Olivier Biquard

POLYCOPIÉ ON DIFFERENTIAL GEOMETRY AND GLOBAL ANALYSIS

Olivier Biquard
UPMC Université Paris 06,
UMR 7586, Institut de Mathématiques de Jussieu.

February 6, 2008

POLYCOPIÉ
ON
DIFFERENTIAL GEOMETRY
AND
GLOBAL ANALYSIS

Olivier Biquard

CONTENTS

1. Hodge theory 1
1.1. The Hodge operator 1
1.2. Adjoint operator 2
1.3. Hodge-de Rham Laplacian 5
1.4. Statement of Hodge theory 6
1.5. Bochner technique 9
1.6. Differential operators 12
1.7. Basic elliptic theory 14
1.8. Proof of the main theorem 17
1.9. Elliptic complexes 18

2. Moduli problems 19
2.1. Finite dimensional quotients 20
2.2. Hölder spaces 21
2.3. Spaces of connections and gauge group 23
2.4. The action of the gauge group 24
2.5. The moduli space of flat unitary connections 26
2.6. The moduli space of instantons 31
2.7. The moduli space of complex flat connections 33
2.8. The diffeomorphism group 34
2.9. Action of diffeomorphisms on metrics 35
2.10. The Einstein equation 38

3. A gluing construction: Dehn surgery for Einstein metrics 45
3.1. Cusp ends 45
3.2. Toral black hole metric 46

3.3. Twisted toral black hole metric 48
3.4. Dependence on the parameter 49
3.5. Dehn surgery and statement of the theorem 50
3.6. Idea of proof of the theorem 52
3.7. The Hölder spaces 53
3.8. Weighted Hölder norms 55
3.9. Estimate for the inverse of the linearisation 57
3.10. Explicit calculations for the toral black hole metric 62

Bibliography 71

CHAPTER 1

HODGE THEORY

THIS chapter is an introduction to Hodge theory, and more generally to the analysis on elliptic operators on compact manifolds. Hodge theory represents De Rham cohomology classes (that is topological objects) on a compact manifold by harmonic forms (solutions of partial differential equations depending on a Riemannian metric on the manifold). It is a powerful tool to understand the topology from the geometric point of view.

In this chapter we mostly follow reference [**4**], which contains a complete concise proof of Hodge theory, as well as applications in Kähler geometry.

1.1. The Hodge operator

Let V be a n-dimensional oriented euclidean vector space (it will be later the tangent space of an oriented Riemannian n-manifold). Therefore there is a canonical volume element $\mathrm{vol} \in \Omega^n V$. The exterior product $\Omega^p V \wedge \Omega^{n-p} V \to \Omega^n V$ is a non degenerate pairing. Therefore, for a form $\beta \in \Omega^p V$, one can define $*\beta \in \Omega^{n-p} V$ by its wedge product with p-forms:

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \,\mathrm{vol} \tag{1.1.1}$$

for all $\beta \in \Omega^p V$. The operator $* : \Omega^p \to \Omega^{n-p}$ is called the *Hodge $*$ operator.*

In more concrete terms, if $(e_i)_{i=1\dots n}$ is a direct orthonormal basis of V, then $(e^I)_{I \subset \{1,\dots,n\}}$ is an orthonormal basis of ΩV. One checks easily that

$$\begin{aligned} *1 &= \mathrm{vol}, & *e^1 &= e^2 \wedge e^3 \wedge \cdots \wedge e^n, \\ *\,\mathrm{vol} &= 1, & *e^i &= (-1)^{i-1} e^1 \wedge \cdots \wedge \widehat{e^i} \cdots e^n. \end{aligned}$$

More generally,

$$*e^I = \epsilon(I, \complement I) e^{\complement I}, \tag{1.1.2}$$

where $\epsilon(I, \complement I)$ is the signature of the permutation $(1, \dots, n) \to (I, \complement I)$.

1.1.3. Exercise. — Suppose that in the basis (e_i) the quadratic form is given by the matrix $g = (g_{ij})$, and write the inverse matrix $g^{-1} = (g^{ij})$. Prove that for a 1-form $\alpha = \alpha_i e^i$ one has

$$*\alpha = (-1)^{i-1} g^{ij} \alpha_j e^1 \wedge \dots \wedge \widehat{e^i} \wedge \dots \wedge e^n. \tag{1.1.4}$$

1.1.5. Exercise. — Prove that $*^2 = (-1)^{p(n-p)}$ on Ω^p.

If n is even, then $* : \Omega^{n/2} \to \Omega^{n/2}$ satisfies $*^2 = (-1)^{n/2}$. Therefore, if $n/2$ is even, the eigenvalues of $*$ on $\Omega^{n/2}$ are ± 1, and $\Omega^{n/2}$ decomposes accordingly as

$$\Omega^{n/2} = \Omega_+ \oplus \Omega_-. \tag{1.1.6}$$

The elements of Ω_+ are called *selfdual forms*, and the elements of Ω_- *antiselfdual forms*. For example, if $n = 4$, then $\Omega_\pm$ is generated by the forms

$$e^1 \wedge e^2 \pm e^3 \wedge e^4, \quad e^1 \wedge e^3 \mp e^2 \wedge e^4, \quad e^1 \wedge e^4 \pm e^2 \wedge e^3. \tag{1.1.7}$$

1.1.8. Exercise. — If $n/2$ is even, prove that the decomposition (1.1.6) is orthogonal for the quadratic form $\Omega^{n/2} \wedge \Omega^{n/2} \to \Omega^n \simeq \mathbb{R}$, and

$$\alpha \wedge \alpha = \pm |\alpha|^2 \operatorname{vol} \qquad \text{if } \alpha \in \Omega_\pm. \tag{1.1.9}$$

1.1.10. Exercise. — If u is an orientation-preserving isometry of V, that is $u \in SO(V)$, prove that u preserves the Hodge operator. This means the following: u induces an isometry of $V^* = \Omega^1$, and an isometry $\Omega^p u$ of $\Omega^p V$ defined by $(\Omega^p u)(x^1 \wedge \dots \wedge x^p) = u(x^1) \wedge \dots \wedge u(x^p)$. Then for any p-form $\alpha \in \Omega^p V$ one has

$$*(\Omega^p u)\alpha = (\Omega^{n-p} u) * \alpha.$$

This illustrates the fact that an orientation-preserving isometry preserves every object canonically attached to a metric and an orientation.

1.2. Adjoint operator

Suppose (M^n, g) is an oriented Riemannian manifold, and $E \to M$ a unitary bundle. Then on sections of E with compact support, one can define the L^2 scalar product and the L^2 norm:

$$(s, t) = \int_M \langle s, t \rangle_E \operatorname{vol}^g, \qquad \|s\|^2 = \int_M \langle s, s \rangle_E \operatorname{vol}^g. \tag{1.2.1}$$

If E and F are unitary bundles and $P : \Gamma(E) \to \Gamma(F)$ is a linear operator, then a *formal adjoint of* P is an operator $P^* : \Gamma(F) \to \Gamma(E)$ satisfying

$$(Ps, t)_E = (s, P^*t)_F \tag{1.2.2}$$

for all sections $s \in C_c^\infty(E)$ and $t \in C_c^\infty(F)$.

1.2.3. Example. — Consider the differential of functions,

$$d : C^\infty(M) \to C^\infty(\Omega^1).$$

Choose local coordinates (x^i) in an open set $U \subset M$ and suppose that the function f and the 1-form $\alpha = \alpha_i dx^i$ have compact support in U; write $\mathrm{vol}^g = \gamma(x) dx^1 \wedge \cdots \wedge dx^n$, then by integration by parts:

$$\begin{aligned}
\int_M \langle df, \alpha \rangle \mathrm{vol}^g &= \int g^{ij} \partial_i f \alpha_j \gamma dx^1 \cdots dx^n \\
&= -\int f \partial_i (g^{ij} \alpha_j \gamma) dx^1 \cdots dx^n \\
&= -\int f \gamma^{-1} \partial_i (g^{ij} \alpha_j \gamma) \mathrm{vol}^g .
\end{aligned}$$

It follows that

$$d^* \alpha = -\gamma^{-1} \partial_i (\gamma g^{ij} \alpha_j). \tag{1.2.4}$$

More generally, one has the following formula.

1.2.5. Lemma. — *The formal adjoint of the exterior derivative* $d : \Gamma(\Omega^p M) \to \Gamma(\Omega^{p+1} M)$ *is*

$$d^* = (-1)^{np+1} * d * .$$

Proof. — For $\alpha \in C_c^\infty(\Omega^p)$ and $\beta \in C_c^\infty(\Omega^{p+1})$ one has the equalities:

$$\begin{aligned}
\int_M \langle d\alpha, \beta \rangle \mathrm{vol}^g &= \int_M du \wedge *v \\
&= \int_M d(u \wedge *v) - (-1)^p u \wedge d * v
\end{aligned}$$

by Stokes theorem, and using exercice 1.1.5:

$$\begin{aligned}
&= (-1)^{p+1+p(n-p)} \int_M u \wedge * * d * v \\
&= (-1)^{pn+1} \int_M \langle u, *d * v \rangle \mathrm{vol}^g .
\end{aligned}$$

$\square$

1.2.6. Remarks. — 1) If n is even then the formula simplifies to $d^* = -*d*$.

2) The same formula gives an adjoint for the exterior derivative $d^\nabla : \Gamma(\Omega^p \otimes E) \to \Gamma(\Omega^{p+1} \otimes E)$ associated to a unitary connection ∇ on a bundle E.

3) As a consequence, for a 1-form α with compact support one has

$$\int_M (d^*\alpha) \operatorname{vol}^g = 0 \tag{1.2.7}$$

since this equals $(\alpha, d(1)) = 0$.

1.2.8. Exercise. — Suppose that (M^n, g) is a manifold with boundary. Note $\vec{n}$ is the normal vector to the boundary. Prove that (1.2.7) becomes:

$$\int_M (d^*\alpha) \operatorname{vol} = -\int_{\partial M} *\alpha = -\int_{\partial M} \alpha_{\vec{n}} \operatorname{vol}^{\partial M} . \tag{1.2.9}$$

For 1-forms we have the following alternative formula for d^*.

1.2.10. Lemma. — *Let E be a vector bundle with unitary connection ∇, then the formal adjoint of $\nabla : \Gamma(M, E) \to \Gamma(M, \Omega^1 \otimes E)$ is*

$$\nabla^*\alpha = -\operatorname{Tr}^g(\nabla u) = -\sum_1^n (\nabla_{e_i}\alpha)(e_i).$$

Proof. — Take a local orthonormal basis (e_i) of TM, and consider an E-valued 1-form $\alpha = \alpha_i e^i$. We have $*\alpha = (-1)^{i-1}\alpha_i e^1 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge e^n$. One can suppose that just at the point p one has $\nabla e_i(p) = 0$, therefore $de^i(p) = 0$ and, still at the point p,

$$d^\nabla * \alpha = \sum_1^n (\nabla_i \alpha_i) e^1 \wedge \cdots \wedge e^n.$$

Finally $\nabla^*\alpha(p) = -\sum_1^n (\nabla_i\alpha_i)(p)$. □

1.2.11. Remark. — Actually the same formula is also valid for p-forms. Indeed, $d^\nabla : \Gamma(M, \Omega^p) \to \Gamma(M, \Omega^{p+1})$ can be deduced from the covariant derivative $\nabla : \Gamma(M, \Omega^p) \to \Gamma(M, \Omega^1 \otimes \Omega^p)$ by the formula[(1)]

$$d^\nabla = (p+1)\mathbf{a} \circ \nabla,$$

where $\mathbf{a}$ is the antisymmetrization of a $(p+1)$-tensor. Also observe that if $\alpha \in \Omega^p \subset \otimes^p \Omega^1$, its norm as a p-form differs from its norm as a p-tensor by

$$|\alpha|^2_{\Omega^p} = p! |\alpha|^2_{\otimes^p \Omega^1}.$$

[(1)] This formula is true as soon as ∇ is a torsion free connection on M.

Putting together this two facts, one can calculate that d^* is the restriction of ∇^* to antisymmetric tensors in $\Omega^1 \otimes \Omega^p$. We get the formula

$$d^*\alpha = -\sum_1^n e_i \lrcorner \nabla_i \alpha. \tag{1.2.12}$$

Of course the formula remains valid for E-valued p-forms, if E has a unitary connection ∇.

1.2.13. Exercise. — Consider the symmetric part of the covariant derivative,

$$\delta^* : \Gamma(\Omega^1) \to \Gamma(S^2\Omega^1).$$

Prove that its formal adjoint is the divergence δ, defined for a symmetric 2-tensor h by

$$(\delta h)_X = -\sum_1^n (\nabla_{e_i} h)(e_i, X).$$

1.3. Hodge-de Rham Laplacian

1.3.1. Definition. — Let (M^n, g) be an oriented Riemannian manifold. The *Hodge-De Rham Laplacian on p-forms* is defined by

$$\Delta\alpha = (dd^* + d^*d)\alpha.$$

Clearly, Δ is a formally selfadjoint operator. The definition is also valid for E-valued p-forms, using the exterior derivative d^∇, where E has a metric connection ∇.

1.3.2. Example. — On functions $\Delta = d^*d$; using (1.2.4), we obtain the formula in local coordinates:

$$\Delta f = -\frac{1}{\sqrt{\det(g_{ij})}} \partial_i \big(g^{ij} \sqrt{\det(g_{ij})} \partial_j f \big). \tag{1.3.3}$$

In particular, for the flat metric $g = \sum_1^n (dx^i)^2$ of $\mathbb{R}^n$, one has

$$\Delta f = -\sum_1^n \partial_i^2 f.$$

In polar coordinates on $\mathbb{R}^2$, one has $g = dr^2 + r^2 d\theta^2$ and therefore

$$\Delta f = -\frac{1}{r}\partial_r(r\partial_r f) - \frac{1}{r^2}\partial_\theta^2 f.$$

More generally on $\mathbb{R}^n$ with polar coordinates $g = dr^2 + r^2 g_{S^{n-1}}$, one has

$$\Delta f = -\frac{1}{r^{n-1}}\partial_r(r^{n-1}\partial_r f) + \frac{1}{r^2}\Delta_{S^{n-1}} f.$$

Similarly, on the real hyperbolic space H^n with geodesic coordinates, $g = dr^2 + \sinh^2(r) g_{S^{n-1}}$ and the formula reads

$$\Delta f = -\frac{1}{\sinh(r)^{n-1}}\partial_r(\sinh(r)^{n-1}\partial_r f) + \frac{1}{r^2}\Delta_{S^{n-1}} f.$$

1.3.4. Exercise. — On p-forms in $\mathbb{R}^n$ prove that $\Delta(\alpha_I dx^I) = (\Delta\alpha_I) dx^I$.

1.3.5. Exercise. — Prove that $*$ commutes with Δ.

1.3.6. Exercise. — If (M^n, g) has a boundary, prove that for two functions f and g one has

$$\int_M (\Delta f) g \,\mathrm{vol} = \int_M \langle df, dg\rangle \,\mathrm{vol} - \int_{\partial M} \frac{\partial f}{\partial \vec{n}} g \,\mathrm{vol}^{\partial M}.$$

Deduce

$$\int_M (\Delta f) g \,\mathrm{vol} = \int_M f \Delta g \,\mathrm{vol} + \int_{\partial M} \big(f\frac{\partial g}{\partial \vec{n}} - \frac{\partial f}{\partial \vec{n}} g\big) \,\mathrm{vol}^{\partial M}.$$

1.3.7. Exercise. — Prove that the radial function defined on $\mathbb{R}^n$ by (V_n being the volume of the sphere S^n)

$$G(r) = \begin{cases} \frac{1}{(n-2)V_{n-1}r^{n-2}} & \text{if } n > 2 \\ \frac{1}{2\pi}\log r & \text{if } n = 2 \end{cases}$$

satisfies $\Delta G = \delta_0$ (Dirac function at 0). Deduce the explicit solution of $\Delta f = g$ for $g \in C_c^\infty(\mathbb{R}^n)$ given by the integral formula

$$f(x) = \int_{\mathbb{R}^n} G(|x-y|) g(y) |dy|^n.$$

The function G is called Green's function.

Similarly, find the Green's function for the real hyperbolic space.

1.4. Statement of Hodge theory

Let (M^n, g) be a closed Riemannian oriented manifold. Consider the De Rham complex

$$0 \to \Gamma(\Omega^0) \xrightarrow{d} \Gamma(\Omega^1) \xrightarrow{d} \cdots \xrightarrow{d} \Gamma(\Omega^n) \to 0.$$

Remind that the De Rham cohomology in degree p is defined by $H^p = \{\alpha \in C^\infty(M, \Omega^p), d\alpha = 0\}/dC^\infty(M, \Omega^{p-1})$.

Other situation: (E, ∇) is a flat bundle, we have the associated complex

$$0 \to \Gamma(\Omega^0 \otimes E) \xrightarrow{d^\nabla} \Gamma(\Omega^1 \otimes E) \xrightarrow{d^\nabla} \cdots \xrightarrow{d^\nabla} \Gamma(\Omega^n \otimes E) \to 0$$

and we can define the De Rham cohomology with values in E in the same way.

In both cases, we have the Hodge-De Rham Laplacian $\Delta = dd^* + d^*d$.

1.4.1. Definition. — A *harmonic form* is a C^∞ form such that $\Delta\alpha = 0$.

1.4.2. Lemma. — *If $\alpha \in C_c^\infty(M, \Omega^p)$, then α is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$. In particular, on a compact connected manifold, any harmonic function is constant.*

Proof. — It is clear that if $d\alpha = 0$ and $d^*\alpha = 0$, then $\Delta\alpha = d^*d\alpha + dd^*\alpha = 0$. Conversely, if $\Delta\alpha = 0$, because

$$(\Delta\alpha, \alpha) = (d^*d\alpha, \alpha) + (dd^*\alpha, \alpha) = \|d\alpha\|^2 + \|d^*\alpha\|^2,$$

we deduce that $d\alpha = 0$ and $d^*\alpha = 0$. □

1.4.3. Remark. — The lemma remains valid on complete manifolds, for L^2 forms α such that $d\alpha$ and $d^*\alpha$ are also L^2. This is proved by taking cut-off functions χ_j, such that $\chi_j^{-1}(1)$ are compact domains which exhaust M, and $|d\chi_j|$ remains bounded by a fixed constant C. Then

$$\begin{aligned}\int_M \langle \Delta\alpha, \chi_j\alpha\rangle \,\mathrm{vol} &= \int_M \big(\langle d\alpha, d(\chi_j\alpha)\rangle + \langle d^*\alpha, d^*(\chi_j\alpha)\rangle\big) \,\mathrm{vol}\\ &= \int_M \big(\chi_j(|d\alpha|^2 + |d^*\alpha|^2) + \langle d\alpha, d\chi_j \wedge \alpha\rangle - \langle d^*\alpha, \nabla\chi_j \lrcorner \alpha\rangle\big) \,\mathrm{vol}\end{aligned}$$

Using $|d\chi_j| \leqslant C$ and taking j to infinity, one obtains $(\Delta\alpha, \alpha) = \|d\alpha\|^2 + \|d^*\alpha\|^2$.

Note $\mathbf{H}^p$ the space of harmonic p-forms on M. The main theorem of this section is:

1.4.4. Theorem. — *Let (M^n, g) be a compact closed oriented Riemannian manifold. Then:*

1. *$\mathbf{H}^p$ is finite dimensional;*
2. *one has a decomposition $C^\infty(M, \Omega^p) = \mathbf{H}^p \oplus \Delta(C^\infty(M, \Omega^p))$, which is orthogonal for the L^2 scalar product.*

This is the main theorem of Hodge theory, and we will prove it later, as a consequence of theorem 1.6.8. Just remark now that it is obvious that $\ker\Delta \perp \operatorname{im}\Delta$, because Δ is formally selfadjoint. Also, general theory of unbounded operators gives almost immediately that $L^2(M,\Omega^p) = \mathbf{H}^p \oplus \overline{\operatorname{im}\Delta}$. What is non trivial is: finite dimensionality of $\mathbf{H}^p$, closedness of $\operatorname{im}\Delta$, and the fact that smooth forms in the L^2 image of Δ are images of smooth forms.

Now we will derive some immediate consequences.

1.4.5. Corollary. — *Same hypothesis. One has the orthogonal decomposition*

$$C^\infty(M,\Omega^p) = \mathbf{H}^p \oplus d\big(C^\infty(M,\Omega^{p-1})\big) \oplus d^*\big(C^\infty(M,\Omega^{p+1})\big),$$

where

$$\ker d = \mathbf{H}^p \oplus d\big(C^\infty(M,\Omega^{p-1})\big), \tag{1.4.6}$$

$$\ker d^* = \mathbf{H}^p \oplus d^*\big(C^\infty(M,\Omega^{p+1})\big). \tag{1.4.7}$$

Note that since harmonic forms are closed, there is a natural map $\mathbf{H}^p \to H^p$. The equality (1.4.6) implies immediately:

1.4.8. Corollary. — *Same hypothesis. The map $\mathbf{H}^p \to H^p$ is an isomorphism.*

Using exercice 1.3.5, we obtain:

***1.4.9. Corollary* (Poincaré duality).** — *Same hypothesis. The Hodge $*$ operator induces an isomorphism $* : \mathbf{H}^p \to \mathbf{H}^{n-p}$. In particular the corresponding Betti numbers are equal, $b_p = b_{n-p}$.*

1.4.10. Remark. — As an immediate consequence, if M is connected then $H^n = \mathbb{R}$ since $H^0 = \mathbb{R}$. Since $*1 = \mathrm{vol}^g$ and $\int_M \mathrm{vol}^g > 0$, an identification with $\mathbb{R}$ is just given by integration of n-forms on M.

1.4.11. Remark. — In Kähler geometry there is a decomposition of harmonic forms using the (p,q) type of forms, $\mathbf{H}^k \otimes \mathbb{C} = \oplus_0^k \mathbf{H}^{p,k-p}$, and corollary 1.4.9 can then be refined as an isomorphism $* : \mathbf{H}^{p,q} \to \mathbf{H}^{m-q,m-p}$, where $n = 2m$.

1.4.12. Remark. — Suppose that n is a multiple of 4. Then by exercises 1.1.8 and 1.3.5, one has an orthogonal decomposition

$$\mathbf{H}^{n/2} = \mathbf{H}_+ \oplus \mathbf{H}_-. \tag{1.4.13}$$

Under the wedge product, the decomposition is orthogonal, $\mathbf{H}_+$ is positive and $\mathbf{H}_-$ is negative, therefore the signature of the manifold is (p,q) with $p = \dim \mathbf{H}_+$ and $q = \dim \mathbf{H}_-$.

1.4.14. Exercise. — Suppose again that n is a multiple of 4. Note $d_\pm : \Gamma(\Omega^{n/2-1}) \to \Gamma(\Omega_\pm)$ the projection of d on selfdual or antiselfdual forms. Prove that on $(n/2-1)$-forms, one has $d_+^* d_+ = d_-^* d_-$. Deduce that the cohomology of the complex

$$0 \to \Gamma(\Omega^0) \xrightarrow{d} \Gamma(\Omega^1) \xrightarrow{d} \cdots \xrightarrow{d} \Gamma(\Omega^{n/2-1}) \xrightarrow{d_+} \Gamma(\Omega_+) \to 0 \tag{1.4.15}$$

is $\mathbf{H}^0$, $\mathbf{H}^1$, …, $\mathbf{H}^{n/2-1}$, $\mathbf{H}_+$.

