# HYPERKÄHLER METRICS ON COTANGENT BUNDLES OF HERMITIAN SYMMETRIC SPACES

OLIVIER BIQUARD AND PAUL GAUDUCHON

## Introduction

The cotangent bundle $M = T^*\Sigma$ of a complex manifold $\Sigma$ is a holomorphic-symplectic manifold. If $\Sigma$ is a generalized flag manifold, then this holomorphic-symplectic structure underlies a hyperkähler metric, whose restriction to $\Sigma$ is the given homogeneous metric. This hyperkähler metric has been constructed using finite dimensional hyperkähler quotients for $S\ell_n(\mathbb{C})$-flags in [9], and in general using Nahm's equations (and infinite dimensional quotients) in [1]. For the simplest case, $M = T^*\mathbb{CP}^n$, there is an explicit formula of Calabi giving the Kähler form of $M$ as the sum of the pull-back of the Kähler form of $\mathbb{CP}^n$ and a term given by an explicit potential [3].

In this paper, we give such a formula for the case where $\Sigma$ is a hermitian symmetric space. To state the formula, identify $T^*\Sigma$ with $T\Sigma$ using the metric, and write $I$ for the complex structure of $\Sigma$. Then, for $\xi \in T_x^*\Sigma \approx T_x\Sigma$, the curvature $R_{I\xi,\xi}$ gives an antiselfadjoint endomorphism of $T_x\Sigma$, and we can take spectral functions $f(IR_{I\xi,\xi})$ of the selfadjoint endomorphism $IR_{I\xi,\xi}$, for a function $f : \mathbb{R} \to \mathbb{R}$.

Note that we use the convention $R_{\xi,\xi'} = \nabla_{[\xi,\xi']} - [\nabla_\xi, \nabla_{\xi'}]$ for the curvature.

**Theorem 1.** *Let $\Sigma = G/H$ be a hermitian symmetric space of compact type, then there is a unique $G$-invariant hyperkähler metric $g$ on $M = T^*\Sigma$ (with its canonical holomorphic-symplectic structure), such that the restriction of $g$ to the zero section is $g_\Sigma$:*

*(i) the Kähler form of $g$ is given by $\omega_I = \pi^*\omega_\Sigma + dd^c\rho$, with*

$$\rho(\xi) = (f(IR_{I\xi,\xi})\xi, \xi), \quad f(x) = \frac{1}{x}\left(\sqrt{1+x} - 1 - \ln\frac{1+\sqrt{1+x}}{2}\right);$$

*(ii) with respect to the decomposition of $TM$ between horizontal and vertical directions, induced by the Levi-Civita connection of $\Sigma$, one has $g(x, y) = (u(x), y)$, with*

$$u_\xi = \begin{pmatrix} A_\xi & 0 \\ 0 & A_\xi^{-1} \end{pmatrix}, \quad A_\xi = 1 + IR_{I\phi(IR_{I\xi,\xi})\xi,\phi(IR_{I\xi,\xi})\xi},$$

$$\text{and } \phi(x) = [2(xf)']^{1/2} = \left(\frac{\sqrt{1+x} - 1}{x}\right)^{1/2}.$$

It is easy to check that this metric is complete (the completeness was already known in [1]). Note that $f(IR_{I\xi,\xi})$ is well defined, since $\Sigma$ is of nonnegative bisectional holomorphic curvature.

Burns [2] has found a formula of type (ii), but case by case using the classification. Using the curvature, we have a unified formula for all hermitian symmetric spaces (including exceptional ones), and of course a formula for the potential.

We now look at other hermitian symmetric spaces.

**Theorem 2.** *Let $\Sigma = G/H$ be a hermitian symmetric space of noncompact type, then there is a unique $G$-invariant hyperkähler metric $g$ defined in a neighborhood $N$ of the zero section $\Sigma$ in $M = T^*\Sigma$, such that the restriction of $g$ to the zero section* *is $g_\Sigma$: it is given by the same formulas as above and $N$ is the set of $\xi$ such that the modulus of the eigenvalues of $R_{I\xi,\xi}$ is less than 1. The metric $g$ is incomplete.*

In particular, there is only one $PS\ell_2(\mathbb{R})$-invariant hyperkähler metric on the cotangent bundle of the Poincaré half-plane, restricting to the standard hyperbolic metric on the zero section, but it is defined only in a neighborhood of the zero section and is incomplete.

**Theorem 3.** *Let $\Sigma$ be a flat complex torus. Then, the translation invariant hyperkähler metrics $g$ on $T^*\Sigma$, restricting to the given flat metric on $\Sigma$, are given by*

$$g = \begin{pmatrix} 1 & B \\ B & 1+B^2 \end{pmatrix}$$

*where $B$, as a function from a fibre to real bilinear forms, is the hessian of the real part of a holomorphic function.*

In fact, this description is valid for any quotient of $\mathbb{C}^n$ by a discrete group of translations. We deduce the following corollary.

**Corollary 4.** *Let $\Sigma$ be any hermitian symmetric space, $\Sigma = \Sigma_f \times \Sigma_c \times \Sigma_{nc}$, where $\Sigma_f$ is flat, $\Sigma_c$ is hermitian symmetric of compact type, and $\Sigma_{nc}$ is hermitian symmetric of noncompact type. Then any hyperkähler metric in a neighborhood of $\Sigma$ in $T^*\Sigma$, invariant under the isometry group of $\Sigma$ and restricting to the given metric on $\Sigma$, must be the product of the metrics described in the previous theorems.*

We have another corollary for nilpotent orbits (see [8] for the hyperkähler metrics of the nilpotent orbits). We simply take the principal part of $f(x)$ when $x$ goes to infinity.

