{\varphi}})(\xi).$$

□

Definition 4.1.4. Let Ω be an open set of $\mathbb{R}^n$ and $m \in \mathbb{R}$. The set of operators $\{a(x,D)\}_{a\in S^m_{\rm loc}(\Omega\times\mathbb{R}^n)}$ as given by the formula (4.1.2) is defined as $\Psi^m(\Omega)$, the set of pseudo-differential operators of order m on Ω.

We have to modify slightly the quantization of our symbols to get an algebra of operators, sending for instance $C^\infty_{\rm c}(\Omega)$ into itself. Let us consider a locally finite partition of unity in Ω, $\mathbf{1}_\Omega(x) = \sum_{j\in\mathbb{N}} \varphi_j(x)$ where each φ_j belongs to $C^\infty_{\rm c}(\Omega)$. Let a be a symbol in $S^m_{\rm loc}(\Omega\times\mathbb{R}^n)$ and A be the operator defined by the formula (4.1.2). We consider the operator

$$\widetilde{A} = \sum_{\substack{j,k \\ \operatorname{supp}\varphi_j\cap\operatorname{supp}\varphi_k\neq\emptyset}} \varphi_j A \varphi_k. \tag{4.1.3}$$

The operator $\varphi_j A\varphi_k$ has a symbol in $S^m_{1,0}$ which is given by $\varphi_j a \diamond \varphi_k$. We consider the finite sets

$$J_j = \{k\in\mathbb{N}, \operatorname{supp}\varphi_j\cap\operatorname{supp}\varphi_k\neq\emptyset\} \tag{4.1.4}$$

and the function

$$\Phi_j = \sum_{k\in J_j}\varphi_k \in C^\infty_{\rm c}(\Omega), \quad \Phi_j = 1 \text{ on a neighborhood of } \operatorname{supp}\varphi_j, \tag{4.1.5}$$

so that for all multi-indices α

$$\varphi_j(x)\partial_x^\alpha(1-\Phi_j)(x) = 0. \tag{4.1.6}$$

We check now the symbol $\tilde a = \sum_j \varphi_j a\diamond\Phi_j$ of $\widetilde{A}$. Given a compact subset of Ω it meets only finitely many $\operatorname{supp}\varphi_j$ and thus $\tilde a$ belongs to $S^m_{\rm loc}(\Omega\times\mathbb{R}^n)$. On the other hand, we have on $\Omega\times\mathbb{R}^n$,

$$a - \tilde a = \sum_j \varphi_j a - \varphi_j a\diamond\Phi_j = \sum_j \varphi_j a\diamond(1-\Phi_j)$$

and we get from Theorem 3.4.8 and (4.1.6) that that each $\varphi_j a\diamond(1-\Phi_j)$ belongs to $S^{-\infty}_{1,0}$; moreover the sum is locally finite, so that $a-\tilde a\in S^{-\infty}_{\rm loc}(\Omega\times\mathbb{R}^n) = \cap_{m\in\mathbb{R}} S^m_{\rm loc}(\Omega\times\mathbb{R}^n)$.

Proposition 4.1.5. *Let Ω be an open set of $\mathbb{R}^n$, let a be a symbol in $S^m_{loc}(\mathbb{R}^n)$. There exists a symbol $\tilde a\in S^m_{loc}(\Omega\times\mathbb{R}^n)$ such that*

(i) $a-\tilde a\in S^{-\infty}_{loc}(\Omega\times\mathbb{R}^n)$, $a(x,D)-\tilde a(x,D)$ sends $\mathscr{E}'(\Omega)$ into $C^\infty(\Omega)$,

(ii) the operator $\tilde a(x,D)$ is properly supported[4], and sends continuously $C^\infty_c(\Omega)$ into itself, $C^\infty(\Omega)$ into itself, $\mathscr{E}'(\Omega)$ into itself, $\mathscr{D}'(\Omega)$ into itself,

[4] A continuous linear operator $A:\mathscr{D}(V)\longrightarrow\mathscr{D}'(U)$ is said to be properly supported when both projections of the support of the kernel k from $\operatorname{supp} k$ in U,V are proper, i.e. for every compact $L\subset V$, there exists a compact $K\subset U$ such that $\operatorname{supp} v\subset L\Longrightarrow \operatorname{supp} Av\subset K$ and for every compact $K\subset U$, there exists a compact $L\subset V$ such that $\operatorname{supp} v\subset L^c\Longrightarrow\operatorname{supp} Av\subset K^c$.

(iii) $\tilde{a}(x, D)$ *defines a continuous linear operator from* $H^{s+m}_{comp}(\Omega)$ *to* $H^s_{comp}(\Omega)$*, from* $H^{s+m}_{loc}(\Omega)$ *to* $H^s_{loc}(\Omega)$*.*

Proof. We have already proven (i), using $\mathscr{E}'(\Omega) = \cup_s H^s_{\mathrm{comp}}(\Omega)$ and Theorem 4.1.3. Using the above notations, we get for $u \in C^\infty_{\mathrm{c}}(\Omega)$,

$$\tilde{a}(x, D)u = \sum_j \varphi_j a(x, D)\Phi_j u \tag{4.1.7}$$

with a finite sum of $C^\infty_{\mathrm{c}}(\Omega)$ functions since $\operatorname{supp} u$ meets only finitely many $\operatorname{supp} \Phi_j$. If $u \in C^\infty(\Omega)$, we have $\Phi_j u \in C^\infty_{\mathrm{c}}(\Omega) \subset C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and $\sum_j \varphi_j a(x, D)\Phi_j u$ is a locally finite sum of $C^\infty_{\mathrm{c}}(\Omega)$ functions, thus a $C^\infty(\Omega)$ function. For $u \in \mathscr{E}'(\Omega)$ with a (compact) support $K \subset \Omega$, the $\Phi_j u$ are all zero, except for a finite set of indices J_K and then $\sum_{j\in J_K} \varphi_j a(x, D)\Phi_j u$ belongs to $\mathscr{E}'(\Omega)$. If $u \in \mathscr{D}'(\Omega)$, we have $\Phi_j u \in \mathscr{E}'(\Omega)$ and $\sum_j \varphi_j a(x, D)\Phi_j u$ is a locally finite sum of distributions in Ω and thus a distribution on Ω, proving *(ii)*. The assertion *(iii)* and the continuity properties are direct consequences of *(ii)* and of Theorem 4.1.3. □

Remark 4.1.6. Let us now consider a symbol a belonging to $S^m_{\mathrm{loc}}(\Omega \times \mathbb{R}^n)$. We can quantify this symbol into a properly supported operator, say $\mathrm{Op}_\Omega(a)$, given by the formula (4.1.7), which has the properties of the operator $\tilde{a}(x, D)$ in the proposition 4.1.5. This quantization defines a linear mapping from $S^m_{\mathrm{loc}}(\Omega \times \mathbb{R}^n)$ to the quotient $\Psi^m_{ps}(\Omega)/\Psi^{-\infty}_{ps}(\Omega)$, where $\Psi^m_{ps}(\Omega)$ stands for the properly supported pseudo-differential operators of order m on Ω, and $\Psi^{-\infty}_{ps}(\Omega) = \cap_{m\in\mathbb{R}} \Psi^m_{ps}(\Omega)$. A change in the choice of the partition of unity (φ_j) will not change this mapping. From the proposition 4.1.5, we see that the operators of $\Psi^m_{ps}(\Omega)$ are continuous from $C^\infty_{\mathrm{c}}(\Omega)$ into itself, from $C^\infty(\Omega)$ into itself, from $\mathscr{E}'(\Omega)$ into itself, from $\mathscr{D}'(\Omega)$ into itself, from $H^{s+m}_{\mathrm{comp}}(\Omega)$ into $H^s_{\mathrm{comp}}(\Omega)$ from $H^{s+m}_{\mathrm{loc}}(\Omega)$ into $H^s_{\mathrm{loc}}(\Omega)$. Note also that if $A \in \Psi^{-\infty}_{ps}(\Omega)$ and $u \in \mathscr{D}'(\Omega)$, if ω is a relatively compact open subset of Ω, u belongs to $H^s_{loc}(\omega)$ for some s and thus $Au \in H^{+\infty}_{loc}(\omega)$ so that $Au \in C^\infty(\omega)$, proving that $\Psi^{-\infty}_{ps}(\Omega)$ sends $\mathscr{D}'(\Omega)$ into $C^\infty(\Omega)$.

Theorem 4.1.7. *Let* Ω *be an open set of* $\mathbb{R}^n$*,* $m_1, m_2 \in \mathbb{R}$*. Let* $a_j \in S^{m_j}_{loc}(\Omega \times \mathbb{R}^n)$*. Then the operator* $Op_\Omega(a_1)Op_\Omega(a_2)$ *belongs to* $\Psi^{m_1+m_2}_{ps}(\Omega)$ *and is such that,*

$$Op_\Omega(a_1)Op_\Omega(a_2) = Op_\Omega(a_1 a_2) \mod \Psi^{m_1+m_2-1}_{ps}(\Omega),$$
$$Op_\Omega(a_1)Op_\Omega(a_2) = Op_\Omega(a_1 a_2 + D_\xi a_1 \cdot \partial_x a_2) \mod \Psi^{m_1+m_2-2}_{ps}(\Omega),$$

and more generally, for all $N \in \mathbb{N}$*,*

$$Op_\Omega(a_1)Op_\Omega(a_2) = Op_\Omega\Big(\sum_{|\alpha|<N} \frac{1}{\alpha!} D^\alpha_\xi a_1 \partial^\alpha_x a_2\Big) \mod \Psi^{m_1+m_2-N}_{ps}(\Omega).$$

Proof. Let $\psi_1, \psi_2 \in C^\infty_{\mathrm{c}}(\Omega)$ with $\psi_2 = 1$ on a neighborhood of the support of ψ_1.

From Theorem 3.4.8, the proposition 4.1.5 and Remark 4.1.6, we have

$$\begin{aligned}\psi_1 \mathrm{Op}_\Omega(a_1)\psi_2 \mathrm{Op}_\Omega(a_2) &= (\psi_1 a_1)(x,D)(\psi_2 a_2)(x,D) \mod \Psi^{-\infty}(\Omega),\\ &= \psi_1\big(\sum_{|\alpha|<N} \frac{1}{\alpha!} D_\xi^\alpha a_1 \partial_x^\alpha a_2\big)(x,D) \mod \Psi^{m_1+m_2-N}(\Omega),\\ &= \psi_1 \mathrm{Op}_\Omega\big(\sum_{|\alpha|<N} \frac{1}{\alpha!} D_\xi^\alpha a_1 \partial_x^\alpha a_2\big) \mod \Psi^{m_1+m_2-N}(\Omega),\end{aligned}$$

and since the lhs and the first term in the rhs are both properly supported, the equality takes place $\mod \Psi_{ps}^{m_1+m_2-N}(\Omega)$. It means that for all $\psi_1 \in C_c^\infty(\Omega)$, we have

$$\psi_1 \mathrm{Op}_\Omega(a_1)\mathrm{Op}_\Omega(a_2) = \psi_1 \mathrm{Op}_\Omega\big(\sum_{|\alpha|<N} \frac{1}{\alpha!} D_\xi^\alpha a_1 \partial_x^\alpha a_2\big) \mod \Psi_{ps}^{m_1+m_2-N}(\Omega).$$

Since the operator $\mathrm{Op}_\Omega(a)$ is properly supported and completely determined modulo $\Psi_{ps}^{-\infty}(\Omega)$ by its definition on $C_c^\infty(\Omega)$, it concludes the proof. □

Let $a \in S_{loc}^m(\Omega \times \mathbb{R}^n)$ and let us consider as above the operator $\mathrm{Op}_\Omega(a) = \sum_{j\sim k} \varphi_j a(x,D)\varphi_k$, $j \sim k$ meaning $\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_k \neq \emptyset$. With Φ_j given by (4.1.5), we have $\mathrm{Op}_\Omega(a) = \sum_{j\sim k} \varphi_j \Phi_j a(x,D)\varphi_k$ and thus the adjoint operator is

$$\sum_{j\sim k} \varphi_k J(\overline{\Phi_j a})(x,D)\varphi_j.$$

Since $J(\overline{\Phi_j a}) = \sum_{|\alpha|<N} \frac{1}{\alpha!} D_\xi^\alpha \partial_x^\alpha (\Phi_j \bar{a}) + r_{N,j}$ with $r_{N,j} \in S_{1,0}^{m-N}$, we get

$$(\mathrm{Op}_\Omega(a))^* = \sum_{j\sim k} \varphi_k \sum_{|\alpha|<N} \frac{1}{\alpha!} D_\xi^\alpha \partial_x^\alpha (\Phi_j \bar{a})(x,D)\varphi_j + \sum_{j\sim k} \varphi_k r_{N,j}(x,D)\varphi_j.$$

Let $a^* \in S_{loc}^m(\Omega \times \mathbb{R}^n)$ such that for all N,

$$a^* - \sum_{|\alpha|<N} \frac{1}{\alpha!} D_\xi^\alpha \partial_x^\alpha \bar{a} \in S_{loc}^{m-N}(\Omega \times \mathbb{R}^n).$$

Since Φ_j is 1 near the support of φ_j, we obtain $(\mathrm{Op}_\Omega(a))^* = (\mathrm{Op}_\Omega(a^*))$ modulo $\Psi_{ps}^{-\infty}(\Omega)$.

4.2 Inversion of (micro)elliptic operators

Definitions

Let Ω be an open subset of $\mathbb{R}^n$ and $(x_0,\xi_0) \in \Omega \times (\mathbb{R}^n\backslash\{0\}) = \dot{T}^*(\Omega)$; a *conic-neighborhood* of (x_0,ξ_0) is defined as a subset of $\Omega \times \mathbb{R}^n\backslash\{0\}$ containing for some positive r the set

$$W_{x_0,\xi_0}(r) = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\backslash\{0\}, |x-x_0| < r, \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| < r, |\xi| > \frac{1}{r}\}. \quad (4.2.1)$$

Definition 4.2.1. Let $a \in S^m_{loc}(\Omega \times \mathbb{R}^n)$ and $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n \backslash \{0\}$. The symbol a is said to be *elliptic at* (x_0, ξ_0), when there exists a conic-neighborhood W of (x_0, ξ_0) such that

$$\inf_{(x,\xi)\in W} |a(x, \xi)||\xi|^{-m} > 0. \tag{4.2.2}$$

The points of $\dot{T}^*(\Omega)$ where a is not elliptic are called characteristic points.

Let us give an example of an elliptic symbol of order 0 at (x_0, ξ_0). Considering a function $\chi_0 \in C^\infty_c(\mathbb{R}), \chi_0(t) = 1$ for $t \le 1$, $\chi_0(t) = 0$ for $t \ge 2$, we define on $\mathbb{R}^{2n}$ for $r > 0$

$$\theta_{r,x_0,\xi_0}(x, \xi) = \chi_0(r^{-2}|x - x_0|^2)\chi_0\Big(r^{-2}\Big|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\Big|^2\Big)\big(1 - \chi_0(2r^2|\xi|^2)\big). \tag{4.2.3}$$

It is easy to check that θ_{r,x_0,ξ_0} belongs to $S^0_{1,0}$, is elliptic at (x_0, ξ_0) (note that $\theta_{r,x_0,\xi_0} \equiv 1$ on $W_{x_0,\xi_0}(r)$ and $\operatorname{supp}\theta_{r,x_0,\xi_0} \subset W_{x_0,\xi_0}(2r)$).

Definition 4.2.2. A function a defined on $\Omega \times \mathbb{R}^n$ will be said positively-homogeneous of degree m when for all $\xi \in \mathbb{R}^n$ with $|\xi| \ge 1$ and all $t \ge 1$, $a_m(x, t\xi) = t^m a_m(x, \xi)$. A function a defined on $\Omega \times \mathbb{R}^n\backslash\{0\}$ will be said positively homogeneous of degree m when for all $\xi \in \mathbb{R}^n\backslash\{0\}$ and all $t > 0$, $a_m(x, t\xi) = t^m a_m(x, \xi)$.

Lemma 4.2.3. *Let $a \in S^m_{loc}(\Omega \times \mathbb{R}^n)$ and $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n\backslash\{0\}$ such that the symbol a is elliptic at (x_0, ξ_0). Then for $b \in S^{m'}_{loc}(\Omega \times \mathbb{R}^n)$ with $m' < m$, the symbol $a + b$ is elliptic at (x_0, ξ_0). In particular, if there exists $a_m \in C^\infty(\Omega \times \mathbb{R}^n)$ positively-homogeneous of degree m such that*

$$a_m(x_0, \xi_0/|\xi_0|) \ne 0, \quad a - a_m \in S^{m-1}_{loc}(\Omega \times \mathbb{R}^n),$$

then the symbol a is elliptic at (x_0, ξ_0). This is the case in particular of a differential operator with $C^\infty(\Omega)$ coefficients $\sum_{|\alpha|\le m} a_\alpha(x)D^\alpha_x$ such that

$$0 \ne a_m(x_0, \xi_0)(= \sum_{|\alpha|=m} a_\alpha(x_0)\xi_0^\alpha).$$

Proof. The first part of the lemma is obvious since for K compact subset of Ω, $\lim_{|\xi|\to+\infty}\big(\sup_{x\in K} |b(x, \xi)|\big)|\xi|^{-m} = 0$. The second part is due to the fact that the property of homogeneity and the smoothness of a imply[5] that $a_m \in S^m_{loc}(\Omega\times\mathbb{R}^n)$. □

Remark 4.2.4. Note that if $\mathrm{Op}_\Omega(a_1) = \mathrm{Op}_\Omega(a_2)$ with $a_j \in S^m_{\mathrm{loc}}(\Omega \times \mathbb{R}^n)$, then $\mathrm{Op}_\Omega(a_1 - a_2) \in \Psi^{-\infty}_{ps}(\Omega)$ and thus

$$(a_1 - a_2)(x, D) = r(x, D) \text{ with } r \in S^{-\infty}_{\mathrm{loc}}(\Omega \times \mathbb{R}^n).$$

[5] For $\omega \in C^\infty_b(\mathbb{R}^n)$ vanishing for $|\xi| \le 1/2$ and equal to 1 for $|\xi| \ge 1$, we have in fact

$$a_m(x, \xi) = \omega(\xi)a_m(x, \xi/|\xi|)|\xi|^m + (1 - \omega(\xi))a_m(x, \xi) \in S^m_{loc}(\Omega \times \mathbb{R}^n) + S^{-\infty}_{loc}(\Omega \times \mathbb{R}^n).$$

A consequence of Remark 3.4.14 is that, for all $\chi \in C_c^\infty(\Omega)$, $\chi(x)(a_1 - a_2)(x,\xi) = \chi(x)r(x,\xi)$ which gives $a_1 - a_2 = r$(as functions of $S^{-\infty}_{\text{loc}}(\Omega \times \mathbb{R}^n)$). As a result, the characteristic points of a_1 and a_2 are the same, and one may define $\operatorname{char} \mathrm{Op}_\Omega(a)$ as the characteristic points of a.

Lemma 4.2.5. *Let $a \in S^m_{loc}(\Omega \times \mathbb{R}^n)$ and $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n \backslash \{0\}$ such that the symbol a is elliptic at (x_0, ξ_0). Then there exists $r > 0$ and $b \in S^{-m}_{loc}(\Omega \times \mathbb{R}^n)$, elliptic at (x_0, ξ_0) such that*

$$Op_\Omega(b) Op_\Omega(a) = Op_\Omega(\theta_{r,x_0,\xi_0}) + Op_\Omega(\rho),$$

with $\rho \in S^{-\infty}_{loc}(\Omega \times \mathbb{R}^n)$ and θ_{r,x_0,ξ_0} is given by (4.2.3).

Proof. Since a is elliptic at (x_0, ξ_0), we may assume that (4.2.2) holds for some conic-neighborhood $W_{x_0,\xi_0}(r_0)$. Let us consider the symbol $\theta_{r_1,x_0,\xi_0} \in S^0_{1,0}$ with $r_1 = r_0/2$ so that $\operatorname{supp} \theta_{r_1,x_0,\xi_0} \subset W_{x_0,\xi_0}(r_0)$. The assumption of Theorem 3.4.15 is verified with $\chi = \theta_{r_1,x_0,\xi_0}$. Considering $r_2 = r_0/4$ so that $\operatorname{supp} \theta_{r_2,x_0,\xi_0} \subset W_{x_0,\xi_0}(r_1) \subset \{\theta_{r_1,x_0,\xi_0} = 1\}$ we can find $b_1 \in S^{-m}_{1,0}$ such that, omitting the subscripts x_0, ξ_0,

$$b_1(x,D)\big(\theta_{r_1} a\big)(x,D) = \theta_{r_2}(x,D) + \rho(x,D), \quad \rho \in S^{-\infty}_{1,0},$$

implying with $r = r_0/8$,

$$\theta_r(x,D) b_1(x,D)\big(\theta_{r_1} a\big)(x,D) = \theta_r(x,D) + \tilde\rho(x,D), \quad \tilde\rho \in S^{-\infty}_{1,0},$$

and thus, with $b = \theta_r \diamond b_1$, which belongs to $S^m_{1,0}$, we have modulo $\Psi^{-\infty}_{ps}(\Omega)$

$$\begin{aligned}
\mathrm{Op}_\Omega(b)\mathrm{Op}_\Omega(a) &\equiv \mathrm{Op}_\Omega(b)\mathrm{Op}_\Omega(\theta_{r_1} a) + \mathrm{Op}_\Omega(\theta_r \diamond b_1)\mathrm{Op}_\Omega((1-\theta_{r_1})a) \\
&\equiv \mathrm{Op}_\Omega(b)\mathrm{Op}_\Omega(\theta_{r_1} a) \\
&\equiv \mathrm{Op}_\Omega(\theta_r).
\end{aligned}$$

□

Definition 4.2.6. Let Ω be an open set of $\mathbb{R}^n$, $a \in S^m_{\text{loc}}(\Omega \times \mathbb{R}^n)$, $A = \mathrm{Op}_\Omega(a)$. We define the essential support of A, denoted by $\operatorname{essupp} A$ as the complement in $\Omega \times \mathbb{R}^n \backslash \{0\}$ of the points (x_0, ξ_0) for which there exists a conic-neighborhood W so that a is of order $-\infty$ in W, i.e.

$$\forall (N, \alpha, \beta) \in \mathbb{N} \times \mathbb{N}^n \times \mathbb{N}^n, \qquad \sup_{(x,\xi) \in W} |(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi)||\xi|^N < \infty.$$

Note that from Remark 4.2.4, this definition depends only on A and the essential support is a closed conic subset of $\dot T^*(\Omega)$. Thanks to Lemma 4.2.3, if $A = \mathrm{Op}_\Omega(a_m + b)$ with $a_m \in C^\infty(\Omega \times \mathbb{R}^n)$ positively-homogeneous of degree m and $b \in S^{m-1}_{loc}(\Omega \times \mathbb{R}^n)$, then $\operatorname{char} A = \{(x,\xi) \in \dot T^*(\Omega), a_m(x,\xi) = 0\}$

Remark 4.2.7. Let Ω be an open set of $\mathbb{R}^n$ and $(A, B) \in \Psi^{m_1}_{ps}(\Omega) \times \Psi^{m_2}_{ps}(\Omega)$. Then we have

$$\operatorname{essupp}(AB) \subset \operatorname{essupp} A \cap \operatorname{essupp} B. \tag{4.2.4}$$

In fact if $\dot T^*(\Omega) \ni (x_0, \xi_0)$ belongs to $(\operatorname{essupp} A)^c \cup (\operatorname{essupp} B)^c$, the composition formula of Theorem 4.1.7 shows that (x_0, ξ_0) is in $(\operatorname{essupp} AB)^c$.

Theorem 4.2.8. *Let Ω be an open set of $\mathbb{R}^n$, $A \in \Psi^m_{ps}(\Omega)$. Let (x_0, ξ_0) be an elliptic point for A, i.e. $(x_0, \xi_0) \notin \operatorname{char} A$. Then there exist $B \in \Psi^{-m}_{ps}(\Omega), R, S \in \Psi^0_{ps}(\Omega)$, such that*

$$BA = \operatorname{Id} + R, \quad AB = \operatorname{Id} + S, \quad (x_0, \xi_0) \notin \operatorname{essupp} R, \quad (x_0, \xi_0) \notin \operatorname{essupp} S. \tag{4.2.5}$$

Proof. Lemma 4.2.5 implies the first result. On the other hand we can prove similarly that there exists $B_1 \in \Psi^{-m}_{ps}(\Omega)$ such that $AB_1 = \operatorname{Id} + S_1, (x_0, \xi_0) \notin \operatorname{essupp} S_1$. Now we see that

$$B = B(AB_1 - S_1) = (\operatorname{Id} + R)B_1 - BS_1 = B_1 + RB_1 - BS_1,$$

so that $AB = \operatorname{Id} + S, (x_0, \xi_0) \notin \operatorname{essupp} S$, (using (4.2.4)). The proof is complete. □

The wave-front-set of a distribution

Definition 4.2.9. Let Ω be an open set of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. The wave-front-set of u, denoted by WFu, is the subset of $\dot{T}^*(\Omega)$ whose complement is given by

$$\begin{aligned}(WFu)^c = \\ \{(x, \xi) \in \dot{T}^*(\Omega), \exists W \text{conic-neighborhood of } (x, \xi) \text{ s.t. } \forall a \in S^m_{loc}(\Omega \times \mathbb{R}^n) \\ \text{with } \operatorname{supp} a \subset W, \text{we have } \operatorname{Op}_\Omega(a)u \in C^\infty(\Omega)\}.\end{aligned} \tag{4.2.6}$$

Proposition 4.2.10. *Let Ω be an open set of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. The wave-front-set of u is a closed conic subset of $\dot{T}^*(\Omega)$ whose canonical projection[6] on Ω is* singsupp u. *Moreover, we have*

$$\begin{aligned}(WFu)^c = \{(x, \xi) \in \dot{T}^*(\Omega), \exists a \in S^0_{loc}(\Omega \times \mathbb{R}^n) \textit{ elliptic at } (x, \xi) \\ \textit{with } Op_\Omega(a)u \in C^\infty(\Omega)\}.\end{aligned} \tag{4.2.7}$$

Proof. The first assertion (closed conic) follows immediately from the definition. Now if $x_0 \notin \operatorname{singsupp} u$, there exists $r_0 > 0$ such that $u_{|B(x_0,r_0)}$ is C^∞, ($B(x, r)$ stands for the open Euclidean ball of $\mathbb{R}^n$ with center x and radius r). As a result if $\xi_0 \in \mathbb{S}^{n-1}$, $a \in S^m_{loc}(\Omega \times \mathbb{R}^n)$ with $\operatorname{supp} a \subset W_{x_0,\xi_0}(r_1), r_1 = r_0/2$, $\chi_0 \in C^\infty_c(B(x_0, r_0))$, $\chi_0 = 1$ on $B(x_0, r_1)$

$$\operatorname{Op}_\Omega(a)u = \overbrace{\operatorname{Op}_\Omega(a) \underbrace{\chi_0 u}_{\in C^\infty_c(\Omega)}}^{\in C^\infty_c(\Omega)} + \overbrace{\underbrace{\operatorname{Op}_\Omega(a)(1 - \chi_0)}_{\in \Psi^{-\infty}_{ps}(\Omega)} u}^{\in C^\infty(\Omega)}$$

since $A \in \Psi^{-\infty}_{ps}(\Omega)$ sends $\mathscr{D}'(\Omega)$ into $C^\infty(\Omega)$, proving that $\{x_0\} \times \mathbb{S}^{n-1} \subset (WFu)^c$. Conversely, if $x_0 \in \operatorname{singsupp} u$, there must exists some $\xi_0 \in \mathbb{S}^{n-1}$ such that $(x_0, \xi_0) \in WFu$, otherwise $\{x_0\} \times \mathbb{S}^{n-1} \subset (WFu)^c$ and using the compactness of $\mathbb{S}^{n-1}$, we could find an open neighborhood ω of x_0 in Ω, such that for all $a \in S^0_{loc}(\Omega \times$

[6] This is the mapping $\dot{T}^*(\Omega) \ni (x, \xi) \mapsto x \in \Omega$.

