EXTENDED CORRESPONDENCE OF KOSTANT-SEKIGUCHI-VERGNE

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Introduction

Let G/H be a symmetric space of noncompact type, so H is a maximal compact subgroup of G and we have a Cartan decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. The Kostant-Sekiguchi-Vergne correspondence [Ver95] gives a diffeomorphism between the nilpotent G-orbits in $\mathfrak{g}$ and the nilpotent $H^{\mathbb{C}}$-orbits in $\mathfrak{m}^{\mathbb{C}}$.

The aim of this article is to extend this correspondence to all G-orbits in $\mathfrak{g}$. It turns out that each G-orbit in $\mathfrak{g}$ is diffeomorphic to each orbit in a family of $H^{\mathbb{C}}$-orbits in $\mathfrak{m}^{\mathbb{C}}$, parametrised by a 'parabolic weight'. These diffeomorphisms provide a set of H-invariant Kähler metrics on any G-orbit in $\mathfrak{g}$ (such that, of course, the Kähler form equals the Kirillov-Kostant-Souriau symplectic form of the orbit).

More precisely, define the compact Lie algebra $\mathfrak{u} = \mathfrak{h} \oplus i\mathfrak{m}$. Let $\mathfrak{a}_0 \subset \mathfrak{m}$ a maximal abelian subalgebra, which we complete into a Cartan subalgebra of $\mathfrak{u}$,

$$\mathfrak{t}_{\mathfrak{u}} = \mathfrak{t}_0 \oplus i\mathfrak{a}_0, \quad \mathfrak{t}_0 \subset \mathfrak{h}.$$

Then $\mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a 'maximally split' Cartan subalgebra of $\mathfrak{g}$. In order to parametrise all semisimple conjugacy classes of $\mathfrak{g}$, one needs to consider a finite number of Cartan subalgebras $\mathfrak{t}_{\mathfrak{g}} \subset \mathfrak{g}$, which can be chosen to satisfy

$$\mathfrak{t}_{\mathfrak{g}} = \mathfrak{t} \oplus \mathfrak{a}, \qquad \mathfrak{t}_0 \subset \mathfrak{t} \subset \mathfrak{h}, \quad \mathfrak{a} \subset \mathfrak{a}_0 \subset \mathfrak{m}. \tag{1}$$

Then any semisimple element in $\mathfrak{g}$ is G-conjugate to $\tau_1 + i\tau_2$, where

$$\tau_1 \in \mathfrak{t}, \quad \tau_2 \in i\mathfrak{a}, \tag{2}$$

and $\mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ as in (1). By the Kostant-Rallis theorem, any nilpotent element in $\mathfrak{g}$ is conjugate to $\sigma_1 + i\sigma_2$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a representation of $\mathfrak{su}_2$ into $\mathfrak{u}$ (so $[\sigma_1, \sigma_2] = -2\sigma_3$, etc.), such that

$$\sigma_1 \in \mathfrak{h}, \quad \sigma_2, \sigma_3 \in i\mathfrak{m}. \tag{3}$$

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Finally, any element of $\mathfrak{g}$ is conjugate to $\tau_1+i\tau_2+\sigma_1+i\sigma_2$, with (τ_1,τ_2) satisfying (2), and the representation $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ satisfies (3) and commutes with the τ_i:

$$[\sigma_i,\tau_j]=0. \tag{4}$$

Theorem 1. *1) For any $\tau_3\in i\mathfrak{a}$ such that $[\tau_3,\sigma_i]=0$ and the regularity assumptions $C_{\mathfrak{g}}(\tau_1,\tau_2)=C_{\mathfrak{g}}(\tau_2,\tau_3)=C_{\mathfrak{g}}(\tau_1,\tau_2,\tau_3)$ are satisfied, there exists an H-invariant diffeomorphism from the G-orbit of $\tau_1+i\tau_2+\sigma_1+i\sigma_2$ in $\mathfrak{g}$ to the $H^{\mathbb{C}}$-orbit of $\tau_2+i\tau_3+\sigma_2+i\sigma_3$ in $\mathfrak{m}^{\mathbb{C}}$. This diffeomorphism gives a H-invariant Kähler metric on the G-orbit of $\tau_1+i\tau_2+\sigma_1+i\sigma_2$ in $\mathfrak{g}$, whose Kähler form is the Kirillov-Kostant-Souriau symplectic form.*