1.4.16. Exercise. — Using exercise 1.3.4, calculate the harmonic forms and the cohomology of a flat torus $\mathbb{R}^n/\mathbb{Z}^n$.

1.4.17. Exercise. — Let (M,g) be a compact oriented Riemannian manifold.

1) If γ is an orientation-preserving isometry of (M,g) and α a harmonic form, prove that $\gamma^*\alpha$ is harmonic.

2) (requires some knowledge of Lie groups) Prove that if a *connected* Lie group G acts on M, then the action of G on $H^\bullet(M,\mathbb{R})$ given by $\alpha \to \gamma^*\alpha$ is trivial[(2)].

3) Deduce that harmonic forms are invariant under $\mathrm{Isom}(M,g)^o$, the connected component of the identity in the isometry group of M. Apply this observation to give a proof that the cohomology of the n-sphere vanishes in degrees $k = 1, \dots, n-1$ (prove that there is no $SO(n+1)$-invariant k-form on S^n using the fact that the representation of $SO(n)$ on $\Omega^k\mathbb{R}^n$ is irreducible and therefore has no fixed nonzero vector).

1.5. Bochner technique

Let (E,∇) be a bundle equipped with a unitary connection over an oriented Riemannian manifold (M^n,g). Then $\nabla : \Gamma(E) \to \Gamma(\Omega^1 \otimes E)$ and we can define

[(2)] If ξ belongs to the Lie algebra of G and X_ξ is the associated vector field on M given by the infinitesimal action of G (that is defined by $X_\xi(x) = \frac{d}{dt} e^{t\xi}x|_{t=0}$), then one has $\frac{d}{dt}(e^{t\xi})^*\alpha|_{t=0} = \mathscr{L}_{X_\xi}\alpha = i_{X_\xi} d\alpha + d i_{X_\xi}\alpha$. Deduce that if α is closed, then the infinitesimal action of G on $H^\bullet(M,\mathbb{R})$ is trivial.

the *rough Laplacian* $\nabla^*\nabla$ acting on sections of E. Using a local orthonormal basis (e_i) of TM, from lemma 1.2.10 it follows that

$$\nabla^*\nabla s = \sum_1^n -\nabla_{e_i}\nabla_{e_i} s + \nabla_{\nabla_{e_i} e_i} s. \tag{1.5.1}$$

If we calculate just at a point p and we choose a basis (e_i) which is parallel at p, then the second term vanishes.

In particular, using the Levi-Civita connection, we get a Laplacian $\nabla^*\nabla$ acting on p-forms. It is not equal to the Hodge-De Rham Laplacian, as follows from:

***1.5.2. Lemma* (Bochner formula).** — *Let (M^n, g) be an oriented Riemannian manifold. Then for any 1-form α on M one has*

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}(\alpha).$$

1.5.3. Remark. — There is a similar formula (Weitzenböck formula) on p-forms: the difference $\Delta\alpha - \nabla^*\nabla\alpha$ is a zero-th order term involving the curvature of M.

Proof of the lemma. — We have $d\alpha_{X,Y} = (\nabla_X\alpha)_Y - (\nabla_Y\alpha)_X$, therefore

$$d^*d\alpha_X = -\sum_1^n (\nabla_{e_i} d\alpha)_{e_i,X} = \sum_1^n -(\nabla_{e_i}\nabla_{e_i}\alpha)_X + (\nabla_{e_i}\nabla_X\alpha)_{e_i},$$

where in the last equality we calculate only at a point p, and we have chosen the vector fields (e_i) and X parallel at p.

Similarly, $d^*\alpha = -\sum_1^n (\nabla_{e_i}\alpha)_{e_i}$, therefore

$$dd^*\alpha_X = -\sum_1^n \nabla_X((\nabla_{e_i}\alpha)_{e_i}) = -\sum_1^n (\nabla_X\nabla_{e_i}\alpha)_{e_i}.$$

Therefore, still at the point p, comparing with (1.5.1),

$$(\Delta\alpha)_X = (\nabla^*\nabla\alpha)_X + \sum_1^n (R_{e_i,X}\alpha)_{e_i} = (\nabla^*\nabla\alpha)_X + \mathrm{Ric}(\alpha)_X. \tag{1.5.4}$$

□

1.5.5. Remark. — There is a similar formula if the exterior derivative is coupled with a bundle E equipped with a connection ∇. The formula for the Laplacian $\Delta = (d^\nabla)^* d^\nabla + d^\nabla (d^\nabla)^*$ becomes

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}(\alpha) + \mathscr{R}^\nabla(\alpha), \tag{1.5.6}$$

where the additional last term involves the curvature of ∇,

$$\mathscr{R}^{\nabla}(\alpha)_X = \sum_1^n R^{\nabla}_{e_i,X}\alpha(e_i). \tag{1.5.7}$$

The proof is exactly the same as above, a difference arises just in the last equality of (1.5.4), when one analyses the curvature term: the curvature acting on α is that of $\Omega^1 \otimes E$, so equals $R \otimes 1 + 1 \otimes R^{\nabla}$, from which:

$$\sum_1^n (R_{e_i,X}\alpha)_{e_i} = \mathrm{Ric}(\alpha)_X + \sum_1^n R^{\nabla}_{e_i,X}\alpha(e_i).$$

Now let us see an application of the Bochner formula. Suppose M is compact. By Hodge theory, an element of $H^1(M)$ is represented by a harmonic 1-form α. By the Bochner formula, we deduce $\nabla^*\nabla\alpha + \mathrm{Ric}(\alpha) = 0$. Taking the scalar product with α, one obtains

$$\|\nabla\alpha\|^2 + (\mathrm{Ric}(\alpha), \alpha) = 0. \tag{1.5.8}$$

If $\mathrm{Ric} \geqslant 0$, this equality implies $\nabla\alpha = 0$ and $\mathrm{Ric}(\alpha) = 0$. If $\mathrm{Ric} > 0$, then $\alpha = 0$; if $\mathrm{Ric} \geqslant 0$ we get only that α is parallel, therefore the cohomology is represented by parallel forms. Suppose that M is connected, then a parallel form is determined by its values at one point p, so we get an injection

$$\mathbf{H}^1 \hookrightarrow \Omega^1_p.$$

Therefore $\dim \mathbf{H}^1 \leqslant n$, with equality if and only if M has a basis of parallel 1-forms. This implies that M is flat, and by Bieberbach's theorem that M is a torus. Therefore we deduce:

1.5.9. Corollary. — *If (M^n, g) is a compact connected oriented Riemannian manifold, then:*

- *if* $\mathrm{Ric} > 0$, *then $b_1(M) = 0$;*
- *if* $\mathrm{Ric} \geqslant 0$, *then $b_1(M) \leqslant n$, with equality if and only if (M, g) is a flat torus.*

This corollary is a typical example of application of Hodge theory to prove vanishing theorems for the cohomology: one uses Hodge theory to represent cohomology classes by harmonic forms, and then a Weitzenböck formula to prove that the harmonic forms must vanish or be special under some curvature assumption. For examples in Kähler geometry see [**4**]. See also the application to the spectral estimate (2.10.16).

1.6. Differential operators

A linear operator $P : \Gamma(M, E) \to \Gamma(M, F)$ between sections of two bundles E and F is a *differential operator of order* d if, in any local trivialisation of E and F over a coordinate chart (x^i), one has

$$Pu(x) = \sum_{|\alpha| \leqslant d} a^\alpha(x) \partial_\alpha u(x),$$

where $\alpha = (\alpha_1, \dots, \alpha_k)$ is a multiindex with each $\alpha_i \in \{1 \dots n\}$, $|\alpha| = k$, $\partial_\alpha = \partial_{\alpha_1} \dots \partial_{\alpha_n}$, and $a^\alpha(x)$ is a matrix representing an element of $\mathrm{Hom}(E_x, F_x)$.

The *principal symbol* of P is defined for $x \in M$ and $\xi \in T^*_x M$ by taking only the terms of order d in P:

$$\sigma_P(x, \xi) = \sum_{|\alpha| = d} a^\alpha(x) \xi_\alpha,$$

where $\xi_\alpha = \xi_{\alpha_1} \cdots \xi_{\alpha_d}$ if $\xi = \xi_i dx^i$. It is a degree d homogeneous polynomial in the variable ξ with values in $\mathrm{Hom}(E_x, F_x)$.

A priori, it is not clear from the formula in local coordinates that the principal symbol is intrinsically defined. But it is easy to check that one has the following more intrinsic definition: suppose $f \in C^\infty(M)$, $t \in \mathbb{R}$ and $u \in \Gamma(M, E)$, then

$$e^{-tf(x)} P\big(e^{tf(x)} u(x)\big)$$

is a polynomial of degree d in the variable t, whose monomial of degree d is a homogeneous polynomial of degree d in $df(x)$. It is actually

$$t^d \sigma_P(x, df(x)) u(x).$$

The following property of the principal symbol is obvious.

1.6.1. Lemma. — $\sigma_{P \circ Q} = \sigma_P \circ \sigma_Q$.

1.6.2. Examples. — 1) If one has a connection $\nabla : \Gamma(E) \to \Gamma(\Omega^1 \otimes E)$, then $e^{-tf} \nabla(e^{tf} u) = t df \otimes u + \nabla u$. Therefore

$$\sigma_\nabla(x, \xi) = \xi \otimes : \Gamma(E_x) \to \Gamma(\Omega^1_x \otimes E_x).$$

2) The principal symbol of the exterior derivative $d^\nabla : \Gamma(\Omega^p \otimes E) \to \Gamma(\Omega^{p+1} \otimes E)$ is $\sigma_d(x, \xi) = \xi \wedge$.

3) The principal symbol of $d^* : \Gamma(\Omega^{p+1} \otimes E) \to \Gamma(\Omega^p \otimes E)$ is $\sigma_{d^*}(x, \xi) = -\xi \lrcorner$.

4) The principal symbol of the composite $\nabla^* \nabla$ is the composite $-(\xi \lrcorner) \circ (\xi \otimes) = -|\xi|^2$.

1.6.3. Exercise. — Prove that the principal symbol of the Hodge-De Rham Laplacian is also $\sigma_\Delta(x,\xi) = -|\xi|^2$.

1.6.4. Lemma. — *Any differential operator $P : \Gamma(E) \to \Gamma(F)$ of order d has a formal adjoint P^*, whose principal symbol is*

$$\sigma_{P^*}(x,\xi) = (-1)^d \sigma_P(x,\xi)^*.$$

1.6.5. Exercise. — Prove the lemma in the following way. In local coordinates, write $\mathrm{vol}^g = v(x)dx^1 \wedge \cdots \wedge dx^n$. Choose orthonormal trivialisations of E and F, and write $P = \sum a^\alpha(x)\partial_\alpha$. Then prove that

$$P^* t = \sum_{|\alpha| \leqslant d} (-1)^{|\alpha|} \frac{1}{v(x)} \partial_\alpha\big(v(x) a^\alpha(x)^* t\big).$$

The proof is similar to that in example 1.2.3.

1.6.6. Remark. — In analysis, the principal symbol is often defined slightly differently: ξ_j corresponds to $D_j = \frac{1}{i}\frac{\partial}{\partial x^j}$. The advantage is that D_j is formally selfadjoint, so with this definition the principal symbol of P^* is always $\sigma_P(x,\xi)^*$ and the principal symbol of the Laplacian becomes positive.

1.6.7. Definition. — A differential operator $P : \Gamma(E) \to \Gamma(F)$ is *elliptic* if for any $x \in M$ and $\xi \neq 0$ in T_xM, the principal symbol $\sigma_P(x,\xi) : E_x \to F_x$ is injective.

Here is our main theorem on elliptic operators. It will be proved in section 1.8.

1.6.8. Theorem. — *Suppose (M^n, g) is a compact oriented Riemannian manifold, and $P : \Gamma(E) \to \Gamma(F)$ is an elliptic operator, with* $\operatorname{rank} E = \operatorname{rank} F$. *Then*

1. $\ker(P)$ *is finite dimensional;*
2. *there is a L^2 orthogonal sum*

$$C^\infty(M,F) = \ker(P^*) \oplus P\big(C^\infty(M,E)\big).$$

The Hodge theorem 1.4.4 follows immediately, by applying to the Hodge-De Rham Laplacian Δ.

Remark that $\ker(P^*)$ is also finite dimensional, since P^* is elliptic if P is elliptic. The difference $\dim\ker P - \dim\ker P^*$ is the *index* of P, defined by

$$\operatorname{ind}(P) = \dim\ker P - \dim\operatorname{coker} P.$$

Operators with finite dimensional kernel and cokernel are called *Fredholm operators*, and the index is invariant under continuous deformation among Fredholm operators. Since ellipticity depends only on the principal symbol, it follows immediately that the index of P depends only on σ_P. The fundamental index theorem of Atiyah-Singer gives a topological formula for the index, see the book [**2**].

A useful special case is that of a formally selfadjoint elliptic operator. Its index is of course zero. The invariance of the index then implies that any elliptic operator with the same symbol (or whose symbol is a deformation through elliptic symbols) has also index zero.

1.7. Basic elliptic theory

In this section we explain the basic results enabling to prove theorem 1.6.8.

Sobolev spaces. — The first step is to introduce the Sobolev space $H^s(\mathbb{R}^n)$ of tempered distributions f on $\mathbb{R}^n$ such that the Fourier transform satisfies

$$\|f\|_s^2 := \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2(1+|\xi|^2)^s|d\xi|^n < +\infty. \tag{1.7.1}$$

Equivalently, $H^s(\mathbb{R}^n)$ is the space of functions $f \in L^2(\mathbb{R}^n)$ which admit s derivatives in distribution sense[(3)] in L^2, and

$$\|f\|_s^2 \sim \sum_{|\alpha|\leqslant s} \|\partial_\alpha s\|_{L^2}^2. \tag{1.7.2}$$

(But observe that the definition (1.7.1) is valid also for any real s).

If M is a compact manifold and E a vector bundle over M, then one can define the space $C^k(M,E)$ of sections of E whose coefficients are of class C^k in any trivialisation of E, and $H^s(M,E)$ the space of sections of E whose coefficients in any trivialisation and any coordinate chart are functions of class H^s in $\mathbb{R}^n$. If M is covered by a finite number of charts (U_j) with trivialisations of $E|_{U_j}$ by a basis of sections $(e_{j,a})_{a=1,\dots,r}$, choose a partition of unity (χ_j) subordinate to (U_j), then a section u of E can be written $u = \sum \chi_j u_{j,\alpha} e_{j,\alpha}$ with $\chi_j u_{j,\alpha}$ a function with compact support in $U_j \subset \mathbb{R}^n$, therefore

$$\|u\|_{C^k} = \sup_{j,\alpha} \|\chi_j u_{j,\alpha}\|_{C^k(\mathbb{R}^n)}, \quad \|u\|_s^2 = \sum \|\chi_j u_{j,\alpha}\|_{H^s(\mathbb{R}^n)}^2. \tag{1.7.3}$$

Up to equivalence of norms, the result is independent of the choice of coordinate charts and trivialisations of E.

(3) Weak derivative: $g = D_\alpha f$ if for any $\phi \in C_c^\infty(\mathbb{R}^n)$ one has $\int_{\mathbb{R}^n}(D_\alpha\phi)f = \int_{\mathbb{R}^n}\phi g$.

There is another approach to define C^k and H^s norms for sections of E. Suppose that M^n has a Riemannian metric, and E is equipped with a unitary connection ∇. Then one can define

$$\|u\|_{C^k} = \sup_{j\leqslant k}\sup_M |\nabla^j u|, \quad \|u\|_s = \sum_0^k \int_M |\nabla^j u|^2 \operatorname{vol}^g . \tag{1.7.4}$$

1.7.5. Remark. — On a noncompact manifold, the definition (1.7.3) does not give a well defined class of equivalent norms when one changes the trivialisations. On the contrary, definition (1.7.4), valid only for integral s, can be useful if (M,g) is non compact; the norms depend on the geometry at infinity of g and ∇.

1.7.6. Example. — If M is a torus T^n, then the regularity can be seen on the Fourier series: $f \in H^s(T^n)$ if and only if

$$\|f\|_s^2 = \sum_{\xi\in\mathbb{Z}^n} (1+|\xi|^2)^s |\hat f(\xi)|^2 < +\infty.$$

From the inverse formula $f(x) = \sum_\xi \hat f(\xi)\exp(i\langle \xi, x\rangle)$, by the Cauchy-Schwartz inequality,

$$|f(x)| \leqslant \sum_{\xi\in\mathbb{Z}^n} |\hat f(\xi)| \leqslant \|f\|_s \big(\sum_\xi (1+|\xi|^2)^{-s}\big)^{1/2} < +\infty \quad \text{ if } s > \frac{n}{2}.$$

It follows that there is a continuous inclusion $H^s \subset C^0$ is $s > \frac{n}{2}$. Similarly it follows that $H^s \subset C^k$ if $s > k + \frac{n}{2}$.

Of course the same results are true on $\mathbb{R}^n$ using Fourier transform, and one obtains the following lemma.

***1.7.7. Lemma* (Sobolev)**. — *If M^n is compact, $k \in \mathbb{N}$ and $s > k + \frac{n}{2}$, then there is a continuous and compact injection $H^s \subset C^k$.*

The fact that the inclusion is compact follows from the following lemma (which is obvious on a torus, and the general case follows):

***1.7.8. Lemma* (Rellich)**. — *If M^n is compact and $s > t$, then the inclusion $H^s \subset H^t$ is compact.*

Action of differential operators. — If $P : \Gamma(M,E) \to \Gamma(M,F)$ is a differential operator of order d, then looking at P in local coordinates it is clear that P induces continuous operators $P : H^{s+d}(M,E) \to H^s(M,F)$.

In general a *weak solution* of the equation $Pu = v$ is a L^2 section u of E such that for any $\phi \in C_c^\infty(M,F)$ one has

$$(u, P^*\phi) = (v, \phi).$$

We can now state the main technical result in this section.

1.7.9. Lemma **(Local elliptic estimate)**. — *Let $P : \Gamma(M,E) \to \Gamma(M,F)$ be an elliptic operator. Fix a ball B in a chart with local coordinates (x^i) and the smaller ball $B_{1/2}$. Suppose that $u \in L^2(B,E)$ and $Pu \in H^s(B,F)$, then $u \in H^{s+d}(B_{1/2},E)$ and*

$$\|u\|_{H^{s+d}(B_{1/2})} \leqslant C\big(\|Pu\|_{H^s(B)} + \|u\|_{L^2(B)}\big). \tag{1.7.10}$$

1.7.11. Remark. — An important addition to the lemma is the fact that for a family of elliptic operators with bounded coefficients and bounded inverse of the principal symbol, one can take the constant C to be uniform.

1.7.12. Remark. — Elliptic regularity is not true in C^k spaces, that is $Pu \in C^k$ does not imply $u \in C^{k+d}$ in general.

We will not prove lemma 1.7.9, which is a difficult result. There are basically two ways to prove it. The first way is to locally approximate the operator on small balls by an operator with constant coefficients on $\mathbb{R}^n$ or T^n, where an explicit inverse is available using Fourier transform: one then glues together these inverses to get an approximate inverse for P which will give what is needed on u. See [**8**] for this method. The second way is more modern and uses microlocal analysis: one inverts the operator "microlocally", that is fiber by fiber on each cotangent space—this is made possible by the theory of pseudodifferential operators. See a nice and concise introduction in [**4**].

This implies immediately the following global result:

1.7.13. Corollary **(Global elliptic estimate)**. — *Let $P : \Gamma(M,E) \to \Gamma(M,F)$ be an elliptic operator. If $u \in L^2(M,E)$ and $Pu \in H^s(M,F)$, then $u \in H^{s+d}(M,E)$ and*

$$\|u\|_{s+d} \leqslant C\big(\|Pu\|_s + \|u\|_{L^2}\big). \tag{1.7.14}$$

From the elliptic estimate and the fact that $\cap_s H^s = C^\infty$, we obtain:

1.7.15. Corollary. — *If P is elliptic and $Pu = 0$, then u is smooth. More generally, if Pu is C^∞ then u is C^∞.*

1.7.16. Exercise. — Prove (1.7.14) for operators with constant coefficients on the torus.

1.8. Proof of the main theorem

We can now prove theorem 1.6.8.

First let us prove the first statement: the kernel of P is finite dimensional. By the elliptic estimate (1.7.14), for $u \in \ker(P)$ one has

$$\|u\|_{s+d} \leqslant C\|u\|_{L^2}.$$

Therefore the first identity map in the following diagram is continuous:

$$(\ker P, L^2) \longrightarrow (\ker P, H^{s+d}) \longrightarrow (\ker P, L^2).$$

The second inclusion is compact by lemma 1.7.8. The composite map is the identity of $\ker P$ equipped with the L^2 scalar product, it is therefore a compact map. This implies that the closed unit ball of $\ker(P)$ is compact, therefore $\ker(P)$ is a finite dimensional vector space.

Now let us prove the theorem in Sobolev spaces. We consider P as an operator

$$P : H^{s+d}(M, E) \longrightarrow H^s(M, F), \tag{1.8.1}$$

and in these spaces we want to prove

$$H^s(M, F) = \ker(P^*) \oplus \operatorname{im}(P). \tag{1.8.2}$$

We claim that for any $\epsilon > 0$, there exists an L^2 orthonormal family $(v_1, \dots, v_N)$ in H^{s+d}, such that

$$\|u\|_{L^2} \leqslant \epsilon\|u\|_{s+d} + \Big(\sum_1^N |(v_j, u)|^2\Big)^{1/2}. \tag{1.8.3}$$

Suppose for the moment that the claim is true. Then combining with the elliptic estimate (1.7.14), we deduce

$$(1 - C\epsilon)\|u\|_{s+d} \leqslant C\|Pu\|_s + C\Big(\sum_1^N |(v_j, u)|^2\Big)^{1/2}.$$

Choose $\epsilon = \frac{1}{2C}$, and let T be the subspace of sections in $H^{s+d}(M, E)$ which are L^2 orthogonal to the $(v_i)_{i=1\dots N}$. Then we obtain

$$2\|u\|_{s+d} \leqslant C\|Pu\|_s \quad \text{for } u \in T.$$

It follows that $P(T)$ is closed in $H^s(M, F)$. But $\operatorname{im}(P)$ is the sum of $P(T)$ and the image of the finite dimensional space generated by the $(v_i)_{i=1\dots N}$, so $\operatorname{im}(P)$ is closed as well in $H^s(M, F)$.

Finally the statement (1.8.1) in the Sobolev spaces H^s implies the statement for the space C^∞, which finishes the proof of the theorem. Indeed, suppose that $v \in C^\infty(M,F)$ is L^2 orthogonal to $\ker(P^*)$. Fix any $s \geqslant 0$ and apply (1.8.2) in H^s: therefore there exists $u \in H^{s+d}(M,E)$ such that $Pu = v$. Then u is C^∞ by corollary 1.7.15.