$\mathbb{R}^n$), $\operatorname{supp} a \subset \omega \times \mathbb{R}^n$, $\operatorname{Op}_\Omega(a)u \in C^\infty(\Omega)$; taking $a(x,\xi) = \chi(x)$ where $\chi \in C_c^\infty(\omega)$ would give $u \in C^\infty(\omega)$, contradicting $x_0 \in \operatorname{singsupp} u$. Calling N_u the complement of the set defined by (4.2.7), we see immediately that $(WFu)^c \subset N_u^c$; conversely, if $(x_0,\xi_0) \in N_u^c$, we can find A such that

$$A \in \Psi^0_{ps}(\Omega), \quad Au \in C^\infty(\Omega), \quad (x_0,\xi_0) \notin \operatorname{char} A.$$

Applying Theorem 4.2.8, we find $B \in \Psi^0_{ps}(\Omega)$ so that (4.2.5) holds and this implies for $c \in S^m_{loc}(\Omega \times \mathbb{R}^n)$,

$$BAu = u + Ru \Longrightarrow \operatorname{Op}_\Omega(c)u = \underbrace{\operatorname{Op}_\Omega(c)BAu}_{\in C^\infty(\Omega)} - \operatorname{Op}_\Omega(c)Ru$$

and since $(x_0,\xi_0) \notin \operatorname{essupp} R$, there exists a conic-neighborhood W of (x_0,ξ_0) such that R is of order $-\infty$ in W so that, taking c supported in W will imply $\operatorname{Op}_\Omega(c)R \in \Psi^{-\infty}_{ps}(\Omega)$ and $\operatorname{Op}_\Omega(c)Ru \in C^\infty(\Omega)$, proving that $(x_0,\xi_0) \notin WFu$. The proof of the proposition is complete. □

Lemma 4.2.11. *Let Ω be an open set of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. Then*

$$\begin{aligned}(WFu)^c = &\{(x,\xi) \in \dot{T}^*(\Omega), \exists W \text{ conic-neighborhood of } (x,\xi) \text{ s.t. } \forall A \in \Psi^m_{ps}(\Omega),\\ &\text{with } \operatorname{essupp} A \subset W, \text{ we have } Au \in C^\infty(\Omega)\}. \end{aligned} \tag{4.2.8}$$

Proof. Calling M_u the complement of the set defined by (4.2.8), we have obviously $M_u^c \subset (WFu)^c$ and conversely if $(x_0,\xi_0) \notin WFu$, there exists $r_0 > 0$ such that for all $a \in S^m_{loc}(\Omega \times \mathbb{R}^n)$ supported in $W_{x_0,\xi_0}(r_0)$, $\operatorname{Op}_\Omega(a)u \in C^\infty(\Omega)$. Now if $B \in \Psi^m_{ps}(\Omega)$, with $\operatorname{essupp} B \subset W_{x_0,\xi_0}(r_0/2)$, we have

$$B = \operatorname{Op}_\Omega(b) = \operatorname{Op}_\Omega(\underbrace{b\theta_{r_0/2}}_{\substack{\text{supported}\\ \text{in } W(r_0)}}) \mod \psi^{-\infty}_{ps}(\Omega)$$

and thus $Bu \in C^\infty(\Omega)$, completing the proof of the lemma. □

The elliptic regularity theorem

Theorem 4.2.12. *Let Ω be an open set of $\mathbb{R}^n$ and $A \in \Psi^m_{ps}(\Omega)$. Then for $u \in \mathscr{D}'(\Omega)$,*

$$WF(Au) \subset WFu \subset \operatorname{char} A \cup WF(Au).$$

Proof. If $(x_0,\xi_0) \notin WFu$, there exists a conic-neighborhood W of (x_0,ξ_0) such that (4.2.8) holds and taking $C \in \Psi^m_{ps}(\Omega)$ with $\operatorname{essupp} C \subset W$, we get from (4.2.4) that $\operatorname{essupp} CA \subset W$, and Lemma 4.2.11 implies that $(x_0,\xi_0) \notin WF(Au)$. On the other hand, if $(x_0,\xi_0) \notin \operatorname{char} A$ and $(x_0,\xi_0) \notin WF(Au)$, Theorem 4.2.8 provides $B \in \Psi^{-m}_{ps}(\Omega)$ satisfying (4.2.5): we get

$$u = BAu - Ru, \quad (x_0,\xi_0) \notin \operatorname{essupp} R \quad \text{i.e. } R \text{ of order } -\infty \text{ on } W,$$

where W is a conic-neighborhood of (x_0, ξ_0). Taking $C \in \Psi^m_{ps}(\Omega)$ with $\operatorname{essupp} C \subset W$ we have

$$Cu = CBAu - \underbrace{CR}_{\in \Psi^{-\infty}_{ps}(\Omega)} u,$$

so that $CRu \in C^\infty(\Omega)$. On the other hand, since $(x_0, \xi_0) \notin WF(Au)$, thanks to Lemma 4.2.11, there exists a conic-neighborhood W_1 of (x_0, ξ_0) such that, for all $P \in \Psi^m_{ps}(\Omega)$ with $\operatorname{essupp} P \subset W_1$, we have $PAu \in C^\infty(\Omega)$. This proves that $CBAu \in C^\infty(\Omega)$, provided $\operatorname{essupp} C \subset W_1$ and with $\operatorname{essupp} C \subset W_1 \cap W$ we get $Cu \in C^\infty(\Omega)$, which implies $(x_0, \xi_0) \notin WFu$, using Lemma 4.2.11. □

Corollary 4.2.13. *Let Ω be an open set of $\mathbb{R}^n$, $A \in \Psi^m_{ps}(\Omega)$. Then for $u \in \mathscr{D}'(\Omega)$,* $\operatorname{singsupp}(Au) \subset \operatorname{singsupp} u \subset \operatorname{singsupp}(Au) \cup \operatorname{pr}(\operatorname{char} A)$ *and in particular, if A is elliptic on Ω, i.e.* $\operatorname{char} A = \emptyset$, *we obtain that* $\operatorname{singsupp} u = \operatorname{singsupp}(Au)$.

Definition 4.2.14 (H^s wave-front-set)**.** Let Ω be an open set of $\mathbb{R}^n$, $s \in \mathbb{R}$ and $u \in \mathscr{D}'(\Omega)$. The H^s wave-front-set of u, denoted by $WF_s u$ is the subset of $\dot{T}^*(\Omega)$ whose complement is given by

$$\begin{aligned}(WF_s u)^c = \{(x, \xi) \in \dot{T}^*(\Omega), \exists W \text{conic-neighborhood of } (x, \xi) \text{ s.t.} \\ \forall A \in \psi^0(\Omega) \text{ with } \operatorname{essupp} A \subset W, \text{we have } Au \in H^s_{loc}(\Omega)\}.\end{aligned} \quad (4.2.9)$$

When $(x, \xi) \notin WF_s u$, we shall say that u is H^s at (x, ξ) and write $u \in H^s_{(x,\xi)}$.

The proof of the following theorem is a simple adaptation of the proof of Theorem 4.2.12.

Theorem 4.2.15. *Let Ω be an open set of $\mathbb{R}^n$, $s, m \in \mathbb{R}$ and $A \in \Psi^m_{ps}(\Omega)$. Then for $u \in \mathscr{D}'(\Omega)$,*

$$WF_s(Au) \subset WF_{s+m} u \subset \operatorname{char} A \cup WF_s(Au). \quad (4.2.10)$$

4.3 Propagation of singularities

Let Ω be an open subset of $\mathbb{R}^n$, $m \in \mathbb{R}$ and $P \in \Psi^m_{ps}(\Omega)$ a pseudo-differential operator with symbol p such that

$$p(x, \xi) = p_m(x, \xi)\omega(\xi) + p_{m-1}(x, \xi), \quad (4.3.1)$$

with p_m positively homogeneous of degree m and $p_{m-1} \in S^{m-1}_{loc}(\Omega \times \mathbb{R}^n)$ and

$$\omega \in C^\infty(\mathbb{R}^n), \begin{cases} \omega(\xi) = 0 & \text{for } |\xi| \le 1/2, \\ \omega(\xi) = 1 & \text{for } |\xi| \ge 1. \end{cases} \quad (4.3.2)$$

We shall say that p_m is the[7] principal symbol of P. Note also that the function $p_m(x, \xi)\omega(\xi)$ is positively-homogeneous with degree m, according to the terminology of Definition 4.2.2. In the sequel, we shall mainly consider operators of that type.

[7] If p_m, q_m are positively homogeneous of degree m on $\Omega \times \mathbb{R}^n \backslash \{0\}$ such that for $|\xi| \ge R > 0$ $|p_m(x, \xi) - q_m(x, \xi)| \le C|\xi|^{m-1}$, this implies $|p_m(x, \xi/|\xi|) - q_m(x, \xi/|\xi|)| \le C|\xi|^{-1}$ and thus $p_m = q_m$ on $\Omega \times \mathbb{R}^n \backslash \{0\}$.

Theorem 4.3.1. *Let P as above, $t_0 < t_1 \in \mathbb{R}$ and $I = [t_0, t_1] \ni t \mapsto \gamma(t) \in \dot{T}^*(\Omega)$ be a null bicharacteristic*[8] *curve of* $\operatorname{Re} p_m$*. Let us assume that* $\operatorname{Im} p_m \geq 0$ *on a conic-neighborhood of $\gamma(I)$. Let $s \in \mathbb{R}$ and $u \in \mathscr{D}'(\Omega)$ such that $Pu \in H^s$ at $\gamma(I)$ (i.e. $WF_s Pu \cap \gamma(I) = \emptyset$). Then*

$$\gamma(t_0) \in WF_{s+m-1} u \Longrightarrow \gamma(t_1) \in WF_{s+m-1} u. \tag{4.3.3}$$

Remark 4.3.2. The property (4.3.3) means that the singularities are propagating *forward* when the imaginary part of p_m is nonnegative A statement equivalent to (4.3.3) is

$$\gamma(t_1) \notin WF_{s+m-1} u \Longrightarrow \gamma(t_0) \notin WF_{s+m-1} u, \tag{4.3.4}$$

meaning that the regularity is propagating backward in that case. If we change the sign condition on $\operatorname{Im} p_m$, we have to reverse the direction of propagation of singularities and we have under $\operatorname{Im} p_m \leq 0$ near $\gamma(I)$,

$$\gamma(t_1) \in WF_{s+m-1} u \Longrightarrow \gamma(t_0) \in WF_{s+m-1} u. \tag{4.3.5}$$

When the imaginary part of p_m is identically 0, the propagation takes place in both directions and $WF_{s+m-1} u \backslash WF_s(Pu)$ is invariant by the Hamiltonian flow of p_m; this implies in particular for a real-valued p_m that $WFu \backslash WF(Pu)$ is invariant by the Hamiltonian flow of p_m.

Proof of Theorem 4.3.1. Multiplying by an elliptic operator of order $1 - m$, we are reduced to the case $m = 1$. We have to prove that

$$u \in H^s_{\gamma(t_1)}, Pu \in H^s_{\gamma(I)} \Longrightarrow u \in H^s_{\gamma(t_0)}. \tag{4.3.6}$$

It is enough to prove that

$$u \in H^s_{\gamma(t_1)}, Pu \in H^s_{\gamma(I)}, u \in H^{s-\frac12}_{\gamma(I)} \Longrightarrow u \in H^s_{\gamma(t_0)}. \tag{4.3.7}$$

In fact, since $\gamma(I)$ is compact, we may assume that $u \in H^{s_0-\frac12}_{\gamma(I)}$ for some s_0. The property $(4.3.7)_{s_0}$ is identical to $(4.3.6)_{s_0}$. Assume now that $u \in H^{s_0+\frac12}_{\gamma(t_1)}, Pu \in H^{s_0+\frac12}_{\gamma(I)}$: this implies that $u \in H^{s_0}_{\gamma(t_1)}, Pu \in H^{s_0}_{\gamma(I)}$ and since $u \in H^{s_0-\frac12}_{\gamma(I)}$, the property $(4.3.7)_{s_0}$ gives eventually $u \in H^{s_0}_{\gamma(I)}$ so that the property $(4.3.7)_{s_0+\frac12}$ gives $u \in H^{s_0+\frac12}_{\gamma(I)}$. Inductively, we assume that for some $k \in \mathbb{N}^*$,

$$u \in H^{s_0+\frac{k}{2}}_{\gamma(t_1)}, Pu \in H^{s_0+\frac{k}{2}}_{\gamma(I)} \Longrightarrow u \in H^{s_0+\frac{k}{2}}_{\gamma(I)}. \tag{4.3.8}$$

[8]For a C^1 on $\Omega \times \mathbb{R}^n$, the Hamiltonian vector field of a is

$$H_a = \sum_{1 \leq j \leq n} \left(\frac{\partial a}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial a}{\partial x_j} \frac{\partial}{\partial \xi_j}\right).$$

The integral curves of H_a are called the bicharacteristic curves of a. Since $H_a(a) = 0$, a is constant along its bicharacteristic curves; the *null* bicharacteristic curves are those on which a vanishes.

Then if $u \in H^{s_0+\frac{k+1}{2}}_{\gamma(t_1)}, Pu \in H^{s_0+\frac{k+1}{2}}_{\gamma(I)}$, (4.3.8) gives $u \in H^{s_0+\frac{k}{2}}_{\gamma(I)}$ and the property $(4.3.7)_{s_0+\frac{k+1}{2}}$ gives $u \in H^{s_0+\frac{k+1}{2}}_{\gamma(I)}$, so that (4.3.7) implies (4.3.8) for all $k \in \mathbb{N}$ and all s_0 such that $u \in H^{s_0-\frac{1}{2}}_{\gamma(I)}$, meaning that(4.3.7) implies (4.3.6). Now to prove (4.3.7), it is enough to get it for $s = 0$: assuming (4.3.7) for $s = 0$ and considering properly supported pseudo-differential operator E_s, E_{-s} of order $s, -s$, elliptic on a neighborhood of $\gamma(I)$, whose symbols have an asymptotic expansion $\sum_{j\in\mathbb{N}} c_{\pm s-j}, c_{\pm s-j}$ positively-homogeneous of degree $\pm s - j$ and such that

$$E_{-s}E_s = \operatorname{Id} + R, \quad \gamma(I) \subset (\operatorname{essupp} R)^c$$

we get under the hypothesis of (4.3.7) that

$$E_s u \in H^0_{\gamma(t_1)}, E_s P E_{-s} E_s u \in H^0_{\gamma(I)}, E_s u \in H^{-1/2}_{\gamma(I)},$$

and since the operator $E_s P E_{-s}$ is of order 1 with the same principal symbol as P, we can then apply (4.3.7) for $s = 0$, entailing $E_s u \in H^0_{\gamma(t_0)}$ which gives $u \in H^s_{\gamma(t_0)}$ using the ellipticity of E_s. The remaining part of the proof is devoted to establishing (4.3.7) for $s = 0$. As a last preliminary remark we note that

$$J = \{t \in [t_0, t_1], u \in H^0_{\gamma(s)} \text{ for } s \in [t, t_1]\}$$

is a nonempty open interval of $[t_0, t_1]$; if $\inf J$ belongs to J it is also closed and thus equal to $[t_0, t_1]$; as a result, we may assume that $J =]t_0, t_1]$. Of course there is no loss of generality setting $t_0 = 0, t_1 = 1$. Summing-up, we have to prove

$$u \in H^0_{\gamma(t)} \text{ for } t \in]0,1], Pu \in H^0_{\gamma([0,1])}, u \in H^{-1/2}_{\gamma([0,1])} \Longrightarrow u \in H^0_{\gamma(0)}. \tag{4.3.9}$$

We may also assume that u is compactly supported: if $\varphi \in C^\infty_c(\Omega)$ is 1 near the first projection of $\gamma([0,1])$, we have $P\varphi u = [P, \varphi]u + \varphi P u$ and since

$$(\operatorname{essupp}[P, \varphi])^c \supset \gamma([0,1])$$

we get that φu satisfies as well the assumptions of (4.3.9). On the other hand, if p_1 is the real part of the principal symbol of P, we may assume that at $\gamma(0)$, $dp_1 \wedge \xi \cdot dx \neq 0$, otherwise $\partial_\xi p_1(\gamma(0)) = 0, \partial_x p_1(\gamma(0)) = \lambda \xi(0)$ and the solution $\gamma(t) = (x(t), \xi(t))$ of

$$\dot{x}(t) = \partial_\xi p_1(x(t), \xi(t)), \quad \dot{\xi}(t) = -\partial_x p_1(x(t), \xi(t)), \quad (x(0), \xi(0)) = \gamma(0),$$

is $x(t) = x(0),\ \xi(t) = e^{-\lambda t}\xi(0)$; since the wave-front-set is conic, (4.3.9) is obvious. Let us consider W_0 a conic neighborhood of $\gamma([0,1])$ such that in W_0 $Pu \in H^0, u \in H^{-1/2}$. Let $m_0 \in S^0$ real-valued. The symbol of P is $p_1 + iq_1 + p_0 + iq_0$ with $p_j, q_j \in S^j$ real-valued, p_1, q_1 positively-homogeneous of degree 1 and $q_1 \geq 0$. We calculate with $M = m_0(x, D)$, $A = \frac{1}{2}(P + P^*), B = \frac{1}{2i}(P - P^*)$, for $v \in C^\infty_c(\Omega)$,

$$\begin{aligned} &2\operatorname{Re}\langle Pv, iM^*Mv\rangle \\ &\qquad = \langle [A, iM^*M]v, v\rangle + 2\operatorname{Re}\langle M^*[M, B]v, v\rangle + 2\operatorname{Re}\langle BMv, Mv\rangle. \end{aligned} \tag{4.3.10}$$

From the Gårding inequality (Theorem 3.5.1) we have

$$2\operatorname{Re}\langle BMv, Mv\rangle + \alpha_0\|Mv\|_0^2 \ge 0, \quad \alpha_0 \text{ a semi-norm of } q_1. \tag{4.3.11}$$

On the other hand, the principal symbol of $M^*[M,B]$ is purely imaginary, belongs to S^0, and so that

$$2\operatorname{Re}\langle M^*[M,B]v, v\rangle + C_1\|v\|_{-1/2}^2 \ge 0. \tag{4.3.12}$$

We have also

$$\langle [A, iM^*M]v, v\rangle + C_2\|v\|_{-1/2}^2 \ge \langle \{p_1, m^2\}(x,D)v, v\rangle \frac{1}{2\pi}. \tag{4.3.13}$$

As a result, we have

$$\|MPv\|_0^2 + (1+\alpha_0)\|Mv\|_0^2 + (C_1+C_2)\|v\|_{-1/2}^2 \ge \langle \{p_1, m^2\}(x,D)v, v\rangle \frac{1}{2\pi}. \tag{4.3.14}$$

We can find $(t,y,\tau,\eta) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}^{n-1}$ as C^∞ local symplectic coordinates near $\gamma(0)$, $(x,\xi) \mapsto (t,y)$ homogeneous functions of degree 0 with respect to ξ, $(x,\xi)\mapsto(\tau,\eta)$ homogeneous functions of degree 1 with respect to ξ, so that $\partial_t = H_{p_1}$. We choose $\theta_0 \in C_c^\infty(\mathbb{R})$ supported on $[-\varepsilon_0, \varepsilon_0]$ with $\varepsilon_0 > 0$, positive on $(-\varepsilon_0, \varepsilon_0)$, with L^2-norm 1 and consider $\theta_1(t) = \theta_0(t - 3\varepsilon_0)$: with

$$\kappa(t) = \int_{-\varepsilon_0}^t \big(\theta_0(s)^2 - \theta_1(s)^2\big)ds = \int_{t-3\varepsilon_0}^t \theta_0(s)^2 ds,$$

we have for the C^∞ function κ supported in $[-\varepsilon_0, 4\varepsilon_0]$,

$$0 \le \kappa \le 1,\ \dot\kappa = \theta_0^2 - \theta_1^2, \quad [-\varepsilon_0, 2\varepsilon_0] \subset \{\dot\kappa \ge 0\}, \quad \operatorname{supp}\kappa \subset [-\varepsilon_0, 4\varepsilon_0],$$

and the following variation table.

t	$-\varepsilon_0$		0		ε_0		$2\varepsilon_0$		$3\varepsilon_0$		$4\varepsilon_0$
$\dot\kappa(t)$	0	$+$	$\dot\kappa(0) > 0$	$+$	0	0	0	$-$	$\dot\kappa(3\varepsilon_0) < 0$	$-$	0
$\kappa(t)$	0	$\nearrow$	$\kappa(0) > 0$	$\nearrow$	1	1	1	$\searrow$	$\kappa(3\varepsilon_0) > 0$	$\searrow$	0

We multiply now the function κ by ν^2 with $\nu \in C^\infty(\mathbb{R}^{2n-1}_{y,\tau,\eta}; [0,1]), \nu(0) = 1$, ν homogeneous with degree 0 with respect to τ, η, and we get that

$$\begin{gathered} 0 \le \kappa(t)\nu^2(Y) \le 1, \quad \partial_t\big(\kappa(t)\nu^2(Y)\big) = \big(\theta_0^2(t) - \theta_1^2(t)\big)\nu^2(Y), \\ [-\varepsilon_0, 2\varepsilon_0] \times \operatorname{supp}\nu \subset \{\dot\kappa\otimes\nu^2 \ge 0\}, \ \operatorname{supp}\kappa\otimes\nu^2 \subset [-\varepsilon_0, 4\varepsilon_0]\times\operatorname{supp}\nu, \\ \kappa(0)\nu^2(0) > 0, \quad \dot\kappa(0)\nu^2(0) > 0, \quad \{\dot\kappa\otimes\nu^2 < 0\} \subset [2\varepsilon_0, 4\varepsilon_0]\times\operatorname{supp}\nu. \end{gathered}$$

The mapping

$$\mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}^{n-1} \ni (t,y,\tau,\eta) \mapsto x(t,y,\tau,\eta), \xi(t,y,\tau,\eta) \in \mathbb{R}^n\times\mathbb{R}^n$$

is a local symplectomorphism Θ, with x homogeneous of degree 0, ξ homogeneous of degree 1, and the push-forward μ of $\kappa \otimes \nu^2$ by Θ given by $\mu \circ \Theta = \kappa \otimes \nu^2$ is homogeneous of degree 0 with respect to τ, η and satisfies

$$\begin{gathered}\mu \in C_c^\infty(\dot{T}^*(\Omega); [0,1]), \quad \operatorname{supp} \mu \subset \Theta([-\varepsilon_0, 4\varepsilon_0] \times \operatorname{supp} \nu), \quad \mu(\gamma(0)) > 0,\\ H_{p_1}(\mu) = \chi_0^2 - \chi_1^2, \quad \chi_0, \chi_1 \in C_c^\infty(\dot{T}^*(\Omega)), \quad \chi_0(\gamma(0)) > 0,\\ \operatorname{supp} \chi_0 = \Theta([-\varepsilon_0, \varepsilon_0] \times \operatorname{supp} \nu), \quad \operatorname{supp} \chi_1 = \Theta([2\varepsilon_0, 4\varepsilon_0] \times \operatorname{supp} \nu),\\ \text{and thus } \operatorname{supp} \chi_0 \cap \operatorname{supp} \chi_1 = \emptyset.\end{gathered}$$

The function μ is homogeneous of degree 0 such that

$$\begin{gathered}\operatorname{supp} \mu \subset W, \quad \text{conic neighborhood of } \gamma([-\varepsilon_0, 4\varepsilon_0]) \subset W_0\\ H_{p_1}(\mu) = \chi_0^2 - \chi_1^2, \quad \operatorname{supp} \chi_1 \subset W_1,\\ \text{where } W_1 \subset W_0 \text{ is a conic neighborhood of } \gamma(3\varepsilon_0) \text{ with } u \in H^0 \text{ on } W_1.\end{gathered}$$

We consider now, with $T = t \circ \Theta^{-1}$ so that $H_{p_1}(T) = 1$, and T is homogeneous with degree 0, the symbol

$$m = \mu e^{\lambda T}$$

where λ is so that $\lambda \geq (1 + \alpha_0)2\pi$ and α_0 is given in (4.3.11). Checking

$$\begin{aligned}H_{p_1}(m^2) - 2\pi(1+\alpha_0)m^2 &= 2\mu e^{\lambda T}\big(H_{p_1}(\mu)e^{\lambda T} + \lambda e^{\lambda T}\mu\big) - 2\pi(1+\alpha_0)\mu^2 e^{2\lambda T}\\ &= 2\mu e^{2\lambda T} H_{p_1}(\mu) + \mu^2 e^{2\lambda T}\big(2\lambda - 2\pi(1+\alpha_0)\big) \geq 2\mu e^{2\lambda T} H_{p_1}(\mu).\end{aligned}$$

Since $\mu e^{2\lambda T} H_{p_1}(\mu)$ is supported in W, positive at $\gamma(0)$ and non-negative except on a neighborhood of $\gamma(3\varepsilon_0)$, in fact such that

$$\mu e^{2\lambda T} H_{p_1}(\mu) = \mu(e^{\lambda T}\chi_0)^2 - \mu(e^{\lambda T}\chi_1)^2,$$

the inequality (4.3.14) gives with $A_0, A_1 \in \Psi^0_{ps}(\Omega)$, A_0 elliptic at $\gamma(0)$, $\operatorname{essupp} A_1 \subset W_1$,

$$\|A_0 v\|_0^2 \leq \|A_1 v\|_0^2 + \|MPv\|_0^2 + (C_1 + C_2)\|v\|_{-1/2}^2. \tag{4.3.15}$$

Replacing in that inequality v by Nv where $N \in \psi^0_{ps}(\Omega), \operatorname{essupp} N \subset W_0$, with a symbol equal to 1 on W gives with $R \in \psi^{-\infty}_{ps}(\Omega)$

$$\|A_0 v\|_0^2 \leq \|A_1 v\|_0^2 + \|MPv\|_0^2 + (C_1 + C_2)\|Nv\|_{-1/2}^2 + \|Rv\|_0^2. \tag{4.3.16}$$

Since $u \in \mathscr{E}'(\Omega)$, we can apply (4.3.16) to $v_\varepsilon = u * \rho_\varepsilon$ where $\rho_\varepsilon(x) = \rho(x/\varepsilon)\varepsilon^{-n}$, with $\rho \in C_c^\infty(\mathbb{R}^n)$ of integral 1 and ε small enough so that $v_\varepsilon \in C_c^\infty(\Omega)$. Since u is H^0 on W_1 and also $H^{-1/2}$ on W_0, Pu is H^0 on W_0, we get that the $\|A_0 v_\varepsilon\|_0^2$ is bounded for $\varepsilon \to 0_+$, implying that the weak limit $A_0 u$ in $\mathscr{E}'(\Omega)$ belongs to H^0, proving that u is H^0 at $\gamma(0)$. The proof of Theorem 4.3.1 is complete. □

4.4 Local solvability

Functional analysis arguments

Definition 4.4.1. Let Ω be an open subset of $\mathbb{R}^n$, $x_0 \in \Omega$, $m \in \mathbb{R}$ and $P \in \Psi^m_{ps}(\Omega)$ a properly supported pseudo-differential operator. We shall say that P is locally solvable at x_0 if there exists an open neighborhood $V \subset \Omega$ of x_0 such that

$$\forall f \in C^\infty(\Omega), \quad \exists u \in \mathscr{D}'(\Omega) \quad \text{with } Pu = f \text{ in } V. \tag{4.4.1}$$

Note that this definition makes sense since P is properly supported (in particular P is an endomorphism of $\mathscr{D}'(\Omega)$) and we can actually restrict a distribution to an open set. Moreover the set of points $x \in \Omega$ such that P is locally solvable at x is open.

Definition 4.4.2. Let Ω, x_0, P be as above and let $\mu \geq 0$. We shall say that P is locally solvable at x_0 with loss of μ derivatives if, for every $s \in \mathbb{R}$, there exists an open neighborhood $V \subset \Omega$ of x_0 such that

$$\forall f \in H^s_{loc}(\Omega), \quad \exists u \in H^{s+m-\mu}_{loc}(\Omega) \quad \text{with } Pu = f \text{ in } V. \tag{4.4.2}$$

Remark 4.4.3. Note that the neighborhood V above may depend on s.

Lemma 4.4.4. *Let Ω be an open subset of $\mathbb{R}^n$, $x_0 \in \Omega$, $m \in \mathbb{R}$ and let $P \in \Psi^m_{ps}(\Omega)$ be a pseudo-differential operator solvable at x_0. Then there exists a neighborhood $V \subset \Omega$ of x_0, $N \in \mathbb{N}$, $C > 0$ such that*

$$\forall v \in C^\infty_c(V), \quad C\|P^* v\|_N \geq \|v\|_{-N}. \tag{4.4.3}$$

Proof. The solvability of P at x_0 gives the existence of a neighborhood V of x_0 such that (4.4.1) holds. We consider now $v_0 \in C^\infty_c(V)$ such that $P^* v_0 = 0$: Then, for all $\varphi \in C^\infty_c(V)$ the solvability property implies the existence of $u \in \mathscr{D}'(\Omega)$ with $Pu = \varphi$ in V. As a result, we have for all $\varphi \in C^\infty_c(V)$

$$\begin{aligned}\int \varphi(x)\overline{v_0(x)}dx = \langle (Pu)_{|V}, v_0\rangle_{\mathscr{D}^*(V),\mathscr{D}(V)} = \langle Pu, v_0\rangle_{\mathscr{D}^*(\Omega),\mathscr{D}(\Omega)} \\ = \langle u, P^* v_0\rangle_{\mathscr{D}^*(\Omega),\mathscr{D}(\Omega)} = 0,\end{aligned} \tag{4.4.4}$$

which gives $v_0 = 0$. Then for any compact subset K of V, the space $C^\infty_K(V)$ is a metrizable topological space for the topology given by the countable family of norms $\{\|P^* v\|_{H^r}\}_{r\in\mathbb{N}}$. Let $\tilde K$ be a compact subset of V, neighborhood of a given compact subset K of V and consider the space $C^\infty_{\tilde K}(V)$, equipped with its standard Fréchet topology, where the semi-norms may be given by the family $\{\|\varphi\|_{H^s}\}_{s\in\mathbb{N}}$. For a fixed $v \in C^\infty_K(V)$, the mapping

$$C^\infty_{\tilde K}(V) \ni \varphi \mapsto \int \varphi(x)\overline{v(x)}dx$$

is obviously continuous since $C^\infty_{\tilde K}(V)$ is equipped with its standard Fréchet topology. For a fixed $\varphi \in C^\infty_{\tilde K}(V)$, the mapping

$$C^\infty_K(V) \ni v \mapsto \int \varphi(x)\overline{v(x)}dx$$

is continuous for the topology on $C^\infty_K(V)$ given by the family $\{\|P^* v\|_{H^r}\}_{r\in\mathbb{N}}$ since $\varphi = Pu$ on V with $u \in \mathscr{D}'(\Omega)$ and thus, as in (4.4.4),

$$\left|\int \varphi(x)\overline{v(x)}dx = \langle u, P^* v\rangle_{\mathscr{D}^*(\Omega),\mathscr{D}(\Omega)}\right| \le C_u \|P^* v\|_r.$$

A separately continuous bilinear form on the product of a Fréchet space with a metrizable space is in fact continous so that

$$\exists C>0, \exists N\in\mathbb{N},\ \forall v\in C^\infty_K(V), \forall \varphi \in C^\infty_{\tilde K}(V),$$
$$\left|\int \varphi(x)\overline{v(x)}dx\right| \le C\|P^* v\|_N \|\varphi\|_N,$$

and since $\tilde K$ is a neighborhood of K, it gives the lemma. □

Lemma 4.4.5. *Let Ω be an open subset of $\mathbb{R}^n$, $x_0 \in \Omega$, $m, s, \mu \in \mathbb{R}$ and let $P \in \Psi^m_{ps}(\Omega)$ be a pseudo-differential operator such that there exists an open neighborhood V of x_0 such that*

$$\forall f \in H^s_{loc}(\Omega), \quad \exists u \in H^{s+m-\mu}_{loc}(\Omega) \quad \textit{with } Pu = f \textit{ in } V. \tag{4.4.5}$$

Then there exists a neighborhood $W \subset \Omega$ of x_0, $C > 0$ such that

$$\forall v \in C^\infty_c(W), \quad C\|P^* v\|_{-s-m+\mu} \ge \|v\|_{-s}. \tag{4.4.6}$$

Proof. We consider $v_0 \in C^\infty_c(V)$ such that $P^* v_0 = 0$: Then, for all $\varphi \in C^\infty_c(V)$ the solvability property (4.4.5) implies (4.4.1) and the proof of Lemma 4.4.4 gives $v_0 = 0$. Then for any compact subset K of V, the space $C^\infty_K(V)$ is a normed space with the norm $\|P^* v\|_{-s-m+\mu}$. Let $\tilde K$ be a compact subset of V, neighborhood of a given compact subset K of V and consider the Hilbert space $H^s_{\tilde K}(V)$. For a fixed $v \in C^\infty_K(V)$, the mapping $H^s_{\tilde K}(V) \ni \varphi \mapsto \langle \varphi, v\rangle$ is obviously continuous since $|\langle \varphi, v\rangle| \le \|\varphi\|_s \|v\|_{-s}$. For a fixed $\varphi \in H^s_{\tilde K}(V)$, the mapping

$$C^\infty_K(V) \ni v \mapsto \langle \varphi, v\rangle$$

is continuous for the topology on $C^\infty_K(V)$ given by the norm $\|P^* v\|_{-s-m+\mu}$ since $\varphi = Pu$ on V with $u \in H^{s+m-\mu}_{loc}(\Omega)$ and thus, as in (4.4.4), with $\chi \in C^\infty_c(\Omega)$, $\chi = 1$ near the support of $P^*(C^\infty_K(V))$,

$$\begin{aligned}\big|\langle \varphi, v\rangle = \langle u, P^* v\rangle_{\mathscr{D}^*(\Omega),\mathscr{D}(\Omega)} = \langle \chi u, P^* v\rangle_{H^{s+m-\mu},\mathscr{D}(\Omega)}\big| \\ \le \|\chi u\|_{s+m-\mu} \|P^* v\|_{-s-m+\mu}.\end{aligned}$$

As before, this bilinear form is continuous,

$$\exists C > 0, \forall v \in C^\infty_K(V), \forall \varphi \in H^s_{\tilde{K}}(V), |\langle \varphi, v\rangle| \le C\|P^*v\|_{-s-m+\mu}\|\varphi\|_s,$$

and since $\tilde{K}$ is a neighborhood of K, it gives $\|v\|_{-s} \le C\|P^*v\|_{-s-m+\mu}$, for all $v \in C^\infty_K(V)$. Choosing K as a compact neighborhood of x_0 included in V, we can take $W = \mathring{K}$ to obtain the lemma. □

Lemma 4.4.6. *Let Ω be an open subset of $\mathbb{R}^n$, $x_0 \in \Omega$, $m \in \mathbb{R}$ and $P \in \Psi^m_{ps}(\Omega)$ a properly supported pseudo-differential operator. Let $s \in \mathbb{R}$. Assume that there exists $\mu \ge 0$ and an open neighborhood $V \subset \Omega$ of x_0 such that,*

$$\exists C > 0, \forall v \in C^\infty_c(V), \quad \|v\|_{-s} \le C\|P^*v\|_{-s-m+\mu}. \tag{4.4.7}$$

Then, for all $f \in H^s_{loc}(\Omega)$, there exists $u \in H^{s+m-\mu}(\Omega)$ such that $Pu = f$ in V.