2) If $\tau_3\in i\mathfrak{a}$ satisfies $[\tau_3,\sigma_i]=0$ but we have only the regularity $C_{\mathfrak{g}}(\tau_1,\tau_2)=C_{\mathfrak{g}}(\tau_1,\tau_2,\tau_3)$, then there still exists a H-invariant diffeomorphism from the G-orbit of $\tau_1+i\tau_2+\sigma_1+i\sigma_2$ in $\mathfrak{g}$ to a $H^{\mathbb{C}}$-space, which gives a H-invariant Kähler structure on the orbit.

In the more special case of a nilpotent orbit ($\tau_1=\tau_2=0$), the only choice is $\tau_3=0$ and we get a diffeomorphism with a $H^{\mathbb{C}}$-orbit in $\mathfrak{m}^{\mathbb{C}}$. This is the Kostant-Sekiguchi-Vergne correspondence [Ver95] between nilpotent G-orbits in $\mathfrak{g}$ and nilpotent $H^{\mathbb{C}}$-orbits in $\mathfrak{m}^{\mathbb{C}}$. Our correspondence is therefore an extension of the correspondence between nilpotent orbits.

The $H^{\mathbb{C}}$-spaces involved in the second part of the theorem are degenerations of $H^{\mathbb{C}}$-orbits in $\mathfrak{m}^{\mathbb{C}}$. See section 2 for details.

The theorem does not depend on the choice of the Cartan subalgebra $\mathfrak{t}_{\mathfrak{g}}=\mathfrak{t}\oplus\mathfrak{a}\subset\mathfrak{g}$. We make the choice in order to make clear the number of parameters in the construction. Therefore, given $\tau_1+i\tau_2$, the choice of $\mathfrak{t}_{\mathfrak{g}}$ which gives all the parameters τ_3 in the construction is the one satisfying (2) with $\mathfrak{a}$ of maximal dimension.

The method gives also results for degenerate G-orbits. For example, let us restrict to the extreme case $\tau_1=\tau_2=0$, $\sigma=0$. An element $\tau^r(=i\tau_3)\in\mathfrak{a}$ defines a parabolic subgroup P of G, with Lie algebra

$$\mathfrak{p}=\mathfrak{l}\oplus\mathfrak{n},\quad \mathfrak{l}=C(\tau^r),$$

and the nilpotent part $\mathfrak{n}$ is the sum of the eigenspaces of $\operatorname{ad}\tau^r$ with negative eigenvalues. Then:

Theorem 2. *The cotangent bundle $T^*(G/P)$ admits a family of H-invariant Kähler metrics, parameterised by elements $\tau^r\in\mathfrak{a}$ such that $C(\tau^r)=\mathfrak{l}$. The corresponding complex structure is that of the $H^{\mathbb{C}}$-orbit of τ^r in $\mathfrak{m}^{\mathbb{C}}$.*

For a complete statement including the degenerate cases, see theorem 3. Finally the reader may find convenient to check the different cases in the simplest case ($G = SU_{1,1}$) in section 4.

The motivation for these results comes from a joint project with Oscar García-Prada and Ignasi Mundet I Riera, to study the moduli spaces of representations of the fundamental group of a punctured Riemann surface into a real group G. A natural condition to obtain symplectic moduli spaces is to fix the conjugacy class of the monodromy around the punctures, or equivalently, the conjugacy class $\mathscr{O}_G(A)$ of a logarithm in $\mathfrak{g}$. A powerful tool of study of these moduli spaces is the bijection with Higgs bundles ([Hit87, Don87] and [Sim90] in the punctured case). In the case of a representation into G, the corresponding Higgs bundle is a $H^{\mathbb{C}}$-holomorphic bundle E with a Higgs field $\theta \in H^0(\Omega^1_{\log D} \otimes E(\mathfrak{m}^{\mathbb{C}}))$, where D is the divisor consisting of the marked points. Here the natural moduli spaces are obtained by fixing the $H^{\mathbb{C}}$-conjugacy class $\mathscr{O}_{H^{\mathbb{C}}}(B)$ in $\mathfrak{m}^{\mathbb{C}}$ of the residue of θ at the points.