It remains to prove the claim (1.8.3). Choose a Hilbertian basis (v_j) of L^2, and suppose that the claim is not true. Then there exists a sequence of $(u_N) \in H^{s+d}(M,E)$ such that

1. $\|u_N\|_{L^2} = 1$,
2. $\epsilon\|u_N\|_{s+d} + \left(\sum_1^N |(v_j,u_N)|^2\right)^{1/2} < 1$.

From the second condition we deduce that (u_N) is bounded in $H^{s+d}(E)$, therefore there is a weakly convergent subsequence in $H^{s+d}(E)$, and the limit satisfies

$$\epsilon\|u\|_{s+d} + \|u\|_0 \leqslant 1.$$

By the compact inclusion $H^{s+d} \subset L^2$ this subsequence is strongly convergent in $L^2(E)$ so by the first condition, the limit u satisfies

$$\|u\|_0 = 1,$$

which is a contradiction.

1.8.4. Remark. — The same proof applies for an elliptic operator $P : \Gamma(E) \to \Gamma(F)$ where the ranks of E and F are not the same. The results are

1. $\ker P$ is finite dimensional (this can be also obtained by identifying $\ker P$ with $\ker P^*P$, and by noting that P^*P is elliptic if P is elliptic);
2. the image of the operator $P : H^{s+d}(M,E) \to H^s(M,F)$ is closed, and there is a L^2 orthogonal decomposition $H^s(M,F) = \ker P^* \oplus \operatorname{im} P$; note that here $\ker P^*$ depend on s as P^* is not elliptic if $\operatorname{rank} F > \operatorname{rank} E$.

1.9. Elliptic complexes

CHAPTER 2

MODULI PROBLEMS

MODULI spaces of solutions of geometric partial differential equations (that is the spaces of solutions modulo the group of symmetries, which is usually an infinite dimensional group) are an important theme in differential geometry: moduli spaces of flat connections, instantons or solutions of the Seiberg-Witten equations, complex structures, holomorphic curves, selfdual metrics, Einstein metrics... This chapter develops the local theory by focusing on two examples: flat or selfdual connections and Einstein metrics.

In more details, we will study the local structure of the space of flat unitary connections modulo gauge transformations on a manifold M. This is governed by a deformation complex, which is the De Rham complex associated to the flat connection ∇:

$$0 \to \Gamma(\mathfrak{u}(E)) \xrightarrow{d^\nabla} \Gamma(\Omega^1 \otimes \mathfrak{u}(E)) \xrightarrow{d^\nabla} \Gamma(\Omega^2 \otimes \mathfrak{u}(E)) \xrightarrow{d^\nabla} \cdots$$

If M is 4-dimensional, there is short version of the sequence

$$0 \to \Gamma(\mathfrak{u}(E)) \xrightarrow{d^\nabla} \Gamma(\Omega^1 \otimes \mathfrak{u}(E)) \xrightarrow{d^\nabla_-} \Gamma(\Omega^2_- \otimes \mathfrak{u}(E))$$

which governs the deformations of selfdual connections ∇ (satisfying $F(\nabla)_- = 0$). This moduli space plays an important role in gauge theory, but we develop here only a small part of the theory—the local theory. For more on the compactification of the moduli spaces and on the applications to 4-dimensional topology, see the book [**5**].

The second main example will be the moduli space of Einstein metrics, for which the book [**3**] is a good reference. The corresponding deformation complex is

$$0 \to \Gamma(TM) \xrightarrow{\delta^*} \Gamma(S^2T^*M) \xrightarrow{d\,\mathrm{Ric}\,-\lambda} \Gamma(S^2T^*M) \xrightarrow{\delta+\frac{1}{2}d\,\mathrm{Tr}} \Gamma(TM) \to 0.$$

Here the theory is less satisfactory, as the complex is obstructed as soon as the dimension of the moduli space is greater than 0, so the description of local deformations of Einstein metrics is an open problem in most cases. The main use of the complex is then to deduce local rigidity in the zero dimensional case.

2.1. Finite dimensional quotients

Let us first review quickly the finite dimensional theory. Let G be a Lie group acting on a manifold M. We want to make M/G a manifold. The properties expected from such a quotient are:

1. the projection map $\pi : M \to M/G$ is a smooth submersion;
2. (universal property): if a map $f : M \to N$ is G-invariant ($f(gx) = f(x)$), then it factors through π: there is a smooth map $\tilde{f} : M/G \to N$ such that $f = \tilde{f} \circ \pi$,

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & \nearrow_{\tilde{f}} & \\ M/G & & \end{array}$$

Such a quotient can be constructed if the action satisfies the two following conditions:

1. the action is proper, that is the graph map
$$\Phi : M \times G \longrightarrow M \times M$$
$$(x, g) \mapsto (x, gx)$$
is proper; an equivalent condition is that for any compact K in M, the set $\{g \in G, g(K) \cap K \neq \emptyset\}$ is compact;
2. the action is free, that is for any $x \in M$ the isotropy group at x,
$$G_x = \{g \in G, g \cdot x = x\},$$
is trivial.

The first condition implies that the quotient M/G is Hausdorff. Under this condition, one has the following general result.

***2.1.1. Theorem* (Slice theorem).** — *Let G act properly on the manifold M. For each $x \in M$ there exists a submanifold $S \subset M$ (the slice), containing x and diffeomorphic to a ball, such that*

1. *if $g \in G_x$ then $g(S) \subset S$;*
2. *if $g \in G$ and $g(S) \cap S \neq \emptyset$ then $g \in G_x$;*

3. *there exists a section $\sigma : G/G_x \to G$ defined in a neighbourhood U of the identity, such that the map $F : U \times S \to M$ defined by $F(u,s) = \sigma(u) \cdot s$ is a diffeomorphism onto a neighborhood of x.*

Note that if G_x is trivial, the second condition says that each orbit near x meets S in only one point, so $M/G \simeq S$ near x, and one recovers the structure of M/G. Also note that in this case, the theorem implies that G_y is trivial for y close to x.

In general, the theorem implies a topological equivalence $M/G \simeq S/G_x$ near x, but S/G_x is usually singular.

Proof of the theorem (exercise). — By properness of the action, G_x is compact so one can choose on M a G_x-invariant Riemannian metric. Then choose for S the image by the Riemannian exponential of (a small ball in) the orthogonal complement of $T_x(Gx)$.

The third condition is a direct application of the inverse function theorem.

The second condition is obtained by contradiction: suppose $y_k, z_k \in S$ converge to x and $z_k = g_k(y_k)$. Use the properness of the action to extract a limit $g \in G_x$ for (g_k), and then decompose $g^{-1}g_k$ on the product $G_x \cdot (\operatorname{im}\sigma)$ to prove that it belongs to G_x. $\square$

2.2. Hölder spaces

A useful tool in nonlinear analysis is Hölder spaces. This is a family of functional spaces $C^{k,\alpha}$ for $\alpha \in (0,1)$ which interpolate between the C^k and C^{k+1} spaces, but have the advantage over C^k spaces that they behave nicely with respect to elliptic operators.

For a function f in $\mathbb{R}^n$, we say that $f \in C^\alpha$ if $f \in C^0$ and

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < +\infty.$$

Then the C^α norm is defined as

$$\|f\|_{C^\alpha} = \sup |f| + \sup \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

It is clear that a C^1 function is C^α for all $\alpha < 1$.

The space of $C^{k,\alpha}$ functions is defined as the space of C^k functions f whose k-th derivatives $\nabla^k f \in C^\alpha$, and the $C^{k,\alpha}$ norm is defined as

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + \|\nabla^k f\|_{C^\alpha}.$$

The reason why Hölder spaces are nice for nonlinear analysis is:

2.2.1. Lemma. — *The spaces $C^{k,\alpha}$ are Banach algebras.*

Proof. — Let us do that only for C^α. If f and g are C^α functions, then

$$\frac{|f(x)g(x)-f(y)g(y)|}{|x-y|^\alpha}\leqslant |f(x)|\frac{|g(x)-g(y)|}{|x-y|^\alpha}+|g(y)|\frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

which immediately implies

$$\sup\frac{|(fg)(x)-(fg)(y)|}{|x-y|^\alpha}\leqslant \|f\|_{C^0}\|g\|_{C^\alpha}+\|g\|_{C^0}\|f\|_{C^\alpha}$$

which proves the lemma. □

2.2.2. Remarks. — 1) The Sobolev spaces H^s are algebras is s is large enough. More precisely, H^s is an algebra as soon as $s>n/2$, that is as soon as $H^s\subset C^0$ (exercise: prove it, using the continuous injection $H^s\subset L^p$ if $\frac{s}{n}\geqslant\frac12-\frac1p$ if $1<p<\infty$).

2) One can prove that there is a continuous inclusion $H^s\subset C^{k,\alpha}$ if $s\geqslant\frac{n}{2}+k+\alpha$ and $0<\alpha<1$. The inclusion is compact if $s>\frac{n}{2}+k+\alpha$.

As for Sobolev spaces (section 1.7), Hölder spaces can be generalized to sections of a bundle E on a compact manifold M. It is clear that a differential operator $P:\Gamma(E)\to\Gamma(F)$ of order d induces a continuous operator

$$P:C^{k+d,\alpha}(E)\longrightarrow C^{k,\alpha}(F). \tag{2.2.3}$$

The Hölder version of the elliptic estimate (lemma 1.7.9) is the Schauder estimate:

2.2.4. Lemma. — *Suppose $P:\Gamma(E)\to\Gamma(F)$ is an elliptic operator of order d. If on a coordinate ball B one has $Pu\in C^{k,\alpha}(F)$, then $u\in C^{k+d,\alpha}(E)$ and one has the estimate*

$$\|u\|_{C^{k+d,\alpha}(B_{1/2},E)}\leqslant C\big(\|Pu\|_{C^{k,\alpha}(B,F)}+\|u\|_{C^0(B,E)}\big).$$

The operator P acting on Hölder spaces (2.2.3) again satisfies

$$C^{k,\alpha}(F)=\ker(P^*)\oplus\operatorname{im}(P), \tag{2.2.5}$$

as follows immediately from the same statement (1.8.2) for Sobolev spaces and from lemma 2.2.4.

2.3. Spaces of connections and gauge group

Fix a flat connection A on a rank r bundle E over M. The holonomy representation gives a representation

$$\rho : \pi_1(M) \longrightarrow GL_r(\mathbb{C}),$$

which sends a loop c starting from x to the parallel transport along c, which is a transformation of $E_x \simeq \mathbb{C}$. If we change the base point x, we obtain the same representation, up to conjugation by an element of G.

If g is a gauge transformation of E (that is a section of the bundle $GL(E)$), then it acts on A by

$$\nabla_{g(A)} = g \circ \nabla_A \circ g^{-1}.$$

Then the holonomy of A along a loop starting from x is conjugated by g_x. So we obtain a map

$$\{\text{flat connections}\}/\text{gauge} \longrightarrow \{\text{rep. } \pi_1(M) \to GL(r,\mathbb{C})\}/_{GL(r,\mathbb{C})} \tag{2.3.1}$$

and it is well-known that this is map is a bijection. An inverse map is constructed by associating to the representation ρ the quotient bundle $E = \tilde{M} \times_\rho \mathbb{C}^r$, obtained by taking the quotient of the flat trivial bundle $M \times \mathbb{C}^r$ by the equivalence relation $(x, e) \sim (xg, \rho(g)^{-1}e)$.

If one restricts to unitary representations $\rho : \pi_1(M) \to U(r)$, then one obtains unitary bundles E with unitary flat connections. This is the space that we are now going to study.

Fix the unitary bundle (E, h) and consider

$$\mathscr{A} = \{C^{1,\alpha} \text{ unitary connections on } E\}$$

and the gauge group

$$\mathscr{G} = \{C^{2,\alpha} \text{ unitary gauge transformations of } E\}.$$

This definition means that a connection $A \in \mathscr{A}$ if the coefficients of its connection 1-form are of regularity $C^{1,\alpha}$ in any smooth local trivialization of E. If one fixes a smooth connection $A_0 \in \mathscr{A}$, then one can describe $\mathscr{A}$ as the affine space

$$\mathscr{A} = \{A_0 + a, \quad a \in C^{1,\alpha}(M, \Omega^1 \otimes \mathfrak{u}(E))\}.$$

The curvature of $A = A_0 + a$ can be written as

$$F(A) = F(A_0) + d_{A_0}a + a \wedge a \in C^\alpha(M, \Omega^2 \otimes \mathfrak{u}(E)).$$

It is clear that the linear map $a \to d_{A_0}a$ is smooth $C^{1,\alpha} \to C^{\alpha}$, and the bilinear map $a \to a \wedge a$ is also smooth as $C^{1,\alpha}$ is an algebra, so we get the first part of the following lemma.

2.3.2. Lemma. — *1) The curvature map $F : \mathscr{A} \to C^{\alpha}(M, \Omega^2 \otimes \mathfrak{u}(E))$ is a smooth map, whose differential at a point A is $a \mapsto d_A a$.*

2) The group $\mathscr{G}$ is a Banach Lie group, with Lie algebra $C^{2,\alpha}(\Omega^2 \otimes \mathfrak{u}(E))$, it acts smoothly on $\mathscr{A}$, and the differential of the action at the identity is $u \mapsto -d_A u$.

Proof. — Locally an element $g \in \mathscr{G}$ is a $C^{2,\alpha}$ application g from an open set to U_r. Since $\exp : \mathfrak{u}_r \to U_r$ is a local diffeomorphism near the origin, if g is C^0 close to the identity, then $g = \exp(u)$ with u of class $C^{2,\alpha}$. This furnishes a chart from a neighbourhood of the identity in $\mathscr{G}$ to $C^{2,\alpha}(\mathfrak{u}(E))$, which gives the structure of a Banach manifold. In this chart, because $C^{2,\alpha}$ is an algebra, the group operations are smooth (use the Campbell-Hausdorff formula to prove that the composition is smooth). In the same way one obtains a chart near any $g_0 \in \mathscr{A}$ by parametrizing nearby transformations by $g = g_0 e^u$ with $u \in C^{2,\alpha}(\mathfrak{u}(E))$.

The action of an element of $\mathscr{G}$ on a connection $A = A_0 + a$ is by

$$g(A) = A - d_A g g^{-1} = A_0 + gag^{-1} - d_{A_0} g g^{-1}.$$

This involves only one derivative of g (so the result is $C^{1,\alpha}$) and algebraic operations, so the action is smooth. □

2.4. The action of the gauge group

We now study the action of the gauge group and we try to mimick the finite dimensional case explained in section 2.1 to construct $\mathscr{A}/\mathscr{G}$. First we must see when the action is free. First observe that the constant homotheties (this is a circle S^1 in the center of $\mathscr{G}$) act always trivially on $\mathscr{A}$, so it is enough to look at the action of $\mathscr{G}/S^1$. If $A \in \mathscr{A}$, note

$$H^0_A = \ker d_A, \quad d_A : C^{2,\alpha}(\mathfrak{u}(E)) \to C^{1,\alpha}(\Omega^1 \otimes \mathfrak{u}(E)).$$

2.4.1. Lemma. — *Let $A \in \mathscr{G}_A$, then the stabilizer of A is reduced to the homotheties if and only if H^0_A is reduced to $\mathbb{R}$.*

Proof. — If $u \in C^{2,\alpha}(\mathfrak{u}(E))$ satisfies $d_A u = 0$, then $g = \exp u \in \mathscr{G}$ also satisfies $d_A g = 0$, therefore

$$g(A) = A - d_A g g^{-1} = A.$$

Conversely, if $g(A) = A$ then $d_A g = 0$, so g is parallel and its eigenvalues are constant. If g is not an homothety, then it must have at least two distinct eigenvalues, so we get an eigendecomposition $E = \oplus_1^k E_j$ of E, and since g is parallel, A also decomposes as $A = \oplus_1^k A_j$, where A_j is a unitary connection on E_i. Consider a transformation $u = \oplus_1^k i x_j$ with $x_j \in \mathbb{R}$: then $u \in C^{2,\alpha}(\mathfrak{u}(E))$ and $d_A u = 0$, so $H_A^0 \neq 0$. □

2.4.2. Remark. — Actually the proof of the lemma contains more:

1. the Lie algebra of the isotropy group $\mathscr{G}_A$ of A is H_A^0;
2. the connections A for which $\mathscr{G}_A$ is not reduced to S^1 are the reducible connections (admitting a direct sum decomposition).

Let $A \in \mathscr{A}$, then by lemma 2.3.2, the tangent space to the $\mathscr{G}$ orbit of A is $\operatorname{im}(d_A) \subset C^{1,\alpha}(\Omega^1 \otimes \mathfrak{u}(E))$. But

$$C^{1,\alpha}(\Omega^1 \otimes \mathfrak{u}(E)) = \operatorname{im}(d_A) \oplus \ker(d_A^*).$$

2.4.3. Exercise. — Check that this equality is indeed true, even for A with non smooth coefficients ($C^{1,\alpha}$).

Therefore we have a candidate for a slice to the orbit of A,

$$S_{A,\epsilon} = A + (\ker d_A^*) \cap B_\epsilon,$$

where B_ϵ is the ball of radius ϵ. Choose a supplementary subspace U for H_A^0:

$$C^{2,\alpha}(\mathfrak{u}(E)) = H_A^0 \oplus U.$$

2.4.4. Lemma. — *If ϵ is small enough then*

1. *$g \in \mathscr{G}_A$ implies $g(S_{A,\epsilon}) \subset S_{A,\epsilon}$;*
2. *if $g \in \mathscr{G}$ and $g(S_{A,\epsilon}) \cap S_{A,\epsilon} \neq \emptyset$, then $g \in \mathscr{G}_A$;*
3. *the map $F : U \times S_{A,\epsilon} \to \mathscr{A}$ given by $(u, A) \mapsto e^u(A)$ is diffeomorphism onto a neighbourhood of A.*

This statement is exactly analogous to the finite dimensional slice theorem 2.1.1. It follows that local structure of $\mathscr{A}/\mathscr{G}$ near A is that of $S_{A,\epsilon}/\mathscr{G}_A$. In particular:

2.4.5. Corollary. — *If A is irreducible, then near A the quotient $\mathscr{A}/\mathscr{G}$ is a manifold, with tangent space*

$$T_A(\mathscr{A}/\mathscr{G}) = \ker(d_A^*) \subset C^{1,\alpha}(\Omega^1 \otimes \mathfrak{u}(E)).$$

Proof of the lemma. — The first statement is left as an exercise. The third statement follows immediately from the inverse function theorem, since the differential of F at $(0, A)$ is

$$d_{(0,A)}F(u, a) = d_A u + a$$

which is an isomorphism $C^{2,\alpha}(\mathfrak{u}(E)) \times \ker(d_A^*) \to C^{1,\alpha}(\mathfrak{u}(E))$ by the definition of $S_{A,\epsilon}$.

There remains to prove the second statement. By contradiction, suppose that there exist two sequences (A_i) and (B_i) of connections in $S_{A,1/i}$, and gauge transformations $g_i \in \mathscr{G}$ such that $g(A_i) = B_i$. Write $A_i = A + a_i$, $B_i = A + b_i$ with $a_i, b_i \to 0$ in $C^{1,\alpha}$. Then

$$b_i = g a_i g^{-1} - d_A g_i g_i^{-1},$$

which we can rewrite

$$d_A g_i = g_i a_i - b_i g_i. \tag{2.4.6}$$

Since the g_i are unitary transformations, they are bounded. We will now use an iteration on the equation (2.4.6) to prove that g_i remains actually bounded in $C^{2,\alpha}$. The RHS of (2.4.6) remains bounded, so $d_A g_i$ remains bounded in C^0 and g_i remains bounded in C^1. In the same way, we deduce that g_i is bounded in C^2, and finally that the RHS of (2.4.6) goes to zero in $C^{1,\alpha}$. It follows that g_i is bounded in $C^{2,\alpha}$ with $\|d_A g_i\|_{C^{1,\alpha}} \to 0$, so we can extract a weak limit g in $C^{2,\alpha}$ such that $d_A g = 0$ (therefore $g \in \mathscr{G}_A$). Because $\|d_A(g - g_i)\|_{C^{1,\alpha}} \to 0$ the convergence of the (g_i) to g is actually strong in $C^{2,\alpha}$. The end of the proof is similar to the finite dimensional case. □

2.5. The moduli space of flat unitary connections

Suppose that A is a unitary connection on the Hermitian bundle $E \to M$. We consider the associated De Rham complex:

$$0 \to \Gamma(\mathfrak{u}(E)) \xrightarrow{d_A} \Gamma(\Omega^1 \otimes \mathfrak{u}(E)) \xrightarrow{d_A} \Gamma(\Omega^1 \otimes \mathfrak{u}(E)) \xrightarrow{d_A} \cdots \tag{2.5.1}$$

and we note H_A^i its cohomology groups. This is a complex precisely when A is flat. We have already seen that the first operator in the complex is the infinitesimal action of the gauge group on the space of connections, and therefore H_A^0 measures the reducibility of A. The second operator in the complex is the differential of the curvature (lemma 2.3.2), so its kernel consists of the infinitesimal solutions of the equation $F(A) = 0$. Dividing by the gauge group, we deduce that infinitesimal flat connections modulo the gauge group are parametrized by H^1_A.

Let us define the moduli space

$$\mathscr{M} = \{A \in \mathscr{A}, F_A = 0\}/\mathscr{G}.$$

Remark that the moduli space does not depend on the precise regularity we have chosen for the connections. Indeed, if the $C^{1,\alpha}$ connection A is flat, then it is gauge equivalent to the smooth connection $M \times_\rho \mathbb{C}^r$, where ρ is its holonomy representation.

Let us analyze the equation $F = 0$ near a flat connection A. Take a connection $B = A + a$, decompose a into its trace part and trace free part: the trace part is just a one form with values in $i\mathbb{R}$ (the homotheties in $\mathfrak{u}(E)$), and we get

$$a = i\beta + a^o, \quad \beta \in \Gamma(\Omega^1), \quad a^o \in \Gamma(\Omega^1 \otimes \mathfrak{su}(E)). \tag{2.5.2}$$

Then $F(A+a) = d_A a + a \wedge a = d_A + \frac{1}{2}[a,a]$, therefore β does not contribute in the bracket, and we get

$$F_{A+a} = id\beta + d_A a^o + a^o \wedge a^o, \tag{2.5.3}$$

where the only trace term is the first. So the equation $F(A+a) = 0$ decouples into

$$d\beta = 0, \qquad F^o_{A+a} = d_A a^o + a^o \wedge a^o = 0. \tag{2.5.4}$$

Similarly the gauge condition $d^*_A a = 0$ decouples into

$$d^*\beta = 0, \qquad d^*_A a^o = 0. \tag{2.5.5}$$

This is seen also on the deformation complex (2.5.1) which decouples into the trace part, giving the usual De Rham complex, and the trace free part, giving the complex

$$0 \to \Gamma(\mathfrak{su}(E)) \xrightarrow{d_A} \Gamma(\Omega^1 \otimes \mathfrak{su}(E)) \xrightarrow{d_A} \Gamma(\Omega^1 \otimes \mathfrak{su}(E)) \xrightarrow{d_A} \cdots. \tag{2.5.6}$$

We will note $(H^i_A)^o$ the cohomology of this complex.