Proof. Let $f_0 \in H^s_{loc}(\Omega)$. The inequality (4.4.7) implies the injectivity of P^* on $C^\infty_c(V)$. Assuming as we may that $V \Subset \Omega$, we get that the space $P^*(C^\infty_c(V))$ is a subspace of $C^\infty_K(\Omega)$, where K is a compact subset of Ω. We consider $P^*(C^\infty_c(V))$ as a subspace of $H_0^{-s-m+\mu}(\Omega)$ and we can define the linear form

$$P^*(C^\infty_c(V)) \ni P^*v \mapsto \langle v, f_0\rangle$$

which satisfies the following estimate: with $\chi \in C^\infty_c(\Omega)$ equal to 1 on $\overline{V}$, we have

$$|\langle v, f_0\rangle| \le \|v\|_{-s}\|\chi f_0\|_s \le C\|P^*v\|_{-s-m+\mu}\|\chi f_0\|_s.$$

We can extend this linear form to the whole $H_0^{-s-m+\mu}(\Omega)$ to a linear form ξ with norm $\le C\|\chi f_0\|_s$ by the Hahn-Banach theorem. This means that there exists $u_0 \in H^{s+m-\mu}(\Omega)$ such that

$$\forall g \in H_0^{-s-m+\mu}(\Omega), \quad \langle g, u_0\rangle = \xi(g), \quad \|u_0\|_{H^{s+m-\mu}(\Omega)} \le C\|\chi f_0\|_s,$$

and in particular for all $v \in C^\infty_c(V)$,

$$\langle v, f_0\rangle = \xi(P^*v) = \langle P^*v, u_0\rangle = \langle v, Pu_0\rangle$$

and thus $Pu_0 = f_0$ on V. □

Remark 4.4.7. Note in particular that if the estimate

$$\|v\|_{\sigma+m-\mu} \le C\|P^*v\|_\sigma$$

is proven true for any $\sigma \in \mathbb{R}$, for $v \in C^\infty_c(V_\sigma)$, where V_σ is a neighborhood of x_0 (which may depend on σ), then the result of the lemma holds and P is locally solvable at x_0 with loss of μ derivatives (see Remark 4.4.3). The estimate above can be true for $\mu = 0$ if and only if P is elliptic at x_0. Moreover the two previous lemmas show that the local solvability questions for a properly supported operator are equivalent to proving an a priori estimate of the type (4.4.3), (4.4.7).

Lemma 4.4.8. *Let Ω be an open subset of $\mathbb{R}^n$, $x_0 \in \Omega$, $m \in \mathbb{R}$ and $P \in \Psi^m_{ps}(\Omega)$ a properly supported pseudo-differential operator. If P is solvable at x_0, then there exists a neighborhood V of x_0 and an integer N such that*

$$\forall f \in C^N(\Omega), \quad \exists u \in \mathscr{E}'^N(\Omega), \quad \text{with } Pu = f \text{ in } V. \tag{4.4.8}$$

Proof. In fact, from (4.4.3), the estimate

$$C\|P^* v\|_N \geq \|v\|_{-N} \tag{4.4.9}$$

holds for all $v \in C^\infty_c(V)$ for some neighborhood V of x_0 and some $N \in \mathbb{N}$. We may assume $V \Subset \Omega$. Let $f_0 \in C^N(\Omega)$. The inequality (4.4.9) implies the injectivity of P^* on $C^\infty_c(V)$. We consider the space $P^*(C^\infty_c(V))$ as a subspace of $C^N_K(\Omega)$, where K is a compact subset of Ω and we can define the linear form

$$P^*(C^\infty_c(V)) \ni P^* v \mapsto \langle v, f_0 \rangle$$

which satisfies the following estimate: with $\chi \in C^\infty_c(\Omega)$, $\chi = 1$ near $\overline{V}$, we have

$$|\langle v, f_0\rangle| \leq \|v\|_{-N}\|\chi f_0\|_N \leq C\|P^* v\|_N \|\chi f_0\|_N \leq C_1 \|\chi f_0\|_N \sup_{\substack{|\alpha|\leq N \\ x \in K}} |\partial^\alpha (P^* v)(x)|.$$

By the Hahn-Banach theorem, we can extend this linear form to a linear form ξ defined on $C^N(\Omega)$ such that

$$\forall g \in C^N(\Omega), \quad |\xi(g)| \leq C_1 \|\chi f_0\|_N \sup_{\substack{|\alpha|\leq N \\ x \in K}} |\partial^\alpha g(x)|.$$

This means that there exists $u_0 \in \mathscr{E}'^N(\Omega)$ such that $\forall g \in C^N(\Omega)$, $\langle g, u_0\rangle = \xi(g)$ and in particular for all $v \in C^\infty_c(V)$,

$$\langle v, f_0\rangle = \xi(P^* v) = \langle P^* v, u_0 \rangle = \langle v, P u_0\rangle$$

and thus $Pu_0 = f_0$ on V. □

Remarks on solvability with loss of μ derivative(s)

To establish local solvability at x_0 with loss of μ derivatives, it is enough to prove (see Lemma 4.4.6 and Remark 4.4.7) that for every $s \in \mathbb{R}$, there exists $r > 0$ and $C > 0$ such that

$$\forall v \in C^\infty_c(B(x_0, r)), \quad C\|P^* v\|_{-s-m+\mu} \geq \|v\|_{-s}. \tag{4.4.10}$$

However, we may be able to prove a weaker estimate only for some s. The next lemma establishes local solvability as a consequence of a weak estimate.

Lemma 4.4.9. *Let Ω be an open subset of $\mathbb{R}^n$, $x_0 \in \Omega, m \in \mathbb{R}, \mu \geq 0, s < n/2$ and $P \in \Psi^m_{ps}(\Omega)$ a properly supported pseudo-differential operator. Let us assume that there exists $r > 0, C > 0$ such that*

$$\forall v \in C_c^\infty(B(x_0, r)), \quad C\|v\|_{-s-1} + C\|P^* v\|_{-s-m+\mu} \geq \|v\|_{-s}. \tag{4.4.11}$$

Then, there exists $r > 0, C > 0$ such that (4.4.10) holds and for all $f \in H^s_{loc}(\Omega)$, there exists $u \in H^{s+m-\mu}(\Omega)$ such that $Pu = f$ in V, where V is some neighborhood of x_0. In particular P is locally solvable at x_0.

Proof. We get that $\|u\|_{-s-1} \leq \phi(r)\|u\|_{-s}$ with $\lim_{r\to 0} \phi(r) = 0$, provided $-s > -n/2$ and this proves that shrinking r leads to (4.4.10) which implies the lemma by applying Lemma 4.4.6. □

On the other hand, we may also want to stick on our definition 4.4.2 of solvability with loss of μ derivatives for which we need to prove an estimate for every $s \in \mathbb{R}$.

Lemma 4.4.10. *Let Ω be an open subset of $\mathbb{R}^n$, $x_0 \in \Omega$, $m \in \mathbb{R}, \mu \geq 0$ and $P \in \Psi^m_{ps}(\Omega)$ a properly supported pseudo-differential operator, with homogeneous principal symbol p_m such that*

$$\mathbb{S}^{n-1} \ni \xi \mapsto p_m(x_0, \xi) \text{ is not identically 0.} \tag{4.4.12}$$

Let us assume that for every $s \in \mathbb{R}$, there exists $r > 0, C_1, C_2$ such that

$$\forall v \in C_c^\infty(B(x_0, r)), \quad C_2\|v\|_{-s-1} + C_1\|P^* v\|_{-s-m+\mu} \geq \|v\|_{-s}. \tag{4.4.13}$$

Then, for every $s \in \mathbb{R}$, there exists $r > 0, C > 0$ such that (4.4.10) holds.

Proof. If (4.4.10) were not true, we could find a sequence $(v_k)_{k\geq 1}$ such that $v_k \in C^\infty_c(B(x_0, k^{-1}))$ with $\|v_k\|_{-s} = 1$ and $\lim_k \|P^* v_k\|_{-s-m+\mu} = 0$. The estimate (4.4.13) implies that $C_2\|v_k\|_{-s-1} \geq 1/2$ for k large enough and since the sequence (v_k) is compact in H^{-s-1}, it has a subsequence strongly convergent in H^{-s-1} towards $v_0 \neq 0$, which is a weak limit in H^{-s}. We have

$$P^* v_0 = 0, \quad \operatorname{supp} v_0 = \{x_0\}, \quad \text{so that } v_0 = Q(D)\delta_{x_0}$$

where Q is a non-zero polynomial. As a consequence, we have for $u \in C^\infty_c(\Omega)$

$$\langle \bar{Q}(D)Pu, \delta_{x_0}\rangle = \langle u, P^* v_0\rangle = 0 \Longrightarrow (\bar{Q}(D)Pu)(x_0) = 0, \text{ for all } u \in C^\infty_c(\Omega),$$

and that, for all $\xi \in \mathbb{R}^n\backslash\{0\}$, $\bar{q}(\xi)p(x_0, \xi) = 0$, where q is the principal part of Q (homogeneous with degree ν) and p the principal symbol of P, since

$$\bar{q}(\xi)p(x_0, \xi)u(x) = \lim_{t\to+\infty} \big((\bar{Q}(D)P)(e^{2i\pi t\langle\cdot,\xi\rangle}u(\cdot))\big)(x_0)t^{-m-\nu}e^{-2i\pi t\langle x_0,\xi\rangle}.$$

Since q is a non-zero polynomial[9], this implies that $\xi \mapsto p(x_0, \xi)$ is identically zero, contradicting the assumption. The proof of the lemma is complete. □

[9]The open set $\{\xi \in \mathbb{R}^n, q(\xi) \neq 0\}$ is dense since the closed set $\{\xi \in \mathbb{R}^n, q(\xi) = 0\}$ cannot have interior points because q is a non-zero polynomial.

Lemma 4.4.11. *Let Ω be an open subset of $\mathbb{R}^n$, $x_0 \in \Omega, m \in \mathbb{R}$ and $P \in \Psi^m_{ps}(\Omega)$ a properly supported pseudo-differential operator. To get* $(4.4.11)_s$ *it suffices to prove that for every $\xi_0 \in \mathbb{S}^{n-1}$, there exists some $\varphi \in S^0_{1,0}$ non-characteristic at (x_0, ξ_0), such that*

$$\begin{aligned}&\exists C, \exists r_0 > 0, \forall r \in]0, r_0], \exists A(r), \exists \varepsilon(r) \text{with} \lim_{r\to 0} \varepsilon(r) = 0, \forall v \in C^\infty_c(B(x_0, r)),\\ &\quad \varepsilon(r)\|v\|_{\sigma+m-\mu} + A(r)\|v\|_{\sigma+m-\mu-1} + C\|P^*v\|_\sigma \ge \|\varphi^w v\|_{\sigma+m-\mu},\end{aligned} \tag{4.4.14}$$

with $\sigma = -s - m + \mu$.

Proof. To get $(4.4.11)_s$ with $s = -\sigma - m + \mu$, it is enough to prove that, for every $\xi_0 \in \mathbb{S}^{n-1}$, there exists some $\varphi \in S^0_{1,0}$ non-characteristic at (x_0, ξ_0), $r > 0, C > 0$, such that

$$\forall v \in C^\infty_c(B(x_0, r)), \quad C\|v\|_{\sigma+m-\mu-1} + C\|P^*v\|_\sigma \ge \|\varphi^w v\|_{\sigma+m-\mu}. \tag{4.4.15}$$

In fact, if (4.4.15) holds, one can find finitely many $\varphi_1, \dots, \varphi_\nu$ such that

$$\sum_{1\le j\le \nu} |\varphi_j|^2 \quad \text{is elliptic at } x_0$$

and for all $v \in C^\infty_c(B(x_0, r_0))$ (r_0 is the minimum of the $r_j > 0$ corresponding to each φ_j),

$$\begin{aligned}\|v\|^2_{\sigma+m-\mu} &\le C_1 \sum_{1\le j\le \nu} \|\varphi_j^w v\|^2_{\sigma+m-\mu} + C_2\|v\|^2_{\sigma+m-\mu-1}\\ &\le C_1\nu 2C^2\|P^*v\|^2_\sigma + (C_2 + C_1\nu 2C^2)\|v\|^2_{\sigma+m-\mu-1},\end{aligned}$$

which gives $C\|v\|_{\sigma+m-\mu-1} + C\|P^*v\|_\sigma \ge \|v\|_{\sigma+m-\mu}$. The same argument as above gives the implication $(4.4.14)_\sigma \Longrightarrow (4.4.11)_s$. □

Remark 4.4.12. Assume in particular that $m = 1$ and that, on a conic neighborhood of $(0; e_n)$ the principal symbol of p is

$$\xi_1 + q(x_1, x', \xi'), \quad q \text{ complex-valued.} \tag{4.4.16}$$

Considering a positively-homogeneous function of n variables χ_0 supported in

$$\{\xi \in \mathbb{R}^n, |\xi_1| \le C_1|\xi'|\},$$

equal to 1 on $\{\xi \in \mathbb{R}^n, |\xi_1| \le C_0|\xi'|, |\xi| \ge 1\}$, and ψ_0 a positively-homogeneous function of $n-1$ variables supported in a conic-neigborhood of $\xi'_0 = (0, \dots, 0, 1) \in \mathbb{R}^{n-1}$ and equal to one on a conic-neighborhood of ξ'_0, the symbol

$$l_1(x, \xi) = \xi_1 + q(x_1, x', \xi')\psi_0(\xi')\chi_0(\xi) \in S^1_{1,0}$$

coincides with p on some conic-neighborhood of $(0, e_n)$ and we have

$$l_1^w = D_1 + \big(\underbrace{q(x_1, x', \xi')\psi_0(\xi')}_{q_1}\big)^w - \big(q(x_1, x', \xi')\psi_0(\xi')(1 - \chi_0(\xi))\big)^w.$$

The symbol q_1 does not belong to $S^1_{1,0}(\mathbb{R}^{2n})$, but only to $S^1_{1,0}(\mathbb{R}^{2n-2}_{x',\xi'})$, uniformly with respect to x_1. Let us assume that for $v \in \mathscr{S}(\mathbb{R}^n), \operatorname{supp} v \subset \{|x_1| \le T\}$,

$$C\|D_1 v + q_1^w v\|_0 \ge T^{-1}\|v\|_0. \tag{4.4.17}$$

We consider χ_1 positively-homogeneous function of n variables supported in $\{\xi \in \mathbb{R}^n, |\xi_1| \le C_0|\xi'|\}$, and we apply (4.4.17) to $\rho(x_1T^{-1})\chi_1^w u$ where $u \in \mathscr{S}(\mathbb{R}^n)$, supported in $|x_1| \le T/2$ and $\rho_1 \in C^\infty_{[-2,2]}(\mathbb{R})$, equal to 1 on $[-1,1]$: we get

$$2C\|l_1^w \rho(x_1T^{-1})\chi_1^w u + \big((1-\chi_0)q_1\big)^w \rho(x_1T^{-1})\chi_1^w u\| \\ \ge T^{-1}\|\rho(x_1T^{-1})\chi_1^w u\|_0.$$

The term $\big((1-\chi_0)q_1\big)^w \rho(x_1T^{-1})\chi_1^w$ has a symbol in $S^{-\infty}_{1,0}(\mathbb{R}^{2n})$ (to be checked directly by the composition formula whose expansion is 0). The term

$$[l_1^w, \rho(x_1T^{-1})]\chi_1^w u = (2i\pi T)^{-1}\underbrace{\rho'(x_1T^{-1})}_{=0 \text{ on } [-T,T]}\ \chi_1^w\ \underbrace{\rho(2x_1T^{-1})}_{\substack{\text{supported in}\\ [-T,T]}} u = r_T^w u$$

$r_T \in S^{-\infty}_{1,0}(\mathbb{R}^{2n})$. The term $[l_1^w, \chi_1^w]$ is L^2-bounded and we have thus

$$\|\chi_1^w l_1^w u\|_0 + \|u\|_0 + \alpha(T)\|u\|_{-1} \ge \frac{c_0}{T}\|\chi_1^w u\|$$

which gives (4.4.14) for $m = 1, \sigma = 0, \mu = 1$. Staying with the case $\mu = 1$ (loss 1), the argument is not different for other values of m, σ. We can of course replace the assumption (4.4.16) by $e(x,\xi)(\xi_1 + q(x_1, x', \xi'))$ where e is elliptic on a conic-neighborhood of $(0; e_n)$.

Remark 4.4.13. On the other hand, when $\mu > 1$, the rhs in the estimate (4.4.17) has to be replaced by $\|v\|_{1-\mu}$ and the fact that this suffices to prove local solvability requires some particular care.

Operators of real principal type

Theorem 4.4.14. *Let Ω be an open subset of $\mathbb{R}^n$, $m \in \mathbb{R}$ and $P \in \Psi^m_{ps}(\Omega)$ a pseudo-differential operator with symbol p such that (4.3.1) holds with p_m real-valued and positively-homogeneous of degree m. We assume that P is of principal type, i.e.*

$$\forall (x,\xi) \in \dot{T}^*(\Omega), \quad p_m(x,\xi) = 0 \Longrightarrow \partial_\xi p_m(x,\xi) \neq 0. \tag{4.4.18}$$

Then the operator P is locally solvable at every point of Ω with loss of one derivative.

Proof. One could of course use the estimate (4.3.16), but the argument for solvability alone is so simple in that case that it may be worthy to look at it. We consider a point $x_0 \in \Omega$. If p_m is elliptic at x_0, the estimate

$$C\|v\|_{\sigma+m-1} + C\|P^*v\|_\sigma \ge \|v\|_{\sigma+m}$$

follows from Theorem 4.2.8 for v supported in $B(x_0, r_0)$ with $r_0 > 0$ small enough. If there exists $\xi_0 \neq 0$ such that $p_m(x_0, \xi_0) = 0$, we may choose the coordinates so that $x_0 = 0, \xi_0 = e_n$ and

$$p_m(x, \xi) = (\xi_1 + a(x_1, x', \xi'))e(x, \xi)$$

with a homogeneous of degree 1 with respect to ξ', e elliptic with degree $m-1$ on a conic-neighborhood of $(0, e_n)$. According to Remark 4.4.12 the question reduces to proving an estimate for the operator $L = D_{x_1} + a(x_1, x', \xi')^w$ where $a \in C^\infty(\mathbb{R}^{2n-1}; \mathbb{R})$ such that, for all α, β,

$$\sup_{(x,\xi')\in\mathbb{R}^n\times\mathbb{R}^{n-1}} |(\partial_{\xi'}^\alpha \partial_x^\beta a)(x, \xi')| \langle\xi'\rangle^{-1+|\alpha|} < \infty.$$

We find

$$2\operatorname{Re}\langle D_{x_1}u + a(x_1, x', \xi')^w u, ix_1 u\rangle_{L^2} = \frac{1}{2\pi}\|u\|_0^2, \tag{4.4.19}$$

so that for $u \in C_c^\infty(\mathbb{R}^n)$, $u = 0$ on $|x_1| \geq T$, we have $2\|Lu\|_0 T\|u\|_0 \geq \frac{1}{2\pi}\|u\|_0^2$ and thus $\|Lu\|_0 \geq \frac{1}{4\pi T}\|u\|_0$. □

Operators of principal type, complex symbols with a nonnegative imaginary part

Theorem 4.4.15. *Let Ω be an open subset of $\mathbb{R}^n$, $m \in \mathbb{R}$ and $P \in \Psi_{ps}^m(\Omega)$ a pseudo-differential operator with symbol p such that (4.3.1) holds with p_m complex-valued and positively-homogeneous of degree m. We assume that P is of principal type (see (4.4.18)) such that the function $\operatorname{Im} p_m$ is nonnegative (resp. nonpositive). Then the operator P is locally solvable at every point of Ω with loss of one derivative.*

Proof. To handle the complex-valued case, we see that the principal type condition (4.4.18) implies that at a non-elliptic point,

$$p_m(x_0, \xi_0) = 0,\ d(\operatorname{Im} p_m)(x_0, \xi_0) = 0,\ \partial_\xi(\operatorname{Re} p_m)(x_0, \xi_0) \neq 0.$$

The first thing that we can do is to use the estimate (4.3.15): assuming $m = 1$, the bicharacteristic curve $\dot\gamma = H_{\operatorname{Re} p_m}(\gamma)$ starting at $\gamma(0) = (x_0, \xi_0)$ we find $A_0, A_1 \in \Psi_{ps}^0(\Omega)$, A_0 elliptic at $\gamma(0)$, $\operatorname{essupp} A_1 \subset W_1$, W_1 conic-neighborhood of $\gamma(3\varepsilon_0)$ where $\varepsilon_0 > 0$, $\gamma(0) \notin W_1$, $M \in \Psi_{ps}^0(\Omega)$,

$$\|A_0 v\|_0^2 \leq \|A_1 v\|_0^2 + \|MPv\|_0^2 + (C_1 + C_2)\|v\|_{-1/2}^2. \tag{4.4.20}$$

Applying this to $v \in C_c^\infty(B(x_0, r))$ with r small enough, the principal-type assumption $\partial_\xi \operatorname{Re} p_m(x_0, \xi_0) \neq 0$ implies, with χ_0 supported in the unit ball of $\mathbb{R}^n$, that $A_1 v = A_1\chi_0\big((\cdot - x_0)/r\big)v$ with $\operatorname{essupp} A_1\chi_0\big((\cdot - x_0)/r\big) = \emptyset$, so that (4.4.14) holds with $\mu = 1, m = 1, \sigma = 0$. The other cases are analogous.

On the other hand, it is also interesting to find directly a multiplier method, as in the real-principal type case. We need only to handle

$$L_\pm = D_{x_1}u + a(x_1, x', \xi')^w u \pm ib(x, \xi)^w + r_0(x, \xi)^w$$

with $b \in S^1_{1,0}, b \ge 0$, $a(x_1, \cdot, \cdot)$ real-valued in $S^1_{1,0}(\mathbb{R}^{2n-2}_{x',\xi'})$ uniformly in x_1 and $r_0 \in S^0_{1,0}$. With $\theta \in C^\infty(\mathbb{R};\mathbb{R})$, we calculate

$$2\,\mathrm{Re}\langle D_{x_1}u + a(x_1,x',\xi')^w u + ib(x,\xi)^w u, i\theta(x_1)^2 u\rangle_{L^2} = \frac{1}{\pi}\langle\theta\theta' u, u\rangle + 2\,\mathrm{Re}\langle b^w u, \theta^2 u\rangle.$$

We have

$$\theta^2 b^w + b^w\theta^2 = \theta\theta b^w + b^w\theta\theta = \theta[\theta, b^w] + 2\theta b^w\theta + [b^w,\theta]\theta = 2\theta b^w\theta + [[b^w,\theta],\theta]$$

and thus, from Gårding inequality (Theorem 3.5.1), we find

$$2\,\mathrm{Re}\langle L_+u, i\theta^2 u\rangle \ge \frac{1}{\pi}\langle\theta\theta' u, u\rangle + \langle[[b^w,\theta],\theta]u,u\rangle - C_0\|\theta u\|^2 - \|[r_0,\theta]u\|^2$$

where C_0 depends on semi-norms of b, r_0; to handle the term r_0, we have used

$$\langle r_0^w u, \theta^2 u\rangle = \langle[\theta, r_0^w]u, \theta u\rangle + \langle r_0^w\theta u, \theta u\rangle.$$

We have with $\lambda > 0$, $\theta(x_1) = e^{\lambda x_1}$, for $u \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ vanishing at $|x_1| \ge 1/\lambda$ (so that with $\chi_0 \in C^\infty_{\mathrm{c}}(\mathbb{R})$ equal to 1 on $[-1,1]$, $u = \chi_0(\lambda x_1)u$)

$$2\,\mathrm{Re}\langle L_+u, ie^{2\lambda x_1}u\rangle \ge (\pi^{-1}\lambda - C_0)\|e^{\lambda x_1}u\|^2 - C(\lambda)\|u\|^2_{-1/2}$$

implying $2\pi\lambda^{-1}\|\chi_0(\lambda x_1)e^{\lambda x_1}L_+u\|_0^2 + C(\lambda)\|u\|^2_{-1/2} \ge (\frac{\lambda}{2\pi} - C_0)\|e^{\lambda x_1}u\|^2$, and assuming χ_0 valued in $[0,1]$, vanishing on $(-2,2)^c$, and $\lambda \ge 4\pi C_0$,

$$2\pi\lambda^{-1}e^4\|L_+u\|_0^2 + C(\lambda)\|u\|^2_{-1/2} \ge \frac{\lambda}{4\pi e^2}\|u\|^2.$$

Choosing $\lambda = 1 + 4\pi C_0$, we find that there exists $r_0 > 0$ such that, for $u \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, with $\mathrm{diameter}(\mathrm{supp}\,u) \le r_0$

$$C_1\|L_+u\|_0 + C_1\|u\|_{-1/2} \ge \|u\|_0,$$

proving the local solvability of L_+^*; the case of L_-^* is analogous. □

4.5 Pseudo-differential operators in harmonic analysis

Singular integrals, examples

The Hilbert transform

A basic object in the classical theory of harmonic analysis is the Hilbert transform, given by the one-dimensional convolution with $pv(1/\pi x) = \frac{d}{\pi dx}(\ln|x|)$, where we consider here the distribution derivative of the $L^1_{\mathrm{loc}}(\mathbb{R})$ function $\ln|x|$. We can also compute the Fourier transform of $pv(1/\pi x)$, which is given by $-i\,\mathrm{sign}\,\xi$ (see e.g.

(1.2.26)). As a result the Hilbert transform $\mathscr{H}$ is a unitary operator on $L^2(\mathbb{R})$ defined by

$$\widehat{\mathscr{H}u}(\xi) = -i \operatorname{sign} \xi \hat{u}(\xi). \tag{4.5.1}$$

It is also given by the formula

$$(\mathscr{H}u)(x) = \lim_{\epsilon \to 0_+} \frac{1}{\pi} \int_{|x-y| \geq \epsilon} \frac{u(y)}{x-y} dy.$$

The Hilbert transform is certainly the first known example of a *Fourier multiplier* ($\mathscr{H}u = F^{-1}(a\hat{u})$ with a bounded a). Since the sign function is bounded, it is obviously bounded on $L^2(\mathbb{R})$, but is is tempting to relate that result to Theorem 3.4.2 of L^2-boundedness of the $S^0_{1,0}$ class; naturally the singularity at 0 of the sign function prevents it to be a symbol in that class.

The Riesz operators, the Leray-Hopf projection

The Riesz operators are the natural multidimensional generalization of the Hilbert transform. We define for $u \in L^2(\mathbb{R}^n)$,

$$\widehat{R_j u}(\xi) = \frac{\xi_j}{|\xi|} \hat{u}(\xi), \quad \text{so that } R_j = D_j/|D| = (-\Delta)^{-1/2} \frac{\partial}{i\partial x_j}. \tag{4.5.2}$$

The R_j are selfadjoint bounded operators on $L^2(\mathbb{R}^n)$ with norm 1.