In this correspondence between G-representations and Higgs bundles, one can ask what is the relation between the conjugacy classes $\mathscr{O}_G(A) \subset \mathfrak{g}$ and $\mathscr{O}_{H^{\mathbb{C}}}(B) \subset \mathfrak{m}^{\mathbb{C}}$? the answer is that the relation is exactly the extended Kostant-Sekiguchi-Vergne correspondence. Moreover, on both sides (representations and Higgs bundles), one can add parameters ('parabolic structures') which correspond also exactly to the parameters which appear in the extended Kostant-Sekiguchi-Vergne correspondence.

Finally let us observe that the relation with representations of π_1 is a motivation, but also it gives the method to prove the theorem. The transcendental correspondence between representations and Higgs bundles is realised using Hitchin's selfduality equations, which specialise to the Nahm's equations already used by Vergne in the nilpotent case [Ver95].

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1. Nahms's equations and the moduli space

We shall consider a space of solutions of Nahm's equations:

$$\begin{aligned} \frac{dT_1}{ds} &= -[T_2, T_3], \\ \frac{dT_2}{ds} &= -[T_3, T_1], \\ \frac{dT_3}{ds} &= -[T_1, T_2], \end{aligned} \tag{5}$$

where s is a real parameter, $s \in (-\infty, 0]$, and $T_i(s)$ takes value into $\mathfrak{u}$, more precisely:

$$T_1(s) \in \mathfrak{h}, \quad T_2(s), T_3(s) \in i\mathfrak{m}. \tag{6}$$

There is an interpretation of these equations in terms of flat connections and Hitchin's selfduality equations. Indeed, one can add an unknown function,

$$T_0(s) \in \mathfrak{h}, \tag{7}$$

and consider the new system:

$$\begin{aligned} \frac{dT_1}{ds} &= -[T_0, T_1] - [T_2, T_3], \\ \frac{dT_2}{ds} &= -[T_0, T_2] - [T_3, T_1], \\ \frac{dT_3}{ds} &= -[T_0, T_3] - [T_1, T_2]. \end{aligned} \tag{8}$$

This system is equivalent to Hitchin's selfduality equations for the G-connection on the disk, written in polar coordinates (r, θ):

$$D = d + (T_0 + iT_3)\frac{dr}{r} + (T_1 + iT_2)d\theta. \tag{9}$$

The two first equations of (8) say that D is flat, and the third equation is the condition the metric be harmonic.

The relation between the systems (5) and (8) is simple: the gauge transformations $g(s) \in H$ act on the solutions of the second system (8) by $(T_0, T_1, T_2, T_3) \to (\mathrm{Ad}(g)T_0 - \frac{dg}{ds}g^{-1}, \mathrm{Ad}(g)T_1, \mathrm{Ad}(g)T_2, \mathrm{Ad}(g)T_3)$, so one can pass from solutions of the second system to the first system by solving the ordinary differential equation $\frac{dg}{ds}g^{-1} = \mathrm{Ad}(g)T_0$ with the initial condition $g(0) = 1$.

Fix a triple $\tau = (\tau_1, \tau_2, \tau_3)$ of commuting elements of $\mathfrak{u}$, with

$$\tau_1 \in \mathfrak{h}, \quad \tau_2, \tau_3 \in i\mathfrak{m}.$$

(As explained in the introduction, we can suppose that $\tau_1 \in \mathfrak{t}$ and $\tau_2, \tau_3 \in i\mathfrak{a}$ for some Cartan subalgebra $\mathfrak{t}_{\mathfrak{g}} = \mathfrak{t} \oplus \mathfrak{a}$ of $\mathfrak{g}$). We also fix a representation $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of $\mathfrak{su}_2$ into $\mathfrak{u}$ (so $[\sigma_1, \sigma_2] = -2\sigma_3$, etc.), such that

$$\sigma_1 \in \mathfrak{h}, \quad \sigma_2, \sigma_3 \in i\mathfrak{m}, \quad [\sigma_i, \tau_j] = 0.$$