**Corollary 5.** *Under the hypothesis of theorem 1, if $R_{I\xi,\xi}$ has generically no zero eigenvalue, then, taking $f(x) = x^{-1/2}$, $\phi(x) = x^{-1/4}$, and removing the contribution from the basis (so that $\omega_I = dd^c\rho$ and $A_\xi = IR_{I\phi(IR_{I\xi,\xi})\xi,\phi(IR_{I\xi,\xi})\xi}$) defines again a hyperkähler metric on the open set $\{\xi, R_{I\xi,\xi}$ is injective$\} \subset T^*\Sigma$; this metric is the hyperkähler metric of a nilpotent orbit of $\mathfrak{g}^{\mathbb{C}}$.*

See section 3 to understand which nilpotent orbit appears.

In the first section, we establish the equations for hyperkähler metrics on cotangent spaces of hermitian symmetric spaces, that are invariant under the isometry group of the basis. We deduce the proof of theorem 3 and a uniqueness statement (lemma 6) when one fixes the restriction on the zero section.

In the second section, we prove *a formula for the potential in symplectic quotients* (theorem 7) which is essentially an amplification of an idea in [6]. The formula contains a term involving a character of the group: this term is needed when taking the inverse image by a normalized moment map of nonzero vectors in the dual of the Lie algebra. We deduce the formula for the potential in the case of complex grassmannians (since their cotangent spaces are obtained by finite dimensional quotients). This section is not necessary for the proofs of the formulas, but we think that it gives a good motivation for the formula for the potential, since we obtain the other hermitian symmetric spaces simply by extrapolating the intrinsic formula for complex grassmannians.

In the third section, we recall some root theory for hermitian symmetric spaces (we use the "restricted root theorem" of Harish-Chandra and Moore), and we prove the above theorems.

## 1. Equations and the example of $T^*\mathbb{T}^n$

We recall some facts on hyperkähler structures (see for example [5]). A triple $(g, I, J)$ formed by a riemannian metric $g$ and two anticommuting $g$-orthogonal almost-complex structures $I$ and $J$ on a manifold $M$ is a hyperkähler structure on $M$ whenever the pairs $(g,I)$ and $(g,J)$ are Kähler. Then the pair $(g, K = IJ)$ is Kähler as well.

For any triple as above, let us denote $\omega_I$, $\omega_J$ and $\omega_K$ the corresponding Kähler forms. Then we have:

$$(g,I,J) \text{ is hyperkähler iff } d\omega_I = d\omega_J = d\omega_K = 0. \tag{1.1}$$

The point here is that the three Kähler forms being closed implies the integrability of the three almost-complex structures.

If $(g,I,J)$ is a hyperkähler structure on $M$, then $(M, I, \omega_c = \omega_2 + i\omega_3)$ is a holomorphic-symplectic manifold. Consider the case that the underlying holomorphic-symplectic manifold is the cotangent bundle $T^*\Sigma$ of some complex manifold $\Sigma$. By (1.1), a riemannian metric $g$ on $T^*\Sigma$ is hyperkähler (with respect to the underlying holomorphic-symplectic structure) if and only if

$$g(I\cdot, I\cdot) = g(\cdot,\cdot), \tag{1.2}$$

$$J^2 = -1, \text{ where } J \text{ is defined by } \operatorname{re}\omega_c = g(J\cdot,\cdot), \tag{1.3}$$

$$d\omega_I = 0, \text{ where } \omega_I = g(I\cdot,\cdot). \tag{1.4}$$

Suppose now that $\Sigma$ is Kähler and consider the decomposition of the tangent bundle $TM = H^\nabla \oplus V$ of $TM$ between the horizontal part, determined by the Levi-Civita connection $\nabla$, and the vertical part, tangent to the fibres of the projection $\pi$. Denote by $g_0$ the riemannian metric on $M$ induced by the metric of $\Sigma$ and the decomposition $H^\nabla \oplus V$. Identify $T\Sigma$ and $T^*\Sigma$ using the metric, so that for any $\xi \in M$, $H^\nabla_\xi$ and $V_\xi$ are identified with $T_{\pi_*\xi}\Sigma$. With respect to $g_0$ and the above decomposition of $TM$, the complex structure $I_M$, the real part $\operatorname{re}\omega_c$ of the symplectic form and the unknown metric $g$ are written as

$$I_M = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \operatorname{re}\omega_c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix},$$

where $A$, $B$, $D$ are endomorphisms of $T_{\pi_*\xi}\Sigma$. Then, the conditions (1.2) and (1.3) are respectively equivalent to

$$AI = IA,\ DI = ID,\ BI = -IB, \tag{1.5}$$

$$AD = 1 + B^2,\ DA = 1 + (B^*)^2,\ AB^* = BA,\ B^*D = DB. \tag{1.6}$$

To compute $d\omega_I$, we introduce, for any vector field $X$ on $\Sigma$, the horizontal lift $X^h$ of $X$ with respect to $\nabla$ and the vertical vector field $X^v$, whose restriction to each fibre $T^*_x\Sigma \approx T_x\Sigma$ is the constant vector field $X_x$. For any two vector fields $X$ an $Y$ on $\Sigma$, one has $[X^v, Y^v] = 0$, $[X^h, Y^v] = (\nabla_X Y)^v$ and $[X^h, Y^h]_\xi = [X,Y]^h_\xi + R_{X,Y}\xi$, where $R$ is the curvature of $\Sigma$ (convention $R_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$). The condition (1.4) is then easily checked to be equivalent to the following system, for any $\xi$ over $x$ and any $X, Y, Z \in T_x\Sigma$,

$$\begin{aligned} d^v(DI) &= 0, \\ (Z, d^v(BI)_{X,Y}) + ((Z^h \cdot (DI)X, Y) &= 0, \\ ((Z^v \cdot (AI)X, Y) + (DIR_{X,Y}\xi, Z) & \\ +((X^h \cdot (BI)Z, Y) - ((Y^h \cdot (BI)Z, X) &= 0, \\ \sum_{XYZ} ((X^h \cdot (AI)Y, Z) + \sum_{XYZ} (BIR_{X,Y}\xi, Z) &= 0. \end{aligned} \tag{1.7}$$

Here, $d^v(DI)$ is the vertical exterior derivative of $DI$, as a 2-form; $BI$ is considered as a 1-form (in fact a (1,0)-form by (1.2)) on $\pi^{-1}(x)$, with values in $T_x\Sigma$; $Z^h \cdot (DI)$ and the like denote the covariant derivative in the horizontal directions with respect to the pull-back of $\nabla$ acting on sections of $\pi^{-1}(\operatorname{End} T\Sigma)$; in the last formula, we sum over the circular permutations of $X$, $Y$, and $Z$.