We can also consider the $n \times n$ matrix of operators given by $Q = R \otimes R = (R_j R_k)_{1 \leq j,k \leq n}$ sending the vector space of $L^2(\mathbb{R}^n)$ vector fields into itself. The operator Q is selfadjoint and is a projection since $\sum_l R_l^2 = \operatorname{Id}$ so that $Q^2 = (\sum_l R_j R_l R_l R_k)_{j,k} = Q$. As a result the operator

$$\mathbb{P} = \operatorname{Id} - R \otimes R = \operatorname{Id} - |D|^{-2}(D \otimes D) = \operatorname{Id} - \Delta^{-1}(\nabla \otimes \nabla) \tag{4.5.3}$$

is also an orthogonal projection, the Leray-Hopf projector (a.k.a. the Helmholtz-Weyl projector); the operator $\mathbb{P}$ is in fact the orthogonal projection onto the closed subspace of L^2 vector fields with null divergence. We have for a vector field $u = \sum_j u_j \partial_j$, the identities $\operatorname{grad} \operatorname{div} u = \nabla(\nabla \cdot u)$, $\operatorname{grad} \operatorname{div} = \nabla \otimes \nabla = (-\Delta)(iR \otimes iR)$, so that

$$Q = R \otimes R = \Delta^{-1} \operatorname{grad} \operatorname{div}, \quad \operatorname{div} R \otimes R = \operatorname{div},$$

which implies $\operatorname{div} \mathbb{P}u = \operatorname{div} u - \operatorname{div}(R \otimes R)u = 0$, and if $\operatorname{div} u = 0$, $\mathbb{P}u = u$. The Leray-Hopf projector is in fact the $(n \times n)$-matrix-valued Fourier multiplier given by $\operatorname{Id} - |\xi|^{-2}(\xi \otimes \xi)$. This operator plays an important role in fluid mechanics since the Navier-Stokes system for incompressible fluids can be written for a given divergence-free v_0,

$$\begin{cases} \partial_t v + \mathbb{P}\big((v \cdot \nabla)v\big) - \nu \triangle v = 0 \\ \mathbb{P}v = v, \\ v_{|t=0} = v_0. \end{cases}$$

As already said for the Riesz operators, $\mathbb{P}$ is *not* a classical pseudo-differential operator, because of the singularity at the origin: however it is indeed a Fourier multiplier with the same functional properties as those of R.

In three dimensions the curl operator is given by the matrix

$$\operatorname{curl} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} = \operatorname{curl}^*, \tag{4.5.4}$$

since we can note that the matrix

$$C(\xi) = 2\pi \begin{pmatrix} 0 & -i\xi_3 & i\xi_2 \\ i\xi_3 & 0 & -i\xi_1 \\ -i\xi_2 & i\xi_1 & 0 \end{pmatrix}$$

is purely imaginary and anti-symmetric, a feature that could not happen for scalar Fourier multiplier. We get also $\operatorname{curl}^2 = -\Delta \operatorname{Id} + \operatorname{grad}\operatorname{div}$ and (the Biot-Savard law)

$$\operatorname{Id} = (-\Delta)^{-1}\operatorname{curl}^2 + \Delta^{-1}\operatorname{grad}\operatorname{div}, \quad \text{also equal to } \ (-\Delta)^{-1}\operatorname{curl}^2 + \operatorname{Id} - \mathbb{P},$$

which gives $\operatorname{curl}^2 = -\Delta\mathbb{P}$, so that

$$[\mathbb{P}, \operatorname{curl}] = \Delta^{-1}\big(\Delta\mathbb{P}\operatorname{curl} - \Delta\operatorname{curl}\mathbb{P}\big) = \Delta^{-1}\big(-\operatorname{curl}^3 + \operatorname{curl}(-\Delta\mathbb{P})\big) = 0,$$
$$\mathbb{P}\operatorname{curl} = \operatorname{curl}\mathbb{P} = \operatorname{curl}(-\Delta)^{-1}\operatorname{curl}^2 = \operatorname{curl}\big(\operatorname{Id} - \Delta^{-1}\operatorname{grad}\operatorname{div}\big) = \operatorname{curl}$$

since $\operatorname{curl}\operatorname{grad} = 0$ (note also the adjoint equality $\operatorname{div}\operatorname{curl} = 0$).

These examples show that some interesting cases of Fourier multipliers are quite close to pseudo-differential operators, with respect to the homogeneity and behaviour for large frequencies, although the singularities at the origin in the momentum space make them slightly different. They belong to the family of singular integrals that we shall review briefly.

Theorem 4.5.1. *Let Ω be a function in $L^1(\mathbb{S}^{n-1})$ such that $\int_{\mathbb{S}^{n-1}} \Omega(\omega)d\sigma(\omega) = 0$. Then the following formula defines a tempered distribution T:*

$$\langle T, \varphi\rangle = \lim_{\epsilon\to 0_+} \int_{|x|\ge\epsilon} \Omega\big(\frac{x}{|x|}\big)|x|^{-n}\varphi(x)dx = -\int (x\cdot\partial_x\varphi(x))\Omega\big(\frac{x}{|x|}\big)|x|^{-n}\ln|x|dx.$$

The distribution T is homogeneous of degree $-n$ on $\mathbb{R}^n$ and, if Ω is odd, the Fourier transform of T is a bounded function.

N.B. We shall use the principal-value notation $T = pv\Big(|x|^{-n}\Omega\big(\frac{x}{|x|}\big)\Big)$; when $n = 1$ and $\Omega = \operatorname{sign}$, we recover the principal value $pv(1/x) = \frac{d}{dx}(\ln|x|)$ which is odd, homogeneous of degree -1, and whose Fourier transform is $-i\pi\operatorname{sign}\xi$ (see e.g. (1.2.26)).

Proof. Let φ be in $\mathscr{S}(\mathbb{R}^n)$ and $\epsilon > 0$. Using polar coordinates, we check

$$\int_{\mathbb{S}^{n-1}} \Omega(\omega)\int_\epsilon^{+\infty} \varphi(r\omega)\frac{dr}{r}d\sigma(\omega)$$
$$= \int_{\mathbb{S}^{n-1}} \Omega(\omega)\Big[\varphi(\epsilon\omega)\ln(\epsilon^{-1}) - \int_\epsilon^{+\infty} \omega\cdot d\varphi(r\omega)\ln r dr\Big]d\sigma(\omega).$$

Since the mean value of Ω is 0, we get the first statement of the theorem, noticing that the function $x \mapsto \Omega(x/|x|)|x|^{-n+1}\ln(|x|)(1+|x|)^{-2}$ is in $L^1(\mathbb{R}^n)$. We have

$$\langle x\cdot\partial_x T, \varphi\rangle = -\langle T, x\cdot\partial_x\varphi\rangle - n\langle T,\varphi\rangle \tag{4.5.5}$$

and we see that

$$\begin{aligned}\langle T, x\cdot\partial_x\varphi\rangle &= \lim_{\epsilon\to 0_+}\int_{\mathbb{S}^{n-1}}\Omega(\omega)\int_\epsilon^{+\infty} r\omega\cdot(d\varphi)(r\omega)\frac{dr}{r}d\sigma(\omega)\\ &= \int_{\mathbb{S}^{n-1}}\Omega(\omega)\int_0^{+\infty}\omega\cdot(d\varphi)(r\omega)drd\sigma(\omega)\\ &= \int_{\mathbb{S}^{n-1}}\Omega(\omega)\int_0^{+\infty}\frac{d}{dr}\big(\varphi(r\omega)\big)drd\sigma(\omega) = -\varphi(0)\int_{\mathbb{S}^{n-1}}\Omega(\omega)d\sigma(\omega)=0\end{aligned}$$

so that (4.5.5) implies that $x\cdot\partial_x T = -nT$ which is the homogeneity of degree $-n$ of T. As a result the Fourier transform of T is an homogeneous distribution with degree 0.

N.B. Note that the formula $-\int(x\cdot\partial_x\varphi(x))\Omega(\frac{x}{|x|})|x|^{-n}\ln|x|dx$ makes sense for $\Omega\in L^1(\mathbb{S}^{n-1}), \varphi\in\mathscr{S}(\mathbb{R}^n)$ and defines a tempered distribution. For instance, if $n=1$ and $\Omega=1$, we get the distribution derivative $\frac{d}{dx}\big(\operatorname{sign} x\ln|x|\big)$. However, the condition of mean value 0 for Ω on the sphere is necessary to obtain T as a principal value, since in the discussion above, the term factored out by $\ln(1/\epsilon)$ is $\int_{\mathbb{S}^{n-1}}\Omega(\omega)\varphi(\epsilon\omega)d\sigma(\omega)$ which has the limit $\varphi(0)\int_{\mathbb{S}^{n-1}}\Omega(\omega)d\sigma(\omega)$. On the other hand, from the defining formula of T, we get with $\Omega_j(\omega)=\frac12(\Omega(\omega)+(-1)^j\Omega(-\omega))$ (Ω_1(resp.Ω_2) is the odd (resp. even) part of Ω)

$$\begin{aligned}\langle T,\varphi\rangle &= \int_{\mathbb{S}^{n-1}}\Omega(\omega)\langle\frac{d}{dt}\big(H(t)\ln t\big),\varphi(t\omega)\rangle_{\mathscr{S}'(\mathbb{R}_t),\mathscr{S}(\mathbb{R}_t)}d\sigma(\omega)\\ &= \int_{\mathbb{S}^{n-1}}\Omega_1(\omega)\langle pv(\frac{1}{2t}),\varphi(t\omega)\rangle_{\mathscr{S}'(\mathbb{R}_t),\mathscr{S}(\mathbb{R}_t)}d\sigma(\omega)\\ &\qquad + \int_{\mathbb{S}^{n-1}}\Omega_2(\omega)\langle\frac{d}{dt}\big(H(t)\ln t\big),\varphi(t\omega)\rangle_{\mathscr{S}'(\mathbb{R}_t),\mathscr{S}(\mathbb{R}_t)}d\sigma(\omega),\end{aligned}\tag{4.5.6}$$

since

$$A_1=\int_{\mathbb{S}^{n-1}}\Omega_1(\omega)\langle pv(\frac{1}{2t}),\varphi(t\omega)\rangle d\sigma(\omega) = -\frac12\int_{\mathbb{S}^{n-1}}\Omega_1(\omega)\langle\ln|t|,\omega\cdot d\varphi(t\omega)\rangle d\sigma(\omega)$$

and $\langle\ln|t|,\omega\cdot d\varphi(t\omega)\rangle = \int_0^{+\infty}\omega\cdot d\varphi(t\omega)\ln t dt + \int_0^{+\infty}\omega\cdot d\varphi(-s\omega)(\ln s)ds$ so that

$$A_1=\int_{\mathbb{S}^{n-1}}\Omega_1(\omega)\langle H(t)\ln t, \underbrace{-\frac12\omega\cdot\big(d\varphi(t\omega)+d\varphi(-t\omega)\big)}_{-\frac12\frac{d}{dt}\left(\varphi(t\omega)-\varphi(-t\omega)\right)}\rangle d\sigma(\omega)$$

and thus since Ω_1 is odd,

$$\begin{aligned}A_1&=\int_{\mathbb{S}^{n-1}}\Omega_1(\omega)\langle\frac{d}{dt}\big(H(t)\ln t\big),\frac12(\varphi(t\omega)-\varphi(-t\omega))\rangle d\sigma(\omega)\\ &= \int_{\mathbb{S}^{n-1}}\Omega_1(\omega)\langle\frac{d}{dt}\big(H(t)\ln t\big),\varphi(t\omega)\rangle d\sigma(\omega).\end{aligned}$$

Let us show that, when Ω is odd, the Fourier transform of T is bounded. Using (4.5.6) and (1.2.26) we get

$$\langle \hat{T}, \psi \rangle = \int_{\mathbb{S}^{n-1}} \Omega(\omega) \langle pv(\frac{1}{2t}), \widehat{\psi}(t\omega) \rangle d\sigma(\omega)$$
$$= -\frac{i\pi}{2} \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \Omega(\omega) \operatorname{sign}(\omega \cdot \xi) \varphi(\xi) d\xi d\sigma(\omega)$$

proving that

$$\hat{T}(\xi) = -\frac{i\pi}{2} \int_{\mathbb{S}^{n-1}} \Omega(\omega) \operatorname{sign}(\omega \cdot \xi) d\sigma(\omega) \tag{4.5.7}$$

which is indeed a bounded function since $\Omega \in L^1(\mathbb{S}^{n-1})$. □

4.6 Remarks on the Calderón-Zygmund theory and classical pseudo-differential operators

It is possible to generalize Theorem 4.5.1 in several directions. In particular the L^p-boundedness ($1 < p < \infty$) of these homogeneous singular integrals can be established, provided some regularity assumptions are made on $\hat{T}$ (see e.g. Theorem 7.9.5 in [5], the reference books on harmonic analysis by E.M. Stein [24] and J. Duoandikoetxea [2]).

Also a Calderón-Zygmund theory of singular integrals with "variable coefficients", given by some kernel $k(x, y)$ satisfying some conditions analogous to homogeneous functions of degree $-n$ of $x - y$, has reached a high level of refinement (see e.g. the book by R. Coifman & Y. Meyer [1] and the developments in [2]). Although that theory is not independent of the theory of classical pseudo-differential operators, the fact that the symbols do have a singularity at $\xi = 0$ make them quite different ; the constrast is even more conspicuous for the L^p theory of Calderón-Zygmund operators, which is very well understood although its analogue for general pseudo-differential operators (see e.g. [21]) has not reached the same level of understanding. We have seen above that the classes $S^0_{\rho,\delta}$ give rise to L^2-bounded operators provided $0 \le \delta \le \rho \le 1, \delta < 1$, and it is possible to prove that the operators with symbol in the class $S^0_{1,0}$ are L^p-bounded, $1 < p < \infty$. The method of proof of that result is not significantly different of the proof of the Calderón-Zygmund theorem of L^p-boundedness for standard homogeneous singular integrals and is based on the weak $(1, 1)$ regularity and the Marcinkiewicz interpolation theorem. However, some operators with symbol in the class $S^0_{1,\delta}$ with $0 < \delta < 1$ are not L^p-bounded for $p \neq 2$.

The present book is almost entirely devoted to the developments of the L^2 theory of pseudo-differential operators, but it is certainly useful to keep in mind that some very natural and useful examples of singular integrals are not pseudo-differential operators. For the very important topic of L^p-theory of pseudo-differential operators, we refer the reader to [20].

Chapter 5

The Huygens principle

5.1 First order real principal type operators

We study here the Cauchy problem for the linear first-order equation

$$D_t u + a(t, x, D_x)u = f \quad \text{on } (0, T) \times \mathbb{R}^n_x, \quad u_{t=0} = u_0, \tag{5.1.1}$$

where T is a positive parameter, $a \in C^\infty([0, T] \times \mathbb{R}^{2n})$ such that for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$,

$$\sup_{(t,x,\xi)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n} |(\partial_x^\alpha \partial_\xi^\beta a)(t, x, \xi)\langle\xi\rangle^{-1+|\beta|}| < +\infty, \tag{5.1.2}$$

and

$$\sup_{(t,x,\xi)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n} \operatorname{Im} a(t, x, \xi) < +\infty. \tag{5.1.3}$$

We expect the Cauchy problem to be well-posed so that looking at the ODE

$$\frac{du}{idt} + \big(a_1(t) + ia_2(t)\big)u = 0, \quad a_j \text{ smooth real-valued,,}$$

we have $u = u(0) \exp \int_0^t (-ia_1(s) + a_2(s))ds$ so that $|u(t)| = |u(0)|e^{\int_0^t a_2(s)ds}$, the latter integral remains bounded for $t \geq 0$ whenever $a_2 = \operatorname{Im} a$ is bounded above. This makes Condition (5.1.3) rather natural.

Lemma 5.1.1. *Let a, T be as above and let $\sigma \in \mathbb{R}$. There exists $\lambda(\sigma)$ such that for any $\lambda \geq \lambda(\sigma)$, we have for every $u \in C^1\big([0, T]; H^\sigma(\mathbb{R}^n)\big) \cap C^0\big([0, T]; H^{\sigma+1}(\mathbb{R}^n)\big)$,*

$$\sup_{t\in[0,T]} e^{-\lambda t}\|u(t)\|_{H^\sigma(\mathbb{R}^n)} \leq \|u(0)\|_{H^\sigma(\mathbb{R}^n)} + \int_0^T e^{-\lambda t}\|D_t u + a(t, x, D_x)u\|_{H^\sigma(\mathbb{R}^n)}dt. \tag{5.1.4}$$

N.B. It should be noted that no better estimate than (5.1.4) is satisfied for the ODE $D_t + a(t)$ where a is a complex-valued function with an imaginary part bounded above.

Proof. We assume first that $\sigma = 0$ and we note that, with $L^2(\mathbb{R}^n)$ norms and dot-products, assuming $A_1^* = A_1, A_2^* = -A_2$, $0 \le t \le T$,

$$2\operatorname{Re}\int_0^t \langle D_s u + \big(A_1(s) + iA_2(s)\big)u, -i\mathbf{1}_{[0,t]}(s)u\rangle = -\|u(0)\|^2 + \|u(t)\|^2 + 2\int_0^t \langle -A_2(s)u, u\rangle ds,$$

and assuming that $A_2 \le 0$ (operator inequality) we get with $\mathcal{L} = D_t + A(t)$

$$2\int_0^t \|(\mathcal{L}u)(s)\|\|u(s)\|ds + \|u(0)\|^2 \ge \|u(t)\|^2,$$

and thus

$$\|u(t)\|^2 \le R(t) = 2\int_0^t \|(\mathcal{L}u)(s)\|\|u(s)\|ds + \|u(0)\|^2,$$

so that

$$\dot R = 2\|(\mathcal{L}u)(t)\|\|u(t)\| \le \|(\mathcal{L}u)(t)\| 2R^{1/2}$$

and thus

$$\frac{d}{dt}R^{1/2} \le \|(\mathcal{L}u)(t)\|,$$

so that (for $t \in [0,T]$)

$$\|u(t)\| \le R^{1/2}(t) \le R^{1/2}(0) + \int_0^t \|(\mathcal{L}u)(s)\|ds = \|u(0)\| + \int_0^t \|(\mathcal{L}u)(s)\|ds,$$

the same estimate as for an ODE. We note however that we do not have a priori $A_2 \le 0$ as required above but that Gårding's inequality shows that

$$A_2 \le \beta, \tag{5.1.5}$$

where β is a semi-norm of a. Using the above discussion, we get for $\lambda \ge \beta$,

$$\|v(t)\| \le \|v(0)\| + \int_0^t \|D_s v + A(s)v - i\lambda v\|ds,$$

and setting $v(t) = u(t)e^{-\lambda t}$, this gives, since $D_s - i\lambda = e^{-\lambda s}D_s e^{\lambda s}$,

$$\|u(t)\|e^{-\lambda t} \le \|u(0)\| + \int_0^t \|e^{-\lambda s}(D_s + A(s))e^{\lambda s}e^{-\lambda s}u\|ds,$$

which is the sought result for $\sigma = 0$. To get the result for arbitrary σ, we note that

$$\|u(t)\|_{H^\sigma} = \|\langle D\rangle^\sigma u(t)\|_{L^2},$$

and replacing u by $\langle D\rangle^\sigma u$ in the above inequality for $A(t)$ replaced by $\langle D\rangle^\sigma A(t)\langle D\rangle^{-\sigma}$ (which is a first-order operator whose symbol satisfies (5.1.3)) yields for $\lambda \ge \lambda(\sigma)$, $0 \le t \le T$,

$$\|u(t)\|_{H^\sigma}e^{-\lambda t} \le \|u(0)\|_{H^\sigma} + \int_0^t e^{-\lambda s}\|\langle D\rangle^{-\sigma}\big(D_s + \langle D\rangle^\sigma A(s)\langle D\rangle^{-\sigma}\big)\langle D\rangle^\sigma u\|_{H^\sigma}ds$$
$$= \|u(0)\|_{H^\sigma} + \int_0^t e^{-\lambda s}\|\big(D_s + A(s)\big)u\|_{H^\sigma}ds,$$

completing the proof of the lemma. □

It might be worthy as well to record the Hilbertian lemma proven above, noting that we have used only (5.1.5).

Lemma 5.1.2. *Let $\mathbb{H}$ be a complex Hilbert space, let $T > 0$ be given and let $[0, T] \ni t \mapsto A(t) \in \mathcal{B}(\mathbb{H})$ be a continuous mapping such that*

$$\forall t \in [0, T], \quad \frac{A(t) - A^*(t)}{2i} \leq \beta < +\infty.$$

Then for $\lambda \geq \beta$ and for $u \in C^1([0, T]; \mathbb{H})$, we have with $D_t = -i\partial_t$,

$$\sup_{t\in[0,T]} e^{-\lambda t}\|u(t)\|_{\mathbb{H}} \leq \|u(0)\|_{\mathbb{H}} + \int_0^T e^{-\lambda t}\|D_t u + A(t)u\|_{\mathbb{H}} dt. \tag{5.1.6}$$

We can prove now an existence and uniqueness result based upon the inequalities in Lemma 5.1.1 and the Hahn-Banach Theorem.

Theorem 5.1.3. *Let $T > 0$ and a satisfying (5.1.2) and (5.1.3). Let $\sigma \in \mathbb{R}$. Then for any $f \in L^1([0, T]; H^\sigma(\mathbb{R}^n))$ and any $u_0 \in H^\sigma(\mathbb{R}^n)$, there exists a unique solution of (5.1.1) in $C^0([0, T], H^\sigma(\mathbb{R}^n))$ and we have as well for $\lambda \geq \lambda(\sigma)$,*

$$\sup_{t\in[0,T]} e^{-\lambda t}\|u(t)\|_{H^\sigma(\mathbb{R}^n)} \leq \|u_0\|_{H^\sigma(\mathbb{R}^n)} + \int_0^T e^{-\lambda t}\|f(t)\|_{H^\sigma(\mathbb{R}^n)} dt. \tag{5.1.7}$$

Proof. We start with the proof of uniqueness. We may thus assume by linearity that $f = 0$ and $u(0) = 0$. We have also $a(t, x, D_x)u \in C^0([0, T], H^{\sigma-1})$ and thus $\partial_t u \in C^0([0, T], H^{\sigma-1})$, implying since $u(0) = 0$ that

$$u \in C^1([0, T], H^{\sigma-1}) \cap C^0([0, T], H^\sigma),$$

and we may apply Inequality (5.1.4) for $\lambda \geq \lambda(\sigma - 1)$, entailing that $u = 0$ on $[0, T]$. Let us prove now the existence part of Theorem 5.1.3. Let σ, f and u_0 be given as in the statement of the theorem. For $\phi \in C_c^\infty([0, T) \times \mathbb{R}^n)$, we define

$$\psi = \mathcal{L}^*\phi = D_t\phi + A^*(t)\phi, \quad A(t) = a(t, x, D_x),$$

and it follows from Lemma 5.1.1 for $\lambda \geq \lambda(-\sigma)$,

$$\sup_{t\in[0,T]} e^{-\lambda(T-t)}\|\phi(t)\|_{H^{-\sigma}(\mathbb{R}^n)} \leq \int_0^T e^{-\lambda(T-t)}\|D_t\phi + A^*(t)\phi\|_{H^{-\sigma}(\mathbb{R}^n)} dt,$$

so that

$$\sup_{t\in[0,T]} \|\phi(t)\|_{H^{-\sigma}(\mathbb{R}^n)} \leq C_0 \int_0^T \|\psi\|_{H^{-\sigma}} dt. \tag{5.1.8}$$

As a result, we have

$$\left|\int_0^T \langle f(t), \phi(t)\rangle dt + \langle u_0, \phi(0)\rangle\right| \leq C_1 \int_0^T \|\psi\|_{H^{-\sigma}} dt.$$

We consider the anti-linear form

$$\mathcal{L}^*\big(C_c^\infty([0,T)\times\mathbb{R}^n)\big)\ni\mathcal{L}^*\phi\mapsto\mathcal{T}(\mathcal{L}^*\phi)=\int_0^T\langle f(t),\phi(t)\rangle dt+\langle u_0,\phi(0)\rangle\in\mathbb{C},$$

which is well-defined since $\mathcal{L}^*(\phi_1-\phi_2)=0$ implies $\phi_1=\phi_2$ from (5.1.8), and is such that

$$|\mathcal{T}(\mathcal{L}^*\phi)|\le C_1\int_0^T\|\mathcal{L}^*\phi\|_{H^{-\sigma}}dt.$$

Using the Hahn-Banach theorem, we may extend $\mathcal{T}$ to a continuous anti-linear form on $L^1([0,T];H^{-\sigma})$ and thus we can find $u\in L^\infty([0,T];H^\sigma)$ such that for every $\phi\in C_c^\infty([0,T)\times\mathbb{R}^n)$,

$$\begin{aligned}\int_0^T\langle f(t),\phi(t)\rangle dt+\langle u_0,\phi(0)\rangle=\mathcal{T}(\mathcal{L}^*\phi)&=\int_0^T\langle u,\mathcal{L}^*\phi\rangle dt\\&=\int_0^T\langle u,D_t\phi+A^*(t)\phi\rangle dt=\langle Hu,D_t\phi+A^*(t)(\phi)\rangle\end{aligned}$$

which means that

$$D_t(Hu)+A(t)Hu=f+\frac{1}{i}\delta_0(t)\otimes u_0\quad\text{on }(-\infty,T)\times\mathbb{R}^n,$$

that is

$$D_tu+A(t)u=f\quad\text{on }(0,T),\quad u(0)=u_0,$$

in the distribution sense. If f belongs to $\mathscr{S}(\mathbb{R}^{n+1})$, we obtain from the equation that

$$\partial_tu\in L^\infty((0,T);H^{\sigma-1}),$$

and thus $u\in C^0([0,T];H^{\sigma-1})$. Using again the equation, we find that

$$D_tu\in C^0([0,T];H^{\sigma-2})\quad\text{and thus}\quad u\in C^1([0,T];H^{\sigma-2}),\quad u(0)=u_0.$$

If $f\in\mathscr{S}(\mathbb{R}^{n+1})$, $u_0\in\mathscr{S}(\mathbb{R}^n)$, replacing in the discussion above σ by $\sigma+2$, we may thus apply the inequality (5.1.4). Now for f,u_0 as in the theorem, we may choose sequences $(f_k)_{k\ge1}$ and $(u_{0,k})_{k\ge1}$ in the relevant Schwartz space with $(f_k)_{k\ge1}$ converging in $L^1([0,T];H^\sigma)$ towards f and $(u_{0,k})_{k\ge1}$ converging towards u_0 in H^σ and we are able to find a sequence $(u_k)_{k\ge1}$ in $C^1([0,T];H^\sigma)$ such that

$$D_tu_k+A(t)u_k=f_k,\quad u_k(0)=u_{0,k},$$

along with Inequality (5.1.4). As a result we find that for $\lambda\ge\lambda(\sigma)$,

$$\begin{aligned}&\sup_{t\in[0,T]}e^{-\lambda t}\|u_k(t)-u_l(t)\|_{H^\sigma(\mathbb{R}^n)}\\&\qquad\le\|u_{0,k}-u_{0,l}\|_{H^\sigma(\mathbb{R}^n)}+\int_0^Te^{-\lambda t}\|f_k-f_l\|_{H^\sigma(\mathbb{R}^n)}dt.\end{aligned}\tag{5.1.9}$$

The Cauchy criterion gives the convergence of the sequence $(u_k)_{k\ge1}$ in $C^0([0,T];H^\sigma)$ towards $u \in C^0([0,T];H^\sigma)$ with

$$D_t(Hu_k) + A(t)Hu_k = f_k + \frac{1}{i}\delta_0(t)\otimes u_{0,k}, \quad \text{on } (-\infty,T)\times\mathbb{R}^n,$$

implying directly

$$D_t(Hu) + A(t)Hu = f + \frac{1}{i}\delta_0(t)\otimes u_0, \quad \text{on } (-\infty,T)\times\mathbb{R}^n,$$

whereas Inequality (5.1.4) for u_k entails the same inequality for u. The proof of Theorem 5.1.3 is complete. □

Corollary 5.1.4. *Let $T>0$ and a satisfying (5.1.2) and (5.1.3) and let u satisfying (5.1.1) with $f \in \cap_{\sigma\in\mathbb{R}} L^\infty([0,T];H^\sigma(\mathbb{R}^n))$ and $u_0 \in H^{+\infty}(\mathbb{R}^n) = \cap_{\sigma\in\mathbb{R}} H^\sigma(\mathbb{R}^n)$. Then $u \in \cap_{\sigma\in\mathbb{R}} C^1([0,T];H^\sigma(\mathbb{R}^n))$. In particular, for any $t\in[0,T]$, we have $u(t,\cdot)\in C^\infty(\mathbb{R}^n)$.*

Proof. Theorem 5.1.3 implies that $u \in \cap_{\sigma\in\mathbb{R}} C^0([0,T];H^\sigma(\mathbb{R}^n))$ and the equation implies that $\partial_t u \in \cap_{\sigma\in\mathbb{R}} L^\infty([0,T];H^{\sigma-1}(\mathbb{R}^n))$, which gives the result. □

5.2 Some Hilbertian lemmas

We want to study the wave-front-set of the solution $u(t,\cdot)$ of (5.1.1), say with $f=0$, knowing the wave-front-set of u_0 and we wish to show that the singularities are indeed propagating backward along the Hamiltonian flow of the real part of principal symbol $\tau + \operatorname{Re} a_1(t,x,\xi)$, providing $\operatorname{Im} a_1(t,x,\xi)\le 0$, where a_1 stands for principal symbol of $a(t,x,D_x)$: we assume that a_1 belongs to S^1 (i.e. satisfies (5.1.2)), $a-a_1$ is bounded in S^0, i.e.

$$\sup_{(t,x,\xi)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n} |(\partial_x^\alpha\partial_\xi^\beta(a-a_1))(t,x,\xi)\langle\xi\rangle^{|\beta|}| < +\infty, \tag{5.2.1}$$

and that a_1 is homogeneous of degree 1 for $|\xi|\ge1$, i.e. is such that

$$\forall\mu\ge1, \forall\xi \text{ such that } |\xi|\ge1, \quad a_1(t,x,\mu\xi) = \mu a_1(t,x,\xi). \tag{5.2.2}$$

In order to motivate the backward-forward story above we reformulate Lemma 5.1.2 with the following lemma.