The trace part of the problem (2.5.4)–(2.5.5) is just the linear problem $d\beta = d^*\beta = 0$, so the solutions are harmonic 1-forms. Geometrically they correspond to the flat unitary connections induced on $\det E$. The space of solutions is $H^1(M,\mathbb{R})$, at least locally[(1)]. The problem on the trace free part is non linear, and the answer is the following result.

[(1)] Actually the space of flat line bundles is $H^1(M,\mathbb{R})/H^1(M,\mathbb{Z})$, because $H^1(M,\mathbb{Z})$ parametrizes the connected components of the gauge group of maps $g : M \to U(1)$.

2.5.7. Theorem. — *If A is irreducible and $(H_A^2)^o$ of the complex (2.5.6) vanishes, then $\mathscr{M}$ is a manifold near A, with tangent space at A equal to H_A^1.*

Note that $H_A^1 = H^1(M,\mathbb{R}) \oplus (H_A^1)^o$. For a statement when $(H_A^2)^o$ does not vanish, see exercise 2.5.11.

Proof. — We have to consider the zero set $\{F_{A+a} = 0\}$ modulo the gauge group near a given connection $A \in \mathscr{A}$. By lemma 2.4.4, this is equivalent to restrict the equation to the slice $S_{A,\epsilon}$. Also we have seen that the equation $F_A = 0$ decomposes into its trace and trace free parts, and that the solutions on the trace part are parametrized by $H^1(M,\mathbb{R})$. There remains to understand the trace free part, so the whole proof we will suppose that a is trace free:

$$a \in C^{1,\alpha}(\Omega^1 \otimes \mathfrak{su}(E)).$$

Let us first consider the case of Riemann surfaces. Then the De Rham complex (2.5.1) stops at degree 2. The condition $(H_A^2)^o = 0$ then means that $d_A : C^{1,\alpha}(\mathfrak{su}(E)) \to C^\alpha(\mathfrak{su}(E))$ is surjective. It follows that the curvature map

$$F^o : S^o_{A,\epsilon} \to C^\alpha(\mathfrak{su}(E))$$

is a submersion, so the zero set is a manifold, whose tangent space at A is the kernel of the differential of F^o, therefore is equal to

$$\{a \in C^{1,\alpha}(\Omega^1 \otimes \mathfrak{su}(E)), d_A^* a = 0, d_A a = 0\} = (H_A^1)^o.$$

Adding the trace part gives the statement of the theorem.

The general case is more difficult. The curvature is no more a submersion, but there will be a replacement: the third operator in the complex (2.5.1) gives a constraint on F_A: the Bianchi identity $d_A F_A = 0$. We will use it to construct, starting from an infinitesimal solution $a_1 \in (H_A^1)^o$, a true solution $A + a \in \mathscr{A}$ of the equation $F_{A+a} = 0$. This will prove that $(H_A^1)^o$ indeed parametrizes $\mathscr{M}$ near A (or more exactly the part with fixed induced connection on $\det E$).

Let us now proceed with the construction. Consider

$$d_A : C^{1,\alpha}(\Omega^1 \otimes \mathfrak{su}(E)) \to C^\alpha(\Omega^2 \otimes \mathfrak{su}(E)).$$

Inside $C^{1,\alpha}(\Omega^1 \otimes \mathfrak{su}(E))$, decompose

$$\ker d_A^* = H_A^1 \oplus W. \tag{2.5.8}$$

Since $H_A^2 = 0$, one has $\operatorname{im} d_A = \ker d_A$ inside $C^\alpha(\Omega^2 \otimes \mathfrak{su}(E))$, and there exists a continuous right inverse

$$G : \ker d_A \to W$$

satisfying $d_A G\gamma = \gamma$ for any $\gamma \in C^\alpha(\Omega^2 \otimes \mathfrak{su}(E))$ such that $d_A\gamma = 0$.

Now start from $a_1 \in (H^1_A)^o$, represented by a harmonic 1-form, and look for a solution

$$A + a = A + a_1 + \Phi(a_1), \quad \Phi(a_1) \in W.$$

The curvature of our first order approximation is

$$F_{A+a_1} = a_1 \wedge a_1$$

which is of order 2. We try to find a correction $a_2 \in W$, such that $F_{A+a_1+a_2} = 0$ up to order 3. The equation gives $da_2 = -a_1 \wedge a_1$ at order 2, so we choose the second order term $a_2 = -G(a_1 \wedge a_1)$. This is possible because $d_A a_1 = 0$ implies $d_A(a_1 \wedge a_1) = 0$. Now $A + a_1 + a_2$ is a better solution of the problem:

$$F_{A+a_1+a_2} = a_1 \wedge a_2 + a_2 \wedge a_1 + a_2 \wedge a_2$$

is of order 3.

The general pattern is now clear. We solve inductively by terms $A + a_1 + a_2 + \cdots + a_k$, taking at each step

$$a_k = -G\big(F^{(k)}_{A+a_1+\cdots+a_{k-1}}\big),$$

where the supscript (k) denotes terms of order k exactly, that is terms involving wedge products $a_{i_1} \wedge \cdots \wedge a_{i_p}$ with $i_1 + \cdots + i_p = k$. This is possible if

$$d_A(F^{(k)}_{A+a_1+\cdots+a_{k-1}}) = 0,$$

which follows from $(d_{A+a_1+\cdots+a_{k-1}} F_{A+a_1+\cdots+a_{k-1}})^{(k)} = 0$. (Indeed a_{k-1} has been constructed so that $F_{A+a_1+\cdots+a_{k-1}}$ is at least of order k). The convergence of the series

$$\Phi(a_1) = \sum_{i \geqslant 2} a_i$$

in $C^{1,\alpha}$ for small a_1 is left to the reader. The reader will also check that if $A + a \in S_{A,\epsilon}$ is a solution with small enough ϵ, then the projection of a on $(H^1_A)^o$ using the decomposition (2.5.8) completely determines a, so we have completely parameterized $\mathscr{M}$ near A by the graph of Φ. □

2.5.9. Remark. — From the slice theorem, it is immediate that if A is not irreducible, than a local model for $\mathscr{M}$ near A is $H^1_A/\mathscr{G}_A$.

2.5.10. Remark. — The wedge product $a \otimes b \mapsto a \wedge b$ induces a map $H^1_A \otimes H^1_A \to H^2_A$, and this contains the first obstruction to solve the problem in the proof of the theorem. If this map vanishes and M is a Kähler manifold, then $\mathscr{M}$ is still smooth at A (Goldman-Millson).

***2.5.11. Exercise* (Kuranishi's model).** — The exercise gives another proof of the theorem, and a description of the moduli space in the case $(H^2_A)^o$ does not vanish. The result is that everything reduces to zero sets of maps between finite dimensional spaces. We can restrict to the nonabelian part of the equation, so we consider only connections $B = A + a$ with $\operatorname{Tr} a = 0$, and the moduli space $\mathscr{M}^o = \{F_B = 0\}/\mathscr{G}^o_A$, where $\mathscr{G}^o = C^{2,\alpha}(SU(E))$. Locally the moduli space $\mathscr{M}$ is the product of $\mathscr{M}^o$ with $H^1(M, \mathbb{R})$.

1) Decompose $\Gamma(\Omega^2 \otimes \mathfrak{su}(E)) = \ker(d_A) \oplus \operatorname{im}(d^*_A)$, and note π the orthogonal projection on $\ker(d_A)$. Prove that for $B \in \mathscr{A}$ close to A, the projection $\pi : \ker(d_B) \to \ker(d_A)$ is injective (prove an estimate $\|d_B f\| \geqslant c\|f\|$ for B close to A and $f \in \operatorname{im} d^*_A$). Maybe here you need $C^{k,\alpha}$ connections and $C^{k+1,\alpha}$ gauge transformations with $k > 1$.

2) Deduce that for B close to A, one has $F_B = 0$ if and only if $\pi F_B = 0$.

3) Still inside $\mathfrak{su}(E)$-valued 2-forms, decompose $\ker(d_A) = (H^2_A)^o \oplus \operatorname{im}(d_A)$, and note π_0 the orthogonal projection on $\operatorname{im}(d_A)$. If A is irreducible, prove that $N = \{\pi_0 F_B = 0\}/\mathscr{G}^o$ is smooth near A, with tangent space $(H^1_A)^o$ at the point A (otherwise locally $N \approx (H^1_A)^o/\mathscr{G}^o_A$). Deduce another proof of the theorem when $(H^2_A)^o = 0$.

4) Suppose now that $(H^2_A)^o \neq 0$. Prove that $\{\pi F_B = 0\}/\mathscr{G}^o$ is the zero set of a map $f : N \to (H^2_A)^o$. Deduce that near A, the moduli space is given as the zero set of a smooth map $f : (H^1_A)^o \to (H^2_A)^o$. If A is reducible, prove that f is $\mathscr{G}^o_A$-equivariant, and a local model for $\mathscr{M}^o$ is given by $f^{-1}(0)/\mathscr{G}^o_A$.

***2.5.12. Remark*.** — On a Riemann surface, by Poincaré duality, $(H^0_A)^o = 0$ implies $(H^2_A)^o = 0$. Therefore $\mathscr{M}$ is smooth at the irreducible points. An application of the index theorem of Atiyah-Singer shows that its dimension is $(2g-2)(r^2-1)+2g$. The identification of the tangent space with the space H^1_A enables to define geometric structures on $\mathscr{M}$: the alternate 2-form $\int_M \operatorname{Tr}(a \wedge b)$ on H^1_A turns out to be a symplectic form[(2)] (construction of Atiyah-Bott); the L^2 norm of harmonic forms gives a Riemannian metric on $\mathscr{M}$, related to the symplectic structure by a complex structure on $\mathscr{M}$ which in turn has its origin in the identification of $\mathscr{M}$ with a space of holomorphic bundles on the Riemann surface (theorem of Narasimhan-Seshadri). The result is a Kähler metric on $\mathscr{M}$.

[(2)] The form is closed because it is obviously closed when seen on $\mathscr{A}$, as it has constant coefficients; then $\mathscr{M}$ is obtained by a process called "symplectic reduction" under the action of $\mathscr{G}$; this implies that there is an induced symplectic form on $\mathscr{M}$.

2.6. The moduli space of instantons

On a four dimensional manifold (M^4, g), we have the decomposition of 2-forms into selfdual and antiselfdual forms:

$$\Omega^2 = \Omega_+ \oplus \Omega_-.$$

In this dimension we can study the selfdual connections, that is unitary connections such that F_A is selfdual. These connections are often called *instantons*.

If A is a unitary connection on a bundle E, then there is a sequence of operators,

$$0 \to \Gamma(\mathfrak{u}(E)) \xrightarrow{d_A} \Gamma(\Omega^1 \otimes \mathfrak{u}(E)) \xrightarrow{d_A^-} \Gamma(\Omega_-^2 \otimes \mathfrak{u}(E)) \to 0. \tag{2.6.1}$$

This is the half De Rham complex considered in (1.4.15), but with values in $\mathfrak{u}(E)$. The composite $d_A^- d_A$ equals F_A^-, so if A is an instanton then the sequence (2.6.1) is a complex. Therefore we have well defined cohomology groups H_A^0, H_A^1 and H_A^-. We have also the trace free part $(H_A^-)^o$.

2.6.2. Theorem. — *If A is irreducible and $(H_A^-)^o = 0$, then the moduli space of instantons on E is smooth at A, with tangent space H_A^1.*

In general, a local model for the moduli space is given by the quotient by $\mathscr{G}_A$ of the zero set of a $\mathscr{G}_A$-equivariant map $H_A^1 \to (H_A^-)^o$.

The proof is the same than that of theorem 2.5.7 and exercice 2.5.11.

Usually one considers only $SU(r)$ instantons, which avoids the problem of distinguishing the abelian part of the equation.

As for the case of flat connections, the moduli space does not depend on the precise regularity of the connections which has been chosen for the constructions (in this case it is more usual to choose Sobolev spaces rather than Hölder spaces), because one can show that any instanton is gauge equivalent to a smooth instanton[(3)].

This result is just the beginning of gauge theory. The moduli spaces of the instanton equation, or other gauge equations like Seiberg-Witten equations, are used to build differential invariants of M^4. This requires killing the obstruction space H_A^2 by a generic deformation of the equation, and understanding a compactification of $\mathscr{M}$. Then the topology of $\mathscr{M} \subset \mathscr{A}/\mathscr{G}$ gives

[(3)] The proof is local, and consists in a local version of the gauge fixing $d_A^* a = 0$: one can prove that on any sufficiently small ball, there exists a trivialisation in which $A = d + a$ and $d^* a = 0$; then the two equations $d^* a = 0$ and $(da + a \wedge a)_- = 0$ form a nonlinear elliptic system from which one can deduce the regularity of a, see [**5**] for more details.

rise to the invariants. For example, in the case of Seiberg-Witten equation, the dimension of the moduli space is often zero and the invariant is just the number of points in $\mathscr{M}$. For more on gauge theory see the book [**5**]. For an introduction to Seiberg-Witten theory see [**7**].

2.6.3. Exercise **(Moduli space near a reducible connection)**
Let (M^4, g) be a compact oriented Riemannian manifold with $b_1(M) = b_2^-(M) = 0$. Let E be a SU_2-fibre bundle on M, so E is a Hermitian rank 2 bundle with trivial determinant: $\det E = \Lambda^2 E = \mathbb{C}$. One gives the second Chern class $c_2(E) = -1$.

Let $\mathscr{M}$ be the moduli space of SU_2-instantons on E. At a connection A, the local structure of $\mathscr{M}$ is therefore governed by the deformation complex

$$0 \to \Gamma(\mathfrak{su}(E)) \to \Gamma(\Omega^1 \otimes \mathfrak{su}(E)) \to \Gamma(\Omega^2_- \otimes \mathfrak{su}(E)) \to 0.$$

It will be admitted that under the previous hypothesis, the index theorem gives $\dim H^0 - \dim H^1 + \dim H^2 = -5$.

Let $\mathscr{G}$ be the gauge group of SU_2-transformations of E, so the elements of $\mathscr{G}$ are the sections of $SU(E)$. Let $A \in \mathscr{M}$ be a reducible connection, therefore $E = L \oplus L^*$ (so that $\det E = L \otimes L^* = \mathbb{C}$), and the connection A decomposes as $A = A_L \oplus (A_L)^*$, where A_L is a connection on L and $(A_L)^*$ the connection induced by A on L^*. Topologically L must satisfy $c_1(L)^2 = -c_2(E) = 1$.

1) Prove that the stabilizer $\mathscr{G}_A$ of A is the group $U(1)$, seen inside $\mathscr{G}_A$ as the matrix $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ in the decomposition $E = L \oplus L^*$. Deduce $\dim H^0_A = 1$.

2) Prove that the bundle $\mathfrak{su}(E)$ can be identified with the bundle $\mathbb{R} \oplus L \otimes L$, where $(v, s) \in \mathbb{R} \oplus L \otimes L$ is identified to the antiselfadjoint endomorphism $\begin{pmatrix} iv & s \\ -s^* & -iv \end{pmatrix}$.

3) One looks at the action of $e^{i\theta} \in U(1) \subset \mathscr{G}_A$ on $\mathfrak{su}(E) = \mathbb{R} \oplus L \otimes L$. Prove that $e^{i\theta}$ acts trivially on the $\mathbb{R}$ part and by $e^{2i\theta}$ on the $L \otimes L$ part.

4) Prove that H^1_A of the deformation complex decomposes into $H^1_A = H^1(\mathbb{R}) \oplus H^1(L \otimes L)$, and $H^1(\mathbb{R}) = 0$.

5) Suppose $H^2_A = 0$. Prove that topologically the moduli space $\mathscr{M}$ near A is a (neighborhood of 0 in the) quotient $\mathbb{C}^3/U(1)$, that is $\mathscr{M} - \{A\}$ is near A a cone $\mathbb{R}_+^* \times \mathbb{C}P^2$.

2.7. The moduli space of complex flat connections

In this section, we will see what happens when we consider arbitrary flat connections, rather than just flat unitary connections. We still fix a Hermitian bundle (E, h), but the connections will be no more unitary, and the metric h will be used only as an auxiliary for the Hodge theory of the deformation complex.

Given a flat connection A, the deformation complex for all flat connections is now

$$0 \to \Gamma(\Omega^0 \otimes \operatorname{End} E) \overset{d_A}{\to} \Gamma(\Omega^1 \otimes \operatorname{End} E) \overset{d_A}{\to} \Gamma(\Omega^2 \otimes \operatorname{End} E) \overset{d_A}{\to} \cdots \tag{2.7.1}$$

We consider the space $\mathscr{A}^c$ of all flat complex connections on E with regularity $C^{1,\alpha}$, the complexified gauge group $\mathscr{G}^c$ consisting of all sections of $GL(E)$ of regularity $C^{2,\alpha}$. Of course again constant homotheties act trivially on $\mathscr{A}^c$, so we really want to look at the action of $\mathscr{G}^c/\mathbb{C}^*$ on $\mathscr{A}^c$.

The theory is completely parallel to that in the case of unitary connections, except that lemma 2.4.4 and its corollary 2.4.5 are no more true. To get a free action on $\mathscr{A}^c$, instead of restricting to irreducible connections, we shall need to restrict to *semisimple connections*, that is flat connections A on E such that there is no nontrivial A-parallel subbundle $F \subset E$.

For such a semisimple connection A, an element $g \in \mathscr{G}^c_A$ satisfies $d_A g = 0$. Then the generalized eigenspaces of g are parallel, which by semisimplicity implies that there is only one: so g has only one constant eigenvalue. Up to composing by a constant homothety, g is therefore unipotent, and again $\ker(g-1)$ is parallel with respect to A: since it is not trivial, it must be the whole of E so $g = 1$.

So we have proved that for a semisimple connection, the stabilizer of A is reduced to $\mathbb{C}^*$, which acts trivially on the whole of $\mathscr{A}^c$. Now the only thing to check is the second statement in the slice lemma 2.4.4. Suppose that we have sequences a_i and b_i going to 0 in $C^{1,\alpha}(\Omega^1 \otimes \operatorname{End} E)$, such that, as in (2.4.6),

$$d_A g_i = g_i a_i - b_i g_i. \tag{2.7.2}$$

Up to composing g_i with a homothety, we can suppose that $\sup_M |g_i| = 1$. Then we can extract exactly as in the proof of lemma 2.4.4 a subsequence $g_i \to g$ in $C^{2,\alpha}$ such that $d_A g = 0$ and $\sup |g| = 0$. It follows that g is a nonzero homothety, and the end of the proof of lemma 2.4.4 remains valid.

Note that here we do not treat the case when the stabilizer is not reduced to homotheties: the proof of the slice theorem in this case fails because the limit g satisfies $d_A g = 0$ but it could be non invertible. This cannot happen for the compact group $U(r)$.

Finally we obtain the analogous result to theorem 2.5.7 and exercice 2.5.11:

2.7.3. Theorem. — *If the flat connection A is semisimple, and the cohomology group H^2_A of the deformation complex (2.7.1) vanishes, then the moduli space of complex flat connections is smooth at the point A, with tangent space H^1_A.*

If $H^2_A \neq 0$, then a local model for the moduli space is given as the zero set of a map $H^1_A \to H^2_A$.

Here all the objects are complex, so it turns out that the moduli space is a complex manifold. The symplectic form alluded to in remark 2.5.12 becomes a holomorphic symplectic form. There is also a special Riemannian metric on $\mathscr{M}$ (it is hyperKähler), its construction requires the theory of harmonic metrics on flat bundles.

2.8. The diffeomorphism group

We now start to study a different problem: the action of diffeomorphisms of a manifold on the space of Riemannian metrics, and the Einstein equation.

If M and N are two compact manifolds, then the space of $C^{k,\alpha}$ maps from M to N is a Banach manifold, whose tangent space at a map $f \in C^{k,\alpha}(M,N)$ is $C^{k,\alpha}(f^*TN)$. Indeed, choose a Riemannian metric on N, then one can parametrize the maps g which are C^0 close to f by

$$g(x) = \exp_{f(x)}(X(x)), \quad X(x) \in T_{f(x)}N.$$

The vector $X(x)$ is a section on M of the bundle f^*TN, and $C^{k,\alpha}$ if and only if g is $C^{k,\alpha}$. This gives a chart $C^{k,\alpha}(M,N) \to C^{k,\alpha}(f^*TN)$. If we have two maps f_1 and f_2, there is a smooth transition function between the two charts, given by $X(x) \mapsto Y(x) = \exp^{-1}_{f_2(x)} \exp_{f_1(x)}(X(x))$.

If $M = N$, we can ask the map f to be a diffeomorphism, and we obtain the diffeomorphism group $\mathscr{D}^{k,\alpha}$ of diffeomorphisms of M of regularity $C^{k,\alpha}$. From the description of the tangent space above, we obtain that its Lie algebra is

$$T_1\mathscr{D}^{k,\alpha} = C^{k,\alpha}(TM),$$

that is vector fields on M of regularity $C^{k,\alpha}$. Here we see a first problem: the bracket of two such vector fields is only $C^{k-1,\alpha}$, so that $\mathscr{D}^{k,\alpha}$ cannot be a Lie group. This comes from the fact that the composition is continuous, but not smooth in $\mathscr{D}^{k,\alpha}$, so $\mathscr{D}^{k,\alpha}$ is a smooth manifold, but only a topological group.

2.8.1. Lemma. — *If $\phi \in \mathscr{D}^{k,\alpha}(M)$, then the right translation by ϕ is smooth; if ϕ is smooth then the left translation by ϕ is smooth.*

Proof. — The right translation by ϕ sends $\psi(x) = \exp_x(X_x)$ to $\psi \circ \phi(x) = \exp_{\phi(x)}(X_{\phi(x)})$, so in the corresponding charts we obtain the map $X \to X \circ \phi$ which is smooth since it is linear.

The left translation by ϕ send $\psi(x) = \exp_x(X_x)$ to $\phi \circ \psi(x) = \phi(\exp_x X(x)) = \exp^{\phi_* g}_{\phi(x)}(\phi_* X(x))$, where in the last term the exponential of the metric $\phi_* g$ is used. If ϕ is smooth then $\phi_* g$ is smooth and can be used to build charts for $\mathscr{D}^{k,\alpha}$: in the corresponding charts the left translation is $X \mapsto \phi_* X$ which is again smooth because it is a continuous linear map. □

Note that it is clear that if ϕ is only $C^{k,\alpha}$ then $\phi_* X$ is $C^{k-1,\alpha}$ so the above map $X \mapsto \phi_* X$ is not well defined in $C^{k,\alpha}$.

The stabilizer of the action of the diffeomorphisms on a metric g is the group of isometries of g: it is a (finite dimensional) Lie group [**6**], whose Lie algebra is the space of Killing vector fields: in the kernel of the symmetrization of the covariant derivative $\delta^* : \Gamma(TM) \to \Gamma(S^2 T^* M)$. Therefore $\mathscr{D}^{k,\alpha} / \operatorname{Isom}(g)$ is a smooth manifold, with tangent space at the identity equal to $C^{k,\alpha}(TM) / \ker \delta^*$.