From now on, we restrict ourselves to the case when $\Sigma$ is a hermitian symmetric space (of any type), and we only consider hyperkähler metrics on $M = T^*\Sigma$, that are invariant under the action of the group $G$ of isometries of $\Sigma$. The horizontal distribution $H^\nabla$ is tangent to the orbits of $G$ in $M$, so that $A$, $B$, $D$ are parallel with respect to $\pi^{-1}\nabla$ in horizontal directions. The last equation in (1.7) is then empty ($B$ commutes with $R_{X,Y}$ since $B$ is parallel), and the second and third equation reduce respectively to:

$$\begin{aligned} d^v B &= 0, \\ (Z^v \cdot (AI))_\xi - R_{\xi,(DI)Z} &= 0. \end{aligned} \tag{1.8}$$

We are now ready to deduce the two following consequences, for torus and for hermitian symmetric spaces without flat factor.

**Proof of theorem 3.** By the second equation in (1.8), $A$ is constant on each fibre, hence equal to 1. By the first equation in (1.8), in a fibre $T_x^*\Sigma$, $B$ is the differential of some 0-form $\alpha$ with values in $T_x\Sigma$; since $B$ is of type (1,0), $\alpha$ is holomorphic. Since $B$ is symmetric by (1.3), $\alpha$ is closed as a 1-form on $T_x^*\Sigma$, hence the differential of some holomorphic function. Then $B$, as a real bilinear form, is the hessian of the real part of this function. Conversely, it is clear that such a function determines, in this way, a translation invariant hyperkähler metric on $T^*\Sigma$. □

**Lemma 6.** *Let $\Sigma = G/H$ be a hermitian symmetric space without flat factor. Then, the $G$-invariant hyperkähler metrics $g$ on $T^*\Sigma$ (with its canonical symplectic structure) are the metrics of the form:*

$$g = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix},$$

*where $A$ is $G$-equivariant, and satisfies ($\xi \in T_x^*\Sigma \approx T_x\Sigma$ and $Z \in T_x\Sigma$):*

$$(Z^v \cdot (AI))_\xi = R_{(AI)^{-1}Z,\xi}. \tag{1.9}$$

*In particular, in any connected neighborhood of the zero section, $T^*\Sigma$ admits at most one $G$-invariant hyperkähler metric restricting to the given hermitian symmetric metric on $\Sigma$.*

*Proof.* As in the proof of the above theorem, we have $B = d\alpha$ on each fibre, where $\alpha$ is holomorphic and can be chosen $H$-invariant on $T_x^*\Sigma$, hence invariant as well under the action of the complexified group $H^{\mathbb{C}}$. Since $\Sigma$ has no flat factor, the group $H^{\mathbb{C}}$ has a dense orbit; it follows that $\alpha$ is constant and $B$ vanishes identically. Then, by (1.3), $D = A^{-1}$. In particular, the second equation in (1.8) implies the first in (1.7) (use the Bianchi identity) and the whole system is then equivalent to (1.9). The last assertion is a direct consequence of (1.9), which, on each radial ray, reduces to an ordinary differential equation. □

## 2. Potential in symplectic quotients and the grassmannian

We give a general formula for the Kähler potential of Kähler quotients, which comes as an amplification of (3.58) in [6], and can be used for hyperkähler quotients when applied to the zero set of the complex part of the hyperkähler moment map. In particular, we will apply this formula to obtain an explicit formulation for the Kähler potential of the hyperkähler metric on the cotangent bundle of the complex grassmannians.

2.1. **Potential in symplectic Kähler quotients.** Let $(W, g, I)$ be a Kähler manifold and $G$ be a compact Lie group acting on $W$, preserving $g$ and $I$. We assume that the action of $G$ extends to a holomorphic action of the complexified group $G^{\mathbb{C}}$. We also assume that $W$ admits a global Kähler potential $K$, which can be assumed $G$-invariant without loss of generality, so that the Kähler form is $\omega = dd^c K$. Then, if $\mathfrak{g}$ is the Lie algebra of $G$, $K$ determines a $G$-equivariant function $\mu : W \to \mathfrak{g}^*$ for the action of $G$, defined by (writing $\mu^X(x) =< \mu(x), X >$):

$$\mu^X = d^c K(\xi^X), \tag{2.1}$$

where $\xi^X$ is the vector field induced on $W$ by $X \in \mathfrak{g}$. Since $d\mu^X = -i_{\xi^X}\omega$, $\mu$ is a moment map for the action of $G$.

For any character $\chi : G \to \mathbb{S}^1$, we write $-2\pi i c$ for the derivative of $\chi$ at the origin of $G$, where $c$ is viewed as a $G$-invariant element of $\mathfrak{g}^*$. In the sequel, we choose $\chi$ such that $c$ is a regular value for the moment map $\mu$, and $M_c = \mu^{-1}(c)/G$ is a well defined Kähler quotient. We also denote by $\chi : G^{\mathbb{C}} \to \mathbb{C}^*$ the extension of $\chi$ to $G^{\mathbb{C}}$. We assume that, for any $x \in W$, there exists an unique element $g_x \in \exp(i\mathfrak{g})$ such that $g_x x \in \mu^{-1}(c)$.

We denote by $q : W \to \mu^{-1}(c)$ the map $q(x) = g_x x$ and by $\hat{\omega}$ the pull-back on $W$ via $q$ of the Kähler form of the Kähler quotient $M_c$.

**Theorem 7.** *We have $\hat{\omega} = dd^c \hat{K}$, where $\hat{K}$ is given, for $x \in W$, by*

$$\hat{K}(x) = K(g_x x) + \frac{1}{4\pi} \ln |\chi(g_x)|^2.$$

*Remark.* In case that $W$ is a hermitian vector space, endowed with the flat Kähler structure, which is the case of interest in the sequel, we may choose $K(x) = |x|^2/4$.