Lemma 5.2.1. *Let $\mathbb{H}$ be a complex Hilbert space, let I be an interval of $\mathbb{R}$ with a non-empty interior and let $I \ni t \mapsto A(t) \in \mathcal{B}(\mathbb{H})$ be a continuous mapping such that there exists $\beta\in\mathbb{R}$ with*

$$\forall t\in I, \quad \operatorname{Im} A(t) = \frac{A(t)-A^*(t)}{2i} \le \beta, \tag{5.2.3}$$

Then for $\lambda\ge\beta$ and for $u\in C^1(I;\mathbb{H})$, we have with $D_t=-i\partial_t$, $t_0\le t_1$ in I,

$$e^{-\lambda(t_1-t_0)}\|u(t_1)\|_{\mathbb{H}} \le \|u(t_0)\|_{\mathbb{H}} + \int_{t_0}^{t_1} e^{-\lambda(t-t_0)}\|D_t u + A(t)u\|_{\mathbb{H}} dt. \tag{5.2.4}$$

If we have for some $\beta \in \mathbb{R}$

$$\forall t \in I, \quad \operatorname{Im} A(t) = \frac{A(t) - A^*(t)}{2i} \geq -\beta \tag{5.2.5}$$

Then for $\lambda \geq \beta$ *and for* $u \in C^1(I; \mathbb{H})$*, we have with* $D_t = -i\partial_t$*,* $t_0 \leq t_1$ *in* I*,*

$$e^{-\lambda(t_1 - t_0)} \|u(t_0)\|_{\mathbb{H}} \leq \|u(t_1)\|_{\mathbb{H}} + \int_{t_0}^{t_1} e^{-\lambda(t_1 - t)} \|D_t u + A(t) u\|_{\mathbb{H}} dt. \tag{5.2.6}$$

Proof. The first statement is already proven. To prove the second one, we note that setting $u(t) = w(-t)$ we find for $t \in [t_0, t_1]$, $s = -t$,

$$D_t u + A(t) u = -(D_s w)(s) + A(-s) w(s) = -\big(D_s - A(-s)\big) w.$$

Assuming for $s \in -I$,

$$\beta \geq \operatorname{Im} -A(-s) = \frac{-A(-s) + A^*(-s)}{2i} = (-1) \frac{A(-s) - A^*(-s)}{2i},$$

which amounts to assume (5.2.5) means $\operatorname{Im}(-A(-s)) \leq \beta$ so that with

$$s_0 = -t_1 \leq s_1 = -t_0,$$

we get from the (already proven) first part of the lemma for $\lambda \geq \beta$,

$$e^{-\lambda(s_1 - s_0)} \|w(s_1)\|_{\mathbb{H}} \leq \|w(s_0)\|_{\mathbb{H}} + \int_{s_0}^{s_1} e^{-\lambda(s - s_0)} \|D_s w - A(-s) w\|_{\mathbb{H}} ds, \tag{5.2.7}$$

i.e.

$$e^{-\lambda(-t_0 + t_1)} \|u(t_0)\|_{\mathbb{H}} \leq \|u(t_1)\|_{\mathbb{H}} + \int_{t_0}^{t_1} e^{-\lambda(-t + t_1)} \|D_t u + A(t) u\|_{\mathbb{H}} dt, \tag{5.2.8}$$

which is the sought result. □

Remark 5.2.2. If $\beta = 0$ in (5.2.3) (resp. (5.2.5)), we can take $\lambda = 0$ and obtain for $t_0 \leq t_1$ in I if $D_t u + A(t) u = 0$,

$$\|u(t_1)\|_{\mathbb{H}} \leq \|u(t_0)\|_{\mathbb{H}}, \quad (\text{resp. } \|u(t_0)\|_{\mathbb{H}} \leq \|u(t_1)\|_{\mathbb{H}}).$$

The first inequality implies that if $\|u(t_0)\|_{\mathbb{H}}$ is small (or finite) then $\|u(t_1)\|_{\mathbb{H}}$ is smaller (or finite) which amounts to a forward propagation of regularity. We can also see that first inequality as a backward propagation of singularity: if $\|u(t_1)\|_{\mathbb{H}}$ is large (or infinite) then $\|u(t_0)\|_{\mathbb{H}}$ is larger (or infinite). The second inequality is reversing the direction of propagation with respect to the first one. Note also that if both (5.2.3) and (5.2.5) are satisfied, which occurs when $A(t)$ is selfadjoint for all t, then the propagation goes in both directions, backward and forward.

5.3 Propagation of singularities

Introductory remarks

To simplify matters with the orientation of the bicharacteristics, we shall assume that

$$a_1 \text{ is real-valued,} \tag{5.3.1}$$

but we shall keep in mind that only minor modifications will be necessary to tackle the case where $\operatorname{Im} a_1 \le 0$ (backward propagation of singularities, forward propagation of regularity) or the case $\operatorname{Im} a_1 \ge 0$ (forward propagation of singularities, backward propagation of regularity). If u satisfies (5.1.1) with $f = 0$ and if $Q(t, x, D)$ is a pseudo-differential operator commuting with the operator $\mathcal{L} = D_t + a(t, x, D_x)$ we shall have

$$\mathcal{L}Qu = 0, \quad Qu_{|t=0} = Q_0 u_0, \; Q_0 = Q(0, x, D_x)$$

so that with $0 \le t \le T$ for $\sigma \in \mathbb{R}$, $\lambda \ge \lambda(\sigma)$,

$$e^{-\lambda t}\|Q(t)u(t)\|_{H^\sigma} \le \|Q_0 u_0\|_{H^\sigma}.$$

If we know that $Q_0 u_0$ belongs to H^σ, we shall obtain that it is also the case of $Q(t)u(t)$ and it is a type of microlocal propagation result. However, the requirement of exact commutation of $Q(t)$ with $\mathcal{L}$ is neither realistic nor necessary and we can implement the same program with some approximate commutation: if $q(t, x, \xi)$ is a symbol in S^0 uniformly in $t \in [0, T]$, $Q(t) = q(t, x, D_x)$, (i.e. (5.1.2) holds true with $\langle \xi \rangle^{-1+|\beta|}$ replaced by $\langle \xi \rangle^{|\beta|}$), we obtain that the commutator

$$[\mathcal{L}, Q(t)] = \frac{1}{2\pi i}\operatorname{Op}(\frac{\partial q}{\partial t} + \{a_1(t), q(t)\}) + \operatorname{Op}(S^{-1}).$$

As a result if we are able to solve the first-order PDE

$$\frac{\partial q}{\partial t} + \{a_1(t), q(t)\} = 0,$$

we will need only to deal with a remainder of order -1.

The vector field $\partial_t + H_{a_1}$

Let $a_1(t, x, \xi)$ be a real-valued smooth fonction, homogeneous of degree 1 with respect to ξ such that

$$\forall(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n, \qquad \sup_{(t,x,\xi)\in[0,T]\times\mathbb{R}^n\times\mathbb{S}^{n-1}} |(\partial_x^\alpha \partial_\xi^\beta a_1)(t, x, \xi)| < +\infty. \tag{5.3.2}$$

To find a first integral of the above vector field with initial value $q_0(x, \xi)$, we need to solve the first-order system of ODE

$$\begin{cases} \dot{x}(t, y, \eta) &= \frac{\partial a_1}{\partial \xi}\big(t, x(t, y, \eta), \xi(t, y, \eta)\big), \quad x(0, y, \eta) = y, \\ \dot{\xi}(t, y, \eta) &= -\frac{\partial a_1}{\partial x}\big(t, x(t, y, \eta), \xi(t, y, \eta)\big), \quad \xi(0, y, \eta) = \eta, \end{cases} \tag{5.3.3}$$

for $y \in \mathbb{R}^n, \eta \in \mathbb{R}^n \backslash\{0\}$. We note that for $\mu > 0, \eta \neq 0$, we have

$$\begin{aligned}\frac{d}{dt}\{x(t,y,\mu\eta)\} &= \dot{x}(t,y,\mu\eta)\\ &= \frac{\partial a_1}{\partial \xi}\big(t, x(t,y,\mu\eta), \xi(t,y,\mu\eta)\big) = \frac{\partial a_1}{\partial \xi}\big(t, x(t,y,\mu\eta), \mu^{-1}\xi(t,y,\mu\eta)\big),\\ &\qquad x(0,y,\mu\eta) = y,\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\{\mu^{-1}\xi(t,y,\mu\eta)\} &= \mu^{-1}\dot{\xi}(t,y,\mu\eta)\\ &= -\mu^{-1}\frac{\partial a_1}{\partial x}\big(t, x(t,y,\mu\eta), \xi(t,y,\mu\eta)\big) = -\frac{\partial a_1}{\partial x}\big(t, x(t,y,\mu\eta), \mu^{-1}\xi(t,y,\mu\eta)\big),\\ &\qquad \mu^{-1}\xi(0,y,\mu\eta) = \eta.\end{aligned}$$

As a result $\big(x(t,y,\mu\eta), \mu^{-1}\xi(t,y,\mu\eta)\big)$ and $\big(x(t,y,\eta), \xi(t,y,\eta)\big)$ solve the same system of ODE with the same initial data and thus coincide so that $x(t,y,\eta)$ is homogeneous with degree 0 in η and $\xi(t,y,\eta)$ is homogeneous with degree 1 in η. From the estimates (5.3.2), we find solutions of (5.3.3) for $t \in [0,T]$ with $|\eta| = 1$ and then by the above homogeneity, we get solutions for $t \in [0,T], y \in \mathbb{R}^n, \eta \in \mathbb{R}^n\backslash\{0\}$. We set

$$\big(x(t,y,\eta), \xi(t,y,\eta)\big) = \Psi(t,y,\eta),$$

and a first integral of the vector field $\partial_t + H_{a_1}$ should satisfy

$$q\big(t, \Psi(t,y,\eta)\big) = q_0(y,\eta). \tag{5.3.4}$$

Inverting Ψ, we can find some first integrals $\Phi(t,x,\xi)$ such that

$$q(t,x,\xi) = q_0\big(\Phi(t,x,\xi)\big) = q_0\big(y(t,x,\xi), \eta(t,x,\xi)\big)$$

where y (resp. η) is homogeneous with degree 0 (resp. 1) with respect to ξ. As a result q is an homogenous symbol with degree 0 satisfying (5.3.2) and defining

$$\tilde{q}(t,x,\xi) = q(t,x,\xi)\omega(\xi)$$

where $\omega \in C^\infty(\mathbb{R}^n)$, $\omega(\xi) = 1$ for $|\xi| \geq 1$, $\omega(\xi) = 0$ for $|\xi| \leq 1/2$, we obtain that $\tilde{q}$ belongs uniformly to S^0 and is such that

$$\partial_t\tilde{q} + \{a_1, \tilde{q}\} \in S^{-\infty}.$$

Microlocalized energy estimates

With $\mathcal{L} = D_t + a(t,x,D_x), 0 \leq t \leq T$, $a_1(t,x,\xi) \in S^1$ uniformly and real-valued such that $a - a_1 \in S^0$ uniformly. Let $q(t,x,\xi)$ uniformly in S^0 such that

$$\partial_t q(t,x,\xi) + \{a_1, q\} = 0, \quad q(0,x,\xi) = q_0(x,\xi),$$

where q_0 is given in S^0. We have for $\sigma \in \mathbb{R}$, $\lambda \geq \lambda(\sigma)$,

$$\sup_{t\in[0,T]} e^{-\lambda t}\|Q(t)u(t)\|_{H^\sigma(\mathbb{R}^n)} \leq \|Q_0 u_0\|_{H^\sigma(\mathbb{R}^n)} + \int_0^T e^{-\lambda t}\|\mathcal{L}Qu\|_{H^\sigma(\mathbb{R}^n)}dt,$$

and since $[\mathcal{L}, Q]$ is uniformly in S^{-1}, we obtain

$$\sup_{t\in[0,T]} e^{-\lambda t}\|Q(t)u(t)\|_{H^\sigma(\mathbb{R}^n)} \leq \|Q_0 u_0\|_{H^\sigma(\mathbb{R}^n)} + \int_0^T e^{-\lambda t}\|Q\mathcal{L}u\|_{H^\sigma(\mathbb{R}^n)}dt + C_0\int_0^T e^{-\lambda t}\|u\|_{H^{\sigma-1}(\mathbb{R}^n)}dt. \quad (5.3.5)$$

Let $(x_0,\xi_0) \in \mathbb{R}^n \times \mathbb{S}^{n-1}, \sigma_0 \in \mathbb{R}$ such that $u_0 \in H^{\sigma_0}_{(x_0,\xi_0)}$, i.e. $Q_0 u_0 \in H^{\sigma_0}(\mathbb{R}^n)$ for a polyhomogeneous symbol q_0 of order 0, non-characteristic at (x_0, ξ_0). Let us assume that $\mathcal{L}u \in L^1([0,T]; H^{\sigma_0}(\mathbb{R}^n)), u \in \cup_{\sigma\in\mathbb{R}} L^1([0,T], H^\sigma(\mathbb{R}^n))$ and $Q_0 u(0) \in H^{\sigma_0}(\mathbb{R}^n)$. Let $\sigma_1 \in \mathbb{R}$ such that $u \in L^1([0,T], H^{\sigma_1}(\mathbb{R}^n))$ with $\sigma_0 - 1 \leq \sigma_1 < \sigma_0$. Then we obtain that

$$\sup_{t\in[0,T]} e^{-\lambda t}\|Q(t)u(t)\|_{H^{\sigma_0}(\mathbb{R}^n)} \leq \|Q_0 u_0\|_{H^{\sigma_0}(\mathbb{R}^n)} + \int_0^T e^{-\lambda t}\|Q\mathcal{L}u\|_{H^{\sigma_0}(\mathbb{R}^n)}dt + C_0\int_0^T e^{-\lambda t}\|u\|_{H^{\sigma_0-1}(\mathbb{R}^n)}dt,$$

so that $\forall t \in [0,T],\ Q(t)u(t) \in H^{\sigma_0}(\mathbb{R}^n)$.

Theorem 5.3.1. *Let $T > 0$ and $a \in C^\infty([0,T]\times\mathbb{R}^{2n})$ such that (5.1.2) holds true. Moreover let us assume that there exists a_1 real-valued in $C^\infty([0,T]\times\mathbb{R}^{2n})$ such that (5.2.1) and (5.2.2) are satisfied. Let $\sigma_0 \in \mathbb{R}$ and let $u \in L^1([0,T], H^{\sigma_0-1}(\mathbb{R}^n))$ be such that*

$$D_t u + a(t,x,D_x)u \in L^1([0,T]; H^{\sigma_0}(\mathbb{R}^n)), \quad (x_0,\xi_0) \notin WF_{\sigma_0}u(0). \quad (5.3.6)$$

Then defining the flow Ψ of the Hamiltonian vector field of a_1 by

$$\dot{\Psi}(t,y,\eta) = H_{a_1(t,\cdot,\cdot)}\big(\Psi(t,y,\eta)\big), \quad \Psi(0,y,\eta) = (y,\eta),$$

we obtain that for $t \in [0,T]$, $\Psi(t,x_0,\xi_0) \notin WF_{\sigma_0}u(t)$.

Proof. We can find Q_0 with a polyhomogeneous symbol in S^0, non-characteristic at (x_0,ξ_0) such that $Q_0 u(0) \in H^{\sigma_0}(\mathbb{R}^n)$ and if $u \in L^1([0,T], H^{\sigma_0-1}(\mathbb{R}^n))$ we obtain from the above reasoning that $Q(t)u(t) \in H^{\sigma_0}(\mathbb{R}^n)$ and from (5.3.4) we see that $Q(t)$ is non-characteristic at $\Psi(t,x_0,\xi_0)$, proving that $\Psi(t,x_0,\xi_0) \notin WF_{\sigma_0}u(t)$. □

Chapter 6

Elements of Spectral Theory

6.1 The Harmonic Oscillator

We use in this section our Appendix, Section 7.2. We have defined the harmonic oscillator on $\mathbb{R}^n$ as

$$\mathcal{H} = \frac{1}{2}\left(-\Delta + |x|^2\right), \tag{6.1.1}$$

and we have proven that

$$\mathcal{H} = \sum_{k\ge 0}\big(\frac{n}{2}+k\big)\mathbb{P}_k, \quad \mathrm{Id} = \sum_{k\ge 0}\mathbb{P}_k, \tag{6.1.2}$$

where $\mathbb{P}_k$ is the orthogonal projection on $\mathcal{E}_k$, which has dimension $\binom{k+n-1}{n-1}$. The eigenvalue $n/2$ is simple in any dimension and $\mathcal{E}_0$ is generated by

$$\Phi_0(x) = \pi^{-\mathrm{d}/4}e^{-|x|^2/2}.$$

Introducing a small parameter $h \in (0,1]$ (Planck constant), we define

$$\mathcal{H}_h = \frac{1}{2}\left(-h^2\Delta + |x|^2\right). \tag{6.1.3}$$

With the unitary operator U_h on $L^2(\mathbb{R}^n)$ given by

$$(U_h w)(x) = h^{n/4}w(h^{1/2}x), \tag{6.1.4}$$

we find that $hU_h^*\big(-\Delta + |x|^2\big)U_h = -h^2\Delta + |x|^2$ and we get that

$$\mathcal{H}_h = \sum_{k\ge 0}\big(\frac{n}{2}+k\big)h\mathbb{P}_{k,h}, \quad \mathrm{Id} = \sum_{k\ge 0}\mathbb{P}_{k,h}, \quad P_{k,h} = U_h^*\mathbb{P}_k U_h. \tag{6.1.5}$$

Lemma 6.1.1. *Let $h \in (0,1]$ and $\mathcal{H}_h$ given by (6.1.3). Let $0 \le a < b$ be given real numbers. Then with $\sigma(\mathcal{H}_h)$ standing for the spectrum of $\mathcal{H}_h$, we have*

$$\operatorname{card}\big(\sigma(\mathcal{H}_h)\cap[a,b]\big) = |\{(x,\xi)\in\mathbb{R}^{2n}, a \le \frac{1}{2}(h^2 4\pi^2|\xi|^2 + |x|^2) \le b\}| + O_{a,b,n}(h^{-n+1}). \tag{6.1.6}$$

Proof. We calculate first with a change of variables $x = \alpha y, \xi = \alpha^{-1}\eta$, $\alpha = \sqrt{2\pi h}$,

$$\begin{aligned} V_h(a,b) &= \iint \mathbf{1}\big(a \le \frac{1}{2}(h^2 4\pi^2 |\xi|^2 + |x|^2) \le b\big) dx d\xi \\ &= \iint \mathbf{1}\big(\frac{a}{\pi h} \le y^2 + \eta^2 \le \frac{b}{\pi h}\big) dy d\eta = |\mathbb{B}^{2n}| (\pi h)^{-n} \big(b^n - a^n\big) = \frac{b^n - a^n}{n! h^n}, \end{aligned}$$

noting that $\frac{1}{2}(h^2 4\pi^2 \xi^2 + x^2)$ is the symbol of the operator $\mathcal{H}_h$. On the other hand, we have

$$\operatorname{card} \sigma(\mathcal{H}_h) \cap [0,b] = \operatorname{card}\{\alpha \in \mathbb{N}^n, \frac{n}{2} + |\alpha| \le \frac{b}{h}\} = \sum_{0 \le k \le \frac{b}{h}} \binom{k+n-1}{n-1},$$

and also

$$\binom{k+n-1}{n-1} = \frac{(k+n-1)\dots(k+1)}{(n-1)!} = \frac{k^{n-1}}{(n-1)!} + O(k^{n-2}),$$

so that with $\lambda = b/h$ (note that for $\lambda \to +\infty$, $\sum_{0 \le k \le \lambda} k^{n-1} = \lambda^n/n + O(\lambda^{n-1})$),

$$\operatorname{card} \sigma(\mathcal{H}_h) \cap [0,b] = \frac{1}{(n-1)!} \left(\frac{\lambda^n}{n} + O(\lambda^{n-1}) \right) = \frac{b^n}{n! h^n} + O(h^{-n+1}),$$

providing the sought result. □

Remark 6.1.2. We note that the unbounded self-adjoint operator $\mathcal{H}_h$ has a compact resolvent and thus a discrete spectrum made with eigenvalues of finite multiplicities. The previous lemma gives an interesting asymptotic equality between a quantum quantity (the number of eigenvalues located in some interval $[a,b]$) and a classical quantity (the symbol of the harmonic oscillator). Formula (6.1.6) proves that the number of eigenvalues between a and b is well-approximated by the volume of the set where the symbol of the operator lies between these values. In the sequel we shall try to prove that law, the so-called Weyl's law, named after the German Mathematician Hermann Weyl (1885–1955, `http://www-history.mcs.st-andrews.ac.uk/Biographies/Weyl.html`) in a more general context.

6.2 Algebra of pseudo-differential operators on $\mathbb{R}^n$

Classes of symbols

Definition 6.2.1. Let $m \in \mathbb{R}$. We define the symbol class Γ^m as the vector space of functions $a \in C^\infty(\mathbb{R}^{2n})$ such that

$$\forall \alpha \in \mathbb{N}^{2n}, \quad \sup_{X \in \mathbb{R}^{2n}} |(\partial_X^\alpha a)(X) \langle X \rangle^{-(2m-|\alpha|)}| < +\infty. \tag{6.2.1}$$

Using the metrics notation due to L. Hörmander we see that

$$\Gamma^m = S\big(\langle X\rangle^{2m}, g = \frac{|dx|^2 + |d\xi|^2}{(1+|x|^2+|\xi|^2)}\big), \tag{6.2.2}$$

so that the inverse Planck constant function λ is defined by

$$\lambda = \langle X\rangle^2, \quad \Gamma_\rho^m = S(\lambda, \frac{|dX|^2}{\lambda}).$$

As an example, we see that $1+|x|^2+|\xi|^2$ belongs to Γ^1 and more generally

$$(1+|x|^2+|\xi|^2)^m \in \Gamma^m,$$

and a polynomial in x, ξ with degree $2m$ belongs to Γ^m.

Algebra of operators

Most of the results of Section 3.4 can be transferred, *mutatis mutandis* to the present framework. Instead of repeating all the arguments, which are almost essentially the same as in that section we summarize the situation by the following theorem.

Theorem 6.2.2. *Let $a_j \in \Gamma^{m_j}, j = 1, 2$. Then we have*

$$a_1 \diamond a_2 \equiv a_1 a_2 \mod \Gamma^{m_1+m_2-1}, \tag{6.2.3}$$

$$a_1 \diamond a_2 - a_2 \diamond a_1 \equiv \frac{1}{2i\pi}\{a_1, a_2\} \mod \Gamma^{m_1+m_2-2}, \tag{6.2.4}$$

where the **Poisson bracket** $\{a_1, a_2\} = \sum_{1\le j\le n} \frac{\partial a_1}{\partial \xi_j}\frac{\partial a_2}{\partial x_j} - \frac{\partial a_1}{\partial x_j}\frac{\partial a_2}{\partial \xi_j}$. (6.2.5)

For $a \in S^m_{1,0}$, $\quad a^* \equiv \bar{a} \mod \Gamma^{m-1}$. (6.2.6)

6.3 The Wick calculus

Anti-Wick quantization

We recall here some facts on the so-called anti-Wick quantization, as used in [10], [11], [12].

Definition 6.3.1. Let $Y = (y, \eta)$ be a point in $\mathbb{R}^n \times \mathbb{R}^n$. The operator Σ_Y is defined as $\big[2^n e^{-2\pi|\cdot - Y|^2}\big]^w$. Let a be in $L^\infty(\mathbb{R}^{2n})$. The Wick quantization of a is defined as

$$a^{\text{Wick}} = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y dY. \tag{6.3.1}$$

Remark 6.3.2. The operator Σ_Y is a rank-one orthogonal projection: we have

$$\Sigma_Y u = (Wu)(Y)\tau_Y\varphi_0 \quad \text{with } (Wu)(Y) = \langle u, \tau_Y\varphi_0\rangle_{L^2(\mathbb{R}^n)}, \tag{6.3.2}$$

where $\varphi_0(x) = 2^{n/4}e^{-\pi|x|^2}$ and $(\tau_{y,\eta}\varphi_0)(x) = \varphi_0(x-y)e^{2i\pi\langle x-\frac{y}{2},\eta\rangle}$. (6.3.3)

In fact we get from the definition of Σ_Y that, for $u \in \mathscr{S}(\mathbb{R}^n)$

$$\begin{aligned}(\Sigma_{y,\eta}u)(x) &= \iint u(z)e^{2i\pi(x-z)\cdot\xi}2^n e^{-2\pi|\frac{x+z}{2}-y|^2}e^{-2\pi|\xi-\eta|^2}dzd\xi \\ &= \int u(z)e^{2i\pi(x-z)\cdot\eta}2^{n/2}e^{-2\pi|\frac{x+z}{2}-y|^2}e^{-\frac{\pi}{2}|x-z|^2}dz \\ &= \int u(z)e^{-2i\pi(z-\frac{y}{2})\cdot\eta}2^{n/4}e^{-\pi|z-y|^2}dz\ 2^{n/4}e^{-\pi|x-y|^2}e^{2i\pi(x-\frac{y}{2})\cdot\eta} \\ &= \langle u, \tau_{y,\eta}\varphi_0\rangle\tau_{y,\eta}\varphi_0.\end{aligned}$$

Proposition 6.3.3.

(1) *Let a be in $L^\infty(\mathbb{R}^{2n})$. Then $a^{Wick} = W^*a^\mu W$ and $1^{Wick} = Id_{L^2(\mathbb{R}^n)}$ where W is the isometric mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ given above, and a^μ the operator of multiplication by a in $L^2(\mathbb{R}^{2n})$. The operator $\pi_{\mathcal{H}} = WW^*$ is the orthogonal projection on a closed proper subspace $\mathcal{H}$ of $L^2(\mathbb{R}^{2n})$ and has the kernel*

$$\Pi(X,Y) = e^{-\frac{\pi}{2}|X-Y|^2}e^{-i\pi[X,Y]}, \tag{6.3.4}$$

where $[,]$ is the symplectic form. Moreover, we have

$$\|a^{\mathrm{Wick}}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le \|a\|_{L^\infty(\mathbb{R}^{2n})}, \tag{6.3.5}$$

$$a(X) \ge 0 \text{ for all } X \text{ implies } a^{Wick} \ge 0. \tag{6.3.6}$$

(2) *Let m be a real number, and $p \in S(\Lambda^m, \Lambda^{-1}\Gamma)$, where Γ is the Euclidean norm on $\mathbb{R}^{2n}$. Then $p^{Wick} = p^w + r(p)^w$, with $r(p) \in S(\Lambda^{m-1}, \Lambda^{-1}\Gamma)$ so that the mapping $p \mapsto r(p)$ is continuous. More precisely, one has*

$$r(p)(X) = \int_0^1\int_{\mathbb{R}^{2n}}(1-\theta)p''(X+\theta Y)Y^2e^{-2\pi\Gamma(Y)}2^n dY d\theta.$$

Note that $r(p) = 0$ if p is affine and $r(p) = \frac{1}{8\pi}\operatorname{trace} p''$ if p is a polynomial with degree ≤ 2.

(3) *For $a \in L^\infty(\mathbb{R}^{2n})$, the Weyl symbol of a^{Wick} is*

$$a * 2^n \exp -2\pi\Gamma, \text{ which belongs to } S(1,\Gamma) \text{ with } k^{th}\text{-seminorm } c(k)\|a\|_{L^\infty}. \tag{6.3.7}$$

(4) *Let $\mathbb{R} \ni t \mapsto a(t,X) \in \mathbb{R}$ such that, for $t \le s$, $a(t,X) \le a(s,X)$. Then, for $u \in C_c^1\big(\mathbb{R}_t, L^2(\mathbb{R}^n)\big)$, assuming $a(t,\cdot) \in L^\infty(\mathbb{R}^{2n})$,*

$$\int_{\mathbb{R}} \operatorname{Re}\langle D_t u(t), ia(t)^{Wick}u(t)\rangle_{L^2(\mathbb{R}^n)}dt \ge 0. \tag{6.3.8}$$

(5) *With the operator Σ_Y given in Definition 6.3.1, we have the estimate*

$$\|\Sigma_Y\Sigma_Z\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le 2^n e^{-\frac{\pi}{2}\Gamma(Y-Z)}. \tag{6.3.9}$$

(6) *More precisely, the Weyl symbol of* $\Sigma_Y \Sigma_Z$ *is, as a function of the variable* $X \in \mathbb{R}^{2n}$*, setting* $\Gamma(T) = |T|^2$

$$e^{-\frac{\pi}{2}|Y-Z|^2} e^{-2i\pi[X-Y,X-Z]} 2^n e^{-2\pi|X-\frac{Y+Z}{2}|^2}. \tag{6.3.10}$$

Remark 6.3.4. Part of this proposition is well summarized by the following diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}^{2n}) & \xrightarrow[\text{(multiplication by } a)]{a} & L^2(\mathbb{R}^{2n}) \\ W \uparrow & & \downarrow W^* \\ L^2(\mathbb{R}^n) & \xrightarrow[a^{\text{Wick}}]{} & L^2(\mathbb{R}^n) \end{array}$$

Proof. For $u, v \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\langle a^{\text{Wick}} u, v\rangle = \int_{\mathbb{R}^{2n}} a(Y) \langle \Sigma_Y u, v\rangle_{L^2(\mathbb{R}^n)} dY = \int_{\mathbb{R}^{2n}} a(Y)(Wu)(Y)\overline{(Wv)(Y)} dY,$$

which gives

$$a^{\text{Wick}} = W^* a^\mu W. \tag{6.3.11}$$

Also we have from (6.3.1) that $1^{\text{Wick}} = \mathrm{Id}$, since

$$1^{\text{Wick}} = \int_{\mathbb{R}^{2n}} \Sigma_Y dY \quad \text{has Weyl symbol} \int_{\mathbb{R}^{2n}} 2^n e^{-2\pi|X-Y|^2} dY = 1.$$