2.9. Action of diffeomorphisms on metrics

The space of Riemannian metrics on a compact manifold M^n is an open set in the vector space of all symmetric 2-tensors, so it is clearly a manifold. We will consider the space $\mathscr{M}et^{k,\alpha}$ of Riemannian metrics of regularity $C^{k,\alpha}$.

2.9.1. Lemma. — *The action of diffeomorphisms on metrics by $(\phi, g) \mapsto \phi^* g$ is continuous $\mathscr{D}^{k+1,\alpha} \times \mathscr{M}et^{k,\alpha} \to \mathscr{M}et^{k,\alpha}$.*

Proof. — In local coordinates,

$$(\phi^* g)_{kl} = g_{ij}(\phi(x)) \frac{\partial \phi^i}{\partial x^k}(x) \frac{\partial \phi^j}{\partial x^l}(x) \tag{2.9.2}$$

so the statement of the lemma is clear. □

Observe that the action is not smooth: indeed, in formula (2.9.2), if we differentiate with respect to ϕ, then we obtain terms $d_{\phi(x)} g_{ij}(\dot{\phi}(x))$ which are only $C^{k-1,\alpha}$ if g is $C^{k,\alpha}$.

2.9.3. Lemma. — *1) If $\phi \in \mathscr{D}^{k+1,\alpha}$ then the map $g \mapsto \phi^* g$ is smooth in $\mathscr{M}et^{k,\alpha}$.*

2) If g is smooth, then the map $\phi \to \phi^* g$ is smooth $\mathscr{D}^{k+1,\alpha} \to \mathscr{M}et^{k,\alpha}$.

Proof. — The map $g \mapsto \phi^* g$ is linear, so the first statement is obvious. The second statement comes immediately from the explicit formula (2.9.2) above. □

Remind that the infinitesimal action of the diffeomorphisms on metrics is given by $\mathscr{L}_X g = \delta^* X$. Let g be a smooth metric on M, since δ^* is elliptic we have an orthogonal decomposition

$$C^{k,\alpha}(S^2T^*M) = \operatorname{im}(\delta^*) \oplus \ker(\delta). \tag{2.9.4}$$

Therefore a possible slice for the action of $\mathscr{D}^{k+1,\alpha}$ on $\mathscr{M}et^{k,\alpha}$ is

$$S_\epsilon = g + \ker(\delta) \cap B_\epsilon, \tag{2.9.5}$$

where B_ϵ is a ball of radius ϵ in $C^{k,\alpha}(S^2T^*M)$. To check that it is a slice, we must see that the orbit of any metric h close to g meets S_ϵ; so we look for a diffeomorphism ϕ such that

$$\delta^g(\phi^* h) = 0. \tag{2.9.6}$$

Problem: the action $\phi \to \phi^* h$ is not smooth. This is overcomed by observing that $\phi_* \delta^g \phi^* h = \delta^{\phi_* g} h$, so we can replace equation (2.9.6) by

$$\delta^{\phi_* g} h = 0. \tag{2.9.7}$$

Because g is smooth, from lemma 2.9.3 we deduce that the map $(\phi, h) \mapsto \Phi(\phi, h) = \delta^{\phi_* g} h$ is now smooth,

$$\Phi : \mathscr{D}^{k+1,\alpha} \times \mathscr{M}et^{k,\alpha} \to C^{k-1,\alpha}(T^*M).$$

The partial derivative with respect to the first variable is the map

$$\partial_1 \Phi : C^{k+1,\alpha}(TM) \to C^{k-1,\alpha}(TM),$$
$$X \mapsto \delta\delta^* X.$$

To analyse equation (2.9.7), we distinguish two cases:

1. $\ker(\delta\delta^*) = 0$, which is equivalent to $\ker \delta^* = 0$ (which means $\operatorname{Isom}(M, g)$ is discrete): then $\partial_1 \Phi$ is an isomorphism and we can apply the implicit function theorem to Φ, so for any h close to g, there exists a unique diffeomorphism ϕ close to the identity and solving the equation;

2. $\ker(\delta\delta^*) \neq 0$: then $\partial_1\Phi$ is not surjective, but the reason is that Φ satisfies the constraint

$$\Phi(\phi, h) \in \operatorname{im}(\delta^{\phi_* g}) = \ker\left((\delta^*)^{\phi_* g}\right)^{\perp_{\phi_* g}};$$

since $\ker((\delta^*)^{\phi_* g}) = \phi^* \ker((\delta^*)^g)$, all these kernels have the same dimension; this implies that for ϕ close to the identity and h close to g one has

$$\delta^{\phi_* g} h = 0 \Longleftrightarrow \pi \delta^{\phi_* g} h = 0,$$

where π is the orthogonal projection on $\operatorname{im}(\delta^g)$ with respect to g; so we can apply the implicit function theorem to $\pi \circ \Phi$ to solve the equation (2.9.7), and the solution is unique if we restrict to a complement of $\operatorname{Isom}(M, g)$ in $\mathscr{D}^{k+1,\alpha}$.

Together this gives the third condition of the following slice result:

2.9.8. Lemma. — *The slice S_ϵ defined by (2.9.5) for the action of $\mathscr{D}^{k+1,\alpha}$ on $\mathscr{M}et^{k,\alpha}$ satisfies:*

1. *if $\phi \in \operatorname{Isom}(M, g)$ then $\phi(S_\epsilon) \subset S_\epsilon$;*
2. *if $\phi \in \mathscr{D}^{k+1,\alpha}$ and $\phi(S_\epsilon) \cap S_\epsilon \neq \emptyset$, then $\phi \in \operatorname{Isom}(M, g)$;*
3. *choose a section $\sigma : \mathscr{D}^{k+1,\alpha}/\operatorname{Isom}(M, g) \to \mathscr{D}^{k+1,\alpha}$, then the map $F : \mathscr{D}^{k+1,\alpha}/\operatorname{Isom}(M, g) \times S_\epsilon \to \mathscr{M}et^{k,\alpha}$ defined by $F(\phi, g) = \sigma(\phi)^* g$ is a homeomorphism in a neighbourhood of $(1, g)$.*

Observe that the map F cannot be smooth since the pullback $\phi^* g$ is not smooth in these spaces.

Proof. — Only the second condition remains to be proved. As in the lemma 2.4.4, the point is to control the diffeomorphism ϕ and its inverse in $C^{k+1,\alpha}$ if $\phi^* g_1 = g_2$ and g_1 and g_2 are controled in $C^{k,\alpha}$. As ϕ is an isometry from (M, g_2) to (M, g_1), it is controled in C^0 and its first derivatives are bounded (and the same is true for ϕ^{-1}). Now in local coordinates one calculates that the coefficients b_{ij}^l of the Levi-Civita connection of g_2 are given in terms of the coefficients a_{ij}^l of the Levi-Civita connection of g_1 by the formula

$$b_{ij}^l = (\partial_r \phi^j)^{-1} (\partial_s \phi^k)^{-1} a_{rs}^t (\partial_t \phi^l) - (\partial_{rs}^2 \phi^l)(\partial_s \phi^i)^{-1} (\partial_r \phi^j)^{-1}. \tag{2.9.9}$$

Since the first derivatives of ϕ are bounded, we deduce from the equation a control on the second derivatives $(\partial_{rs}^2 f^l)$. Now an iteration based on equation (2.9.9) gives a $C^{k+1,\alpha}$ control on ϕ, since the connections forms a and b are controled in $C^{k-1,\alpha}$ if g_1 and g_2 are controled in $C^{k,\alpha}$. The same holds for ϕ^{-1}. □

2.10. The Einstein equation

The linearization of the Ricci tensor is (see [**3**], that we follow closely in this section)

$$d_g \operatorname{Ric}(\dot g) = \frac{1}{2}\Delta_L \dot g - \delta^* \delta \dot g - \frac{1}{2}\nabla d \operatorname{Tr}(\dot g); \tag{2.10.1}$$

here the Lichnerowicz Laplacian Δ_L is defined by

$$\Delta_L \dot g = \nabla\nabla^* \dot g + \operatorname{Ric}\circ \dot g + \dot g \circ \operatorname{Ric} - 2\overset{\circ}{R}\dot g,$$

where the composition of two quadratic forms means the quadratic form corresponding to the composition of the associated selfadjoint endomorphisms, and the action $\overset{\circ}{R}$ of the curvature on symmetric 2-tensors is a symmetric endormorphism of S^2T^*M, defined by

$$(\overset{\circ}{R}h)_{X,Y} = \sum_1^n h(R_{e_i,X}Y, e_i);$$

obviously

$$\overset{\circ}{R}g = \operatorname{Ric}^g,$$

so $\overset{\circ}{R}$ preserves the decomposition $\mathbb{R}g \oplus S_0^2T^*M$ when g is Einstein.

2.10.2. Remark. — As in the case of connections, the precise regularity used to study the Einstein equation is not relevant. Indeed one can prove that a $C^{k,\alpha}$ Einstein metric is always diffeomorphic to a smooth Einstein metric. If h is a small deformation of a smooth Einstein metric g, we can use the gauge $\delta^g h = 0$, so h is a solution of the nonlinear elliptic system (2.10.11), and it is not difficult to prove that h is smooth.

The general case is obtained by a local version: if g is $C^{k,\alpha}$, one can construct locally $C^{k+1,\alpha}$ harmonic coordinates (x^i), that is satisfying $\Delta x^i = 0$. The choice of coordinates breaks down the diffeomorphism invariance, so that the Einstein equation becomes a nonlinear elliptic system. One can then prove smoothness of the metric in the harmonic coordinates.

2.10.3. Exercise. — Consider $S^2T^*M \subset \Omega^1 \otimes \Omega^1$, and prove that the curvature term $\mathscr{R}^{\Omega^1}(h)$ in equation (1.5.7) is $-\overset{\circ}{R}h$.

***2.10.4. Lemma* (Einstein-Hilbert functional).** — *Let* $E(g) = \int_M \operatorname{Scal}^g \operatorname{vol}^g$, *then* $d_gE(\dot g) = -\int_M \langle \dot g, \operatorname{Ric}^g - \frac{1}{2}\operatorname{Scal}^g g\rangle \operatorname{vol}^g$.

Proof. — Left as an exercise: calculate $\dot{\operatorname{Scal}}$ using (2.10.1). □

2.10.5. Exercise. — Prove that for any diffeomorphism invariant Riemannian functional $E(g)$, the gradient of E is divergence free. Deduce another proof of the Bianchi identity: $\operatorname{Ric} - \frac{1}{2}\operatorname{Scal} g$ is divergence free.

2.10.6. Exercise. — Use the Einstein-Hilbert functional to prove that if (g_t) is a one parameter family of Einstein metrics with varying Einstein constant, $\operatorname{Ric}(g_t) = \lambda(t) g_t$, then

$$\frac{d\lambda}{dt} = \frac{(n-1)\lambda}{n \operatorname{Vol} g_t} \frac{d \operatorname{Vol} g_t}{dt}.$$

Deduce that in the family the sign of λ remains constant ($+1$, 0 or -1).

From the exercise, we deduce that when we look at the deformations of Einstein metrics, then (up to rescaling) we can suppose that the Einstein constant remains constant in the family.

So now let us study the deformations of the equation $\operatorname{Ric}(g) = \lambda g$, where λ is a fixed real number. As usual, there is a deformation complex:

$$0 \to \Gamma(TM) \xrightarrow{\delta^*} \Gamma(S^2T^*M) \xrightarrow{d\operatorname{Ric} - \lambda} \Gamma(S^2T^*M) \xrightarrow{B^g} \Gamma(T^*M) \to 0. \tag{2.10.7}$$

Here B^g is the Bianchi operator, $B^g : \Gamma(S^2T^*M) \to \Gamma(\Omega^1)$, given by

$$B^g h = \delta^g h + \frac{1}{2} d \operatorname{Tr}^g h. \tag{2.10.8}$$

The first arrow is the infinitesimal action of the diffeomorphism, the middle arrow is the linearization of the equation. Because of the Bianchi identity,

$$B^g \operatorname{Ric}^g = 0, \tag{2.10.9}$$

the sequence if a complex when the Einstein equation $\operatorname{Ric}(g) = \lambda g$ is satisfied.

2.10.10. Exercise. — Check that (2.10.7) is an elliptic complex.

Here the theory will not be as nice as in the connection case. As we shall see later, the complex of symbols can be deformed into a selfadjoint complex, so the index of the complex is zero. If for example $H^0 = 0$ (that is $\operatorname{Isom}(g)$ is discrete), then $\dim H^2 = \dim H^1 + \dim H^3$, so there are obstructions (H^2) as soon as there are infinitesimal deformations (H^1). So we will be unable to use the deformation complex to provide a family of deformations. Instead, our main theorem in this section (theorem 2.10.19) will only state a rigidity result: if $H^1 = 0$, then every nearby solution is diffeomorphic to g.

By the slice lemma 2.9.8, for h close to g we can also suppose $\delta^g h = 0$, so we want to solve the system

$$\begin{aligned} \delta^g h &= 0 \\ \operatorname{Ric}(h) &= \lambda h \end{aligned} \tag{2.10.11}$$

To solve both equations together, we define

$$\Phi^g(h) = \operatorname{Ric}(h) - \lambda h + (\delta^h)^* \delta^g h. \tag{2.10.12}$$

Then the system (2.10.11) implies $\Phi^g(h) = 0$. Let us study the equation $\Phi^g(h) = 0$ at an Einstein metric g. The linearization is

$$d_g \Phi^g(\dot g) = \frac{1}{2}\nabla^*\nabla \dot g - \overset{\circ}{R}\dot g - \frac{1}{2}\nabla d \operatorname{Tr} \dot g. \tag{2.10.13}$$

On the trace part, we obtain

$$\operatorname{Tr} d_g \Phi^g(\dot g) = \Delta \operatorname{Tr}(\dot g) - \lambda \operatorname{Tr}(\dot g). \tag{2.10.14}$$

Then:

2.10.15. Lemma. — *If* $\lambda \neq 0$, *then* $\ker(\Delta - \lambda) = 0$.

Proof. — If $\lambda < 0$ it is obvious. If $\lambda > 0$, it is a direct consequence of the following eigenvalue estimate : if $\operatorname{Ric}^g \geqslant \lambda g$, then the first (nonzero) eigenvalue of Δ on functions satisfies

$$\lambda_1 \geqslant \frac{n}{n-1}\lambda. \tag{2.10.16}$$

To prove this estimate, suppose that $\Delta f = \lambda_1 f$ with $\lambda_1 > 0$. Then $\Delta df = \lambda_1 df$. The Bochner formula (lemma 1.5.2) reads

$$(\nabla^*\nabla - \lambda_1) df + \operatorname{Ric}(df) = 0.$$

Taking the scalar product with df, we obtain

$$\|\nabla df\|^2 - \lambda_1 \|df\|^2 \leqslant -\lambda \|df\|^2.$$

But $\Delta f = -\operatorname{Tr}(\nabla df)$ so $\|\nabla df\|^2 \geqslant \frac{1}{n}\|\Delta f\|^2 = \frac{\lambda_1}{n}\|df\|^2$, and the lemma follows. ∎

Coming back to the trace (2.10.14) of our infinitesimal equation, the lemma implies that the solutions are zero if $\lambda \neq 0$, or constants if $\lambda = 0$ (which correspond to scaling g). In the $\lambda = 0$ case we forget the trivial solutions, so the infinitesimal solutions of $\Phi^g(h) = 0$ satisfy

$$\begin{aligned} \operatorname{Tr} \dot g &= 0 \\ \frac{1}{2}\nabla^*\nabla \dot g - \overset{\circ}{R}\dot g &= 0 \end{aligned} \tag{2.10.17}$$

In addition, we look for metrics in the slice to g, so we suppose that we have the gauge condition

$$\delta^g \dot{g} = 0. \tag{2.10.18}$$

The system (2.10.17)–(2.10.18) characterizes infinitesimal Einstein deformations of g. Here is the main theorem of this section.

Call an Einstein metric g *rigid* if any small Einstein deformation of g is diffeomorphic to g.

2.10.19. Theorem. — *If the Einstein metric g has no infinitesimal deformations solving the system (2.10.17)–(2.10.18), then g is rigid as an Einstein metric. This is satisfied in particular when the sectional curvature of g is negative and $n > 2$.*

Proof. — By the Bianchi identity, one has

$$B^h \Phi^g(h) = B^h (\delta^h)^* \delta^g h. \tag{2.10.20}$$

Infinitesimally, this gives

$$B^g \big(d_g \Phi^g(\dot{g}) \big) = B^g (\delta^g)^* \delta^g \dot{g}. \tag{2.10.21}$$

In particular, observe that

$$\delta^g \dot{g} = 0 \quad \text{implies} \quad B^g(d_g \Phi^g(\dot{g})) = 0. \tag{2.10.22}$$

We will note

$$H = \ker B^g, \quad H \subset C^{k-2,\alpha}(S^2 T^* M), \tag{2.10.23}$$

and π the orthogonal projection on H with respect to the L^2 scalar product.

If h is an Einstein metric close to g, then by a diffeomorphism we can put h in the gauge $\delta^g h = 0$, then it satisfies $\Phi^g(h) = 0$ and therefore is in the zero set of the map

$$P : \ker \delta^g \to H, \quad P(h) = \pi \Phi^g(h). \tag{2.10.24}$$

We will prove that under the assumption of the theorem, the map P is a local diffeomorphism, so that $P^{-1}(0) = \{g\}$ and there is no other Einstein metric near g.

The differential of P at g is just

$$d_g P = \pi \circ d_g \Phi^g = d_g \Phi^g \tag{2.10.25}$$

since we restrict to $\ker \delta^g$, see (2.10.22). The kernel of $d_g P$ consists of the infinitesimal Einstein deformations, so it is trivial by hypothesis. The orthogonal of the image is exactly

$$H \cap \ker\big((d_g \Phi^g)^*\big), \tag{2.10.26}$$

and we have to check that it vanishes. An element $u \in H \cap \ker(d_g\Phi^g)^*$ satisfies the system

$$\begin{aligned} B^g u = \delta u + \frac{1}{2} d \operatorname{Tr} u = 0, \\ \frac{1}{2}\nabla^*\nabla u - \overset{\circ}{R} u - \frac{1}{2}(\delta\delta u) g = 0. \end{aligned} \tag{2.10.27}$$

Using the first equation to replace in the second one, we get

$$\frac{1}{2}\nabla^*\nabla u - \overset{\circ}{R} u + \frac{1}{4}(\Delta \operatorname{Tr} u) g = 0. \tag{2.10.28}$$

The trace of this equation is

$$(\frac{1}{2} + \frac{n}{4})\Delta \operatorname{Tr} u - \lambda \operatorname{Tr} u = 0.$$

Since $\frac{1}{2} + \frac{n}{4} \geqslant 1$, lemma 2.10.15 implies that $\operatorname{Tr} u = 0$. Then the system (2.10.27) implies $\delta u = 0$ and $\frac{1}{2}\nabla^*\nabla u - \overset{\circ}{R} u = 0$, so the solutions are again the infinitesimal Einstein deformations, therefore are trivial.

The second part of the theorem involves studying the kernel of the operator $P = \frac{1}{2}\nabla^*\nabla - \overset{\circ}{R}$ in the system (2.10.17), in the case g has negative sectional curvature. By the exercice 2.10.3 and the Bochner-Weitzenböck formula (1.5.6), if we consider $h \in \Gamma(S^2T^*M)$ as a section of $\Omega^1 \otimes \Omega^1$, then

$$Ph = \frac{1}{2}\big(\Delta h - \overset{\circ}{R} h - \frac{\operatorname{Scal}}{n} h\big). \tag{2.10.29}$$

We claim that if $\operatorname{Ric}^g \geqslant \lambda g$, then for trace free symmetric 2-tensors h,

$$\frac{\langle \overset{\circ}{R} h, h\rangle}{|h|^2} + \lambda \leqslant (n-2)\sup K. \tag{2.10.30}$$

Suppose this claim is true. Integrating equation (2.10.29) against h, we get

$$(Ph, h) \geqslant -\frac{1}{2}\left((\overset{\circ}{R} h, h) + \lambda \|h\|^2 \right) \geqslant \frac{1}{2}(n-2)\|h\|^2(-\sup K). \tag{2.10.31}$$

If g has $K < 0$ and $n > 2$, we deduce that $Ph = 0$ implies $h = 0$.

There remains to prove (2.10.30). Suppose a is an eigenvalue of $\overset{\circ}{R}$ on $S^2_0 T^*M$, with associated eigenvector η. Diagonalize η in an orthonormal basis, with eigenvalues $\lambda_1 \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|$ satisfying $\sum \lambda_i = 0$. Then, denoting by K_{max} the largest sectional curvature at the point,

$$\begin{aligned} a\lambda_1 = (\mathring{R}\eta)_{11} &= \sum_{i>1} R_{1ii1}\lambda_i = \sum_{i>1} K_{max}\lambda_i - \sum_{i>1}(K_{max} - R_{1ii1})\lambda_i \\ &\leqslant -K_{max}\lambda_1 + \lambda_1 \sum_{i>1}(K_{max} - R_{1ii1}) \\ &\leqslant \lambda_1\big((n-2)K_{max} - \mathrm{Ric}_{11}\big) \end{aligned}$$

so $a \leqslant (n-2)K_{max} - \lambda$, which proves the claim. □

In particular, note that the theorem proves that the quotients of the hyperbolic space H^n are rigid as Einstein metrics for $n > 2$. Of course this is not true for $n = 2$, as the hyperbolic metrics on a Riemann surface of genus g form a $(6g-6)$-dimensional manifold (Teichmüller space).

There are only few situations in which the moduli space of Einstein metrics is known. The main example is the case of Kähler-Einstein metrics: in that case one can parametrize the moduli space by the nearby complex structures, see [**3**].

Given an infinitesimal solution of Einstein equations, the question of integrating it into a true solution is very difficult, as one has to check high order obstructions. There exist explicit examples where an infinitesimal deformation cannot be integrated. As a result, almost nothing is known on the dimension of the moduli space of Einstein metrics.

2.10.32. Exercise. — The aim of the exercice is to prove that if $\mathrm{Ric}^h < 0$, then the equation $\Phi^g(h) = 0$ defined in (2.10.12) is equivalent to the system (2.10.11).

1) Decompose the covariant derivative on 1-forms as $\nabla = \delta^* + \frac{1}{2}d$, and deduce that on 1-forms.

$$\nabla^*\nabla = \delta\delta^* + \frac{1}{2}d^*d \tag{2.10.33}$$

Use the Bochner formula to deduce, still on 1-forms,

$$(\delta + \frac{1}{2}d\,\mathrm{Tr})\delta^* = \frac{1}{2}\big(\nabla^*\nabla - \mathrm{Ric}\big). \tag{2.10.34}$$

2) Prove that

$$B^h\Phi^g(h) = (\delta^h + \frac{1}{2}d\,\mathrm{Tr}^h)(\delta^h)^*\delta^g h.$$

Deduce that if $\mathrm{Ric}^h < 0$, then $\Phi^g(h) = 0$ implies $\delta^g h = 0$ and therefore h satisfies the system (2.10.11).