Observe that $T_i(s) = \tau_i - \frac{\sigma_i}{2s}$ is an exact solution of Nahms's equations. More generally, we define $\mathscr{M}(\tau, \sigma)$ to be the space of solutions of Nahm's equations (5) on $(-\infty, 0]$, with

$$T_1(s) \in \mathfrak{h}, \quad T_2(s), T_3(s) \in i\mathfrak{m}, \tag{10}$$

such that there exists $h \in H$ with

$$T_i(s) = \mathrm{Ad}(h)\Big(\tau_i - \frac{\sigma_i}{2s}\Big) + O\Big(\frac{1}{s^{1+\epsilon}}\Big) \tag{11}$$

when $s \to -\infty$. There is an obvious action of H by Ad on $\mathscr{M}(\tau, \sigma)$. (Note that fixing in that way the asymptotic behaviour of the solution corresponds to fixing the monodromy of the corresponding flat connection on the punctured disk).

2. The main result

Before stating the main theorem in this section, let us note $C_{\mathfrak{g}}(\tau)$ the centraliser in $\mathfrak{g}$ of τ_1, τ_2 and τ_3, or $C_{\mathfrak{g}}(\tau_1, \tau_2)$ for the centraliser of τ_1 and τ_2 only.

Theorem 3. *The manifold $\mathscr{M}(\tau, \sigma)$ admits a H-invariant complex structure and Kähler metric. It is complete if $\sigma = 0$.*

If $C_{\mathfrak{g}}(\tau_1, \tau_2) = C_{\mathfrak{g}}(\tau)$ then the H-equivariant map $\mathscr{M}(\tau, \sigma) \to \mathfrak{g}$ defined by

$$(T_i(s)) \to T_1(0) + iT_2(0)$$

identifies, as symplectic manifolds, $\mathscr{M}(\tau, \sigma)$ with the G-(co)adjoint orbit of $\tau_1 + i\tau_2 + \sigma_1 + i\sigma_2$ in $\mathfrak{g}$.

If $C_{\mathfrak{g}}(\tau_2, \tau_3) = C_{\mathfrak{g}}(\tau)$ then the H-equivariant map $\mathscr{M}(\tau, \sigma) \to \mathfrak{m}^{\mathbb{C}}$ defined by

$$(T_i(s)) \to T_2(0) + iT_3(0)$$

identifies, as complex manifolds, $\mathscr{M}(\tau, \sigma)$ with the $H^{\mathbb{C}}$-orbit of $\tau_2 + i\tau_3 + \sigma_2 + i\sigma_3$ in $\mathfrak{m}^{\mathbb{C}}$.

Theorem 1 is an immediate consequence of this result.

Conversely, another interpretation of the theorem can be to construct a family of H-invariant Kähler metrics (indexed by regular τ_1's) on a fixed $H^{\mathbb{C}}$-orbit in $\mathfrak{m}^{\mathbb{C}}$.

It is interesting to understand the symplectic or the complex structure of $\mathscr{M}(\tau, \sigma)$ when the regularity conditions on the centralisers are not satisfied. On the symplectic side, this gives degenerations of G-orbits in $\mathfrak{g}$ and leads to theorem 2. On the complex side, the extreme case is $\tau_2 = \tau_3 = 0$, $\sigma = 0$. The element $\tau^r = i\tau_1$ defines a parabolic subgroup $P_{H^{\mathbb{C}}}$ of $H^{\mathbb{C}}$, which we extend into a parabolic subgroup $P_{G^{\mathbb{C}}}$ of $G^{\mathbb{C}}$ (with Levi part the centraliser of τ_1). We denote the Lie algebra of $P_{G^{\mathbb{C}}} \cap G$ by

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n},$$

where $\mathfrak{l}$ is the Levi part and $\mathfrak{n}$ the nilpotent part. Then one gets :

Theorem 4. *The conormal bundle $N^*(H^{\mathbb{C}}/P_{H^{\mathbb{C}}}, G^{\mathbb{C}}/P_{G^{\mathbb{C}}})$ admits a family of H-invariant Kähler metrics, parameterised by the $\tau_1 \in \mathfrak{t}_0$ such that $C_{\mathfrak{g}}(\tau_1) = \mathfrak{l}$. The corresponding symplectic manifold is the G-orbit of τ_1 in $\mathfrak{g}$.*

Note that the corollary is stated for a *real* group G, but the conormal bundle always makes sense as $H^{\mathbb{C}} \times_{P_{H^{\mathbb{C}}}} ((\mathfrak{m}^{\mathbb{C}})^* \cap \mathfrak{p}^{\perp})$.