This implies that

$$W^* W = \mathrm{Id},$$

i.e. W is isometric from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^{2n})$. The operator WW^* is bounded selfadjoint and is a projection since $WW^*WW^* = WW^*$. Defining $\mathcal{H}$ as range W, we get that WW^* is the orthogonal projection onto $\mathcal{H}$, since the range of WW^* is included in the range of W, and for $\Phi \in \mathcal{H}$, we have

$$\Phi = Wu = WW^*Wu \in \mathrm{range}(WW^*).$$

Moreover range W is closed since W is isometric, that latter property implying also, using (6.3.11), the property (6.3.5), whereas (6.3.6) follows from (6.3.1) and $\Sigma_Y \geq 0$ as an orthogonal projection. The kernel of the operator WW^* is, from (6.3.2), (6.3.3), with $X = (x, \xi), Y = (y, \eta)$,

$$\begin{aligned}
\Pi(X,Y) &= \langle \tau_Y \varphi_0, \tau_X \varphi_0 \rangle_{L^2(\mathbb{R}^n)} \\
&= 2^{n/2} \int_{\mathbb{R}^n} e^{-\pi|t-x|^2} e^{-\pi|t-y|^2} e^{2i\pi(t-\frac{y}{2})\cdot\eta} e^{-2i\pi(t-\frac{x}{2})\cdot\xi} dt \\
&= e^{-\frac{\pi}{2}|x-y|^2} 2^{n/2} \int_{\mathbb{R}^n} e^{-\frac{\pi}{2}|2t-x-y|^2} e^{2i\pi t\cdot(\eta-\xi)} dt e^{i\pi(x\cdot\xi-y\cdot\eta)} \\
&= e^{-\frac{\pi}{2}|x-y|^2} 2^{n/2} \int_{\mathbb{R}^n} e^{-2\pi|t|^2} e^{2i\pi(t+\frac{x+y}{2})\cdot(\eta-\xi)} dt e^{i\pi(x\cdot\xi-y\cdot\eta)} \\
&= e^{-\frac{\pi}{2}|x-y|^2} e^{-\frac{\pi}{2}|\xi-\eta|^2} e^{i\pi(x+y)\cdot(\eta-\xi)} e^{i\pi(x\cdot\xi-y\cdot\eta)} \\
&= e^{-\frac{\pi}{2}|x-y|^2} e^{-\frac{\pi}{2}|\xi-\eta|^2} e^{i\pi(x\eta-y\xi)} = e^{-\frac{\pi}{2}|X-Y|^2} e^{-i\pi[X,Y]},
\end{aligned}$$

which is (6.3.4). Postponing the proof of $\mathcal{H} \neq L^2(\mathbb{R}^{2n})$ until after the proof of (2), we have proven (1). To obtain (2), we note that (6.3.1) gives directly that

$$a^{\text{Wick}} = (a * 2^n \exp -2\pi\Gamma)^w$$

and the second order Taylor expansion gives (2) while (3) is obvious from the convolution formula. Note also that $u \in \mathscr{S}(\mathbb{R}^n)$ implies $Wu \in \mathscr{S}(\mathbb{R}^{2n})$ since $e^{-i\pi y\cdot\eta}(Wu)(y,\eta)$ is the partial Fourier transform with respect to x of $\mathbb{R}^n \times \mathbb{R}^n \ni (x,y) \mapsto u(x)2^{n/4}e^{-\pi|x-y|^2}$: this gives also another proof of W isometric since

$$\iint |u(x)|^2 2^{n/2} e^{-2\pi|x-y|^2} dxdy = \|u\|^2_{L^2(\mathbb{R}^n)}.$$

We calculate now, for $u \in \mathscr{S}(\mathbb{R}^n)$ with L^2 norm 1, using the already proven (2) on the Wick quantization of linear forms,

$$\begin{aligned} 2\operatorname{Re}\langle \pi_{\mathcal{H}}\xi_1 Wu, ix_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} &= 2\operatorname{Re}\langle W^*\xi_1 Wu, iW^*x_1 Wu\rangle_{L^2(\mathbb{R}^n)} \\ &= 2\operatorname{Re}\langle \xi_1^{\text{Wick}} u, ix_1^{\text{Wick}} u\rangle_{L^2(\mathbb{R}^n)} = 2\operatorname{Re}\langle D_1 u, ix_1 u\rangle_{L^2(\mathbb{R}^n)} = 1/2\pi. \end{aligned}$$

If $\mathcal{H}$ were the whole $L^2(\mathbb{R}^{2n})$, the projection $\pi_{\mathcal{H}}$ would be the identity and we would have

$$0 = 2\operatorname{Re}\langle \xi_1 Wu, ix_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} = 2\operatorname{Re}\langle \pi_{\mathcal{H}}\xi_1 Wu, ix_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} = 1/2\pi.$$

Let us prove (4). We have from the Lebesgue dominated convergence theorem,

$$\begin{aligned} \alpha &= \int_{\mathbb{R}} \operatorname{Re}\langle D_t u(t), ia(t)^{\text{Wick}} u(t)\rangle_{L^2(\mathbb{R}^n)} dt \\ &= -\lim_{h\to 0_+} \int_{\mathbb{R}} \frac{1}{2\pi h} \operatorname{Re}\langle u(t+h)-u(t), a(t)^{\text{Wick}} u(t+h)\rangle_{L^2(\mathbb{R}^n)} dt \\ &= \lim_{h\to 0_+} \frac{1}{2\pi h}\Big(-\int_{\mathbb{R}} \operatorname{Re}\langle u(t), a(t-h)^{\text{Wick}} u(t)\rangle_{L^2(\mathbb{R}^n)} dt \\ &\qquad\qquad + \int_{\mathbb{R}} \operatorname{Re}\langle u(t), a(t)^{\text{Wick}} u(t+h)\rangle_{L^2(\mathbb{R}^n)} dt\Big) \\ &= \lim_{h\to 0_+} \Big\{ \underbrace{\frac{1}{2\pi h}\int_{\mathbb{R}} \operatorname{Re}\langle \big(a(t)-a(t-h)\big)^{\text{Wick}} u(t), u(t)\rangle_{L^2(\mathbb{R}^n)} dt}_{=\beta(h)} \\ &\qquad\qquad + \underbrace{\int_{\mathbb{R}} \operatorname{Re}\langle \frac{-1}{2\pi hi}\big(u(t+h)-u(t)\big), ia(t)^{\text{Wick}} u(t)\rangle_{L^2(\mathbb{R}^n)} dt}_{\text{with limit } -\alpha} \Big\}. \end{aligned}$$

The previous calculation shows that $\beta(h)$ has a limit when $h \to 0_+$ and $2\alpha = \lim_{h\to 0_+}\beta(h)$. Since the function $a(t)-a(t-h)$ is non-negative, the already proven (6.3.6) implies that the operator $(a(t)-a(t-h))^{\text{Wick}}$ is also non-negative, implying $\beta(h) \geq 0$ which gives $\alpha \geq 0$, i.e. (6.3.8)[1]. Since for the Weyl quantization, one has

[1] Note that (6.3.8) is simply a way of writing that $\frac{d}{dt}\big(a(t)^{\text{Wick}}\big) \geq 0$, which is a consequence of (6.3.6) and of the non-decreasing assumption made on $t \mapsto a(t,X)$.

$\|a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le 2^n \|a\|_{L^1(\mathbb{R}^{2n})}$, we get the result (6.3.9) from (6.3.10). Let us finally prove the latter formula. From the composition formula, we obtain that the Weyl symbol ω of $\Sigma_Y \Sigma_Z$ is

$$\begin{aligned}
\omega(X) &= 2^{2n} \iint e^{-4i\pi[X-X_1, X-X_2]} 2^{2n} e^{-2\pi|X_1-Y|^2} e^{-2\pi|X_2-Z|^2} dX_1 dX_2 \\
&= 2^{4n} \iint e^{-4i\pi[X-Y, X-X_2]} e^{-2i\pi\langle X_1, 2\sigma(X-X_2)\rangle} e^{-2\pi|X_1|^2} e^{-2\pi|X_2-Z|^2} dX_1 dX_2 \\
&= 2^{3n} \int e^{-4i\pi[X-Y, X-X_2]} e^{-2\pi|X-X_2|^2} e^{-2\pi|X_2-Z|^2} dX_2 \\
&= 2^{3n} e^{-\pi|X-Z|^2} \int e^{-4i\pi[X-Y, X-X_2]} e^{-\pi|X+Z-2X_2|^2} dX_2 \\
&= 2^{3n} e^{-\pi|X-Z|^2} e^{-2i\pi[X-Y, X-Z]} \int e^{-4i\pi[X-Y, -X_2]} e^{-4\pi|X_2|^2} dX_2 \\
&= 2^n e^{-\pi|X-Z|^2} e^{-2i\pi[X-Y, X-Z]} e^{-\pi|X-Y|^2} \\
&= 2^n e^{-2i\pi[X-Y, X-Z]} e^{-2\pi|X-\frac{Y+Z}{2}|^2} e^{-\frac{\pi}{2}|Y-Z|^2}.
\end{aligned}$$

□

Fock-Bargmann spaces

There are also several links with the so-called Fock-Bargmann spaces (the space $\mathcal{H}$ above), that we can summarize with the following definitions and properties.

Proposition 6.3.5. *With $\mathcal{H}$ defined in Proposition 6.3.3 we have*

$$\mathcal{H} =\{\Phi \in L^2(\mathbb{R}^{2n}_{y,\eta}), \quad \Phi = f(z) \exp -\frac{\pi}{2}|z|^2,\ z = \eta + iy\ ,\ f \text{ entire}\}, \tag{6.3.12}$$

i.e. $\mathcal{H} = \operatorname{ran} W = L^2(\mathbb{R}^{2n}) \cap \ker(\bar\partial + \frac{\pi}{2}z)$.

Proof. For $v \in L^2(\mathbb{R}^n)$, we have, with the notation $z^2 = \sum_{1\le j\le n} z_j^2$ for $z \in \mathbb{C}^n$,

$$\begin{aligned}(Wv)(y,\eta) &= \int_{\mathbb{R}^n} v(x)2^{n/4}e^{-\pi(x-y)^2}e^{-2i\pi(x-\frac{y}{2})\eta}dx\\ &= \int_{\mathbb{R}^n} v(x)2^{n/4}e^{-\pi(x-y+i\eta)^2}dx e^{-\frac{\pi}{2}(y^2+\eta^2)}e^{-\frac{\pi}{2}(\eta+iy)^2}\end{aligned} \tag{6.3.13}$$

and we see that $Wv \in L^2(\mathbb{R}^{2n})\cap\ker(\bar\partial+\frac{\pi}{2}z)$. Conversely, if $\Phi \in L^2(\mathbb{R}^{2n})\cap\ker(\bar\partial+\frac{\pi}{2}z)$, we have $\Phi(x,\xi) = e^{-\frac{\pi}{2}(x^2+\xi^2)}f(\xi+ix)$ with $\Phi \in L^2(\mathbb{R}^{2n})$ and f entire. This gives

$$\begin{aligned}(WW^*\Phi)(x,\xi) &= \iint e^{-\frac{\pi}{2}\left((\xi-\eta)^2+(x-y)^2+2i\xi y-2i\eta x\right)}\Phi(y,\eta)dyd\eta\\
&= e^{-\frac{\pi}{2}(\xi^2+x^2)}\iint e^{-\frac{\pi}{2}(\eta^2-2\xi\eta+y^2-2xy+2i\xi y-2i\eta x)}\Phi(y,\eta)dyd\eta\\
&= e^{-\frac{\pi}{2}(\xi^2+x^2)}\iint e^{-\frac{\pi}{2}\left(\eta^2+y^2+2iy(\xi+ix)-2\eta(\xi+ix)\right)}\Phi(y,\eta)dyd\eta\\
&= e^{-\frac{\pi}{2}(\xi^2+x^2)}\iint e^{-\pi(y^2+\eta^2)}e^{\pi(\eta-iy)(\xi+ix)}f(\eta+iy)dyd\eta\\
&= e^{-\frac{\pi}{2}|z|^2}\iint e^{-\pi|\zeta|^2}e^{\pi\bar\zeta z}f(\zeta)dyd\eta \quad (\zeta=\eta+iy,\ z=\xi+ix)\\
&= e^{-\frac{\pi}{2}|z|^2}\iint f(\zeta)\prod_{1\le j\le n}\frac{1}{\pi(z_j-\zeta_j)}\frac{\partial}{\partial\bar\zeta_j}\left(e^{-\pi|\zeta|^2}e^{\pi\bar\zeta z}\right)dyd\eta\\
&= e^{-\frac{\pi}{2}|z|^2}\langle f(\zeta)\prod_{1\le j\le n}\frac{\partial}{\partial\bar\zeta_j}\left(\frac{1}{\pi(\zeta_j-z_j)}\right), e^{-\pi|\zeta|^2}e^{\pi\bar\zeta z}\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}\\
&= e^{-\frac{\pi}{2}|z|^2}f(z),\end{aligned}$$

since f is entire. This implies $WW^*\Phi = \Phi$ and $\Phi \in \operatorname{range} W$, completing the proof of the proposition. □

Proposition 6.3.6. *Defining*

$$\mathscr{H} = \ker(\bar\partial + \frac{\pi}{2}z) \cap \mathscr{S}'(\mathbb{R}^{2n}), \tag{6.3.14}$$

the operator W given by (6.3.2) can be extended as a continuous mapping from $\mathscr{S}'(\mathbb{R}^n)$ onto $\mathscr{H}$ (the $L^2(\mathbb{R}^n)$ dot-product is replaced by a bracket of (anti)duality).

The operator $\widetilde{\Pi}$ *with kernel* Π *given by* (6.3.4) *defines a continuous mapping from* $\mathscr{S}(\mathbb{R}^{2n})$ *into itself and can be extended as a continuous mapping from* $\mathscr{S}'(\mathbb{R}^{2n})$ *onto* $\mathscr{H}$. *It verifies*

$$\widetilde{\Pi}^2 = \widetilde{\Pi}, \quad \widetilde{\Pi}_{|\mathscr{H}} = \mathrm{Id}_{\mathscr{H}}. \tag{6.3.15}$$

Proof. As above we use that $e^{-i\pi y\eta}(Wv)(y,\eta)$ is the partial Fourier transform w.r.t. x of the tempered distribution on $\mathbb{R}^{2n}_{x,y}$

$$v(x)2^{n/4}e^{-\pi(x-y)^2}.$$

Since $e^{\pm i\pi y\eta}$ are in the space $\mathscr{O}_M(\mathbb{R}^{2n})$ of multipliers of $\mathscr{S}(\mathbb{R}^{2n})$, that transformation is continuous and injective from $\mathscr{S}'(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^{2n})$. Replacing in (6.3.13) the integrals by brackets of duality, we see that $W(\mathscr{S}'(\mathbb{R}^n)) \subset \mathscr{H}$. Conversely, if $\Phi \in \mathscr{H}$, the same calculations as above give (6.3.15) and (6.3.14). □

Theorem 6.3.7. *Let* $A \in Op(\Gamma^m)$ *with* $m < 0$. *Then* A *is a compact operator on* $L^2(\mathbb{R}^n)$.

Proof. Let $a_m \in \Gamma^m$. Then, we may consider the symbol

$$a_{m-1} = a_m - \big(a_m * 2^n \exp -2\pi|\cdot|^2\big)$$

which belongs to Γ^{m-1}. We may then consider

$$a_{m-2} = a_{m-1} - \big(a_{m-1} * 2^n \exp -2\pi|\cdot|^2\big) \in \Gamma^{m-2},$$

so that

$$\begin{aligned}
a_m &= \big(a_m * 2^n \exp -2\pi|\cdot|^2\big) + a_{m-1} \\
&= \big(a_m * 2^n \exp -2\pi|\cdot|^2\big) + \big(a_{m-1} * 2^n \exp -2\pi|\cdot|^2\big) + a_{m-2}, \\
&\dots \\
&= \big(a_m * 2^n \exp -2\pi|\cdot|^2\big) + \dots + \big(a_{m-N} * 2^n \exp -2\pi|\cdot|^2\big) + a_{m-N-1},
\end{aligned}$$

with $a_j \in \Gamma^j$. As a result, if N is large enough, the symbol a_{m-N-1} belongs to $L^2(\mathbb{R}^{2n})$ and thus the kernel of its Weyl quantization is also in $L^2(\mathbb{R}^{2n})$, so is a Hilbert-Schmidt operator, thus a compact operator. We need now to look at the operator with anti-Wick symbol $\tilde{a} = a_m + \dots + a_{m-N} \in \Gamma^m$ with $m < 0$. Let $\chi \in C_c^\infty(\mathbb{R}^{2n}; [0,1])$, equal to 1 on the unit Euclidean ball $\mathbb{B}^{2n}$. For $\lambda > 0$, we define

$$b_\lambda(X) = \tilde{a}(X)\chi(X/\lambda).$$

Since $m < 0$, we have $\lim_{\lambda\to+\infty} \|b_\lambda - \tilde{a}\|_{L^\infty(\mathbb{R}^{2n})} = 0$: in fact we have

$$|b_\lambda(X) - \tilde{a}(X)| \le \sup_{|X|\ge\lambda} |\tilde{a}(X)| \le C_0\lambda^{2m} \Longrightarrow \|b_\lambda - \tilde{a}\|_{L^\infty(\mathbb{R}^{2n})} \le C_0\lambda^{2m}.$$

This implies from (6.3.5) that $\lim_{\lambda\to+\infty}\big(\tilde{a} - b_\lambda\big)^{\mathrm{Wick}} = 0$, in operator-norm. Now the operator $b_\lambda^{\mathrm{Wick}}$ is obviously compact since its symbol is $b_\lambda * 2^n \exp -2\pi|\cdot|^2$, thus in $L^2(\mathbb{R}^{2n})$ since b_λ belongs to $L^2(\mathbb{R}^{2n})$, proving the compactness of the operator with Weyl symbol a_m. □

6.4 Ellipticity and Sobolev spaces

Ellipticity

Definition 6.4.1. Let $m \in \mathbb{R}$ and let $a \in \Gamma^m$. The symbol a is said to be elliptic in Γ^m whenever

$$\exists R > 0, \exists c > 0, \forall X \text{ with } |X| \ge R \quad \text{we have } |a(X)| \ge c\langle X\rangle^{2m}.$$

Lemma 6.4.2. *Let $m \in \mathbb{R}$ and let $a \in \Gamma^m$ be elliptic in Γ^m. Then there exists b in Γ^{-m} such that*

$$b \diamond a = 1 + r_1, \quad a \diamond b = 1 + r_2, \qquad r_j \in \Gamma^{-\infty} = \cap_{s\in\mathbb{R}}\Gamma^s.$$

Proof. Let $\chi \in C_c^\infty(\mathbb{R}^{2n}; [0,1])$, equal to 1 on the unit Euclidean ball $\mathbb{B}^{2n}$, supported in $2\mathbb{B}^{2n}$ and let $\tilde\chi = 1 - \chi$. We define

$$b_{-m}(X) = \frac{\tilde\chi(X/R)}{a(X)} = \frac{\tilde\chi_R(X)}{a(X)}.$$

Since a is bounded below by $c\langle X\rangle^{2m}$ for $|X| \ge R$, that is on the support of $\tilde\chi(\cdot/R)$, then b_{-m} is a smooth function on $\mathbb{R}^{2n}$. Moreover an application of the Faà de Bruno formula shows that $b_{-m} \in \Gamma^{-m}$. As a result, we have

$$a \diamond b_{-m} = \tilde\chi_R + \rho_{-1} = 1 \underbrace{-\chi_R + \rho_{-1}}_{r_{-1}}, \quad \rho_{-1}, r_{-1} \in \Gamma^{-1}.$$

We can find $b_{-m-1} \in \Gamma^{-m-1}$ such that

$$a \diamond \big(b_{-m} + b_{-m-1}\big) \in 1 + \Gamma^{-2},$$

since it is enough to get

$$1 + r_{-1} + a \diamond b_{-m-1} \in 1 + \Gamma^{-2},$$

and we may choose $b_{-m-1} = -r_{-1}\chi_R a^{-1}$. Following the proof of Lemma 3.4.13, we obtain the result. □

Sobolev spaces

The Sobolev spaces $\mathscr{H}^s$ are defined in (7.2.53).

Theorem 6.4.3. *Let $a \in \Gamma^m$. Then the operator a^w with domain $\mathscr{S}(\mathbb{R}^n)$ is closable.*

Proof. Let us assume that $(u_k)_{k\ge1}$ is a sequence of $\mathscr{S}(\mathbb{R}^n)$, converging in $L^2(\mathbb{R}^n)$, with limit u such that the sequence $(v_k = a^w u_k)_{k\ge1}$ converges in $L^2(\mathbb{R}^n)$ with limit v. For $\phi \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\langle v - a^w u, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \lim_k \langle a^w u_k - a^w u, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \lim_k \langle u_k - u, \bar a^w \phi\rangle_{L^2} = 0,$$

so that $a^w u = v$. □

Theorem 6.4.4. *Let $a \in \Gamma^m$ with $m \geq 0$ be a real-valued globally elliptic symbol. Then the operator a^w with domain $\mathscr{H}^m$ is self-adjoint.*

Proof. In the first place we know that a^w is a continuous linear operator from $\mathscr{H}^m$ into $\mathscr{H}^0 = L^2$ so that we can consider the operator $A = a^w$ with domain $D_A = \mathscr{H}^m$. Let us now check A^* and its domain

$$D_{A^*} = \{v \in \mathscr{H}^0, \exists u \in \mathscr{H}^0, \forall w \in \mathscr{H}^m, \quad \langle v, Aw\rangle_{\mathscr{H}^0} = \langle u, w\rangle_{\mathscr{H}^0}\}.$$

In particular, we note that for $v \in D_{A^*}, \phi \in \mathscr{S}$, we have, since a is real-valued,

$$\langle A^*v, \phi\rangle_{\mathscr{S}^*,\mathscr{S}} = \langle A^*v, \phi\rangle_{\mathscr{H}^0} = \langle v, A\phi\rangle_{\mathscr{H}^0} = \langle v, a^w\phi\rangle_{\mathscr{S}^*,\mathscr{S}} = \langle a^w v, \phi\rangle_{\mathscr{S}^*,\mathscr{S}},$$

which implies that $A^*v = a^w v$, so that for $v \in D_{A^*}$, $a^w v \in L^2$. Since a is elliptic, there exists $b \in \Gamma^{-m}$ such that

$$a^w b^w = I + r^w, \quad b^w a^w = I + s^w, \quad r, s \in \Gamma^{-\infty}.$$

Let $v \in D_{A^*}$: we have

$$v = \overbrace{b^w \underbrace{a^w v}_{\in L^2}}^{\in \mathscr{H}^m} - \overbrace{s^w v}^{\in \mathscr{S}} \in \mathscr{H}^m,$$

and thus $D_{A^*} \subset \mathscr{H}^m$, with $A^* = a^w$ on D_{A^*}. On the other hand if $v \in \mathscr{H}^m \subset L^2$ (since $m \geq 0$), we have

$$\forall w \in D_A = \mathscr{H}^m, \quad \langle v, Aw\rangle_{\mathscr{H}^0} = \langle v, a^w w\rangle_{\mathscr{H}^0} = \langle a^w v, w\rangle_{\mathscr{H}^0},$$

since the latter identity is true for $w \in \mathscr{S}$ and thus if $w = \lim_k w_k$ in $\mathscr{H}^m$, $w_k \in \mathscr{S}$, we find by continuity of a^w from $\mathscr{H}^m$ into L^2

$$\begin{aligned}\langle v, a^w w\rangle_{\mathscr{H}^0} = \lim_k \langle v, a^w w_k\rangle_{\mathscr{H}^0} = \lim_k \langle \underbrace{a^w v}_{\in L^2}, w_k\rangle_{\mathscr{S}^*,\mathscr{S}} \\ = \lim_k \langle a^w v, w_k\rangle_{\mathscr{H}^0} = \langle a^w v, w\rangle_{\mathscr{H}^0}.\end{aligned}$$

This implies that $\mathscr{H}^m \subset D_{A^*}$ and thus $\mathscr{H}^m = D_{A^*}$ with

$$A^* = a^w = A \text{ on } D_{A^*} = D_A,$$

and the self-adjointness of A. □

Chapter 7

Appendix

7.1 On the Faà di Bruno formula

That formula[1] is dealing with the iterated derivative of a composition of functions. First of all, let us consider (smooth) functions of one real variable

$$U \xrightarrow{f} V \xrightarrow{g} W, \qquad U, V, W \text{ open sets of } \mathbb{R}.$$

With $g^{(r)}$ always evaluated at $f(x)$, we have

$$\begin{aligned}
&(g \circ f)' = g' f' \\
&(g \circ f)'' = g'' f'^2 + g' f'' \\
&(g \circ f)''' = g''' f'^3 + g'' 3 f'' f' + g' f''' \\
&(g \circ f)^{(4)} = g^{(4)} (f')^4 + 6 g^{(3)} f'^2 f'' + g''(4 f''' f' + 3 f''^2) + g' f^{(4)} \\
&\text{i.e. } \frac{1}{4!} (g \circ f)^{(4)} = \\
&\frac{g^{(4)}}{4!} \left(\frac{f'}{1!}\right)^4 + 3 \frac{g^{(3)}}{3!} \left(\frac{f''}{2!}\right) \left(\frac{f'}{1!}\right)^2 + \frac{g^{(2)}}{2!} \Big[\left(\frac{f''}{2!}\right)^2 + 2 \frac{f'''}{3!} f' \Big] + \frac{g^{(1)}}{1!} \frac{f^{(4)}}{4!}.
\end{aligned}$$

More generally we have the remarkably simple

$$\frac{(g \circ f)^{(k)}}{k!} = \sum_{\substack{1 \le r \le k \\ k_j \ge 1}} \frac{g^{(r)} \circ f}{r!} \prod_{k_1 + \cdots + k_r = k} \frac{f^{(k_j)}}{k_j!} \tag{7.1.1}$$

· There is only one multi-index $(1, 1, 1, 1) \in \mathbb{N}^{*4}$ such that $\sum_{1 \le j \le 4} k_j = 4$.
· There are 3 multi-indices $(1, 1, 2), (1, 2, 1), (2, 1, 1) \in \mathbb{N}^{*3}$ with $\sum_{1 \le j \le 3} k_j = 4$.
· There is 1 multi-index $(2, 2) \in \mathbb{N}^{*2}$ with $\sum_{1 \le j \le 2} k_j = 4$ and 2 multiindices $(1, 3), (3, 1)$ such that $\sum_{1 \le j \le 2} k_j = 4$.

[1] Francesco Faà di Bruno (1825–1888) was an italian mathematician and priest, born at Alessandria. He was beatified in 1988, probably the only mathematician to reach sainthood so far. The "Chevalier François Faà di Bruno, Capitaine honoraire d'État-Major dans l'armée Sarde", defended his thesis in 1856, in the Faculté des Sciences de Paris in front of the following jury: Cauchy (chair), Lamé and Delaunay.

· There is 1 index $4 \in \mathbb{N}^*$ with $\sum_{1\le j\le 1} k_j = 4$.

Usually the formula is written in a different way with the more complicated

$$\frac{(g\circ f)^{(k)}}{k!} = \sum_{\substack{l_1+2l_2+\cdots+kl_k=k\\ r=l_1+\cdots+l_k}} \frac{g^{(r)}\circ f}{l_1!\ldots l_k!} \prod_{1\le j\le k} \left(\frac{f^{(j)}}{j!}\right)^{l_j}. \qquad (7.1.2)$$

Let us show that the two formulas coincide. We start from (7.1.1)

$$\frac{(g\circ f)^{(k)}}{k!} = \sum_{1\le r\le k} \frac{g^{(r)}\circ f}{r!} \prod_{\substack{k_1+\cdots+k_r=k\\ k_j\ge 1}} \frac{f^{(k_j)}}{k_j!}.$$

If we consider a multi-index

$$(k_1,\ldots,k_r) = (\underbrace{1,\ldots,1}_{l_1\text{times}},\underbrace{2,\ldots,2}_{l_2\text{times}},\ldots,\underbrace{j,\ldots,j}_{l_j\text{times}},\ldots,\underbrace{k,\ldots,k}_{l_k\text{times}})$$

we get in factor of $g^{(r)}/r!$ the term $\prod_{1\le j\le k}\left(\frac{f^{(j)}}{j!}\right)^{l_j}$ with $l_1 + 2l_2 + \cdots + kl_k = k, \quad l_1+\cdots+l_k = r$ and since we can permute the $(k_1,\ldots,k_r)$ above, we get indeed a factor $\frac{r!}{l_1!\ldots l_k!}$ which gives (7.1.2).

The proof above can easily be generalized to a multidimensional setting with

$$U \xrightarrow{f} V \xrightarrow{g} W, \qquad U,V,W \text{ open sets of } \mathbb{R}^m,\mathbb{R}^n,\mathbb{R}^p,\ f,g \text{ of class } C^k.$$

Since the derivatives are multilinear symmetric mappings, they are completely determined by their values on the "diagonal" $T\otimes\cdots\otimes T$: the symmetrized products of $T_1\otimes\cdots\otimes T_k$, noted as $T_1\ldots T_k$, can be written as a linear combination of k-th powers. In fact, in a commutative algebra on a field with characteristic 0, using the polarization formula, the products $T_1\ldots T_k$ are linear combination of k-th powers

$$T_1T_2\ldots T_k = \frac{1}{2^k k!}\sum_{\epsilon_j=\pm1} \epsilon_1\ldots\epsilon_k(\epsilon_1T_1+\cdots+\epsilon_kT_k)^k. \qquad (7.1.3)$$

For $T\in\mathcal{T}_x(U)$, we have

$$\frac{(g\circ f)^{(k)}}{k!}T^k = \sum_{1\le r\le k} \frac{g^{(r)}\circ f}{r!} \prod_{\substack{k_1+\cdots+k_r=k\\ k_j\ge 1}} \frac{f^{(k_j)}}{k_j!}T^{k_j},$$

which is consistent with the fact that $f^{(k_j)}(x)T^{k_j}$ belongs to the tangent space $\mathcal{T}_{f(x)}(V)$ of V at $f(x)$ and $\otimes_{1\le j\le r} f^{(k_j)}(x)T^{k_j}$ is a tensor product in $\mathcal{T}^{r,0}(\mathcal{T}_{f(x)}(V))$ on which $g^{(r)}(f(x))$ acts to send it on $\mathcal{T}_{g(f(x))}(W)$.

7.2 The Harmonic Oscillator

Polynômes d'Hermite

Présentation à l'aide d'une fonction génératrice

Soit $x \in \mathbb{C}$. La fonction $\mathbb{C} \ni t \mapsto e^{-t^2+2tx} = G(x,t)$ est entière et par suite

$$G(x,t) = e^{-t^2+2tx} = \sum_{n\geq 0} \frac{t^n}{n!} H_n(x), \quad H_n(x) = \frac{\partial^n G}{\partial t^n}(x,0), \tag{7.2.1}$$

avec un rayon de convergence infini pour tout $x \in \mathbb{C}$. Notons que

$$H_0(x) = G(x,0) = 1, \quad H_1(x) = \frac{\partial G}{\partial t}(x,0) = e^{-t^2+2tx}(-2t+2x)_{|t=0} = 2x,$$
$$H_2(x) = \frac{\partial^2 G}{\partial t^2}(x,t)_{|t=0} = e^{-t^2+2tx}\big\{(-2t+2x)^2 - 2\big\}_{|t=0} = 4x^2 - 2.$$

Lemma 7.2.1. *Pour $n \in \mathbb{N}$, H_n est un polynôme de degré n, de même parité que n, dont le monôme de plus haut degré est $2^n X^n$. On a également, pour $n \in \mathbb{N}^*, m \in \mathbb{N}$,*

$$H_{n+1}(X) = 2XH_n(X) - 2nH_{n-1}(X), \tag{7.2.2}$$
$$H_n'(X) = 2nH_{n-1}(X), \tag{7.2.3}$$
$$H_{2m}(0) = (-1)^m \frac{(2m)!}{m!}. \tag{7.2.4}$$

N.B. On dira que la fonction G est une fonction génératrice pour la suite des polynômes d'Hermite H_n.