2.10.35. Exercise. — Let g be a smooth metric.

1) Prove that if $\operatorname{Ric}^g < 0$, then g has no Killing vector field, so the isometry group of g is discrete. *Hint:* use formula (2.10.34), which reads $B^g\delta^* = \frac{1}{2}(\nabla^*\nabla - \operatorname{Ric})$.

2) Prove that if $\operatorname{Ric}^g < 0$, then instead of the gauge $\delta^g h = 0$ to find a slice at g for the action of the diffeomorphisms, one can use the "Bianchi gauge"

$$B^g(h) = \delta^g h + \frac{1}{2} d \operatorname{Tr}^g h = 0. \tag{2.10.36}$$

3) One can use this gauge to study the Einstein equation near g. Prove that if $\operatorname{Ric}^h < 0$, the system

$$B^g h = 0, \quad \operatorname{Ric}^h = \lambda h \tag{2.10.37}$$

is equivalent to

$$\Phi^g(h) := \operatorname{Ric}^h - \lambda h + (\delta^h)^* B^g h = 0. \tag{2.10.38}$$

Prove that

$$d_g \Phi^g = \frac{1}{2}\nabla^*\nabla - \overset{\circ}{R}. \tag{2.10.39}$$

Suppose g is Einstein with $\operatorname{Ric}^g < 0$. Prove that if g has no infinitesimal Einstein deformation (that is no $\dot g$ satisfying (2.10.17) and (2.10.18)), then $d_g\Phi^g$ is an isomorphism. Deduce a simpler proof of theorem 2.10.19 in the case $\operatorname{Ric}^g < 0$.

CHAPTER 3

A GLUING CONSTRUCTION: DEHN SURGERY FOR EINSTEIN METRICS

GLUING constructions for solutions of geometric non linear elliptic problems are frequently used now. The examples include gauge theory (instantons, Seiberg-Witten theory), Gromov-Witten theory (holomorphic curves), minimal surfaces, selfdual metrics, extremal Kähler metrics... In all cases one starts from solutions defined on two manifolds X_1 and X_2, which have some coincidence along a common Y which can be found in X_1 and X_2, and one tries to construct solutions on a manifold X obtained from X_1 and X_2 by surgery along Y. The main steps are

1. construction of a family $(\phi_t)_{t>0}$ of approximate solutions on X, which converge when $t \to 0$ to the two given solutions on the disjoint sum $X_1 \amalg X_2$;
2. for small enough t, deform the approximate solution ϕ_t to a true solution on X; the main technical step here is usually to prove a uniform estimate for the norm of a right inverse of the linearization of the problem.

We will illustrate this technique on the particular example of the construction of new Einstein metrics from cusp hyperbolic metrics by Dehn surgery. In dimension 3, this is a classical result of Thurston: in that case, the result is the construction of compact hyperbolic 3-manifolds, and the method of Thurston was group theoretic. In higher dimension, it is a recent result of Anderson [**1**], the proof is analytic, and it gives another proof of the 3-dimensional result.

3.1. Cusp ends

Remind the half space model of hyperbolic n-space: one has

$$H^n = \{x^1 > 0\} \subset \mathbb{R}^n,$$

with the metric
$$g = \frac{(dx^1)^2 + \cdots + (dx^n)^2}{(x^1)^2}.$$
Take a lattice $\mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}_{x^2 \cdots x^n}$, so the quotient is a torus
$$T = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1},$$
with flat metric g_T. We have an induced hyperbolic metric on $(0, +\infty) \times T = H^n/\mathbb{Z}^{n-1}$. Replacing the coordinate x^1 by $r = 1/x^1$, we can write this metric as
$$g = \frac{dr^2}{r^2} + r^2 g_T. \tag{3.1.1}$$
This metric is complete and has two ends:

1. $r \to 0$: then the diameter of the torus slices is $O(r)$ so goes to zero, and the volume $r^{n-2} dr \wedge \mathrm{vol}^T$ has finite integral—it is a *cusp end*;
2. $r \to +\infty$: here the diameter of the torus slices blows up and the volume is infinite—it is a *hyperbolic end*.

The theory of Fuchsian groups enable to construct a lot of complete hyperbolic manifolds H^n/Γ for discrete subgroups $\Gamma \subset SO(1, n)$ with only cusp ends. Each end, near infinity, can be described by the formula (3.1.1).

3.2. Toral black hole metric

Let us now consider the metric
$$g_{BH} = \frac{dr^2}{V(r)} + V(r) d\theta^2 + r^2 g_{T^{n-2}}, \qquad V(r) = r^2 - \frac{a}{r^{n-3}}. \tag{3.2.1}$$
Here T^{n-2} is a torus, θ is a circular variable on a circle of length $\beta > 0$, r is a real variable, and $a \geqslant 0$ is a real parameter. Obviously we need
$$r > r_+ = a^{\frac{1}{n-1}}, \tag{3.2.2}$$
so the metric is defined on $(r_+, +\infty) \times S^1 \times T^{n-2}$. An easy calculation (see section 3.10) shows that g_{BH} is Einstein, with
$$\mathrm{Ric}(g_{BH}) = -(n-1) g_{BH}. \tag{3.2.3}$$

Actually the whole curvature tensor of g_{BH} is easily calculated (see again the details in section 3.10): the submanifold $(r_+, +\infty) \times \{0\} \times T^{n-2}$ is totally geodesic as the fixed point set of the isometry $\theta \to -\theta$. More generally, if we fix any subspace $E \subset T^{n-2}$, then $(r_+, +\infty) \times E$ is totally geodesic. In the same way, $(r_+, +\infty) \times S^1 \times E$ is totally geodesic. It follows that in the basis $(\partial_r, \partial_\theta, \partial_{x^2}, \dots, \partial_{x^{n-1}})$ (or more precisely in the corresponding basis of 2-forms) the curvature is diagonal, so it is determined by the following sectional curvatures (whose calculation is straightforward):

$$\begin{aligned} K_{r\theta} &= -1 + (n-3)(n-2)\frac{a}{2r^{n-1}}, \\ K_{rx^i} &= -1 - (n-3)\frac{a}{2r^{n-1}}, \\ K_{\theta x^i} &= -1 - (n-3)\frac{a}{2r^{n-1}}, \\ K_{x^i x^j} &= -1 + \frac{a}{r^{n-1}}. \end{aligned} \tag{3.2.4}$$

If $n = 3$, of course all the sectional curvatures are -1. If $n \geqslant 4$, observe that $K_{r\theta}$ always takes nonnegative values, so g_{BH} has never negative sectional curvature.

Let us examine the behaviour of g_{BH} at the ends. First, when r goes to infinity, we have $V(r) \sim r^2$ and therefore

$$g_{BH} \sim \frac{dr^2}{r^2} + r^2(d\theta^2 + g_{T^{n-2}}), \quad r \to +\infty. \tag{3.2.5}$$

This is a hyperbolic end. This is confirmed by the formulas (3.2.4): when $r \to +\infty$, all curvatures go to -1 at speed $O(\frac{1}{r^{n-1}})$.

When $r \to r_+$, we have $V(r) \sim (n-1)r_+(r - r_+)$ and therefore

$$g_{BH} \sim \frac{1}{(n-1)r_+}\frac{dr^2}{r - r_+} + (n-1)r_+ d\theta^2 + r_+^2 g_T, \quad r \to r_+, \tag{3.2.6}$$

so after the change of coordinates $u = \sqrt{r - r_+}$, we get

$$g_{BH} \sim \frac{4}{(n-1)r_+}\big(du^2 + \frac{r_+^2(n-1)^2}{4}u^2 d\theta^2\big) + r_+^2 g_T. \tag{3.2.7}$$

In this formula the coordinate u goes to zero, so we can think of the metric as being defined on the product of a punctured disk $D - \{0\}$ with a torus T, where the disk D is equipped with polar coordinates (u, θ). It is clear that the metric g_{BH} on $(D - \{0\}) \times T$ is incomplete. There is an obvious compactification of $(D - \{0\}) \times T$ by $D \times T$, adding a 'core torus' $\{0\} \times T^{n-2}$. The formula (3.2.7) defines a continuous extension of g_{BH} on $D \times T$ provided that the circular variable $\frac{1}{2}r_+(n-1)\theta$ is defined on an interval of length 2π[(1)].

[(1)]In general, the metric $dr^2 + r^2 d\theta^2$ for $\theta \in [0, \alpha]$ has a cone singularity at zero with angle α. If the angle is 2π, the metric is of course the smooth flat metric.

Here this gives the condition that $\theta \in [0, \beta]$ with

$$\beta = \frac{4\pi}{(n-1)r_+}. \tag{3.2.8}$$

The metric g_{BH} actually extends smoothly over $D \times T$. To see this, it suffices to perform the exact calculation for g_{BH} with respect to the coordinate u instead of the approximation (3.2.7). This is left to the reader.

In dimension $n = 3$, there is a useful coordinate change: we have $V(r) = r^2 - a$, so we choose a coordinate t so that

$$dt = \frac{dr}{r^2 - a}, \qquad \text{that is} \qquad t = \operatorname{argcosh} \frac{r}{\sqrt{a}}. \tag{3.2.9}$$

The torus T^{n-2} is now a circle, with coordinate φ, and the formula (3.2.1) becomes

$$g_{BH} = dt^2 + \sinh^2(t) a d\theta^2 + \cosh^2(t) a d\varphi^2. \tag{3.2.10}$$

Observe that with the choice (3.2.8), the variable $\sqrt{a}\theta$ of course varies between 0 and 2π. This formula is the standard form of a hyperbolic metric around the core geodesic $t = 0$.

3.3. Twisted toral black hole metric

There is a twisted version of the toral black hole metric that will be useful later. Observe that the slices $\{r = \text{cst}\}$ are tori $T^{n-1} = S^1 \times T^{n-2}$, and we made the circle S^1 and the torus T^{n-2} orthogonal. This is arbitrary, and can be generalized in the following way. Fix a torus $T^{n-1} = \mathbb{R}^{n-1}/\Lambda$, where Λ is a lattice $\mathbb{Z}^{n-1}$, and a simple circle $S^1 \subset T^{n-1}$. If $(v_1, \dots, v_{n-1})$ is a basis of Λ, such circle is given by a vector

$$v = \sum n_i v_i, \quad n_i \in \mathbb{Z}, \quad (n_i) \text{ prime}. \tag{3.3.1}$$

By composing by an element of $SL_{n-1}(\mathbb{Z})$, we can suppose that $v = v_1$. Then $\langle v_2, \dots, v_{n-1} \rangle$ generates an action of $\mathbb{Z}^{n-2}$ on $\mathbb{R}^{n-1}$ which descends to an action on

$$\mathbb{R}/\langle v \rangle \times \mathbb{R}^{n-2} = S^1 \times \mathbb{R}^{n-2}, \tag{3.3.2}$$

whose quotient is exactly our torus T^{n-1}. We can write the toral black hole metric (3.2.1) on $(r_+, +\infty) \times S^1 \times \mathbb{R}^{n-2}$,

$$g_{BH} = \frac{dr^2}{V(r)} + V(r) d\theta^2 + r^2 g_{\mathbb{R}^{n-2}}, \tag{3.3.3}$$

and observe that this is invariant under the residual $\mathbb{Z}^{n-2}$ action on $S^1 \times \mathbb{R}^{n-1}$, so descends to a metric on the quotient $(r_+, +\infty) \times T^{n-1}$. As before, this metric extends smoothly on the compactification obtained by adding a core torus T^{n-2} at $r = r_+$ (just compactify $(r_+, +\infty) \times S^1 \times \mathbb{R}^{n-2}$ by adding a $\mathbb{R}^{n-2}$ at $r = r_+$, and observe that the $\mathbb{Z}^{n-2}$ action extends as a free action on the compactification).

Another way to write the same twisted metric is the following: the torus T^{n-1} is equipped with the direction of the circles generated by v, but it is not a metric product. Denote η the 1-form dual to the vector field $\frac{v}{|v|}$ (this is our form $d\theta$ above, but we note it differently to emphasize that the metric of T^{n-1} is not a product). From the explicit form of $V(r)$ we can rewrite the metric on $(r_+, +\infty) \times T^{n-1}$ as

$$g_{BH} = \frac{dr^2}{V(r)} + r^2 g_{T^{n-1}} - \frac{a}{r^{n-3}}\eta^2. \tag{3.3.4}$$

Locally, the twisted metric (3.3.3) is the same as our initial metric, so it shares exactly the same properties. In particular it is Einstein.

3.4. Dependence on the parameter

The toral black hole metric depends on the parameter a. Nevertheless, as we shall see, this dependence is artificial. Denote (r_j, θ_j, x_j^i) for $j = 1, 2$ the coordinates for the two metric g_1 and g_2 defined by the parameters a_1 and a_2. Write $a_2 = \lambda^{n-1} a_1$ and perform the change of variables $r_2 = \lambda r_1$. Then the two corresponding functions V are related by $V_2(r_2) = \lambda^2 V_1(r_1)$, and we get

$$g_2 = \frac{dr_1^2}{V_1(r_1)} + V_1(r_1)\lambda^2 d\theta_2^2 + r_1^2 \lambda^2 \big((dx_2^2)^2 + \cdots\big). \tag{3.4.1}$$

Taking

$$\theta_1 = \lambda\theta_2, \quad x_1^j = \lambda x_2^j, \tag{3.4.2}$$

we exactly recover g_1. So all the toral black hole metrics for different values of a are actually the same metric (up to a scaling of the T^{n-2} factor).

On the other hand, it is interesting to note that, at least formally, when $a \to 0$, then $V(r) \to r^2$ so g_{BH} converges to the hyperbolic metric $\frac{dr^2}{r^2} + r^2 d\theta^2 + r^2 g_{T^{n-1}}$. Of course in the process of making a go to zero, we obtain longer and longer circle (coordinate θ) and torus T^{n-2}, see (3.4.2), so we do not get a geometric convergence towards the model cusp metric (3.1.1). To obtain this in the next section, we will need a subtler construction with more and more twisted toral black hole metrics.

3.5. Dehn surgery and statement of the theorem

Here we describe the surgery. Start from a hyperbolic manifold (M^n, g) with cusps. For simplicity we will assume that M has only one end, but if there are several cusp ends, one can do the same on each end. Then we have coordinates such that near the end, the metric has the form

$$g = \frac{dr^2}{r^2} + r^2 g_T. \tag{3.5.1}$$

Up to changing $r \to \lambda r$ and $g_T \to \lambda^{-2} g_T$, we can suppose that the formula is valid for $r \leqslant 2$. Therefore M is the union of a compact part $M_1 = \{r \geqslant 1\}$ and of a cusp end $(0,1] \times T^{n-1}$ along their common boundary: $M = M_1 \amalg_{T^{n-1}} (0,1] \times T^{n-1}$.

The surgery consists in cutting the noncompact cusp end and gluing a compact end $D \times T^{n-2}$ instead. The two pieces M_1 and $D \times T^{n-2}$ will be identified along their common boundary T^{n-1} in the following way. Choose a simple closed geodesic σ in T^{n-1}: this will be the boundary of the disk D that will be glued. As used in section 3.3, one can choose a basis $(v_1 = \sigma, v_2, \dots, v_{n-1})$ of the lattice Λ such that $T^{n-1} = \mathbb{R}^{n-1}/\Lambda$. Then $T^{n-1} = S^1 \times T^{n-2}$, where T^{n-2} is generated by $v_2, \dots, v_n$. We can then perform the topological sum

$$M_\sigma = M_1 \amalg_{T^{n-1}} (D \times T^{n-2}). \tag{3.5.2}$$

The result is a manifold whose topology depends only on the homotopy class of σ, that is only on the class $[\sigma] \in H_1(T^{n-1}, \mathbb{Z})$.

Of course we also want to perform surgery on the metric. Here we will use the twisted toral black hole metric of section 3.3. We know that the torus T^{n-1} is not a Riemannian product $S^1 \times T^{n-2}$, with the S^1 corresponding to σ, but it is twisted: $T^{n-1} = (S^1 \times \mathbb{R}^{n-2})/\mathbb{Z}^{n-2}$, where $\mathbb{Z}^{n-2}$ is generated by $v_2, \dots, v_n$. Denote by ℓ the length of σ. Choose the parameter a so that the associated function $V(r) = r^2 - \frac{a}{r^{n-3}}$ satisfies

$$\beta V(1) = \ell, \quad \text{that is} \quad \frac{4\pi(1-a)}{(n-1)a^{\frac{1}{n-1}}} = \ell. \tag{3.5.3}$$

This makes the length at $r = 1$ of the circle of the toral black hole metric coincide with ℓ. Observe that when ℓ becomes big, then

$$a \sim \left(\frac{4\pi}{(n-1)\ell}\right)^{n-1} \tag{3.5.4}$$

so in particular $a \to 0$. With this choice of a, the twisted toral black hole metric

$$b_\sigma = V(r)^{-1}dr^2 + V(r)d\theta^2 + r^2 g_{\mathbb{R}^{n-2}} \tag{3.5.5}$$

on $(S^1 \times \mathbb{R}^{n-1})/\mathbb{Z}^{n-2}$ coincides exactly on the slice $\{r = 1\}$ with the given metric on T^{n-1}. This is the metric that we will glue with g in order to get a Riemannian metric on M_σ.

In order to obtain a smooth metric on M_σ, we choose a cut-off function $\chi(r) \geqslant 0$ such that

$$\chi(r) = \begin{cases} 0 & \text{if } r \leqslant 1/2 \\ 1 & \text{if } r \geqslant 2. \end{cases} \tag{3.5.6}$$

Then we define on M_σ the metric

$$g_\sigma = \chi g + (1-\chi)b_\sigma. \tag{3.5.7}$$

This metric coincides with g on the compact part $\{r \geqslant 2\}$ of M, and with b_σ for $r \leqslant 1/2$.

Since $V(r) = r^2 - \frac{a}{r^{n-3}}$, from (3.5.4) we deduce that for $1/2 \leqslant r \leqslant 2$ one has when $\ell \to +\infty$

$$V(r) = r^2 + O(\frac{1}{\ell^{n-1}}). \tag{3.5.8}$$

The same is true for all the derivatives of V. It follows that, still on the region $1/2 \leqslant r \leqslant 2$, one has

$$|b_\sigma - g| = O(\frac{1}{\ell^{n-1}}) \text{ and } |\nabla^k(b_\sigma - g)| = O(\frac{1}{\ell^{n-1}}) \tag{3.5.9}$$

for all $k \geqslant 0$. (Here we take the covariant derivative ∇ and the norm with respect to the fixed metric g; this is not an important choice since b_σ is closer and closer to g, so it would be equivalent to use b_σ instead of g). In particular it follows that

$$\mathrm{Ric}(g_\sigma) + (n-1)g_\sigma = O(\frac{1}{\ell^{n-1}}) \tag{3.5.10}$$

and the same is true for all covariant derivatives. Therefore the metrics g_σ are a family of metrics defined on the (distinct) compact manifolds M_σ, which are closer and closer to being Einstein when the length of σ goes to infinity. The family converges to the cusp metric g on M.

We can now state the main theorem of this chapter.

3.5.11. Theorem. — *For ℓ large enough, there is a small perturbation $\tilde{g}_\sigma$ of g_σ on M_σ which satisfies the Einstein equation*

$$\mathrm{Ric}(\tilde{g}_\sigma) + (n-1)\tilde{g}_\sigma = 0.$$

The perturbation $\tilde{g}_\sigma$ is unique (up to diffeomorphism) in a small neighbourhood of g_σ.

If there are several cusp ends, the same result is true, provided that one chooses a long enough geodesic at each end.

The meaning of the 'small neighbourhood' will be made precise in section 3.8, where the theorem is proved.

3.6. Idea of proof of the theorem

We will use the deformation theory of Einstein metrics, studied in section 2.10, to deform g_σ into an Einstein metric. We want to solve the Einstein equation for a metric h close to g_σ. Therefore it is natural to search for h in a good gauge with respect to the action of the diffeomorphisms on the metrics. We choose the condition

$$B^{g_\sigma} h = 0. \tag{3.6.1}$$

It is then natural to pose

$$\Phi(h) = \mathrm{Ric}(h) + (n-1)h + (\delta^h)^* B^{g_\sigma} h. \tag{3.6.2}$$

It has been proved in exercice 2.10.35 that, as far as $\mathrm{Ric}(h) < 0$ (which will be the case by (3.5.10) if h is close to g_σ), the equation $\Phi(h) = 0$ is equivalent to the system

$$B^{g_\sigma} h = 0, \quad \mathrm{Ric}(h) + (n-1)h = 0. \tag{3.6.3}$$

Furthermore, the linearization at g_σ of Φ is

$$L := d_{g_\sigma}\Phi = \frac{1}{2}\nabla^*\nabla - \overset{\circ}{R}. \tag{3.6.4}$$

We can decompose Φ as

$$\Phi(g_\sigma + h) = \Phi(0) + L(h) + Q(h), \tag{3.6.5}$$

where Q is of order 2 in h. The theorem will be proved by using the following abstract lemma.

3.6.6. Lemma. — *Let $\Phi : E \to F$ be a smooth map between Banach spaces, such that $\Phi(x) = \Phi(0) + L(x) + Q(x)$, where $L = d_0\Phi$ is linear and Q satisfies*

$$\|Q(x) - Q(y)\| \leqslant k(\|x\| + \|y\|)\|x - y\| \quad \textit{for } \|x\| \leqslant r_0, \|y\| \leqslant r_0. \tag{3.6.7}$$

Suppose that L is invertible and

$$\|L^{-1}\| \leqslant c \tag{3.6.8}$$

Then if $r \leqslant r_0$ satisfies the two conditions

$$r \leqslant \frac{1}{2kc}, \qquad \|\Phi(0)\| \leqslant \frac{r}{c} - kr^2, \tag{3.6.9}$$

there exists a unique solution of the equation $\Phi(x) = 0$ such that $\|x\| \leqslant r$.

For example, suppose that $c \geqslant 1$, the hypothesis is true for $r = \frac{1}{2kc}$ if $\|\Phi(0)\| \leqslant \frac{1}{4k}$. Using the freedom to fix r, we see that if furthermore $\|\Phi(0)\|$ is small, then the solution will be also small.

Proof. — The equation can be written $x = -L^{-1}(\Phi(0) + Q(x))$. The result is a direct application of the fixed point theorem for contractant mappings. □

Our application of the abstract lemma to the Einstein equation $\Phi(0) = 0$ is not straight forward. We will consider Φ as a map between Hölder spaces, but the manifolds M_σ vary, so some care has to be taken in the definition of the Hölder spaces. Then the estimate (3.6.7) on Q is easy because Q is an explicit nonlinear operator. The norm of $\Phi(0)$ is $O(\frac{1}{\ell^{n-1}})$ so goes to zero, so the main remaining ingredient to apply the lemma is a uniform estimate on the norm of the inverse of the linearization L when $\ell \to +\infty$. As is typical in gluing problems, this last step is the most difficult part of the proof.