One can describe in general the complex structure of $\mathscr{M}(\tau, \sigma)$. The notations are the same as above, with $P_{H^{\mathbb{C}}}$ the parabolic subgroup of $H^{\mathbb{C}}$ associated to $i\tau_1$, and $\mathscr{O}_{P_{H^{\mathbb{C}}}}(X)$ denotes the $P_{H^{\mathbb{C}}}$-orbit in $\mathfrak{m}^{\mathbb{C}}$ of an element $X \in \mathfrak{m}^{\mathbb{C}}$. The complex structure of $\mathscr{M}(\tau, \sigma)$ is that of

$$H^{\mathbb{C}} \times_{P_{H^{\mathbb{C}}}} \Big(\mathscr{O}_{P_{H^{\mathbb{C}}}}(\tau_2 + i\tau_3 + \sigma_2 + i\sigma_3) + \mathfrak{n}^{\mathbb{C}} \cap \mathfrak{m}^{\mathbb{C}} \Big).$$

3. Proofs

First remark that if the group G is a complex semisimple group, then the results of section 2 are statements about complex coadjoint orbits, and are contained in [Kro90a] for regular semisimple orbits, in [Kro90b] for nilpotent orbits, and in [Biq96] for other orbits or degenerate cases. Therefore we shall consider the case where G is a real semisimple group. As in [Ver95], we will realise our space of solutions as the fixed point set of an involution in a larger space of solutions of Nahm's equations.

Let us consider the full moduli space $\mathscr{M}_c(\tau, \sigma)$ of solutions $(T_i(s))_{i=1\dots 3}$ of Nahm's equations (5), where the $T_i(s)$ take now values in $\mathfrak{u}$. We require the asymptotic behaviour

$$T_i(s) = \mathrm{Ad}(u)\Big(\tau_i - \frac{\sigma_i}{2s}\Big) + O\Big(\frac{1}{s^{1+\epsilon}}\Big) \tag{12}$$

as in (11), but now u lies in the larger compact group U. The Cartan involution induces an involution of $\mathscr{M}_c(\tau, \sigma)$,

$$\Theta(T_1, T_2, T_3) = (\theta(T_1), -\theta(T_2), -\theta(T_3)),$$

whose fixed point set $\mathscr{M}_c^{\Theta}(\tau, \sigma)$ contains the set of solutions satisfying (10), that is $\mathscr{M}(\tau, \sigma)$. It is important here to note that $\mathscr{M}(\tau, \sigma)$ is only *one* connected component of $\mathscr{M}_c^{\Theta}(\tau, \sigma)$: indeed, in order for the equation (12) to define an asymptotic behaviour with $T_1 \in \mathfrak{h}$ and $T_2, T_3 \in i\mathfrak{m}$, the element u of the group U does not necessarily belong to H, but might belong to a larger subgroup of U keeping the data (τ_1, σ_1) in $\mathfrak{h}$ and $(\tau_2, \sigma_2, \tau_3, \sigma_3)$ in $i\mathfrak{m}$.

The larger moduli space $\mathscr{M}_c(\tau, \sigma)$ is a U-invariant hyperKähler manifold [Biq96]: it has a Riemannian metric and three complex structures (J_1, J_2, J_3) satisfying $J_1 J_2 J_3 = -1$. The metric is Kähler with respect to the three complex structures, and we denote $\omega_1, \omega_2, \omega_3$ the corresponding Kähler forms. Then $\Omega_1 = \omega_2 + i\omega_3$ is a holomorphic symplectic form for J_1, and similarly by circular permutation we get Ω_2 and Ω_3.