*Proof.* We introduce the trivial complex line bundle $L = W \times \mathbb{C}$, we denote by $\sigma$ the canonical section $\sigma(x) = (x, 1)$, and we consider the hermitian metric $h$ on $L$ determined by $h(\sigma) = \exp(4\pi K)$, so that the Kähler form $\omega$ can be written $\omega = -R^D/2\pi i$, where $R^D$ is the curvature of the Chern connection $D$ of $(L, h)$. We make $G^{\mathbb{C}}$ act holomorphically on $L$ by $g(x, u) = (gx, \chi(g)u)$. This action induces a linear action of $G^{\mathbb{C}}$ on the space $\Gamma(L)$ of sections of $L$, by putting: $(gs)(x) = g(s(g^{-1}x))$ for any $s \in \Gamma(L)$; then the induced action of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ on $\Gamma(L)$ reads as follows, for $X \in \mathfrak{g}^{\mathbb{C}}$:

$$X\, s = -D_{\xi^X} s + 2\pi i \mu^X - 2\pi i c(X). \tag{2.2}$$

Via the action of $G^{\mathbb{C}}$ on $L$, we define a new hermitian metric $\hat{h}$ on $L$ by $\hat{h}(u) = h(g_x u)$, for any element $u$ of the fibre $L_x$, where $g_x$ is defined as above. In particular, for the canonical section $\sigma$, we have

$$\hat{h}(\sigma)(x) = |\chi(g_x)|^2 h(\sigma)(g_x x) = |\chi(g_x)|^2 e^{4\pi K(g_x x)}.$$

Let $\hat{R}$ be the curvature of the Chern connection of the hermitian bundle $(L, \hat{h})$. The theorem follows from the above formula and the fact that $\hat{R} = -2\pi i \hat{\omega}$.

To prove this fact, note that, by the very definition of the Kähler quotient structure on $M_c$, we have $\hat{\omega} = q^*(\omega|_{\mu^{-1}(c)})$. It follows that $2\pi i q^*(\omega|_{\mu^{-1}(c)})$ coincides with the curvature of the Chern connection $\hat{D}$ of the hermitian bundle $\hat{L} = (q^*(L|_{\mu^{-1}(c)}), q^* h)$. On the other hand, the action of $G^{\mathbb{C}}$ on $L$ induces an isomorphism $\psi : L \to \hat{L}$, which identifies the hermitian bundle $\hat{L}$ with $(L, \hat{h})$. It thus remains to prove that the induced connection $D' = \psi^{-1} \circ \hat{D} \circ \psi$ coincides with the Chern connection of $(L, \hat{h})$. Since $D'$ clearly preserves the metric $\hat{h}$, it is sufficient to show that $D'$ is compatible with the holomorphic structure of $L$, that is $D'_\xi \sigma$ is $\mathbb{C}$-linear in $\xi$. This, in turn, follows directly from the expression:

$$D'_\xi \sigma(x) = g_x^{-1} \left[ \left( D_{(g_x)_* \xi} g_x \sigma \right) (g_x x) \right],$$

which we obtain by a direct computation using (2.2), and the following obvious expression for the differential of $q$ at the point $x$: $q_*\xi = (g_x)_*\xi + \xi^{X_x}_{g_x x}$, where $X_x$ is the element of $i\mathfrak{g}$ defined by: $X_x = (\xi \cdot g)g_x^{-1}$. □

2.2. **The cotangent bundle of the complex grassmannian.** In this paragraph, we apply the formula of theorem 7 to the case of the cotangent space $T^*G_{r,N}$ of the complex grassmannian of $r$-subspaces of $\mathbb{C}^N$, realized as a hyperkähler quotient (see [9]) of the flat quaternionic space $V \oplus V^*$, where $V = \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^N)$ and the dual $V^*$ is identified with $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^r)$ via the trace. The unitary group $G = U_r$ acts on $V \oplus V^*$ by $g(x, \xi) = (x \circ g^{-1}, g \circ \xi)$ which clearly preserves the flat hyperkähler structure. To perform the quotient by $G$, we restrict the action to the open subset $V^{reg} \oplus V^*$, where $V^{reg}$ is the set of elements of maximal rank $r$ in $V$. We denote by $W$ the submanifold of $V^{reg} \oplus V^*$ defined by

$$W = \{(x, \xi) \in V^{reg} \oplus V^*,\ \xi x = 0\},$$

endowed with the induced structure. Then, $W$ is $G$-invariant and is the zero set of the complex part of the hyperkähler moment map for the action of $G$.

Any character of $U_r$ has the form $\chi_k(g) = (\det g)^\ell$, where $\ell$ is some integer and $k = \ell/2\pi$. The corresponding $G$-invariant element of $\mathfrak{g}^*$ is $c_k(X) = k\,\mathrm{tr}(iX)$. The square norm induces a $U_r$-invariant global potential $K(x, \xi) = (|x|^2 + |\xi|^2)/4$ on $W$. Then, the associated moment map (2.1) is the real part of the hyperkähler moment map for $U_r$. For any $k > 0$, $c_k$ is a regular value for $\mu$ and we have

$$\mu^{-1}(c_k) = \{(x, \xi) \in W,\ x^*x - \xi\xi^* = 2k\,1\}.$$

On the other hand, the quotient $\mu^{-1}(c_k)/U_r$, as a holomorphic-symplectic manifold, coincides with the quotient $W/G^{\mathbb{C}}$, where $G^{\mathbb{C}} = GL_r\mathbb{C}$ is the full linear group, acting on $W$ in the natural way, and $W/G^{\mathbb{C}}$ is naturally identified, still as a holomorphic-symplectic manifold, to the cotangent space $T^*G_{r,N}$ by identifying the class of the pair $(x, \xi)$ with the pair $(P, \alpha)$, where $P \in G_{r,N}$ is the image of $x$ and $\alpha$ is the element of $T^*_P G_{r,N} = \mathrm{Hom}(\mathbb{C}^N/P, P)$ defined by $\alpha = x\xi$.