3.7. The Hölder spaces

Our first step is to define correctly the Hölder spaces needed for the analysis, in a uniform way for all manifolds M_σ. We cannot choose balls, because when σ goes to infinity the injectivity radius goes to zero. Nevertheless there is an alternative.

It is easy to check that the second fundamental form of the slices $\{x = \mathrm{cst}\}$ for g_{BH} is bounded, and all its covariant derivatives. Now the calculation (3.2.4) for the curvature tensor shows that the curvature and all its covariant derivatives remain bounded indepently of σ.

The fact that the curvature is uniformly bounded implies that the conjugacy radius remains bounded below by a uniform constant ρ. Therefore the exponential map remains a local diffeomorphism on all balls $B_x(\rho) \subset T_x M_\sigma$. Furthermore, because all the covariant derivatives of the curvature are uniformly bounded, we get that on each such ball the metric

$$\exp_x^* g_\sigma = g_{ij} dx^i dx^j \tag{3.7.1}$$

has coefficients (g_{ij}) mutually bounded with those of the flat metric (δ_{ij}),

$$C_0^{-1}(\delta_{ij}) \leqslant (g_{ij}) \leqslant C_0(\delta_{ij}) \tag{3.7.2}$$

where the constant C is uniform with respect to σ, and all the derivatives of g_{ij} are bounded,

$$|\nabla^k g_{ij}| \leqslant C_k \tag{3.7.3}$$

on $B(\rho)$ for a uniform constant C_k. We can say that the metrics g_σ have 'bounded local geometry'.

For each σ, cover M_σ by (a finite number of) images by the exponential map of balls $B_{x_i}(\rho/2)$. The Hölder norm of a tensor f is then defined as the supremum of the Hölder norms of $\exp_{x_i}^* f$ on each ball. Because of the uniform control (3.7.2)–(3.7.3) on the coefficients of the metrics, the elliptic estimate for the geometric operator $L = \frac{1}{2}\nabla^*\nabla - \overset{\circ}{R}$ in these balls have controled constants, see remark 1.7.11 which is valid also for Hölder spaces. Therefore there is a constant $C > 0$, such that the following estimate is valid in every ball $B_{x_i}(\rho)$ and for all σ:

$$\|u\|_{C^{2,\alpha}(B(\rho/2))} \leqslant C\big(\|Lu\|_{C^\alpha(B(\rho))} + \|u\|_{C^0(B(\rho))}\big). \tag{3.7.4}$$

Of course the uniform global elliptic estimate follows:

$$\|u\|_{C^{2,\alpha}(M_\sigma)} \leqslant C\big(\|Lu\|_{C^\alpha(M_\sigma)} + \|u\|_{C^0(M_\sigma)}\big). \tag{3.7.5}$$

We are now in a better position to apply lemma 3.6.6. Indeed we have an operator between two well defined Banach spaces (depending on σ),

$$\Phi : C^{2,\alpha}(S^2T^*M) \to C^\alpha(S^2T^*M), \tag{3.7.6}$$

defined by

$$\Phi(h) = \mathrm{Ric}(g_\sigma + h) + (n-1)(g_\sigma + h) + (\delta^*)^{g+h} B^g h. \tag{3.7.7}$$

In particular,

$$\Phi(0) = \mathrm{Ric}(g_\sigma) + (n-1)g_\sigma, \tag{3.7.8}$$

so by equation (3.5.10) we have

$$\|\Phi(0)\|_{C^\alpha} \leqslant \frac{C}{\ell^{n-1}} \tag{3.7.9}$$

for some constant C independent of σ. Also, $Q(h) = \Phi(h) - \Phi(0) - L(h)$ is a nonlinear explicit expression depending on two derivatives of h, with zero linearisation at 0. It follows that in each uniform ball satisfying (3.7.2)–(3.7.3), if $\|x\|_{C^{2,\alpha}} \leqslant r_0$ and $\|y\|_{C^{2,\alpha}} \leqslant r_0$, then there exists a constant k such that

$$\|Q(x) - Q(y)\|_{C^\alpha} \leqslant k\big(\|x\|_{C^{2,\alpha}} + \|y\|_{C^{2,\alpha}}\big)\|x - y\|_{C^{2,\alpha}}. \tag{3.7.10}$$

Of course, the whole point again is that the constants r_0 and k do not depend on σ. The estimates (3.7.9) and (3.7.10) are precisely the needed ingredients for the application of lemma 3.6.6, the last missing ingredient is a uniform control of the norm of $\|L^{-1}\|$.

Unfortunately, this turns out to be not true between Hölder spaces, as we shall see later in the proof. This is the motivation to introduce modified Hölder spaces in the next section.

3.8. Weighted Hölder norms

On the same spaces $C^{k,\alpha}$ of sections of S^2T^*M we define a slightly different norm. Remind that on the end of M_σ we have a smooth function $r \geqslant r_+$ that we can extend smoothly over the whole M_σ, so that $r > 2$ on the fixed 'compact part' of M_σ. Then we define

$$\|f\|_{C^{k,\alpha}_\delta} = \|\big(\frac{r}{r_+}\big)^\delta f\|_{C^{k,\alpha}}, \tag{3.8.1}$$

where δ is a fixed real number. The quotient by r_+ is just a normalization which ensures that the weight $\frac{r}{r_+}$ satisfies the lower bound $\frac{r}{r_+} \geqslant 1$ if $\delta \geqslant 0$. As r is smooth and bounded below on each M_σ, the $C^{k,\alpha}_\delta$ norm is equivalent to the $C^{k,\alpha}$ norm, but the equivalence is not uniform with respect to σ, because $r_+ \to 0$.

Observe that, with respect to the metrics g_σ, one has a uniform estimate

$$\left|\frac{dr}{r}\right| \leqslant C, \tag{3.8.2}$$

and more generally, all derivatives are uniformly bounded:

$$|\nabla^k \ln r| \leqslant C_k. \tag{3.8.3}$$

It follows that on each ball $B(x,\rho)$ with uniform geometry (3.7.2)–(3.7.3), the function r has only uniform variation: $C^{-1} \leqslant \frac{r}{r(x)} \leqslant C$, where C does not depend on σ; using the bound on derivatives, we get

$$C^{-1}\frac{r(x)^\delta}{r_+^\delta}\|f\|_{C^{k,\alpha}(B(x,\rho))} \leqslant \|f\|_{C^{k,\alpha}_\delta(B(x,\rho))} \leqslant C\frac{r(x)^\delta}{r_+^\delta}\|f\|_{C^{k,\alpha}(B(x,\rho))} \tag{3.8.4}$$

for a constant which does not depend on σ. We deduce that the elliptic estimate (3.7.4) still holds with uniform constants in the ball $B(x,\rho)$:

$$\|u\|_{C^{2,\alpha}_\delta(B(x,\rho/2))} \leqslant C\big(\|Lu\|_{C^\alpha_\delta(B(x,\rho))} + \|u\|_{C^0_\delta(B(x,\rho))}\big). \tag{3.8.5}$$

As usual, this implies the global elliptic estimate

$$\|u\|_{C^{2,\alpha}_\delta} \leqslant C\big(\|Lu\|_{C^\alpha_\delta} + \|u\|_{C^0_\delta}\big). \tag{3.8.6}$$

We can check whether we are still in a position to apply the fixed point theorem (lemma 3.6.6) in these modified Hölder spaces. First, because of the weight, the estimate (3.7.9) on $\Phi(0)$ becomes

$$\|\Phi(0)\|_{C^\alpha_\delta} \leqslant \frac{C}{r_+^\delta \ell^{n-1}}. \tag{3.8.7}$$

Using the formula (3.2.2), we know that $r_+ \sim \ell^{-(n-1)}$ so we get

$$\|\Phi(0)\|_{C^\alpha_\delta} \leqslant \frac{C}{\ell^{n-1-\delta}}. \tag{3.8.8}$$

This is still good if $\delta < n-1$.

Second, the estimate (3.7.10) is a local estimate on each ball $B(x,\rho)$. As above we pass to an estimate by multiplying the norms involved by $(\frac{r(x)}{r_+})^\delta$: we obtain

$$\begin{aligned}(\frac{r(x)}{r_+})^\delta &\|Q(x)-Q(y)\|_{C^\alpha(B(x,\rho))} \\ &\leqslant k(\frac{r(x)}{r_+})^\delta\big(\|x\|_{C^{2,\alpha}(B(x,\rho))} + \|y\|_{C^{2,\alpha}(B(x,\rho))}\big)\|x-y\|_{C^{2,\alpha}(B(x,\rho))}.\end{aligned} \tag{3.8.9}$$

As observed above, if $\delta \geqslant 0$ then $\frac{r}{r_+} \geqslant 1$, so $\frac{r}{r_+} \leqslant (\frac{r}{r_+})^2$. Using that $(\frac{r(x)}{r_+})^\delta\|\cdot\|^{k,\alpha}_C$ is (uniformly) equivalent to $\|\cdot\|_{C^{k,\alpha}_\delta}$, we obtain

$$\|Q(x)-Q(y)\|_{C^\alpha_\delta} \leqslant k\big(\|x\|_{C^{2,\alpha}_\delta} + \|y\|_{C^{2,\alpha}_\delta}\big)\|x-y\|_{C^{2,\alpha}_\delta}. \tag{3.8.10}$$

The third and last ingredient is the following lemma, which will be proved in the next section:

3.8.11. Lemma. — *For $0 < \delta < n-1$, the operator $L : C^{2,\alpha}_\delta \to C^\alpha_\delta$ is invertible, and the norm of the inverse is bounded uniformly with respect to σ:*

$$\|L^{-1}\| \leqslant C.$$

Given lemma 3.8.11, the theorem 3.5.11 now follows directly from the fixed point theorem: the constant C and k are fixed, and for ℓ large enough, the norm $\|\Phi(0)\|_{C^\alpha}$ goes to zero by (3.8.8). So lemma 3.6.6 provides a unique solution h for ℓ large enough. Moreover, because of the bound on $\|\Phi(0)\|$, we can be sure that, when ℓ goes to infinity, the solution satisfies

$$\|h\|_{C^{2,\alpha}_\delta} = O(\ell^{-(n-1-\delta)}), \tag{3.8.12}$$

which means that the Einstein metric $g_\sigma + h$ is closer and closer to the approximate solution g_σ.

3.8.13. Exercise. — Let (M^n, g) be a complete hyperbolic metric, with finite volume and a finite number k of cusps. The manifold M can be written as

$$M = M_0 \amalg E_1 \amalg \cdots \amalg E_k,$$

where M_0 is a compact manifold, whose boundary has k torus components, and each end $E_i = (0,2) \times T_i^{n-1}$ with metric $g|_{E_i} = \frac{dr^2}{r^2} + r^2\gamma_i$, where γ_i is a flat metric on the torus T_i. The end E_i is glued to M_0 along its boundary $\{2\} \times T_i = T_i$.

For a choice of simple closed geodesics σ_i in T_i, with length $\ell(\sigma_i) > L$ for L large enough, one produces by Dehn surgery an Einstein metric $g_{\sigma_1,\dots,\sigma_k}$ on a compact manifold $M_{\sigma_1,\dots,\sigma_k}$.

1) Prove that when the lengths of all the geodesics σ_i go to infinity, then $(M_{\sigma_1,\dots,\sigma_k}, g_{\sigma_1,\dots,\sigma_k})$ converges to (M, g).

2) One fixes the first r geodesics $\sigma_1, \dots, \sigma_r$ so that $\ell(\sigma_i) > L$ for $1 \leqslant i \leqslant r$, and one takes a limit when the $\ell(\sigma_i) \to +\infty$ for $i > r$. Prove that the Einstein metrics $g_{\sigma_1,\dots,\sigma_k}$ converge to a metric $g_{\sigma_1,\dots,\sigma_r}$ on a noncompact manifold $M_{\sigma_1,\dots,\sigma_r}$, obtained topologically from M by Dehn surgery on the first k ends. Give an approximation of the metric $g_{\sigma_1,\dots,\sigma_r}$ and precise a bound on the error term. Prove that the metric $g_{\sigma_1,\dots,\sigma_r}$ is complete with finite volume, and Einstein.

3.9. Estimate for the inverse of the linearisation

Lemma 3.8.11 is the main technical result needed for the proof of the gluing theorem. The proof is by contradiction. Suppose that we have no uniform estimate

$$\|u\|_{C^{2,\alpha}_\delta} \leqslant C\|Lu\|_{C^\alpha_\delta}, \tag{3.9.1}$$

then there exists a sequence σ_i and corresponding u_i such that

$$\|u_i\|_{C^{2,\alpha}_\delta} = 1, \quad \|Lu_i\|_{C^\alpha_\delta} \to 0. \tag{3.9.2}$$

By the elliptic estimate (3.8.6), the norm

$$\|u_i\|_{C^0_\delta} = \sup \big(\frac{r}{r_+}\big)^\delta |u| \tag{3.9.3}$$

remains bounded below, so there is a sequence of points $x_i \in M_{\sigma_i}$ such that

$$\big(\frac{r(x_i)}{r_+}\big)^\delta |u(x_i)| \geqslant \eta > 0. \tag{3.9.4}$$

Three cases may occur:

1. there is a subsequence such that $r(x_i)$ remains bounded below by a positive constant, $r(x_i) \geqslant A^{-1} > 0$; this means that x_i remains at bounded distance of the compact part of M;
2. for a subsequence, $\frac{r(x_i)}{r_+} \leqslant A$; this means that x_i remains at finite distance of the core torus T^{n-2} of the black hole metric;
3. for a subsequence, $r(x_i) \to 0$ and $\frac{r(x_i)}{r_+} \to +\infty$; this is a case where x_i remains in the transition part between the black hole metric and the hyperbolic metric on M.

In each case, we will see that because of (3.9.2) we can obtain a convergence of u_i towards a nonzero solution u of $Lu = 0$ on some limit of the M_σ, and in each case we will see that this is not allowed. This will prove the lemma.

3.9.5. Remark. — This method of proof is known as 'blowup analysis': in order to control the solutions, one looks at the possible places where they could blow up, and extracts limits on certain limiting manifolds. The limits are then ruled out using case by case arguments.

First case. Here $x_i \in M_\sigma$ converges to $x \in M$. Consider

$$\tilde u_i = \frac{u_i}{r_+^\delta}. \tag{3.9.6}$$

Then by (3.9.2) the sequence $\tilde u_i$ satisfies

$$\|r^\delta \tilde u_i\|_{C^{2,\alpha}} \leqslant 1, \quad \|r^\delta L\tilde u_i\|_{C^\alpha} \to 0, \quad r(x_i)^\delta |\tilde u_i(x_i)| \geqslant \eta > 0. \tag{3.9.7}$$

Fix some $R > 0$. Let us consider x_i as a point of M, then on the ball around x of radius R for g, the metrics g_σ converge to the hyperbolic metric g. Because

$\|u_i\|_{C^{2,\alpha}(B_R)}$ remains bounded, we can extract weakly in $C^{2,\alpha}_\delta$ a limit u on the ball B_R, which satisfies $Lu = 0$ because of (3.9.7). Because of the compact embedding $C^{2,\alpha} \subset C^0$, the convergence is strong in C^0 so the limit satisfies again $r(x)^\delta |u(x)| \geqslant \eta$. Now taking R larger and larger, we can extract a subsequence which converges on each ball B_R towards a limit u defined on the whole M; on (M, g) the limit now satisfies

$$\|r^\delta u\|_{C^{2,\alpha}} \leqslant 1, \quad Lu = 0, \quad u(x) \neq 0. \tag{3.9.8}$$

Remind that g is hyperbolic, and $L = \frac{1}{2}\nabla^*\nabla - \overset{\circ}{R}$ has no L^2 kernel for a hyperbolic metric by formula (2.10.31), see also remark 1.4.3 to handle the non compactness. If $\delta < \frac{n-1}{2}$, then the first condition in (3.9.8) implies that $u \in L^2$, so u vanishes, which is a contradiction[(2)].

Second case. The points x_i remain at bounded distance of the core torus T^{n-2}, in particular $r(x_i) \to 0$. We then rewrite the metrics g_σ in coordinates where the point x_i converges. We take

$$s = \frac{r}{r_+}, \tag{3.9.9}$$

which by section 3.4 takes the toral black hole metric part of g_σ to the metric

$$\tilde{g}_\sigma = V^{-1}(s) ds^2 + V(s) r_+^2 d\theta^2 + s^2 r_+^2 g_{\mathbb{R}^{n-2}} \tag{3.9.10}$$

on $(1, +\infty) \times (S^1 \times \mathbb{R}^{n-2})/\mathbb{Z}^{n-2}$. Here $V(s)$ is the function obtained for the value $a = 1$ of the parameter,

$$V(s) = s^2 - \frac{1}{s^{n-3}}. \tag{3.9.11}$$

Of course the formula (3.9.10) coincides with g_σ only on the part $\{r \leqslant \frac{1}{2}\}$, that is $s \leqslant \frac{1}{2r_+}$ (which is of order $\ell^{-(n-1)}$).

Observe now that:

- $s(x_i)$ remains bounded, and $u(x_i) \geqslant \eta$;
- $\|(\frac{r}{r_+})^\delta u_i\|_{C^{2,\alpha}} = \|s^\delta u_i\|_{C^{2,\alpha}} \leqslant 1$;
- similarly $\|s^\delta L u_i\|_{C^\alpha} \to 0$.

Here one must be careful, because the manifold varies for each i. It is useful to rewrite the metric (3.9.10) as in formula (3.3.4):

$$\tilde{g}_\sigma = V(s)^{-1} ds^2 + r_+^2 \big(s^2 g_{T^{n-1}} - \frac{1}{s^{n-3}} \eta^2\big), \tag{3.9.12}$$

(2) Actually one can prove that for δ satisfying $0 < \delta < \frac{n-1}{2}$, there is no solution of $Lu = 0$ on M with $r^\delta |u| \leqslant C$.

where $g_{T^{n-1}}$ is now the fixed given metric on T^{n-1}, and η is the unit 1-form dual to the circle action associated to the choice of the geodesic σ in T^{n-1}.

The following observation is useful.

3.9.13. Lemma. — *Suppose that f is a function on a torus T with flat coordinates (x^i). Denote by $\bar{f}$ the mean value of f on T. Suppose now that g_ϵ is a metric on T with diameter smaller than ϵ. If $\|f\|_{C^{2,\alpha}(g_\epsilon)} \leqslant c$, then*

$$|f(x)-\bar{f}| + \sum |\partial_{x^i} f| + \sum |\partial^2_{x^i x^j} f| \leqslant c\epsilon^{2+\alpha}.$$

Proof. — First if $\|f\|_{C^\alpha(g_\epsilon)} \leqslant c$, then writing

$$|f(x)-f(x_0)| \leqslant \|f\|_{C^\alpha(g_\epsilon)} d_{g_\epsilon}(x_0,x)^\alpha,$$

and using that $d_{g_\epsilon}(x_0,x)$ is smaller that the diameter ϵ, we obtain that the difference $f-f(x_0)$ is controled by $c\epsilon^\alpha$. Noting $\bar{f}$ the mean value of f on T, we can rewrite this as

$$|f(x)-\bar{f}| \leqslant c\epsilon^\alpha.$$

Now apply this to the second derivatives $\partial^2_{x^i x^j} f$: because of the scale factor, the hypothesis implies $\|\partial^2_{x^i x^j} f\|_{C^\alpha(g_\epsilon)} \leqslant c\epsilon^2$, and their mean value is 0, so we obtain

$$|\partial^2_{x^i x^j} f| \leqslant c\epsilon^{2+\alpha}.$$

The result now follows easily. □

The lemma proves that if we have a more and more collapsed torus, the functions with bounded derivatives tend to be constant. Of course, the same is true for sections of flat bundles.

Now come back to our metrics $\tilde{g}_\sigma$ and the solutions u_i. Take fixed coordinates (x^i) on T^{n-1}. The form (3.9.10) of the metric and the bound on the derivatives of u_i implies that u_i is close to a tensor $\bar{u}_i(r)$ depending only on r:

$$u_i = \bar{u}_i(r) + v_i, \quad |v_i| + \sum |\partial_{x^j} v_i| + \sum |\partial^2_{x^j x^k} v_i| \leqslant c(s) r_+^{2+\alpha}. \tag{3.9.14}$$

It follows immediately that we can extract a subsequence (u_i) which converges on every compact subset of $(1,+\infty)\times T^{n-1}$ to a u which depends only on the variable s; also $x_i \to x$ and $u(x) \geqslant \eta > 0$ so the solution is not zero; finally we have $\sup s^\delta |u| \leqslant 1$.

Another way to see this limit is to pullback the sequence (u_i) to a sequence $\tilde{u}_i$ on the fixed manifold $(1,+\infty)\times\mathbb{R}^{n-1}$, where the second factor has coordinates $\tilde{x}_i = r_+ x_i$; the metrics g_σ become the fixed metric

$$g_{BH} = V(s)^{-1}ds^2 + V(s)(d\tilde{x}^2)^2 + s^2\big((d\tilde{x}^3)^2 + \cdots + (d\tilde{x}^n)^2\big), \tag{3.9.15}$$

compactified at $s = 1$ by adding a $\mathbb{R}^{n-2}$. The control (3.9.14) on the derivatives of u_i then gives a control

$$r_+^{-2}|v_i| + r_+^{-1} \sum |\partial_{\tilde{x}^j} v_i| + \sum |\partial^2_{\tilde{x}^j \tilde{x}^k} v_i| \leqslant c(s) r_+^{\alpha}. \tag{3.9.16}$$

All the derivatives along $\mathbb{R}^{n-1}$ go to zero (in particular the second derivatives involved in $L\tilde{u}_i$), so $\tilde{u}_i$ converges to a limit u which depends only on s and is a solution of $Lu = 0$ for g_{BH}.

Now the contradiction follows from:

3.9.17. Lemma. — *Let $0 < \delta < n-1$. On $\mathbb{R}^2 \times \mathbb{R}^{n-2}$ with the black hole metric (3.9.15), there is no solution u of the equation*

$$Lu = \frac{1}{2}\nabla^*\nabla u - \mathring{R} u = 0$$

with $u = u(s)$ depending on s only, and satisfying $|u| = O(s^{-\delta})$ when $s \to +\infty$.

The proof relies on a painful calculation of the ODE system satisfied by u. This is done in section 3.10.

It is interesting to note that this lemma would be wrong with $\delta = 0$, there does exist bounded solutions of $Lu = 0$, and this is the main motivation for introducing the weight r^{δ}. Indeed the metric (3.9.15) comes in a family of Einstein metrics, obtained by varying the metric $g_{T^{n-1}}$ of the torus. The corresponding infinitesimal deformations provide bounded solutions of $Lu = 0$.