To describe this structure, it is necessary to consider the extended system (8): as we have seen, it is invariant under the action of gauge transformations $g : (-\infty, 0] \to U$, acting by $g(T_0, T_1, T_2, T_3) = (\mathrm{Ad}(g)T_0 - \frac{dg}{ds}g^{-1}, \mathrm{Ad}(g)T_1, \mathrm{Ad}(g)T_2, \mathrm{Ad}(g)T_3)$. Now we have the following alternative description for $\mathscr{M}_c(\tau, \sigma)$: here we will just give the facts we need, and refer to [Biq96] for the technical details. We denote by $\mathscr{B}$ the space of solutions of the system (8), with asymptotic behaviour when $s \to -\infty$ given by:

$$T_i(s) = \tau_i - \frac{\sigma_i}{2s} + O\Big(\frac{1}{s^{1+\epsilon}}\Big) \text{ for } i \geqslant 1, \qquad T_0(s) = O\Big(\frac{1}{s^{1+\epsilon}}\Big). \tag{13}$$

There is a group $\mathscr{G}$ of gauge transformations acting on $\mathscr{B}$, satisfying the boundary condition $g(0) = 1$. Then $\mathscr{M}_c(\sigma, \tau) = \mathscr{B}/\mathscr{G}$. This construction is actually a hyperKähler quotient of the affine space $\mathscr{A}$ of all configurations $(T_0(s), T_1(s), T_2(s), T_3(s))$ with asymptotic behaviour (13), by the group $\mathscr{G}$; the equations (8) are the hyperKähler moment map. Considering $(T_0 + iT_1, T_2 + iT_3)$ as complex variables leads to the complex structure J_1, the choice $(T_0 + iT_2, T_1 - iT_3)$ to J_2, and $(T_0 + iT_3, T_1 + iT_2)$ to J_3. (From (9) it is clear that J_3 corresponds to the natural complex structure on the space of flat connections, and $\Omega_3 = \omega_1 + i\omega_2$ is the natural holomorphic symplectic form for this complex structure).

The involution Θ acts on $\mathscr{A}$ by

$$\Theta(T_0, T_1, T_2, T_3) = (\theta(T_0), \theta(T_1), -\theta(T_2), -\theta(T_3)).$$

It preserves J_1 and the L^2 metric, and therefore $\mathscr{M}(\tau, \sigma)$ is a J_1-complex submanifold of $\mathscr{M}_c(\tau, \sigma)$. As a fixed point set of an isometry, it is totally geodesic, and therefore complete whenever $\mathscr{M}_c(\tau, \sigma)$ is, that is when $\sigma = 0$. The metric inherited from $\mathscr{M}_c(\tau, \sigma)$ is a H-invariant Kähler metric, with Kähler form $\omega_1 = \Re\Omega_3$ (this reflects the fact that Θ is a real structure for the (J_3, Ω_3) holomorphic symplectic structure). Identifying J_1 and $\omega_1 = \Re\Omega_3$ for $\mathscr{M}(\tau, \sigma)$ is now just a matter of remembering the interpretations of $\mathscr{M}_c(\tau, \sigma)$ for the two different holomorphic symplectic structures (J_1, Ω_1) and (J_3, Ω_3).

We begin by the symplectic structure. In the regular case $C_{\mathfrak{g}}(\tau_1, \tau_2) = C_{\mathfrak{g}}(\tau)$, the map $(T_1, T_2, T_3) \to T_1(0) + iT_2(0)$ identifies $(\mathscr{M}_c(\tau, \sigma), J_3, \Omega_3)$ with the $G^{\mathbb{C}}$-coadjoint orbit $\mathscr{O}_c$ of $\tau_1 + i\tau_2 + \sigma_1 + i\sigma_2$ in $\mathfrak{g}^{\mathbb{C}}$. The conditions (10) say that $\mathscr{M}_c^{\Theta}(\tau, \sigma)$ is characterised by the condition

$T_1(0) + iT_2(0) \in \mathfrak{g}$, that is

$$\mathscr{M}_c^\Theta(\tau,\sigma) = \mathscr{O}_c \cap \mathfrak{g}.$$

(Indeed, by uniqueness, the solution $(T_1(s), T_2(s), T_3(s))$ with given value of $T_1(0) + iT_2(0) \in \mathfrak{g}$ is invariant under Θ.) Now $\mathscr{M}(\tau,\sigma)$ is the component containing $\tau_1 + i\tau_2 + \sigma_1 + i\sigma_2$, that is the G-coadjoint orbit of $\tau_1 + i\tau_2 + \sigma_1 + i\sigma_2$.