For any $(x, \xi)$, we look for a positive self-adjoint $g = g_{x,\xi} \in GL_r$ such that $g^{-1}x^*xg^{-1} - g\xi\xi^*g = 2k\,1$. Writing $\gamma = \sqrt{x^*x}\,g^{-1}$, $g$ is determined by

$$\gamma\gamma^* = k\left(1 + (1 + k^{-2}\sqrt{x^*x}\,\xi\xi^*\sqrt{x^*x})^{1/2}\right).$$

By theorem 7, we get $4\hat{K}(x, \xi) = |xg^{-1}|^2 + |g\xi|^2 + 2k\ln|\det g|^2$. Since $g(x, \xi) \in \mu^{-1}(c_k)$, the two first square norms are equal up to an additive constant, so that, up to an additive constant,

$$\hat{K}(x, \xi) = \frac{k}{2}\ln\det(x^*x) + \frac{k}{2}\,\mathrm{tr}\left(\frac{1}{k}\gamma\gamma^* - \ln(\gamma\gamma^*)\right).$$

The first term is easily recognized as the Kähler potential, pulled back to $V^{reg}$, of the $U_r$-invariant metric on $G_{r,N}$ defined by $(X, X)_k = 2k\,\mathrm{tr}(X^*X)$ for $X \in T_P G_{r,N} = \mathrm{Hom}(P, \mathbb{C}^N/P)$. To interpret the second term, note that $\gamma\gamma^*$ is conjugate to $1 + (1 + k^{-2}\alpha\alpha^*)^{1/2} = 1 + (1 + 4X^*X)^{1/2}$, where $X = 1/(2k)\alpha^*$ is the dual vector of $\alpha$. Let $R$ denote the curvature of $(\cdot, \cdot)_k$, which is independent of $k$, and for each vector $X \in T_P G_{r,N}$ consider the symmetric endomorphism $IR_{IX,X}(Y) = 2(X^*XY + YX^*X)$. For any integer $j$, we infer $((IR_{IX,X})^j X, X)_k = k/2\ \mathrm{tr}((4X^*X)^{j+1})$, hence

$$k/2\,\mathrm{tr}(k^{-1}\gamma\gamma^* - \ln(\gamma\gamma^*)) = (f(IR_{IX,X})X, X)_k,$$

where $f$ is the function defined on $\mathbb{R}_+$ by

$$f(x) = \frac{1}{x}\left(\sqrt{1+x} - 1 - \ln\frac{1 + \sqrt{1+x}}{2}\right).$$

This proves formula (i) of theorem 1 for the case of the grassmannian $\Sigma = G_{r,N}$ with the metric $(\cdot, \cdot)_k$.

*Remark.* Though the $U_N$-invariant metrics $(\cdot,\cdot)_k$ differ from each other by a mere homothety, the corresponding hyperkähler metrics on $T^*G_{r,N}$ are substantially distinct.
*Example.* When $\Sigma = \mathbb{P}^n$, $(\cdot,\cdot)_k$ is the metric with constant holomorphic sectional curvature $c = 2/k$ and the operator $IR_{IX,X}$ acts on $X$ by multiplication by $c|X|_k^2$, so that the formula for the potential reduces to the formula of Calabi [3].

## 3. Root theory and proof of the theorems

In this section, we prove the theorems of the introduction. First, we recall some root theory for hermitian symmetric spaces (see for example [10] and [4]).

3.1. **Root theory for hermitian symmetric spaces.** Let $G/H$ be an irreducible hermitian symmetric space, $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of $G$ and $H$,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h};$$

the curvature of $\Sigma$ is given by $R_{\xi,\xi'} = [\xi, \xi'] \in \mathfrak{h} \subset \operatorname{End}\mathfrak{m}$ for $\xi, \xi' \in \mathfrak{m}$; let $I \in \mathfrak{h}$ be the complex structure and decompose $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ according to the decomposition of $\mathfrak{m}^{\mathbb{C}}$ in eigenspaces for the eigenvalues $\pm i$ of $I$: $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are abelian subalgebras of $\mathfrak{g}^{\mathbb{C}}$; let $\mathfrak{t} \subset \mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $I$ and $\Delta$ be the $\mathfrak{t}^{\mathbb{C}}$-root system in $\mathfrak{g}^{\mathbb{C}}$, so $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \oplus_{\alpha\in\Delta}\mathfrak{g}^\alpha$; let $\Delta_H$ be the set of $\mathfrak{t}^{\mathbb{C}}$-roots of $\mathfrak{h}^{\mathbb{C}}$ and $\Delta_M$ be the set of roots $\alpha$ with $\mathfrak{g}^\alpha \subset \mathfrak{m}^{\mathbb{C}}$; choose an ordering of the set $\Delta$ such that

$$\mathfrak{m}^\pm = \oplus_{\alpha\in\Delta^\pm_{\mathfrak{M}}}\mathfrak{g}^\alpha;$$

two roots $\alpha, \beta \in \Delta$ are strongly orthogonal if neither $\alpha \pm \beta$ is a root; in that case $\alpha$ and $\beta$ are orthogonal; consider a maximal strongly orthogonal set of $\Delta_M^+$:

$$\Psi = \{\psi_1, \dots, \psi_r\},$$

where $\psi_{i+1}$ is the lowest element of $\Delta_M^+$ strongly orthogonal to each of $\psi_1, \dots, \psi_i$; choose for $\alpha \in \Delta^+$ a triple $(h_\alpha, n_\alpha, n_{-\alpha}) \in i\mathfrak{t} \times \mathfrak{g}^\alpha \times \mathfrak{g}^{-\alpha}$, such that $[h_\alpha, n_{\pm\alpha}] = \pm 2n_{\pm\alpha}$, $[n_\alpha, n_{-\alpha}] = h_\alpha$; this choice can be made so that $\mathfrak{g}$ has a basis consisting of a basis of $\mathfrak{t}$ and