Proof. On a pour $n \in \mathbb{N}$, avec $g(y) = e^{-y^2}$, l'identité $G(x,t) = g(x-t)e^{x^2}$, et donc

$$H_n(x) = \frac{\partial^n G}{\partial t^n}(x,0) = e^{x^2}(-1)^n g^{(n)}(x) = e^{x^2}(-1)^n \big(\frac{d}{dx}\big)^n \{e^{-x^2}\}. \tag{7.2.5}$$

Démontrons par récurrence sur $n \in \mathbb{N}$ que H_n est un polynôme de degré n, de même parité que n, dont le monôme de plus haut degré est $2^n X^n$. C'est vérifié pour $n = 0, 1, 2$. Supposons que cette propriété est vérifiée pour un entier $n \geq 0$. On a

$$\begin{aligned} H_{n+1}(x) &= e^{x^2}(-1)^{n+1}\frac{d}{dx}\Big\{e^{-x^2}e^{x^2}\big(\frac{d}{dx}\big)^n\{e^{-x^2}\}\Big\} \\ &= (-1)^{n+1}\Big(\frac{d}{dx} - 2x\Big)\Big\{(-1)^n H_n(x)\Big\} = -H_n'(x) + 2xH_n(x), \end{aligned} \tag{7.2.6}$$

et l'on trouve que H_{n+1} est un polynôme de monôme de plus haut degré $2X2^nX^n = 2^{n+1}X^{n+1}$. En outre comme H_n est de la parité de n, H_n' et XH_n sont de la parité de $n+1$ ainsi donc que H_{n+1}, ce qui achève notre raisonnement par récurrence. On a en outre

$$\frac{\partial G}{\partial x}(x,t) = 2tG(x,t) = 2\sum_{n\geq 0}(n+1)\frac{t^{n+1}}{(n+1)!}H_n(x) = \sum_{k\geq 1}\frac{t^k}{k!}2kH_{k-1}(x), \tag{7.2.7}$$

avec un rayon de convergence infini pour tout $x \in \mathbb{C}$. On a également (cf. (7.2.1))

$$H_k'(x) = \frac{\partial^{k+1} G}{\partial x \partial t^k}(x, 0).$$

et comme la fonction $t \mapsto (\partial G/\partial x)(x, t)$ est entière pour tout x, il vient

$$\frac{\partial G}{\partial x}(x, t) = \sum_{k \geq 0} \frac{t^k}{k!} H_k'(x),$$

qui donne avec (7.2.7), $H_k'(x) = 2kH_{k-1}(x)$ pour $k \geq 1$. Comme nous avons démontré en (7.2.7) que pour $n \geq 0$, $H_{n+1}(x) = 2xH_n(x) - H_n'(x)$, il vient pour $n \geq 1$,

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad H_n'(x) = 2nH_{n-1}(x),$$

ce qui donne (7.2.2), (7.2.3). La propriété (7.2.4) est vraie pour $m = 0$ et si on la suppose vérifiée pour un entier $m \geq 0$, il vient de (7.2.2) (déjà démontré!) pour $2m + 1$,

$$\begin{aligned} H_{2m+2}(0) &= -(4m+2)H_{2m}(0) \\ &= (-1)^{m+1} \frac{(2m+2)!}{(m+1)!} \frac{m+1}{(2m+1)(2m+2)}(4m+2) = (-1)^{m+1} \frac{(2m+2)!}{(m+1)!}, \end{aligned}$$

soit le résultat cherché. □

Une présentation plus explicite

En utilisant la formule de *Faà di Bruno* sur la dérivation des fonctions composées, on peut obtenir une expression plus explicite des polynômes d'Hermite. Rappelons que pour $g, f \in C^\infty(\mathbb{R})$, on a pour $n \geq 1$

$$\frac{(g \circ f)^{(n)}}{n!} = \sum_{1 \leq r \leq n} \frac{g^{(r)} \circ f}{r!} \prod_{\substack{n_1 + \cdots + n_r = n \\ n_j \geq 1}} \frac{f^{(n_j)}}{n_j!}. \tag{7.2.8}$$

On définit H_n par la formule

$$H_n(x) = e^{x^2}(-1)^n \left(\frac{d}{dx}\right)^n \{e^{-x^2}\}. \tag{7.2.9}$$

On se propose maintenant de calculer explicitement H_n en utilisant la formule (7.2.8) : il vient avec $g(y) = e^y$, $f(x) = -x^2$, pour $n \geq 1$

$$H_n(x) = e^{x^2}(-1)^n n! \sum_{1 \leq r \leq n} \frac{g^{(r)} \circ f}{r!} \prod_{\substack{n_1 + \cdots + n_r = n \\ n_j \geq 1}} \frac{f^{(n_j)}}{n_j!}.$$

Les valeurs possibles de n_j dans la formule ci-dessus sont $1, 2$: il faut choisir un sous-ensemble de $\{1, \ldots, r\}$ à s éléments, $1 \leq s \leq r$ sur lequel $n_j = 1$. Il vient par conséquent

$$H_n(x) = e^{x^2}(-1)^n n! \sum_{\substack{1 \leq r \leq n \\ s + 2(r-s) = n}} \frac{e^{-x^2}}{r!}(-2x)^s \left(\frac{(-2)}{2!}\right)^{r-s} C_r^s.$$

On remarque que $n-s=2k$ (un entier pair ≥ 0) et pour $n\geq 1$,

$$r-s=k, 1\leq k+s\leq n, 2k\leq n, \quad \text{i.e. } 1\leq k+n-2k\leq n, 2k\leq n,$$

soit $0\leq 2k\leq n$, pour $n\geq 2$. Il vient, pour $n\geq 2$

$$\begin{aligned} H_n(x) = e^{x^2}(-1)^n n! \sum_{\substack{1\leq r\leq n \\ s+2(r-s)=n}} \frac{e^{-x^2}}{r!}(-2x)^s\Big(\frac{(-2)}{2!}\Big)^{r-s}\frac{r!}{(r-s)!s!} \\ = n! \sum_{0\leq k\leq n/2} \frac{(-2x)^{n-2k}(-1)^k}{(n-2k)!k!}, \end{aligned}$$

formule également valable pour $n=0,1$ car de (7.2.9) vient

$$H_0(x)=1, \quad H_1(x)=2x.$$

On a donc pour tout $n\in\mathbb{N}$,

$$H_n(x) = n! \sum_{0\leq k\leq E(n/2)} \frac{(2x)^{n-2k}(-1)^k}{(n-2k)!k!}, \tag{7.2.10}$$

ce qui montre immédiatement que H_n est un polynôme de degré n, de même parité que n, dont le monôme de plus haut degré est 2^nX^n. De plus si $n=2m$ est pair on redémontre (7.2.4). En outre pour $n\geq 1$, on peut calculer directement

$$\begin{aligned} H_n'(x) &= n! \sum_{0\leq k<n/2} \frac{(2x)^{n-2k-1}2(n-2k)(-1)^k}{(n-2k)!k!} \\ &= 2n\ (n-1)! \sum_{0\leq k<n/2} \frac{(2x)^{n-1-2k}(-1)^k}{(n-1-2k)!k!} \\ &= 2n\ (n-1)! \sum_{0\leq 2k\leq n-1} \frac{(2x)^{n-1-2k}(-1)^k}{(n-1-2k)!k!} \\ &= 2nH_{n-1}(x). \end{aligned} \tag{7.2.11}$$

De plus, la formule (7.2.6) est prouvée directement par récurrence, et l'on a donc

$$H_{n+1}(x) = -H_n'(x) + 2xH_n(x),$$

de sorte qu'avec le calcul (7.2.11), on obtient le Lemme 7.2.1 sans utiliser la fonction génératrice, avec en outre l'expression explicite (7.2.10).

Quelques calculs explicites

La commande *Mathematica* `HermiteH[n, x]` permet d'obtenir le nième polynôme d'Hermite. En écrivant `HermiteH[n, x] // TraditionalForm,` on obtient

$$
\begin{aligned}
H_0(x) &= 1\\
H_1(x) &= 2x\\
H_2(x) &= 4x^2 - 2\\
H_3(x) &= 8x^3 - 12x\\
H_4(x) &= 16x^4 - 48x^2 + 12\\
H_5(x) &= 32x^5 - 160x^3 + 120x\\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120\\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x\\
H_8(x) &= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680\\
H_9(x) &= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x\\
H_{10}(x) &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
H_{20}(x) &= 1048576x^{20} - 99614720x^{18} + 3810263040x^{16} - 76205260800x^{14}\\
&\quad + 866834841600x^{12} - 5721109954560x^{10} + 21454162329600x^8\\
&\quad - 42908324659200x^6 + 40226554368000x^4 - 13408851456000x^2\\
&\quad + 670442572800
\end{aligned}
$$

Équation différentielle

Lemma 7.2.2. *Soit $n \in \mathbb{N}$. Alors le polynôme d'Hermite H_n vérifie*

$$H_n''(X) - 2XH_n'(X) + 2nH_n(X) = 0. \tag{7.2.12}$$

Proof. Démontrons par récurrence pour $n \geq 2$ que

$$2nH_n(X) - 4XnH_{n-1}(X) + 4n(n-1)H_{n-2}(X) = 0. \tag{7.2.13}$$

Cela est vérifié pour $n = 2$ car

$$4H_2(X) - 8XH_1(X) + 8H_0(X) = 4(4X^2 - 2) - 8X\ 2X + 8 = 0.$$

En supposant (7.2.13) vérifié pour un entier $n \geq 2$, on calcule, en utilisant (7.2.2),

$$\begin{aligned}
&(2n+2)H_{n+1}(X) - 4X(n+1)H_n(X) + 4(n+1)nH_{n-1}(X) \\
&= (2n+2)\big(2XH_n(X) - 2nH_{n-1}(X)\big) - 4X(n+1)H_n(X) + 4(n+1)nH_{n-1}(X) \\
&= H_n(X)\big((4n+4)X - 4(n+1)X\big) + H_{n-1}(X)\big(-2n(2n+2) + 4(n+1)n\big) = 0,
\end{aligned}$$

ce qui achève la récurrence. Utilisant le Lemme 7.2.1, il vient pour $n \geq 2$

$$\begin{aligned}
&H_n''(X) - 2XH_n'(X) + 2nH_n(X) \\
&\qquad = 2n2(n-1)H_{n-2}(X) - 2X2nH_{n-1}(X) + 2nH_n(X) = 0,
\end{aligned}$$

d'après (7.2.13), ce qui démontre le résultat cherché pour $n \geq 2$. Pour $n = 0$, on a $H_0 = 1$ et l'équation (7.2.12) est trivialement vérifiée. Pour $n = 1$, on a $H_1 = 2X$ et le membre de gauche de (7.2.12) vaut $-4X + 2 \times 2X = 0$, terminant la démonstration. □

Fonctions d'Hermite

Proposition 7.2.3. *Pour $n, m \in \mathbb{N}$, on a*

$$\int_{\mathbb{R}} H_n(x)H_m(x)e^{-x^2}dx = \delta_{n,m}n!2^n\sqrt{\pi}. \tag{7.2.14}$$

Proof. On a en effet pour $n \geq m$,

$$\int_{\mathbb{R}} H_n(x)H_m(x)e^{-x^2}dx = (-1)^n \int_{\mathbb{R}} \Big(\frac{d}{dx}\Big)^n\{e^{-x^2}\}H_m(x)dx = \int e^{-x^2}H_m^{(n)}(x)dx,$$

qui vaut 0 si $n > m$ (H_m est un polynôme de degré m) et pour $m = n$, on obtient

$$\int_{\mathbb{R}} H_n(x)^2e^{-x^2}dx = \int_{\mathbb{R}} e^{-x^2}n!2^ndx = n!2^n\sqrt{\pi},$$

soit le résultat cherché. □

Les polynômes d'Hermite sont à coefficients réels (cf. e.g. (7.2.10)), de sorte que les fonctions ϕ_n, dites *fonctions d'Hermite* définies sur $\mathbb{R}$ par

$$\phi_n(x) = H_n(x)e^{-x^2/2}(2^n n!)^{-1/2}\pi^{-1/4} \tag{7.2.15}$$

$$= (-1)^n(2^n n!)^{-1/2}\pi^{-1/4}e^{x^2/2}\Big(\frac{d}{dx}\Big)^n\{e^{-x^2}\}, \tag{7.2.16}$$

$$\text{vérifient} \quad \langle\phi_n, \phi_m\rangle_{L^2(\mathbb{R})} = \delta_{n,m}. \tag{7.2.17}$$

Theorem 7.2.4. *La suite des fonctions d'Hermite $\{\phi_n\}_{n\in\mathbb{N}}$ forme une base hilbertienne de $L^2(\mathbb{R})$. Chaque fonction d'Hermite ϕ_n appartient à la classe de Schwartz $\mathscr{S}(\mathbb{R})$.*

Proof. La dernière assertion est triviale car

$$\phi_n(x) = H_n(x)e^{-x^2/2}(2^n n!)^{-1/2}\pi^{-1/4}$$

et donc $\phi_n \in C^\infty(\mathbb{R})$ et, par récurrence sur $k \in \mathbb{N}$,

$$x^l\phi_n^{(k)}(x) = P_{n,k,l}(x)e^{-x^2/2}, \quad P_{n,k,l} \text{ polynôme},$$

ce qui implique que $\sup_{x\in\mathbb{R}} |x^l\phi_n^{(k)}(x)| = C_{n,k,l} < +\infty$. Au vu de (7.2.17), il suffit de démontrer que l'orthogonal de l'espace engendré par $\{\phi_n\}_{n\in\mathbb{N}}$ est réduit à $\{0\}$. Soit f une fonction de $L^2(\mathbb{R})$ telle que,

$$\text{pour tout } n \in \mathbb{N}, \qquad \int_{\mathbb{R}} f(x)\phi_n(x)dx = 0.$$

Comme chaque H_n est un polynôme de degré n, l'espace vectoriel engendré par $\{H_n\}_{0\le n\le N}$ est l'espace des polynômes de degré $\le N$ (récurrence sur N). Par suite on a

$$\text{pour tout } n \in \mathbb{N}, \qquad \int_{\mathbb{R}} f(x)x^n e^{-x^2/2}dx = 0.$$

Considérons la fonction F, donnée pour $z \in \mathbb{C}$ par

$$F(z) = \int_{\mathbb{R}} f(x)e^{-x^2/2}e^{zx}dx.$$

On a pour K compact de $\mathbb{C}$, $M_K = \sup_{z\in K} |\operatorname{Re} z|$, l'estimation

$$\sup_{z\in K} |f(x)e^{-x^2/2}e^{zx}| \le \underbrace{|f(x)|}_{L^2}\underbrace{e^{-x^2/2}e^{|x|M_K}}_{L^2} \in L^1(\mathbb{R}),$$

et comme $z \mapsto f(x)e^{-x^2/2}e^{zx}$ est entière, la fonction F est entière. En outre, pour $n \in \mathbb{N}$, il vient

$$F^{(n)}(0) = \int_{\mathbb{R}} f(x)e^{-x^2/2}x^n dx = 0,$$

ce qui implique que F est identiquement nulle. La fonction $\mathbb{R} \ni x \mapsto h(x) = f(x)e^{-x^2/2}$ appartient à $L^1(\mathbb{R})$ comme produit de deux fonctions de $L^2(\mathbb{R})$ (inégalité de Cauchy-Schwarz). On a de plus

$$\hat{h}(\xi) = F(-2i\pi\xi) = 0,$$

de sorte que la transformée de Fourier de h est nulle, et donc $h = 0$. On a donc, pour presque tout $x \in \mathbb{R}$,

$$f(x)e^{-x^2/2} = 0,$$

ce qui implique $f(x) = 0$ presque partout et $f = 0$ comme fonction de $L^2(\mathbb{R})$. $\square$

Oscillateur harmonique

Equation différentielle

Lemma 7.2.5. *Soit $n \in \mathbb{N}$. Alors la fonction d'Hermite ϕ_n définie par (7.2.15) vérifie l'équation différentielle*

$$-\phi_n''(x) + x^2\phi_n(x) = (2n+1)\phi_n(x). \tag{7.2.18}$$

Proof. En utilisant (7.2.15) et (7.2.12), il vient

$$\begin{aligned}
\phi_n'(x) &= \{H_n'(x) - xH_n(x)\}e^{-x^2/2}(2^n n!)^{-1/2}\pi^{-1/4},\\
\phi_n''(x) &= \{H_n''(x) - xH_n'(x) - H_n(x) - x\big(H_n'(x) - xH_n(x)\big)\}e^{-x^2/2}(2^n n!)^{-1/2}\pi^{-1/4}\\
&= \{H_n''(x) - 2xH_n'(x) + (x^2-1)H_n(x)\}e^{-x^2/2}(2^n n!)^{-1/2}\pi^{-1/4}\\
&= \{-2nH_n(x) + (x^2-1)H_n(x)\}e^{-x^2/2}(2^n n!)^{-1/2}\pi^{-1/4}\\
&= -(2n+1)\phi_n(x) + x^2\phi_n(x),
\end{aligned}$$

ce qui donne le résultat cherché. $\square$

Création, annihilation

Definition 7.2.6. L'opérateur de création (resp. annihilation) A_+ (resp. A_-) est l'opérateur différentiel de $\mathscr{S}(\mathbb{R})$ dans lui-même donné par

$$A_+ = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right), \quad A_- = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right). \tag{7.2.19}$$

L'oscillateur harmonique $\mathcal{H}$ est l'opérateur différentiel de $\mathscr{S}(\mathbb{R})$ dans lui-même donné par

$$\mathcal{H} = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right). \tag{7.2.20}$$

Remark 7.2.7. Du lemme 7.2.5, il vient pour $n \in \mathbb{N}$,

$$\mathcal{H}\phi_n = (\frac{1}{2} + n)\phi_n. \tag{7.2.21}$$

Lemma 7.2.8. *Sur $\mathscr{S}(\mathbb{R})$, on a*

$$\mathcal{H} = A_+A_- + \frac{1}{2}, \tag{7.2.22}$$

$$[A_-, A_+] = A_-A_+ - A_+A_- = I. \tag{7.2.23}$$

Proof. Pour $\psi \in \mathscr{S}(\mathbb{R})$, on a

$$2(A_+A_-\psi)(x) = (-\frac{d}{dx} + x)\{\psi'(x) + x\psi(x)\}$$
$$= -\big(\psi''(x) + x\psi'(x) + \psi(x)\big) + x\psi'(x) + x^2\psi(x),$$

ce qui donne $2A_+A_- = 2\mathcal{H} - I$ et (7.2.22). De plus, on a

$$\begin{aligned}
&2(A_-A_+ - A_+A_-)\psi \\
&= \left(\frac{d}{dx} + x\right)(-\psi'(x) + x\psi(x)) - \left(-\frac{d}{dx} + x\right)(\psi'(x) + x\psi(x)) \\
&= -\psi''(x) + \psi(x) + x\psi'(x) - x\psi'(x) + x^2\psi(x) \\
&\quad + \psi''(x) + \psi(x) + x\psi'(x) - x\psi'(x) - x^2\psi(x) = 2\psi(x),
\end{aligned}$$

soit le résultat cherché. □

Lemma 7.2.9. *Soit $n \in \mathbb{N}$. On a*

$$\phi_n = \frac{1}{\sqrt{n!}} A_+^n \phi_0 \tag{7.2.24}$$

$$A_+\phi_n = \sqrt{n+1}\,\phi_{n+1}, \tag{7.2.25}$$

$$A_-\phi_{n+1} = \sqrt{n+1}\,\phi_n, \quad A_-\phi_0 = 0. \tag{7.2.26}$$

Proof. En utilisant (7.2.16), calculons

$$(A_+\phi_n)(x) = 2^{-1/2}(-1)^n(2^n n!)^{-1/2}\pi^{-1/4}(x - \frac{d}{dx})\Big\{e^{x^2/2}\big(\frac{d}{dx}\big)^n\{e^{-x^2}\}\Big\}.$$

Comme (sur $\mathscr{S}(\mathbb{R})$), on a

$$\frac{d}{dx} - x = e^{x^2/2}\frac{d}{dx}e^{-x^2/2}, \tag{7.2.27}$$

il vient

$$(A_+\phi_n)(x) = (-1)^{n+1}(2^{n+1}n!)^{-1/2}\pi^{-1/4}e^{x^2/2}\big(\frac{d}{dx}\big)^{n+1}\{e^{-x^2}\} = \sqrt{n+1}\,\phi_{n+1},$$

soit (7.2.25). La propriété (7.2.24) est vérifiée pour $n = 0$, et si on la suppose vraie pour un entier $n \geq 0$, il vient

$$\phi_{n+1} \underbrace{=}_{(7.2.25)} (n+1)^{-1/2}A_+\phi_n = (n+1)^{-1/2}(n!)^{-1/2}A_+A_+^n\phi_0 = \frac{A_+^{n+1}\phi_0}{\sqrt{(n+1)!}},$$

et donc (7.2.24). En outre, en utilisant (7.2.25), il vient

$$A_-\phi_{n+1} = (n+1)^{-1/2}A_-A_+\phi_n \underbrace{=}_{(7.2.23)} (n+1)^{-1/2}\big(A_+A_- + 1\big)\phi_n$$
$$\underbrace{=}_{(7.2.22)} (n+1)^{-1/2}\big(\mathcal{H} + \frac{1}{2}\big)\phi_n \underbrace{=}_{(7.2.21)} (n+1)^{-1/2}(n+1)\phi_n = (n+1)^{1/2}\phi_n.$$

De plus, on a

$$\pi^{1/4}\sqrt{2}A_-\phi_0 = (\frac{d}{dx} + x)(e^{-x^2/2}) = 0,$$

ce qui termine la démonstration du lemme. □

Opérateurs sur $\ell^2(\mathbb{N})$

Grâce au Théorème 7.2.4, l'application

$$\begin{array}{ccc} \ell^2(\mathbb{N}) & \longrightarrow & L^2(\mathbb{R}) \\ (a_n)_{n\in\mathbb{N}} & \mapsto & \sum_{n\in\mathbb{N}} a_n\phi_n \end{array}$$

est un isomorphisme isométrique d'espaces de Hilbert d'application réciproque

$$\Psi: \begin{array}{ccc} L^2(\mathbb{R}) & \longrightarrow & \ell^2(\mathbb{N}) \\ u & \mapsto & (\langle u, \phi_n\rangle)_{n\in\mathbb{N}} \end{array}. \tag{7.2.28}$$

On peut donc *identifier* $L^2(\mathbb{R})$ à $\ell^2(\mathbb{N})$ via ces applications. Considérons le sous-espace vectoriel E de $L^2(\mathbb{R})$ défini par

$$E = \{\sum_{n\in\mathbb{N}} a_n\phi_n\}_{(a_n)_{n\in\mathbb{N}}\in\tilde{E}}, \tag{7.2.29}$$

avec le sous-espace $\tilde{E}$ de $\ell^2(\mathbb{N})$ défini par

$$E = \{(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}), \sum_{n\in\mathbb{N}} n^2|a_n|^2 < +\infty\}. \tag{7.2.30}$$

L'oscillateur harmonique $\mathcal{H}$ s'identifie à l'opérateur $\widetilde{\mathcal{H}} : \tilde{E} \to \ell^2(\mathbb{N})$ défini par

$$\widetilde{\mathcal{H}}\big((a_n)_{n\in\mathbb{N}}\big) = \big((n+\frac{1}{2})a_n\big)_{n\in\mathbb{N}},$$

donné par la *matrice diagonale infinie*

$$\begin{pmatrix} \frac{1}{2} & 0 & \dots & & & \\ 0 & \frac{3}{2} & 0 & \dots & & \\ 0 & 0 & \frac{5}{2} & 0 & \dots & \\ \vdots & \vdots & & & & \\ & & & \ddots & & \\ 0 & \dots & \dots & 0 & \frac{1}{2}+n & 0\dots \\ & & & & & \end{pmatrix}.$$

L'opérateur de création A_+ vérifie

$$A_+\phi_n = \sqrt{n+1}\phi_{n+1},$$

et avec Ψ donné par (7.2.28), il vient avec $\tilde{A}_+ = \Psi A_+ \Psi^{-1}$,

$$\begin{aligned} \tilde{A}_+\big((a_n)_{n\in\mathbb{N}}\big) &= \Psi A_+\big(\sum_n a_n\phi_n\big) \\ &= \Psi\big(\sum_n a_n\sqrt{n+1}\phi_{n+1}\big) = \big(0, a_0, a_1\sqrt{2}, a_2\sqrt{3}, \dots\big) \\ &= (b_n)_{n\in\mathbb{N}}, \qquad b_0 = 0, \quad b_n = a_{n-1}\sqrt{n} \text{ pour } n \geq 1, \end{aligned}$$

avec un résultat dans $\ell^2(\mathbb{N})$ si $\sum_n n|a_n|^2 < +\infty$. Notons que l'opérateur borné de $\ell^2(\mathbb{N})$ dans lui-même donné par

$$S\big((a_n)_{n\in\mathbb{N}}\big) = (b_n)_{n\in\mathbb{N}}, \quad b_0 = 0, \ b_n = a_{n-1} \text{ pour } n \ge 1, \tag{7.2.31}$$

est isométrique, injectif et non surjectif avec une image de codimension 1. Si $(e_n)_{n\in\mathbb{N}}$ est la base hilbertienne standard de $\ell^2(\mathbb{N})$, on a

$$Se_n = e_{n+1}, \quad \operatorname{range} S = S(\ell^2(\mathbb{N})) = e_0^{\perp}.$$

L'opérateur d'annihilation A_- vérifie $A_-\phi_n = \sqrt{n}\phi_{n-1}$ pour $n \ge 1$ et $A_-\phi_0 = 0$, et avec Ψ donné par (7.2.28), il vient avec $\tilde{A}_- = \Psi A_- \Psi^{-1}$

$$\begin{aligned}\tilde{A}_-\big((a_n)_{n\in\mathbb{N}}\big) = \Psi A_-\big(\sum_{n\ge1} a_n\phi_n\big) = \Psi\big(\sum_{n\ge1} a_n\sqrt{n}\phi_{n-1}\big) &= \big(a_1, a_2\sqrt{2}, a_3\sqrt{3}, \dots\big)\\ &= (b_n)_{n\in\mathbb{N}}, \qquad b_n = a_{n+1}\sqrt{n+1} \text{ pour } n \ge 0,\end{aligned}$$

avec un résultat dans $\ell^2(\mathbb{N})$ si $\sum_n n|a_n|^2 < +\infty$. Notons que l'opérateur borné de $\ell^2(\mathbb{N})$ dans lui-même donné par

$$S'\big((a_n)_{n\in\mathbb{N}}\big) = (b_n)_{n\in\mathbb{N}}, \quad b_n = a_{n+1} \text{ pour } n \ge 0, \tag{7.2.32}$$

est surjectif, non injectif avec un noyau de dimension 1 égal à $\mathbb{C}e_0$. Si $(e_n)_{n\in\mathbb{N}}$ est la base hilbertienne standard de $\ell^2(\mathbb{N})$, on a

$$S'e_0 = 0, \ S'e_n = e_{n-1}, \text{ pour } n \ge 1, \quad \ker S = \mathbb{C}e_0.$$

On peut remarquer que $S^* = S'$ car

$$\langle S^*e_m, e_n\rangle_{\ell^2(\mathbb{N})} = \langle e_m, Se_n\rangle_{\ell^2(\mathbb{N})} = \langle e_m, e_{n+1}\rangle_{\ell^2(\mathbb{N})} = \delta_{m,n+1},$$

soit $S^*e_0 = 0, \quad S^*e_m = e_{m-1}$ pour $m \ge 1$. On peut résumer une partie des résultats précédents par le résultat suivant.

Theorem 7.2.10. *L'oscillateur harmonique $\mathcal{H}$ défini par (7.2.20) vérifie*

$$\mathcal{H} = \sum_{n\ge0}(\frac{1}{2} + n)\mathbb{P}_n, \quad \mathrm{Id} = \sum_{n\ge0}\mathbb{P}_n, \tag{7.2.33}$$

où $\mathbb{P}_n$ est la projection orthogonale sur $\mathbb{C}\phi_n$, se prolonge en un opérateur continu de l'espace E (défini en (7.2.29)) dans $L^2(\mathbb{R})$.

La dimension supérieure

Soit $\mathtt{d} \ge 1$. On définit pour $\alpha = (\alpha_j)_{1\le j\le d} \in \mathbb{N}^{\mathtt{d}}, x \in \mathbb{R}^{\mathtt{d}}$,

$$\Phi_\alpha(x) = \prod_{j=1}^{\mathtt{d}} \phi_{\alpha_j}(x_j), \quad \mathcal{E}_n = \operatorname{Vect}\{\Phi_\alpha\}_{\alpha\in\mathbb{N}^d, |\alpha|=n}, \tag{7.2.34}$$

avec $|\alpha| = \alpha_1 + \dots + \alpha_d$. On dira que les fonctions Φ_α sont les fonctions d'Hermite en dimension $\mathtt{d}$.

Lemma 7.2.11. *La dimension de* $\mathcal{E}_n$ *est* $C^{\mathtt{d}-1}_{n+\mathtt{d}-1}$.