In the degenerate case of theorem 2, one has $\mathscr{M}_c(\tau,\sigma) = T^*(G^{\mathbb{C}}/P_{G^{\mathbb{C}}})$, but the description from Nahm's equations is more involved and we let the reader to check that $\mathscr{M}(\tau,\sigma) = T^*(G/P)$. This can be also obtained by making a semisimple orbit degenerate to $T^*(G/P)$.

Now pass to the complex structure. If the regularity condition $C_{\mathfrak{g}}(\tau_2,\tau_3) = C_{\mathfrak{g}}(\tau)$ is satisfied, then, as above, the map $(T_1,T_2,T_3) \to T_2(0) + iT_3(0)$ identifies $\mathscr{M}_c(\tau,\sigma)$ with the $G^{\mathbb{C}}$-coadjoint orbit $\mathscr{O}^c$ of $\tau_2 + i\tau_3 + \sigma_2 + i\sigma_3$ in $\mathfrak{g}^{\mathbb{C}}$. Now from (10) it is clear that in this case

$$\mathscr{M}_c^\Theta(\tau,\sigma) = \mathscr{O}_c \cap \mathfrak{m}^{\mathbb{C}},$$

and $\mathscr{M}(\tau,\sigma)$ is the component containing $\tau_2 + i\tau_3 + \sigma_2 + i\sigma_3$, that is its $H^{\mathbb{C}}$-orbit.

Again we leave the degenerate case of theorem 4 to the reader. □

4. The simplest example

Finally we illustrate the results in the simplest case: $G = SU_{1,1}$. The Lie algebra $\mathfrak{g}$ is represented by matrices

$$A = \begin{pmatrix} ix & \bar z \\ z & -ix \end{pmatrix}, \qquad x \in \mathbb{R}, \quad z \in \mathbb{C}.$$

Then $\mathfrak{h} = \{\left(\begin{smallmatrix} ix & 0 \\ 0 & -ix \end{smallmatrix}\right)\}$ and $\mathfrak{m} = \{\left(\begin{smallmatrix} 0 & \bar z \\ z & 0 \end{smallmatrix}\right)\}$. The complexified action of $H = SO_2\mathbb{R}$ on $\mathfrak{m} = \mathbb{R}^2$ is that of $SO_2\mathbb{C}$ on $\mathbb{C}^2$, so the nonzero $H^{\mathbb{C}}$-orbits in $\mathfrak{m}^{\mathbb{C}}$ are copies of $\mathbb{C}^*$. The nonzero G-orbits in $\mathfrak{g}$ are the connected components of

$$\det A = x^2 - |z|^2 = \lambda, \quad A \neq 0.$$

Of course the Cartan subalgebra of $\mathfrak{g}$ is one dimensional, so either it is compact ($\mathfrak{t}_{\mathfrak{g}} = \mathfrak{h}$), either it is noncompact ($\mathfrak{t}_{\mathfrak{g}} = \mathfrak{a} \subset \mathfrak{m}$). There are three cases, in each case the diffeomorphism with a complex $H^{\mathbb{C}}$-space gives a H-invariant Kähler structure on the orbit:

- $\lambda < 0$: here $\mathfrak{t}_{\mathfrak{g}} \subset \mathfrak{m}$, the orbit is a hyperboloid, and is diffeomorphic to the semisimple orbits in $\mathfrak{m}^{\mathbb{C}}$ (so the complex structure is that of $\mathbb{C}^*$); the corresponding orbits in $\mathfrak{m}^{\mathbb{C}}$ depend on a parameter $\tau_3 \in i\mathfrak{a} = i\mathfrak{t}$;

- $\lambda = 0$: the two nilpotent orbits $x = \pm|Z| \neq 0$ are diffeomorphic to the two nilpotent orbits in $\mathfrak{m}^{\mathbb{C}}$ (this is the classical Kostant-Sekiguchi-Vergne correspondence);
- $\lambda > 0$: here $\mathfrak{t} \subset \mathfrak{h}$, we have two orbits $x = \pm\sqrt{\lambda + |z|^2}$ diffeomorphic to $\mathbb{C}$; the parameters $\tau_2 = \tau_3 = 0$ since $\mathfrak{a} = 0$, so we are in the situation of theorem 4: $H^{\mathbb{C}}/P$ is a point and the conormal bundle is just a complex line $\mathbb{C}$.

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