$$e_\alpha = n_\alpha - n_{-\alpha}, \quad f_\alpha = i(n_\alpha + n_{-\alpha}), \quad \alpha \in \Delta^+;$$

one has, for $\alpha \in \Delta_M^+$,

$$Ie_\alpha = f_\alpha, \quad If_\alpha = -e_\alpha, \quad [e_\alpha, f_\alpha] = 2ih_\alpha;$$

from strong orthogonality,

$$\mathfrak{a} = \oplus_{\psi\in\Psi}\mathbb{R}\mathfrak{e}_\psi$$

is a maximal abelian subalgebra in $\mathfrak{m}$, so

$$\mathfrak{m} = \cup_{\mathfrak{x}\in\mathfrak{H}} \operatorname{Ad}(\mathfrak{x})\mathfrak{a} = \operatorname{Ad}(\mathfrak{H})\mathfrak{a}, \tag{3.1}$$

and $r$ is the rank of the symmetric space. If the centralizer in $\mathfrak{m}$ of $\xi \in \mathfrak{a}$ is $\mathfrak{a}$, then $\xi$ is a regular element of $\mathfrak{a}$; in that case, $\mathfrak{m} = \mathfrak{a} + [\mathfrak{h}, \xi]$.

Strong orthogonality of $\Psi$ implies

$$\mathfrak{t}^- = [\mathfrak{a}, \mathfrak{I}\mathfrak{a}] = \oplus_\Psi \mathbb{R}i\mathfrak{h}_\psi;$$

let $\mathfrak{t}^+$ be the orthogonal complement of $\mathfrak{t}^-$ in $\mathfrak{t}$ with respect to the Killing form; let $\mu$ be the restriction of roots from $\mathfrak{t}^{\mathbb{C}}$ to $\mathfrak{t}^-$. The restricted root theorem of Harish-Chandra and Moore asserts that two cases may occur: identify $\psi_i$ with $\mu(\psi_i)$, then

$$\mu(\Delta) \cup \{0\} = \{(\pm\psi_s \pm \psi_t)/2, \quad 1 \le s, t \le r\} \quad \text{or}$$
$$\mu(\Delta) \cup \{0\} = \{(\pm\psi_s \pm \psi_t)/2, \quad 1 \le s, t \le r\} \cup \{\pm\psi_t/2, \quad 1 \le t \le r\}\,.$$

In the first case,

$$\mu(\Delta_H^+) = \{(\psi_s - \psi_t)/2, \quad 1 \le t < s \le r\}\,,$$
$$\mu(\Delta_M^+) = \{(\psi_s + \psi_t)/2, \quad 1 \le t \le s \le r\}\,,$$

in the second case

$$\mu(\Delta_H^+) = \{(\psi_s - \psi_t)/2, \quad 1 \le t < s \le r\} \cup \{-\psi_t/2, \quad 1 \le t \le r\}\,,$$
$$\mu(\Delta_M^+) = \{(\psi_s + \psi_t)/2, \quad 1 \le t \le s \le r\} \cup \{\psi_t/2, \quad 1 \le t \le r\}\,.$$

**Lemma 8.** *If $\alpha \in \Delta_H^+$, then*
*(i) there exists at most one $\psi_+ \in \Psi$, such that $\psi_+ + \alpha$ is a root;*
*(ii) there exists at most one $\psi_- \in \Psi$, such that $\psi_- - \alpha$ is a root;*
*(iii) if both $\psi_+$ and $\psi_-$ exist, then $\psi_+ \neq \psi_-$;*
*(iv) one has $\alpha(h_\psi) = \delta_{\psi\psi_-} - \delta_{\psi\psi_+}$ for $\psi \in \Psi$.*

*Proof.* Suppose that we are in the first case of the restricted root theorem and write $\mu(\alpha) = (\psi_s - \psi_t)/2$, with $t < s$. If $\psi_+ + \alpha$ is a root, then $\mu(\psi_+ + \alpha) = \psi_+ + (\psi_s - \psi_t)/2$, which, according to the description of $\mu(\Delta_M^+)$, implies $\psi_+ = \psi_t$. Similarly, if $\psi_- - \alpha$ is a root, then $\psi_- = \psi_s$. So $\psi_+$ and $\psi_-$ are unique and distinct. The same is true in the second case of the restricted root theorem.

If $\psi_+$ exists, then $\alpha$ and $\alpha + \psi_+$ are roots, but $\alpha - \psi_+$ and $\alpha + 2\psi_+$ are not roots, since $\psi_+ \neq \psi_-$ and $[\mathfrak{m}^+, \mathfrak{m}^+] = \mathfrak{o}$. The fourth assertion follows. □

Suppose that $\psi_+ + \alpha$ and $\psi_- - \alpha$ are roots, then $(\psi_- - \alpha, \psi_+) = -(\alpha, \psi_+) > 0$, so $\beta = \psi_+ + \alpha - \psi_-$ is a root in $\Delta_H$. This implies that

$$[n_\alpha, n_{\psi_+}] = c[n_\beta, n_{\psi_-}] \quad \text{for some } c \in \mathbb{R}.$$

Using (iv) of the above lemma, one gets $[n_\alpha, n_{-\psi_-}] = c[n_\beta, n_{-\psi_+}]$. From this we deduce

$$[f_\alpha, e_{\psi_\pm}] = \pm I[e_\alpha, e_{\psi_\pm}], \quad [cf_\beta, e_{\psi_\pm}] = \mp I[e_\alpha, e_{\psi_\mp}], \tag{3.2}$$

and, for $\xi = \sum_\Psi \xi_\psi e_\psi \in \mathfrak{a}$,

$$[f_\alpha, \xi] = I(\xi_{\psi_+}[e_\alpha, e_{\psi_+}] - \xi_{\psi_-}[e_\alpha, e_{\psi_-}]), \quad [cf_\beta, \xi] = I(\xi_{\psi_-}[e_\alpha, e_{\psi_+}] - \xi_{\psi_+}[e_\alpha, e_{\psi_-}]).$$

As $I[e_\alpha, \xi] = I(\xi_{\psi_+}[e_\alpha, e_{\psi_+}] + \xi_{\psi_-}[e_\alpha, e_{\psi_-}])$, the following lemma is obvious.