Proof. Démontrons que $\operatorname{card}\{\alpha \in \mathbb{N}^{\mathtt{d}}, |\alpha| = l\} = C^{d-1}_{l+d-1}$. Commençons par prouver par récurrence sur l que

$$C^{d-1}_{l+d-1} = \sum_{0 \le j \le l} C^{d-2}_{j+d-2}, \tag{7.2.35}$$

ce qui est vérifié pour $l = 0$, et comme $C^{d-1}_{l+d} = C^{d-1}_{l+d-1} + C^{d-2}_{l+d-1}$, on obtient

$$C^{d-1}_{l+1+d-1} = C^{d-1}_{l+d-1} + C^{d-2}_{l+d-1} = \sum_{0 \le j \le l+1} C^{d-2}_{j+d-2},$$

ce qui démontre (7.2.35). On a par ailleurs

$$\operatorname{card}\{\alpha \in \mathbb{N}^{\mathtt{d}}, |\alpha| = l\} = \sum_{0 \le j \le l} \operatorname{card}\{\beta \in \mathbb{N}^{\mathtt{d}-1}, |\beta| = j\}, \tag{7.2.36}$$

ce qui permet de démontrer par récurrence sur $\mathtt{d}$ que

$$\operatorname{card}\{\alpha \in \mathbb{N}^{\mathtt{d}}, |\alpha| = l\} = C^{\mathtt{d}-1}_{l+\mathtt{d}-1},$$

car cette propriété est vraie pour $d = 1$ et si elle est vérifiée pour un entier $d \ge 1$, il vient de (7.2.36), (7.2.35),

$$\operatorname{card}\{\alpha \in \mathbb{N}^{\mathtt{d}+1}, |\alpha| = l\} = \sum_{0 \le j \le l} \operatorname{card}\{\beta \in \mathbb{N}^{\mathtt{d}}, |\beta| = j\} = \sum_{0 \le j \le l} C^{\mathtt{d}-1}_{l+\mathtt{d}-1} = C^{\mathtt{d}}_{l+\mathtt{d}}.$$

Montrons par récurrence sur $\mathtt{d}$ que les $\sum_{0 \le k \le n} C^{\mathtt{d}-1}_{k+\mathtt{d}-1}$ fonctions $\{\Phi_\alpha\}_{|\alpha|=k, 0 \le k \le n}$ sont indépendantes. C'est vrai pour $d = 1$. Supposons que cette propriété est vérifiée pour un entier $d \ge 1$. Supposons que

$$\sum_{\alpha \in \mathbb{N}^{d+1}, |\alpha| \le m} c_\alpha \Phi_\alpha = 0.$$

On obtient alors l'identité

$$\sum_{0 \le k \le m} \Big\{ \sum_{\substack{\beta \in \mathbb{N}^d \\ |\beta| = m-k}} c_{(\beta,k)} \Big(\prod_{j=1}^{\mathtt{d}} \phi_{\beta_j}(x_j) \Big) \Big\} \phi_k(x_{d+1}) = 0,$$

et de l'indépendance des fonctions $\{\phi_k(x_{d+1})\}_{0 \le k \le m}$, il vient l'identité sur $\mathbb{R}^d$

$$\sum_{\substack{\beta \in \mathbb{N}^d \\ |\beta| = m-k}} c_{(\beta,k)} \prod_{j=1}^{\mathtt{d}} \phi_{\beta_j}(x_j) = 0.$$

L'hypothèse de récurrence démontre que tous les $c_{\beta,k}$ sont nuls. □

Remark 7.2.12. En posant pour $1 \le j \le \mathtt{d}$,

$$A_{+,j} = \frac{1}{\sqrt{2}}\big(-\frac{\partial}{\partial x_j} + x_j\big), \quad A_{-,j} = \frac{1}{\sqrt{2}}\big(\frac{\partial}{\partial x_j} + x_j\big), \tag{7.2.37}$$

il vient de (7.2.24), avec $\alpha! = \prod_{1\le j\le d} \alpha_j!$,

$$\Phi_\alpha(x) = \prod_{j=1}^{\mathtt{d}} \phi_{\alpha_j}(x_j) = \frac{1}{\sqrt{\alpha!}} \prod_{j=1}^{\mathtt{d}} \big(A_{+,j}^{\alpha_j}\phi_0\big)(x_j).$$

Theorem 7.2.13. *Les $(\Phi_\alpha)_{\alpha\in\mathbb{N}^{\mathtt{d}}}$ forment une base hilbertienne de $L^2(\mathbb{R}^{\mathtt{d}})$ composée par les vecteurs propres de l'oscillateur harmonique en dimension $\mathtt{d}$:*

$$\mathcal{H} = \frac{1}{2}\big(-\Delta_x + |x|^2\big) = \sum_{n\ge 0}\big(\frac{\mathtt{d}}{2} + n\big)\mathbb{P}_n, \quad \mathrm{Id} = \sum_{n\ge 0}\mathbb{P}_n, \tag{7.2.38}$$

où $\mathbb{P}_n$ est la projection orthogonale sur $\mathcal{E}_n$, espace de dimension $C_{n+\mathtt{d}-1}^{\mathtt{d}-1}$. La valeur propre $\mathtt{d}/2$ est simple en toute dimension et $\mathcal{E}_0$ est engendré par

$$\Phi_0(x) = \pi^{-\mathtt{d}/4} e^{-|x|^2/2}.$$

Proof. Remarquons tout d'abord que pour $\alpha, \beta \in \mathbb{N}^d$, Φ_α, Φ_β appartiennent à la classe de Schwartz $\mathscr{S}(\mathbb{R}^d)$ et que

$$\langle \Phi_\alpha, \Phi_\beta\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \prod_{j=1}^{\mathtt{d}} \phi_{\alpha_j}(x_j) \prod_{j=1}^{\mathtt{d}} \phi_{\beta_j}(x_j) dx$$
$$= \prod_{j=1}^{\mathtt{d}} \langle \phi_{\alpha_j}, \phi_{\beta_j}\rangle_{L^2(\mathbb{R})} = \prod_{j=1}^{\mathtt{d}} \delta_{\alpha_j,\beta_j} = \delta_{\alpha,\beta}.$$

Démontrons que l'orthogonal de l'espace engendré par $\{\Phi_\alpha\}_{\alpha\in\mathbb{N}^d}$ est réduit à $\{0\}$. Soit f une fonction de $L^2(\mathbb{R}^d)$ telle que,

$$\text{pour tout } \alpha \in \mathbb{N}^d, \qquad \int_{\mathbb{R}} f(x)\Phi_\alpha(x)dx = 0.$$

Comme chaque H_n est un polynôme de degré n, l'espace vectoriel engendré par $\{H_n\}_{0\le n\le N}$ est l'espace des polynômes de degré $\le N$ (récurrence sur N). Par suite on a

$$\text{pour tout } \alpha \in \mathbb{N}^d, \qquad \int_{\mathbb{R}} f(x) x^\alpha e^{-|x|^2/2} dx = 0.$$

Considérons la fonction F, donnée pour $z \in \mathbb{C}^d$ par

$$F(z) = \int_{\mathbb{R}} f(x) e^{-|x|^2/2} e^{\sum_{1\le j\le d} z_j x_j} dx.$$

On a pour K compact de $\mathbb{C}^d$, $M_K = \sup_{z\in K} |\operatorname{Re} z|$, l'estimation

$$\sup_{z\in K} |f(x)e^{-|x|^2/2}e^{z\cdot x}| \le \underbrace{|f(x)|}_{L^2} \underbrace{e^{-|x|^2/2}e^{|x|M_K}}_{L^2} \in L^1(\mathbb{R}^d),$$

et comme $z \mapsto f(x)e^{-|x|^2/2}e^{zx}$ est entière, la fonction F est entière. En outre, pour $\alpha \in \mathbb{N}^d$, il vient

$$F^{(\alpha)}(0) = \int_{\mathbb{R}} f(x)e^{-|x|^2/2}x^\alpha dx = 0,$$

ce qui implique que F est identiquement nulle. La fonction

$$\mathbb{R}^d \ni x \mapsto h(x) = f(x)e^{-|x|^2/2}$$

appartient à $L^1(\mathbb{R}^d)$ comme produit de deux fonctions de $L^2(\mathbb{R}^d)$ (inégalité de Cauchy-Schwarz). On a de plus

$$\hat{h}(\xi) = F(-2i\pi\xi) = 0,$$

de sorte que la transformée de Fourier de h est nulle, et donc $h = 0$. On a donc, pour presque tout $x \in \mathbb{R}^d$,

$$f(x)e^{-|x|^2/2} = 0,$$

ce qui implique $f(x) = 0$ presque partout et $f = 0$ comme fonction de $L^2(\mathbb{R}^d)$. Nous avons donc démontré que

$$\mathrm{Id} = \sum_{n\geq 0} \mathbb{P}_n.$$

En outre de (7.2.37), il vient

$$\mathcal{H} = \sum_{1\leq j\leq d} \mathcal{H}_j, \quad \mathcal{H}_j = \frac{1}{2}\left(-\frac{\partial^2}{\partial x_j^2} + x_j^2\right),$$

et donc pour $\alpha \in \mathbb{N}^d$,

$$\mathcal{H}\Phi_\alpha = \sum_{1\leq j\leq d} \left(\frac{1}{2} + \alpha_j\right)\Phi_\alpha = \left(\frac{d}{2} + |\alpha|\right)\Phi_\alpha,$$

ce qui démontre $\mathcal{H} = \sum_{n\geq 0}\left(\frac{d}{2} + n\right)\mathbb{P}_n$. □

Remark 7.2.14. Bien entendu, l'opérateur $\mathcal{H}$ n'est pas borné sur $L^2(\mathbb{R}^d)$, mais peut se définir sur l'espace

$$E = \{\sum_{n\geq 0} \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|=n}} a_\alpha \Phi_\alpha\}_{(a_\alpha)_{\alpha\in\mathbb{N}^d}\in\tilde{E}}, \tag{7.2.39}$$

$$\text{avec } \tilde{E} = \{(a_\alpha)_{\alpha\in\mathbb{N}^d} \in \ell^2(\mathbb{N}^d), \sum_{\alpha\in\mathbb{N}^d} |\alpha|^2|a_\alpha|^2 < +\infty\}. \tag{7.2.40}$$

La restriction de $\mathcal{H}$ à E est complètement déterminée par les restrictions de $\mathcal{H}$ à $\operatorname{range}\mathbb{P}_n = \mathcal{E}_n$ et l'on vient de voir que

$$\mathcal{H}_{|\mathcal{E}_n} = \left(\frac{d}{2} + n\right)\mathrm{Id}\,. \tag{7.2.41}$$

Harmonic oscillator, another normalization

The Harmonic oscillator $\mathcal{H}_n$ in n dimensions is defined as the operator with Weyl symbol $\pi(|x|^2+|\xi|^2)$ and thus we find that

$$\mathcal{H} = U_{\sqrt{2\pi}}\,\frac{1}{2}\big(|x|^2+4\pi^2|\xi|^2\big)^w U^*_{\sqrt{2\pi}} = U_{\sqrt{2\pi}}\,\frac{1}{2}\big(-\Delta+|x|^2\big)U^*_{\sqrt{2\pi}}.$$

We shall define in one dimension the Hermite function of level $k\in\mathbb{N}$, by

$$\psi_k(x) = \frac{(-1)^k}{2^k\sqrt{k!}}2^{1/4}e^{\pi x^2}\left(\frac{d}{\sqrt{\pi}dx}\right)^k(e^{-2\pi x^2}), \tag{7.2.42}$$

and we find that $(\psi_k)_{k\in\mathbb{N}}$ is a Hilbertian orthonormal basis on $L^2(\mathbb{R})$. The one-dimensional harmonic oscillator can be written as

$$\mathcal{H} = \sum_{k\ge 0}(\frac{1}{2}+k)\mathbb{P}_k, \tag{7.2.43}$$

where $\mathbb{P}_k$ is the orthogonal projection onto ψ_k.

In n dimensions, we consider a multi-index $(\alpha_1,\dots,\alpha_n)=\alpha\in\mathbb{N}^n$ and we define on $\mathbb{R}^n$, using the one-dimensional (7.2.42),

$$\Psi_\alpha(x) = \prod_{1\le j\le n}\psi_{\alpha_j}(x_j), \quad \mathcal{E}_k = \operatorname{Vect}\big\{\Psi_\alpha\big\}_{\alpha\in\mathbb{N}^n,|\alpha|=k}, \quad |\alpha| = \sum_{1\le j\le n}\alpha_j. \tag{7.2.44}$$

We note that the the dimension of $\mathcal{E}_k$ is $\binom{k+n-1}{n-1}$ and that (7.2.43) holds with $\mathbb{P}_k$ standing for the orthogonal projection onto $\mathcal{E}_k$; the lowest eigenvalue of $\mathcal{H}$ is $n/2$ and the corresponding eigenspace is one-dimensional in all dimensions, although in two and more dimensions, the eigenspaces corresponding to the eigenvalue $\frac{n}{2}+k, k\ge 1$ are multi-dimensional with dimension $\binom{k+n-1}{n-1}$. The n-dimensional harmonic oscillator can be written as

$$\mathcal{H}_n = \sum_{k\ge 0}(\frac{n}{2}+k)\mathbb{P}_{k,n}, \tag{7.2.45}$$

where $\mathbb{P}_{k,n}$ stands for the orthogonal projection onto $\mathcal{E}_k$ defined above.

Mehler's formula

Lemma 7.2.15. *For* $\operatorname{Re} t\ge 0$*, we have in n dimensions,*

$$\big(\cosh(t/2)\big)^n \exp -t\pi(|x|^2+|\xi|^2)^w = \Big(e^{-2\tanh(\frac{t}{2})\pi(x^2+\xi^2)}\Big)^w. \tag{7.2.46}$$

Proof. By tensorisation, it is enough to prove that formula for $n=1$, which we assume from now on. To prove that formula, we need only to consider the one-dimensional case. We define

$$L=\xi+ix, \quad \bar{L}=\xi-ix, \quad M(t)=\beta(t)\big(e^{-\alpha(t)\pi L\bar{L}}\big)^w,$$

where α, β are smooth functions of t to be determined. Assuming $\beta(0) = 1, \alpha(0) = 0$, we find that $M(0) = \mathrm{Id}$ and

$$\dot{M} + \pi(|L|^2)^w M = \Big(\dot\beta e^{-\alpha\pi|L|^2} - \beta\dot\alpha\pi|L|^2 e^{-\alpha\pi|L|^2} + \pi(|L|^2)\sharp\beta e^{-\alpha\pi|L|^2}\Big)^w.$$

We have

$$\begin{aligned}
|L|^2\sharp e^{-\alpha\pi|L|^2} &= |L|^2 e^{-\alpha\pi|L|^2} + \frac{1}{4i\pi}\overbrace{\Big\{|L|^2, e^{-\alpha\pi|L|^2}\Big\}}^{=0} \\
&\quad + \frac{1}{(4i\pi)^2}\frac{1}{2}\Big(\partial_\xi^2(|L|^2)\partial_x^2 e^{-\alpha\pi|L|^2} + \partial_x^2(|L|^2)\partial_\xi^2 e^{-\alpha\pi|L|^2}\Big) \\
&= |L|^2 e^{-\alpha\pi|L|^2} \\
&\quad + \frac{1}{(4i\pi)^2}\frac{1}{2} e^{-\alpha\pi|L|^2}\Big(2\big((-2\alpha\pi x)^2 - 2\alpha\pi\big) + 2\big((-2\alpha\pi\xi)^2 - 2\alpha\pi\big)\Big) \\
&= |L|^2 e^{-\alpha\pi|L|^2}\Big(1 - \frac{4\alpha^2\pi^2}{16\pi^2}\Big) + \frac{\alpha\pi}{4\pi^2} e^{-\alpha\pi|L|^2},
\end{aligned}$$

so that

$$\begin{aligned}
&\dot M + \pi(|L|^2)^w M \\
&= \Big(\dot\beta e^{-\alpha\pi|L|^2} - \beta\dot\alpha\pi|L|^2 e^{-\alpha\pi|L|^2} + \pi\beta|L|^2 e^{-\alpha\pi|L|^2}\Big(1 - \frac{4\alpha^2\pi^2}{16\pi^2}\Big) + \frac{\alpha\pi\beta}{4\pi} e^{-\alpha\pi|L|^2}\Big)^w \\
&\qquad = \Big(e^{-\alpha\pi|L|^2}\Big\{|L|^2\big(-\pi\dot\alpha\beta + \pi\beta(1 - \frac{\alpha^2}{4})\big) + \dot\beta + \frac{\alpha\beta}{4}\Big\}\Big)^w.
\end{aligned}$$

We solve now

$$\dot\alpha = 1 - \frac{\alpha^2}{4},\ \alpha(0) = 0 \Longleftrightarrow \alpha(t) = 2\tanh(t/2),$$

and

$$4\dot\beta + \alpha\beta = 0, \beta(0) = 1 \Longleftrightarrow \beta(t) = \frac{1}{\cosh(t/2)}.$$

We obtain that $\dot M + \pi(|L|^2)^w M = 0,\ M(0) = \mathrm{Id}$, and this implies

$$\beta(t)\big(e^{-\alpha(t)\pi L\bar L}\big)^w = M(t) = \exp -t\pi(|L|^2)^w,$$

which proves (7.2.46). □

In particular, for $t = -2is, s \in \mathbb{R}$, we have in n dimensions

$$(\cos s)^n \exp\big(2i\pi s(|x|^2 + |\xi|^2)^w\big) = \Big(e^{2i\pi\tan s(|x|^2+|\xi|^2)}\Big)^w. \tag{7.2.47}$$

Lemma 7.2.16. *For any $z \in \mathbb{C}$, $\operatorname{Re} z \ge 0$, we have*

$$\Big[\exp -\Big(2z\pi\big(|\xi|^2 + |x|^2\big)\Big)\Big]^w = \frac{1}{(1+z)^n}\sum_{k\ge 0}\Big(\frac{1-z}{1+z}\Big)^k \mathbb{P}_k, \tag{7.2.48}$$

where $\mathbb{P}_k$ is defined in (7.2.43) and the equality holds between $L^2(\mathbb{R}^n)$-bounded operators.

Proof. Starting from (7.2.47), we get for $\tau \in \mathbb{R}$, in n dimensions,

$$(\cos(\arctan\tau))^n \exp\bigl(2i\pi \arctan\tau(|x|^2+|\xi|^2)^w\bigr) = \Bigl(e^{2i\pi\tau(|x|^2+|\xi|^2)}\Bigr)^w,$$

so that using the spectral decomposition of the (n-dimensional) Harmonic Oscillator, we get

$$(1+\tau^2)^{-n/2}\sum_{k\ge 0} e^{2i(\arctan\tau)(k+\frac{n}{2})}\mathbb{P}_k = \Bigl(e^{2i\pi\tau(|x|^2+|\xi|^2)}\Bigr)^w,$$

which implies

$$(1+\tau^2)^{-n/2}\sum_{k\ge 0} \frac{(1+i\tau)^{2k+n}}{(1+\tau^2)^{k+\frac{n}{2}}}\mathbb{P}_k = \Bigl(e^{2i\pi\tau(|x|^2+|\xi|^2)}\Bigr)^w,$$

entailing

$$\sum_{k\ge 0} \frac{(1+i\tau)^{k}}{(1-i\tau)^{k+n}}\mathbb{P}_k = \Bigl(e^{2i\pi\tau(|x|^2+|\xi|^2)}\Bigr)^w,$$

proving the lemma by analytic continuation. □

Sobolev spaces based upon the Harmonic Oscillator

For $\theta \in (0,1)$ and $a \in \mathbb{C}, \operatorname{Re} a > 0$, we have

$$a^\theta = \int_0^{+\infty} a e^{-ta} t^{-\theta} dt \frac{1}{\Gamma(1-\theta)},$$

since

$$\int_0^{+\infty} a e^{-ta} t^{-\theta} dt = \int_0^{+\infty} e^{-s} s^{1-\theta-1} a^\theta ds = a^\theta \Gamma(1-\theta),$$

so that

$$\mathcal{H}^\theta = \int_0^{+\infty} e^{-t\mathcal{H}} t^{-\theta} dt \frac{\mathcal{H}}{\Gamma(1-\theta)},$$

and thus

$$\mathcal{H}^\theta = \Bigl(\int_0^{+\infty} \bigl(\cosh(t/2)\bigr)^{-n} e^{-2\tanh(\frac{t}{2})\pi(x^2+\xi^2)} t^{-\theta} dt\Bigr)^w \frac{\mathcal{H}}{\Gamma(1-\theta)},$$

entailing that $\mathcal{H}^\theta = \mu_\theta^w$ with

$$\mu_\theta(x,\xi) = \int_0^{+\infty} \cosh(t/2)^{-n}\bigl(e^{-2\tanh(\frac{t}{2})\lambda}\sharp\lambda\bigr) t^{-\theta} dt \frac{1}{\Gamma(1-\theta)}, \qquad \lambda = \pi(x^2+\xi^2).$$

The Weyl composition formula is

$$(a\sharp b)(x,\xi) = \sum_{0\le k<\nu} 2^{-k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha!\beta!} D_\xi^\alpha \partial_x^\beta a \; D_\xi^\beta \partial_x^\alpha b + r_\nu(a,b), \tag{7.2.49}$$

$$\text{with} \quad r_\nu(a,b)(X) = R_\nu\bigl(a(X)\otimes b(Y)\bigr)_{|X=Y}, \tag{7.2.50}$$

$$R_\nu = \int_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!} \exp \frac{\theta}{4i\pi}[\partial_X, \partial_Y] d\theta \Bigl(\frac{1}{4i\pi}[\partial_X,\partial_Y]\Bigr)^\nu, \tag{7.2.51}$$

and defining

$$\omega_k(a,b) = 2^{-k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha!\beta!} D_\xi^\alpha \partial_x^\beta a \; D_\xi^\beta \partial_x^\alpha b, \tag{7.2.52}$$

we get that

$$\omega_0 = ab, \quad \omega_1 = \frac{1}{4i\pi}\left\{a,b\right\},$$

so that the beginning of this expansion is thus

$$ab + \frac{1}{4i\pi}\{a,b\},$$

where the Poisson bracket $\{a,b\}$ is given by (3.4.12). The $\omega_k(a,b)$ with k even are symmetric in a,b and skew-symmetric for k odd: this is obvious from the above expression coming from $[\partial_X, \partial_Y]^k$. Also, when a,b are real-valued the $\omega_k(a,b)$ with k even are real and purely imaginary for k odd.

We see that

$$\begin{aligned}
&\frac{1}{4(2i\pi)^2} \sum_{|\alpha|+|\beta|=2} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \partial_\xi^\alpha \partial_x^\beta e^{-2\lambda \tanh t/2} \; \partial_\xi^\beta \partial_x^\alpha \pi(x^2+\xi^2) \\
&= \frac{\pi}{4(2i\pi)^2} \sum_{1\le j\le n} \Big(\frac{1}{2}\partial_{\xi_j}^2 \big(e^{-2\lambda \tanh t/2}\big) 2 + \frac{1}{2}\partial_{x_j}^2 \big(e^{-2\lambda \tanh t/2}\big) 2 \Big) \\
&= \frac{-1}{16\pi} \operatorname{div}_{x,\xi} \nabla_{x,\xi} e^{-2\lambda \tanh t/2} = \frac{-1}{16\pi} \operatorname{div}_{x,\xi} e^{-2\lambda \tanh t/2} (-2\tanh t/2) \nabla\lambda \\
&= \frac{-1}{16\pi} e^{-2\lambda \tanh t/2} \Big\{ (-2\tanh t/2)^2 |\nabla\lambda|^2 + (-2\tanh t/2)\Delta\lambda \Big\} \\
&= \frac{-1}{16\pi} e^{-2\lambda \tanh t/2} \Big\{ 4(\tanh t/2)^2 4\pi\lambda + (-2\tanh t/2) 4\pi n \Big\},
\end{aligned}$$

that is

$$e^{-2\tanh(\frac{t}{2})\lambda} \sharp \lambda = e^{-2\tanh(\frac{t}{2})\lambda} \Big(\lambda\big(1-(\tanh t/2)^2\big) + \frac{n \tanh t/2}{2} \Big)$$

and thus

$$\mu_\theta(x,\xi) = \int_0^{+\infty} \cosh(t/2)^{-n} e^{-2\tanh(\frac{t}{2})\lambda} \Big(\lambda\big(1-(\tanh t/2)^2\big) + \frac{n \tanh t/2}{2} \Big) t^{-\theta} dt \frac{1}{\Gamma(1-\theta)},$$

that is

$$\mu_\theta(x,\xi) = \int_0^{+\infty} \cosh(t/2)^{-n-2} e^{-2\tanh(\frac{t}{2})\lambda} \Big(\lambda + \frac{n \sinh t}{4} \Big) t^{-\theta} dt \frac{1}{\Gamma(1-\theta)}.$$

We have

$$\Gamma(1-\theta)|\mu_\theta| \le \int_0^1 e^{-\lambda t/2} C_0 \lambda t^{-\theta} dt + O(e^{-\varepsilon_0 \lambda}), \quad \varepsilon_0 > 0,$$

and a change of variables shows that the first term in the rhs of the above inequality is

$$\int_0^{\lambda} e^{-s} s^{-\theta} \lambda^{\theta} ds = O(\lambda^{\theta}).$$

An easy calculation of derivatives shows that we have in fact

$$\mu_\theta \in S(\lambda^\theta, \frac{dX^2}{\lambda}), \quad |\mu_\theta| \approx \lambda^\theta \quad \text{for large } \lambda \quad \text{(elliptic symbol)}.$$

We have similarly

$$\mathcal{H}^{-1} = \int_0^{+\infty} e^{-t\mathcal{H}} dt,$$

and Mehler's formula provides a symbol in $S(\lambda^{-1}, \frac{dX^2}{\lambda})$ for that operator and for any integer $m \in \mathbb{Z}$, we get that $\mathcal{H}^m$ is a pseudo-differential operator with an (elliptic) symbol in $S(\lambda^m, \frac{dX^2}{\lambda})$. As a result, for any $s \in \mathbb{R}$, $\mathcal{H}^s$ is a pseudo-differential operator with an (elliptic) symbol in $S(\lambda^s, \frac{dX^2}{\lambda})$.

We define for $s \in \mathbb{R}$ the Sobolev spaces based upon the harmonic oscillator

$$\mathscr{H}^s = \mathcal{H}^{-s}\big(L^2(\mathbb{R}^n)\big), \quad \|u\|_{\mathscr{H}^s} = \|\mathcal{H}^s u\|_{L^2}. \tag{7.2.53}$$

The Hilbertian structure and duality properties are obvious, we have explicit pseudo-differential isomorphisms with L^2 for all $\mathscr{H}^s$, and we get now for free the fact that a pseudo-differential operator with symbol in $S(\lambda^t, \frac{dX^2}{\lambda})$ sends continuously $\mathscr{H}^s$ into $\mathscr{H}^{s-t}$. We have indeed for $a \in S(\lambda^t, \frac{dX^2}{\lambda})$, $u \in \mathscr{S}(\mathbb{R}^n)$,

$$\|a^w u\|_{\mathscr{H}^{s-t}} = \|\underbrace{\mathcal{H}^{s-t} a^w \mathcal{H}^{-s}}_{\text{order } 0} \mathcal{H}^s u\|_{L^2} \lesssim \|\mathcal{H}^s u\|_{L^2} = \|u\|_{\mathscr{H}^s}.$$

This means that the algebraic computations with these Sobolev spaces can be made completely similar to what happens for the standard Sobolev spaces in $\mathbb{R}^n$, replacing the Fourier multiplier $\langle D\rangle^s$ by $\mathcal{H}^s$.

7.3 Elements of operator theory

Let H be a Hilbert space, let D be a dense subspace of H and let $A : D \to H$ be a linear operator. The pair (A, D) will be called the operator A with domain D and D will be denoted by D_A. We define

$$D^* = \{v \in H, \exists u \in H, \forall w \in D, \quad \langle v, Aw\rangle_H = \langle u, w\rangle_H\}. \tag{7.3.1}$$

Note that u is uniquely determined by v since if $u, \tilde u$ satisfy for all $w \in D$,

$$\langle v, Aw\rangle_H = \langle u, w\rangle_H, \quad \langle v, Aw\rangle_H = \langle \tilde u, w\rangle_H,$$

we obtain $\langle \tilde u - u, w\rangle_H = 0$ so that $\tilde u - u \in D^\perp = \{0\}$. We define then the adjoint operator $A^* : D^* \to H_1$ by $A^* v = u$ where u is the unique vector in H_1 such that

$$\forall w \in D, \langle v, Aw\rangle_H = \langle u, w\rangle_H.$$

As a result, for $v \in D^*$, the vector A^*v is uniquely determined by the identity

$$\forall w \in D, \quad \langle A^*v, w\rangle_H = \langle v, Aw\rangle_H. \tag{7.3.2}$$

Note also that D^* is a vector space since $v, \tilde{v} \in D^*$ imply that

$$\forall w \in D, \quad \langle A^*v, w\rangle_H = \langle v, Aw\rangle_H, \quad \langle A^*\tilde{v}, w\rangle_H = \langle \tilde{v}, Aw\rangle_H,$$

and thus for $a, \tilde{a} \in \mathbb{C}$, we get

$$\forall w \in D, \langle av + \tilde{a}\tilde{v}, Aw\rangle_{H_2} = \langle aA^*v + \tilde{a}A^*\tilde{v}, w\rangle_{H_1},$$

entailing from (7.3.1) that $av + \tilde{a}\tilde{v} \in D^*$ and from (7.3.2)

$$aA^*v + \tilde{a}A^*\tilde{v} = A^*(av + \tilde{a}\tilde{v}).$$

The pair (A^*, D^*) will be called the adjoint of A.

Definition 7.3.1. Let H, A, D be as above. The operator A with domain D is said to be symmetric whenever

$$\forall u, v \in D, \quad \langle Au, v\rangle_H = \langle u, Av\rangle_H. \tag{7.3.3}$$

The operator A with domain D is said to be self-adjoint whenever $A = A^*$ on $D = D^*$.

Note that a self-adjoint operator is obviously symmetric, whereas the converse is not always true. In particular if an operator A with dense domain D is symmetric, we have $D \subset D^*$: in fact if $v \in D$, we do have for all $w \in D$

$$\langle v, Aw\rangle = \langle Av, w\rangle,$$

so that from (7.3.1), we get $v \in D^*$ with $A^*v = Av$.

Definition 7.3.2. Let H, A, D be as above. The operator A is said to be closed whenever the graph

$$G_A = \{(u, Au)\}_{u \in D},$$

is closed in $H_1 \oplus H_2$.

Remark 7.3.3. Let H, D, A be as above. Then the operator (A^*, D^*) is closed. In fact the graph of A^* is

$$\{v \oplus A^*v\}_{v \in D^*} \subset H \oplus H,$$

and if $(v_k)_{k\ge1}, (A^*v_k)_{k\ge1}$ are converging sequences in H, with

$$v = \lim_k v_k, \quad y = \lim_k A^*v_k,$$

we have for all $w \in D$, $\langle A^*v_k, w\rangle = \langle v_k, Aw\rangle$, and thus

$$\langle y, w\rangle = \lim_k \langle A^*v_k, w\rangle = \langle v, Aw\rangle,$$

so that by definition $y = A^*v$, proving the closedness of the graph.

Remark 7.3.4. Let H be a Hilbert space. The Closed Graph Theorem says that an operator A with domain H is bounded iff its graph is closed. Let us consider an operator (A, D) which is bounded, i.e. such that

$$\sup_{u \in D, \|u\|_H = 1} \|Au\|_H < +\infty. \tag{7.3.4}$$

Then the operator A is closed iff D is a closed subspace of H. The condition is sufficient since if D is closed, A appears as a bounded operator from the Hilbert space D into the Hilbert space H and thus is closed. Conversely, if (7.3.4) is satisfied and A is closed, the graph $\{u \oplus Au\}_{u \in D}$ is closed, entailing that if $(u_k)_{k \geq 1}$ is a sequence of D converging in H, the sequence $(Au_k)_{k \geq 1}$ is a Cauchy sequence in H since from (7.3.4)

$$\|Au_k - Au_l\|_H \leq C\|u_k - u_l\|_H,$$

and thus $\lim_k u_k = u$, $\lim_k Au_k = v$, so that the closeness of the graph of A implies $u \in D$ with $v = Au$, thus the closedness of D.

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