Third case. We suppose that x_i goes infinitely far from the compact part ($r(x_i) \to 0$), but goes also infinitely far from the core torus ($\frac{r(x_i)}{r_+} \to +\infty$). Then we proceed similarly to the previous case, by introducing the coordinate

$$s = \frac{r}{r(x_i)}. \tag{3.9.18}$$

The effect is to send $r(x_i)$ to the fixed $s(x_i) = 1$. In these coordinates, the metric is the toral black hole metric (3.9.15), but with parameter

$$\tilde{a}_i = \frac{a_i}{r(x_i)^{n-1}} = \frac{a_i}{r_+^{n-1}} \frac{r_+^{n-1}}{r(x_i)^{n-1}} \to 0 \tag{3.9.19}$$

since a_i and r_+^{n-1} have the same order. This means that the metrics g_σ on the coverings $(\frac{r_+}{r(x_i)}, +\infty) \times \mathbb{R}^{n-1}$ converge to the model hyperbolic metric

$$g = \frac{ds^2}{s^2} + s^2 g_{\mathbb{R}^{n-1}} \tag{3.9.20}$$

Let $x = \lim x_i$. It follows that we can extract a convergent subsequence (u_i) on balls $B(x, R)$, and the limit u satisfies

1. u depends on s only;
2. $Lu = 0$ for the limit metric g;
3. $\sup s^\delta |u| < +\infty$;
4. $u(x) \neq 0$.

Of course, we can as well consider u as a solution for the complete cusp metric on $(0, +\infty) \times T^{n-1}$. Here the contradiction follows from the following lemma, which will be proved in section 3.10:

3.9.21. Lemma. — *Let $0 < \delta < n-1$. On $\mathbb{R}_+ \times T^{n-1}$ with the cusp hyperbolic metric $\frac{ds^2}{s^2} + s^2 g_{T^{n-1}}$, there is no solution of the equation*

$$Lu = \frac{1}{2}\nabla^*\nabla u - \overset{\circ}{R} u = 0$$

with $u = u(s)$ depending on s only, and satisfying $\sup s^\delta |u| < +\infty$.

3.10. Explicit calculations for the toral black hole metric

In this section we provide the explicit calculations promised in the surgery construction, and we prove lemmas 3.9.17 and 3.9.21. We consider the black hole toral metric

$$g = \frac{dr^2}{V(r)} + V(r) d\theta^2 + r^2 g_{T^{n-2}}. \tag{3.10.1}$$

Our aim is to understand the kernel of the infinitesimal Einstein operator

$$L = \frac{1}{2}\nabla^*\nabla - \overset{\circ}{R} \tag{3.10.2}$$

on tensors depending on the variable r only. Therefore, *in the whole section we consider functions (or tensors) depending on the variable r only, that is constant on each slice $\{r\} \times T^{n-1}$.*

Let $\vec{n} = \sqrt{V}\partial_r$ denote the normal vector of the slices $\{r\} \times T^{n-1}$. The second fundamental form of the slices, $\mathbb{I} = -\frac{1}{2}\mathscr{L}_{\vec{n}}(V d\theta^2 + r^2 g_{T^{n-2}})$, is calculated as

$$\mathbb{I} = -\frac{1}{2}\sqrt{V}\partial_r V d\theta^2 - r\sqrt{V} g_{T^{n-2}}, \tag{3.10.3}$$

or

$$g^{-1}\mathbb{I} = -\begin{pmatrix} q_1 & & & \\ & q_2 & & \\ & & \ddots & \\ & & & q_{n-1} \end{pmatrix} \tag{3.10.4}$$

with

$$q_1 = -\frac{\partial_r V}{2\sqrt{V}}, \quad q_2 = \cdots = q_{n-1} = \frac{\sqrt{V}}{r}. \tag{3.10.5}$$

Also we shall denote

$$H = -\operatorname{Tr}^g \mathbb{I} = \sum_1^{n-1} q_i = q_1 + (n-2)q_2. \tag{3.10.6}$$

Choose orthonormal coordinates $(x^1 = \theta, x^2, \dots, x^{n-1})$ on $T^{n-1} = S^1 \times T^{n-2}$. We shall calculate in the orthonormal frame

$$e_0 = \sqrt{V}\partial_r, e_1 = \frac{\partial_\theta}{\sqrt{V}}, e_2 = \frac{\partial_2}{r}, e_3 = \frac{\partial_3}{r}, \dots \tag{3.10.7}$$

and the dual frame

$$e^0 = \frac{dr}{\sqrt{V}}, e^1 = \sqrt{V}d\theta, e^2 = r dx^2, e^3 = r dx^3, \dots \tag{3.10.8}$$

Levi-Civita connection. — The Levi-Civita connection is given by

$$\begin{cases} \nabla e^0 = \sum_1^{n-1} q_i e^i \otimes e^i, \\ \nabla e^j = -q_j e^j \otimes e^0, \quad j \geqslant 1. \end{cases} \tag{3.10.9}$$

From these formulas we deduce the formulas on symmetric 2-tensors; since the basis (e^j) is ∇_0-parallel, it suffices to write ∇_j for $j \geqslant 1$. The result is below, with the convention that we write only the list of nonzero derivatives:

$$\begin{aligned} \nabla_j (e^0)^2 &= q_j (e^0 e^j + e^j e^0), \\ \nabla_j (e^j)^2 &= -q_j (e^0 e^j + e^j e^0), \\ \nabla_j (e^0 e^j + e^j e^0) &= 2q_j ((e^j)^2 - (e^0)^2), \\ \nabla_j (e^0 e^k + e^k e^0) &= q_j (e^j e^k + e^k e^j), \quad k \neq j, k \geqslant 1, \\ \nabla_j (e^j e^k + e^k e^j) &= -q_j (e^0 e^j + e^j e^0), \quad k \neq j, k \geqslant 1. \end{aligned} \tag{3.10.10}$$

Rough Laplacian. — We deduce the rough Laplacian on symmetric 2-tensors:

$$\nabla^* \nabla = -\sum_0^{n-1} (\nabla_i^2 - \nabla_{\nabla_i e_i}) = -\nabla_0^2 - H\nabla_0 + \mathscr{Q} \tag{3.10.11}$$

with $\mathscr{Q} = -\sum_1^{n-1} \nabla_i^2$ given by the formulas (here as above $j,k \geqslant 1$ and $j \neq k$):

$$\begin{aligned}
\mathscr{Q}(e^0)^2 &= -2\sum_1^{n-1} q_i^2((e^i)^2 - (e^0)^2),\\
\mathscr{Q}(e^j)^2 &= 2q_j^2((e^j)^2 - (e^0)^2),\\
\mathscr{Q}(e^0e^j + e^je^0) &= (3q_j^2 + \sum_1^{n-1} q_i^2)(e^0e^j + e^je^0),\\
\mathscr{Q}(e^je^k + e^ke^j) &= (q_j^2 + q_k^2)(e^je^k + e^ke^j).
\end{aligned} \tag{3.10.12}$$

Since the frame (e^i) is ∇_0-parallel, the term involving derivatives with respect to r, that is $-\nabla_0^2 - H\nabla_0$ acts on each coefficient as the usual scalar Laplacian:

$$-\nabla_0^2 - H\nabla_0 = \Delta = -\frac{1}{r^{n-2}}\partial_r(r^{n-2}V\partial_r). \tag{3.10.13}$$

Riemannian curvature. — The surfaces obtained by fixing all variables x^i but one fixed variable x^j are totally geodesic. It follows that $e_0 \wedge e_j$ is an eigenvector of the curvature operator: $R(e_0 \wedge e_j) = -K_{0j}e_0 \wedge e_j$. Also, the torus slices $\{r\} \times T^{n-1}$ are flat and the second fundamental form is diagonal, so it follows that similarly for $j,k \geqslant 1$ one has $R(e_j \wedge e_k) = -K_{jk}e_j \wedge e_k$, where K_{jk} is determined by the second fundamental form:

$$K_{jk} = -q_jq_k. \tag{3.10.14}$$

The other coefficients K_{0j} are given by

$$K_{0j} = -\sqrt{V}\partial_r q_j - q_j^2. \tag{3.10.15}$$

A useful remark is that for $j \geqslant 2$, one has

$$K_{0j} - K_{1j} = -\sqrt{V}\partial_r(\frac{\sqrt{V}}{r}) - \frac{V}{r^2} + \frac{\sqrt{V}}{r}\frac{1}{2}\frac{\partial_r V}{\sqrt{V}} = 0. \tag{3.10.16}$$

From the form of the curvature tensor, the Ricci tensor is diagonal is the basis (e_i), and we get

$$\begin{aligned}
\mathrm{Ric}_{00} &= \sum_{i\geqslant 1} K_{0i} = -\sqrt{V}\partial_r H - \sum_{i\geqslant 1} q_i^2\\
\mathrm{Ric}_{jj} &= \sum_{i\neq i} K_{ij} = -\sqrt{V}\partial_r q_j - Hq_j
\end{aligned} \tag{3.10.17}$$

From equation (3.10.16) one has $\mathrm{Ric}_{00} = \mathrm{Ric}_{11}$. A calculation gives, for $j \geqslant 2$,

$$\mathrm{Ric}_{jj} = -\frac{\partial_r V}{r} - (n-3)\frac{V}{r^2} \tag{3.10.18}$$

so the Einstein equation $\mathrm{Ric}_{jj} = -(n-1)$ gives immediately the familiar form

$$V(r) = r^2 - \frac{a}{r^{n-3}} \tag{3.10.19}$$

for some constant a. For this V it is easy to check that $\mathrm{Ric}_{00} = \mathrm{Ric}_{11} = -(n-1)$ as well, so we indeed get an Einstein metric.

Finally we calculate the action $\mathring{R}$ of the curvature on symmetric 2-tensors. From the definition

$$(\mathring{R}h)_{X,Y} = \sum h(R_{e_i,X}Y, e_i) \tag{3.10.20}$$

and the fact the curvature operator is diagonal, we deduce immediately that, for $j \neq k$,

$$\mathring{R}(e^j e^k + e^k e^j) = -K_{jk}(e^j e^k + e^k e^j), \quad \mathring{R}((e^j)^2) = \sum_{i\neq j} K_{ij}(e^i)^2. \tag{3.10.21}$$

We now suppose that the metric is Einstein, that is V is given by the formula (3.10.19), which we have not used previously. We can then give the complete eigendecomposition of $\mathring{R}$: here is the list of the eigenvalues and of the corresponding eigenvectors:

1. $-(n-1)$ for $g = \sum_0^{n-1}(e^i)^2$;
2. $-K_{01}$ for $(e^0)^2 - (e^1)^2$ and $e^0e^1 + e^1e^0$;
3. $-K_{02} = -K_{12}$ for $e^0e^j + e^je^0$ and $e^1e^j + e^je^1$ for $j \geqslant 2$;
4. $-K_{23}$ for $(e^j)^2 - (e^k)^2$ and $e^je^k + e^ke^j$ for $j, k \geqslant 2$;
5. $K_{01} - 2K_{02}$ for $\frac{(e^0)^2+(e^1)^2}{2} - \frac{(e^2)^2+\cdots+(e^{n-1})^2}{n-2}$.

Hyperbolic case. — Let us now do some explicit calculations for the case $a = 0$, that is g is the complete hyperbolic cusp metric. We begin by a scalar Laplacian,

$$P = \Delta + \lambda = -\frac{1}{r^n}\partial_r(r^{n-2}\partial_r) + \lambda. \tag{3.10.22}$$

There are two radial solutions, $f_i(r) = r^{\delta_i}$, with

$$\delta_1, \delta_2 = -\frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} + \lambda}. \tag{3.10.23}$$

We choose $\delta_1 > -\frac{n-1}{2}$ and $\delta_2 < -\frac{n-1}{2}$. Then there is an obvious consequence, which is at the same time a useful observation: if a radial solution satisfies $|f| = o(r^{\delta_1})$ when r goes to infinity, then $|f| = O(r^{\delta_2})$ which is a much better decay. A similar statement holds at $r = 0$: if $|f| = o(r^{\delta_2})$ when r goes to zero, then $|f| = O(r^{\delta_1})$.

3.10.24. Remark. — This observation is actually valid for all solutions, and can be proved using Fourier decomposition along the torus T^{n-1}.

Now pass to the operator (3.10.2). Here $q_1 = q_2 = 1$ and all sectional curvatures equal -1, so from equation (3.10.12) and the calculation (3.10.21) and below we deduce the following form of L in the frame $(e^j e^k + e^k e^j)$:

$$L = \frac{1}{2}\Delta + \mathscr{S}, \tag{3.10.25}$$

where Δ is the scalar Laplacian, that is $\Delta(f_{jk}(e^j e^k + e^k e^j)) = (\Delta f_{jk})(e^j e^k + e^k e^j)$, and $\mathscr{S}$ is a linear operator with the following eigenvalues:

1. $(n-1)$ on $\mathbb{R}g$;
2. 0 on $S_0^2 T^* T^{n-1}$ (the directions in T^{n-1} are the $(e_j)_{j\geqslant 1}$);
3. $\frac{n}{2}$ on $(e^0 e^j + e^j e^0)_{j\geqslant 1}$;
4. n on $(e^0)^2 - \frac{1}{n-1}((e^1)^2 + \cdots + (e^{n-1})^2)$.

The important point here is that all the eigenvalues λ of $\mathscr{S}$ are nonnegative, and the zero eigenspace is exactly $S_0^2 T^* T^{n-1}$. Since on all the components, L is of the form $\frac{1}{2}\Delta + \lambda$, it follows from the above analysis that the only bounded infinitesimal Einstein deformations lie in the kernel of $\mathscr{S}$, that is $S_0^2 T^* T^{n-1}$: on this subspace, one has $L = \frac{1}{2}\Delta$, so the bounded solutions are just the constants, since the other solutions blow up at $r = 0$. This corresponds to deforming the flat metric $g_{T^{n-1}}$ of the torus in the formula

$$g = \frac{dr^2}{r^2} + r^2 g_{T^{n-1}}. \tag{3.10.26}$$

Proof of lemma 3.9.21. — Suppose that we have a radial solution of $Lu = 0$ with $\sup r^\delta |u| < +\infty$ for some δ satisfying $0 < \delta < n-1$. We look at the equation on each eigenspace of $\mathscr{S}$: the solutions are r^{δ_1} and r^{δ_2} with $\delta_1 \geqslant 0$ and $\delta_2 \leqslant -(n-1)$. The condition that $u = O(r^{-\delta})$ at infinity rules out r^{δ_1}, and the same condition at $r = 0$ rules out r^{δ_2}, so there is no solution, and the lemma is proved. □

It turns out that the lemma remains true for general (rather than radial) deformations: the general proof relies on remark 3.10.24 to prove that the solution is actually L^2, and then on the usual integration by parts.

Black hole metric. — We now pass to the case of the black hole toral metric. As before we start with the case of the scalar Laplacian (3.10.13). There are two fundamental solutions depending on r only: the first solution is of course the constant solution $f_1(r) = 1$, and the second one is

$$f_2(r) = \int_r^{+\infty} \frac{du}{u^{n-2}V(u)}. \tag{3.10.27}$$

At infinity, $V(r) \sim r^2$ so $f_2(r) \sim \frac{1}{nr^{n-1}}$ which is the solution of the hyperbolic space. When $r \to r_+$, one has $V(r) \sim (n-1)r_+(r-r_+)$ so $f_2(r) \to +\infty$. In particular we see that the only bounded radial solutions are the constants. Furthermore, there is no solution so that $|f(r)| \leqslant cr^{-\delta}$ for some $\delta > 0$.

It is not surprising that the behavior at infinity is the same as the behavior on the hyperbolic space, since when $r \to +\infty$ the coefficients of Δ are asymptotic to that of the hyperbolic Laplacian.

Before attacking the infinitesimal Einstein operator L, it is useful to understand the equation for the infinitesimal action of the diffeomorphisms. Recall that the infinitesimal action of a vector field X is $\delta^* X$; this satisfies of course the infinitesimal Einstein equation, so it lies in the kernel of L if it satisfies the gauge condition $B\delta^* X = 0$. Actually $L\delta^* = \delta^* B \delta^* = \frac{1}{2}\delta^*(\nabla^*\nabla - \mathrm{Ric})$, see section 2.10. So we have to understand the solutions of the equation

$$B\delta^* X = \frac{1}{2}(\nabla^*\nabla - \mathrm{Ric})X = 0. \tag{3.10.28}$$

We use the explicit calculation of the Levi-Civita connection to get:

1. $X = f^j e_j$ with $j \geqslant 1$: then we obtain

$$\delta^* X = \frac{1}{2}(\sqrt{V}\partial_r f^j - q_j f^j)(e^0 e^j + e^j e^0), \tag{3.10.29}$$

$$B\delta^* X = \frac{1}{2}(\Delta f^j + (n-1+q_j^2)f^j)e_j. \tag{3.10.30}$$

2. $X = f^0 e_0$: then

$$\delta^* X = \sqrt{V}\partial_r f^0 (e^0)^2 + q_1 f^0 (e^1)^2 + q_2 f^0\big((e^2)^2 + \cdots + (e^{n-1})^2\big), \tag{3.10.31}$$

$$\mathrm{Tr}(\delta^* X) = \sqrt{V}\partial_r f^0 + H f^0, \tag{3.10.32}$$

$$B\delta^* X = \frac{1}{2}\big(\Delta f^0 + (n-1+q_1^2+(n-2)q_2^2)f\big)e_0. \tag{3.10.33}$$

Observe that in all cases, the equation can be written $(\Delta + \lambda_j(r))f^j = 0$, where $\lambda_j(r)$ is a positive function with limit n at infinity for $j \geqslant 1$, or $2(n-1)$ for $j = 0$. For each j, we therefore have again two solutions with asymptotic behaviour r^{δ_1} and r^{δ_2}, where $\delta_1 > 0$ and $\delta_2 < -(n-1)$ are given by (3.10.23). In particular, for $j \geqslant 1$ one obtains

$$\delta_1 = 1, \quad \delta_2 = -n. \tag{3.10.34}$$

In all cases, observe that if a solution is bounded near infinity then it is $O(r^{-n})$. Of course it must blow up at $r = r_+$ since there is no L^2 solution of the equation (3.10.28).

We can now calculate the (radial) solutions of the infinitesimal Einstein operator L satisfying $\sup |r^\delta u| < +\infty$ for some $\delta > 0$. Again the form of the operator in the basis $(e^j e^k + e^k e^j)$ is

$$L = \frac{1}{2}\Delta + \mathscr{S}, \tag{3.10.35}$$

where $\mathscr{S} = \frac{1}{2}\mathscr{A} - \overset{\circ}{R}$ is a linear operator, which we can calculate explicitly from (3.10.12) and (3.10.21). The asymptotics when $r \to +\infty$ are the same as for the hyperbolic metric, so the solutions of this ODE are asymptotic to that of the hyperbolic space which were calculated above: we have seen that the condition $|u| = O(r^{-\delta})$ at infinity implies $|u| = O(r^{-(n-1)})$, and in particular $u \in L^2$. Then (see section 2.10) the equation $Lu = 0$ implies

$$\operatorname{Tr}(u) = 0, \quad \delta u = 0. \tag{3.10.36}$$

3.10.37. Remark. — The corresponding radial operator for the hyperbolic space, $-\frac{1}{r^n}\partial_r r^{n-2}\partial_r + \mathscr{S}_{hyp}$, where $\mathscr{S}_{hyp}$ is given by the formulas below (3.10.25), is called the *indicial operator* of L. Its solutions completely govern the behaviour of the solutions of our operator L for the metric g_{BH}. We have used this fact for radial solutions (it is then easy, relying on ODE analysis), but it remains true for all solutions. This is more difficult and requires analysis on 'asymptotically hyperbolic manifolds'.

Now write the solution u as

$$u = \sum_0^{n-1} u_{jj}(e^j)^2 + \sum_{0 \leqslant j < k \leqslant n-1} u_{jk}(e^j e^k + e^k e^j). \tag{3.10.38}$$

We have the following cases, corresponding to the different eigenvalues of $\mathscr{S}$. In each case we will show that there is no solution:

1. $2 \leqslant j < k$: then $\mathscr{S}(e^j e^k + e^k e^j) = (q_2^2 + K_{23})(e^j e^k + e^k e^j) = 0$ and similarly $\mathscr{S}((e^j)^2 - (e^k)^2) = 0$; on these components one has $L = \frac{1}{2}\Delta$, the only bounded solutions are the constants, they correspond to changing the flat metric $g_{T^{n-2}}$ in formula (3.10.1), but of course they are not $O(r^{-\delta})$ at infinity;
2. $1 < j$: then $\mathscr{S}(e^1 e^j + e^j e^1) = \frac{(q_1 - q_2)^2}{2}(e^1 e^j + e^j e^1)$, in particular $\mathscr{S} > 0$. So a L^2 solution must vanish.

3. $0 < j$: then $\mathscr{S}(e^0e^j + e^je^0) = (\frac{3}{2}q_j^2 + \sum_1^{n-1} q_i^2 + K_{0j})(e^0e^j + e^je^0)$; here for each j there are two solutions with growth rate r or r^{-n} at infinity; L^2 (or bounded) solutions must be $O(r^{-n})$, and we observe that these solutions must coincide with the solutions (3.10.29) coming from the infinitesimal action of the diffeomorphisms, which blow up at $r = r_+$, so again there is no solution. This can also be proved by writing down the gauge condition $\delta u = 0$ on the tensor $u_{0j}(e^0e^j + e^je^0)$, which results in a first order ODE on u_{0j}.
4. Since $\mathrm{Tr}(u) = 0$, there remains to consider the space spanned by $(e^0)^2 - (e^1)^2$ and $\frac{(e^0)^2+(e^1)^2}{2} - \frac{(e^2)^2+\cdots+(e^{n-1})^2}{n-2}$. The tensors in this space are determined by the coefficients u_{00} and u_{11}. The gauge condition $\delta u = 0$ implies

$$-\sqrt{V}\partial_r u_{00} - (H + q_2)u_{00} = (q_2 - q_1)u_{11}. \tag{3.10.39}$$

This relation implies that the space of solutions (u_{00}, u_{11}) with the decay $O(r^{-(n-1)})$ is 1-dimensional. But we have such a space: the trace free part of the solution (3.10.31), and we know that it blows up at $r = r_+$. So again there is no solution.

This finishes the proof of lemma 3.9.17. This proof is more difficult than in the hyperbolic case: we have proved, as in the hyperbolic case, that the solutions must be L^2, but it is not sufficient to prove that they vanish, because the toral black hole metric has some nonnegative sectional curvatures, so the usual integration by parts is not available. So we had to push further the explicit calculations to indeed rule out all possible solutions.

BIBLIOGRAPHY

[1] M. T. Anderson. Dehn filling and Einstein metrics in higher dimensions. *J. Differ. Geom.*, 73(2):219–261, 2006.

[2] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. Corrected reprint of the 1992 original.

[3] A. L. Besse. *Einstein manifolds*. Springer-Verlag, Berlin, 1987.

[4] J.-P. Demailly. Théorie de Hodge L^2 et théorèmes d'annulation. In *Introduction à la théorie de Hodge*, volume 3 of *Panor. Synthèses*, pages 3–111. Soc. Math. France, Paris, 1996.

[5] S. Donaldson and P. Kronheimer. *The geometry of four-manifolds*. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1990.

[6] S. Kobayashi. *Transformation groups in differential geometry*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1972 edition.

[7] L. I. Nicolaescu. *Notes on Seiberg-Witten theory*, volume 28 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000.

[8] F. W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1983. Corrected reprint of the 1971 edition.