**Lemma 9.** *Suppose $\alpha \in \Delta_H^+$ as above and $\xi = \sum_\Psi \xi_\psi e_\psi \in \mathfrak{a}$; write $\xi_{\psi_+} = 0$ (respectively $\xi_{\psi_-} = 0$) if $\psi_+$ (respectively $\psi_-$) does not exist; if $\xi_{\psi_+} \neq \xi_{\psi_-}$, then*

$$I[e_\alpha, \xi] = [xf_\alpha + ycf_\beta, \xi],$$

*with*

$$x = \frac{\xi_{\psi_+}^2 + \xi_{\psi_-}^2}{\xi_{\psi_+}^2 - \xi_{\psi_-}^2}, \quad y = -\frac{2\xi_{\psi_+}\xi_{\psi_-}}{\xi_{\psi_+}^2 - \xi_{\psi_-}^2}.$$

We will use later this lemma to write $I[e_\alpha, \xi]$ as an element of the infinitesimal orbit of the isotropy group $H$.

3.2. **Proof of the formulas.** Now, let $\Sigma = G/H$ be an irreducible hermitian symmetric space, and $\pi : M = T^*\Sigma \to \Sigma$ its cotangent bundle. We use the notations of the first section.

**Lemma 10.** *Let $g$ be a Kähler metric on $M$, with Kähler form $\omega = \pi^*\omega_\Sigma + dd^c\rho$, where $\rho$ is a $G$-invariant potential on $M$, then*

$$g(x, y) = (u(x), y), \quad \text{with } u = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

*with, at a point $\xi$,*

$$A_\xi = 1 + IR_{I\xi, \operatorname{grad}_\xi \rho}, \quad D_\xi = ((d_\xi \operatorname{grad} \rho) - I(d_\xi \operatorname{grad} \rho)I).$$

*Proof.* This is an easy exercise, using the techniques of the first section. □

**Lemma 11.** *If* $\rho(\xi) =< f(IR_{I\xi,\xi})\xi, \xi >$, *then* $\operatorname{grad}_\xi \rho = 2(xf)'(IR_{I\xi,\xi})\xi$.

*Proof.* From (3.1) and the $H$-invariance of $\rho$, we may suppose $\xi = \sum \xi_\psi e_\psi \in \mathfrak{a}$. By density, it is enough to prove the formula at a regular $\xi \in \mathfrak{a}$, that is $\xi$ such that $\mathfrak{m} = \mathfrak{a} + [\mathfrak{h}, \xi]$. From $H$-invariance, $\operatorname{grad}_\xi \rho$ is orthogonal to $[\mathfrak{h}, \xi]$, so $\operatorname{grad}_\xi \rho \in \mathfrak{a}$. Now we simply differentiate in $\mathfrak{a}$:

$$R_{I\xi,\xi} = \sum \xi_\psi^2 [Ie_\psi, e_\psi] = -2i \sum \xi_\psi^2 h_\psi, \quad \rho(\xi) = \sum \xi_\psi^2 f(4\xi_\psi^2)|e_\psi|^2,$$

$$d_\xi \rho = 2 \sum (xf)'(4\xi_\psi^2)|e_\psi|^2 \xi_\psi d\xi_\psi,$$

which gives the formula of the lemma. □

**Lemma 12.** *For* $\operatorname{grad}_\xi \rho = F(IR_{I\xi,\xi})\xi$, *one has the following formulas at a point* $\xi = \sum \xi_\psi e_\psi$:

*(i) in horizontal directions:*

$$A_\xi = 1 + IR_{IF^{1/2}(IR_{I\xi,\xi})\xi, F^{1/2}(IR_{I\xi,\xi})\xi},$$

*(ii) in vertical directions: for* $\zeta \in \mathfrak{a} \oplus \mathfrak{I}\mathfrak{a}$,

$$D_\xi(\zeta) = 2(xF)'(IR_{I\xi,\xi})(\zeta),$$

*(iii) in vertical isotropic directions: for* $\alpha \in \Delta_H$, *with the notations of lemma 9, if* $\xi_{\psi_+} \neq \xi_{\psi_-}$,

$$D_\xi([e_\alpha, \xi]) = 2 \frac{\xi_{\psi_+}^2 F(4\xi_{\psi_+}^2) - \xi_{\psi_-}^2 F(4\xi_{\psi_-}^2)}{\xi_{\psi_+}^2 - \xi_{\psi_-}^2} (\xi_{\psi_+}[e_\alpha, e_{\psi_+}] + \xi_{\psi_-}[e_\alpha, e_{\psi_-}]),$$

*and the same formula is true replacing* $e_\alpha$ *by* $f_\alpha$.

*Remark.* For formula (iii) when $\xi_{\psi_+} = \xi_{\psi_-}$, it is clear by density that the same formula is true with a coefficient $2(xF)'(4\xi_{\psi_+}^2)$.

*Proof.* The formula (i) is a consequence of

$$R_{I\xi, F(IR_{I\xi,\xi})\xi} = R_{IF^{1/2}(IR_{I\xi,\xi})\xi, F^{1/2}(IR_{I\xi,\xi})\xi}.$$

This is easy for $\xi \in \mathfrak{a}$, whence the general case by (3.1).

Now we prove (ii). Clearly $D_\xi(I\zeta) = ID_\xi(\zeta)$, so we may suppose $\zeta \in \mathfrak{a}$. Since $\mathfrak{a}$ is abelian, $d_\xi R_{I\xi,\xi}(I\zeta) = 0$, so

$$-I(d_\xi \operatorname{grad} \rho)(I\zeta) = F(IR_{I\xi,\xi})\zeta;$$

since $R_{I\xi,\xi}$ and $(d_\xi R_{I\xi,\xi})(\zeta) = 2R_{I\xi,\zeta}$ commute,

$$(d_\xi \operatorname{grad} \rho)(\zeta) = F'(IR_{I\xi,\xi})(2R_{I\xi,\zeta}\xi) + F(IR_{I\xi,\xi})\zeta;$$

using $R_{I\xi,\zeta}\xi = R_{I\xi,\xi}\zeta$, we get the second formula.

Finally, we prove (iii). Since $\operatorname{grad} \rho$ is $H$-equivariant, we get $(d_\xi \operatorname{grad} \rho)[X, \xi] = [X, \operatorname{grad}_\xi \rho]$ for $X \in \mathfrak{h}$. We deduce

$$\begin{aligned}(d_\xi \operatorname{grad} \rho)[e_\alpha, \xi] &= [e_\alpha, F(IR_{I\xi,\xi})\xi] \\ &= \xi_{\psi_+} F(4\xi_{\psi_+}^2)[e_\alpha, e_{\psi_+}] + \xi_{\psi_-} F(4\xi_{\psi_-}^2)[e_\alpha, e_{\psi_-}].\end{aligned}$$

On the other hand, $[Ie_\alpha, \xi] = [xf_\alpha + ycf_\beta, \xi]$ from lemma 9, so using (3.2)

$$\begin{aligned}-I(d_\xi \operatorname{grad} \rho)(I[e_\alpha, \xi]) &= -I[xf_\alpha + ycf_\beta, F(IR_{I\xi,\xi})\xi] \\ &= (x\xi_{\psi_+} F(4\xi_{\psi_+}^2) + y\xi_{\psi_-} F(4\xi_{\psi_-}^2))[e_\alpha, e_{\psi_+}] \\ &\quad - (x\xi_{\psi_-} F(4\xi_{\psi_-}^2) + y\xi_{\psi_+} F(4\xi_{\psi_+}^2))[e_\alpha, e_{\psi_-}];\end{aligned}$$

replacing $x$ and $y$ according to lemma 9 gives the formula. The proof for $f_\alpha$ is the same. □

**Proof of theorem 1.** First we prove that the description (ii) of the theorem follows from the formula for the potential in (i). We use lemmas 10, 11 and 12. The statement for horizontal directions is clear since $F = 2(xf)' = \phi^2$. Thus we have only to check that $D_\xi$ is the inverse of $A_\xi$. It suffices to compare the eigenvalues on the eigenspaces $\mathfrak{g}^\psi$ for $\psi \in \Delta_M$. The eigenvalues on $\mathfrak{g}^{\pm\psi}$ are the same, so we may assume $\psi \in \Delta_M^+$. There are two cases: $\psi \in \Psi$ or $\psi = \alpha + \psi_+$ for $\alpha \in \Delta_H^+$ and $\psi_+ \in \Psi$. For example, in the second case, the eigenvalue of $A_\xi$ on $\mathfrak{g}^{\alpha+\psi_+}$ is

$$1 + 2\xi_{\psi_+}^2 \phi^2(4\xi_{\psi_+}^2) + 2\xi_{\psi_-}^2 \phi^2(4\xi_{\psi_-}^2)$$

and the eigenvalue of $D_\xi$ is

$$2((x\phi^2)(4\xi_{\psi_+}^2) - (x\phi^2)(4\xi_{\psi_-}^2))/(4\xi_{\psi_+}^2 - 4\xi_{\psi_-}^2);$$

since $x\phi^2(x) = \sqrt{1+x} - 1$, the result follows from the trivial identity $1/(\sqrt{1+x} + \sqrt{1+y}) = (\sqrt{1+x} - \sqrt{1+y})/(x-y)$.

Now it is clear from conditions (1.4), (1.5) and (1.6) that the metric is hyperkähler. The uniqueness comes from lemma 6. □

**Proof of theorem 2.** This theorem is now clear. □

**Proof of corollary 4.** From the proof of lemma 6, we see that the only possible non diagonal terms in the metric must come from the $\Sigma_f$ part. □

**Proof of corollary 5.** It is easy to check that these functions define a hyperkähler structure on $\{\xi, R_{I\xi,\xi} \text{ is injective}\} \subset M$. It is not difficult to identify this open subset of $M$, as a holomorphic-symplectic manifold, with the nilpotent orbit in $\mathfrak{g}^{\mathbb{C}}$ of a regular element $X$ of $\mathfrak{m}^+$. Thus we get an explicit formula for its hyperkähler metric. It is easy to verify that the formula does not depend on the choice of the hermitian symmetric metric on $\Sigma$. □

*Remark.* These nilpotent orbits are very special ones: they are the most degenerate ones, since $\mathrm{ad}(X)^3 = 0$. For example they do not cover the case of the regular $\mathfrak{sl}_3$-orbit of [7].

## References


[1] O. Biquard, *Sur les équations de Nahm et la structure de Poisson des algèbres de Lie semi-simples complexes*, Math. Ann., to appear.

[2] D. Burns, *Some examples of the twistor construction*, in “Contributions to several complex variables: in honor of Wilhelm Stoll” (eds. A. Howard and P. M. Wong), Vieweg (1986).

[3] E. Calabi, *Métriques kählériennes et fibrés holomorphes*, Ann. Ec. Norm. Sup. **12** (1979), 269–294.

[4] D. Drucker, *Exceptional Lie algebras and the structure of hermitian symmetric spaces*, Memoirs of the A.M.S. **208** (1978).

[5] N. J. Hitchin, *Hyperkähler manifolds*, Séminaire Bourbaki, exposé **748** (1991).

[6] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, *Hyperkähler metrics and supersymmetry*, Commun. Math. Phys. **108** (1987), 535–559.

[7] P. Z. Kobak and A. Swann, *Quaternionic geometry of a nilpotent variety*, Math. Ann. **297** (1993), 747–764.

[8] P. B. Kronheimer, *Instantons and the geometry of the nilpotent variety*, J. Differential Geometry **32** (1990), 473–490.

[9] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), 365–416.

[10] J. A. Wolf, *Fine structure of hermitian symmetric spaces*, in “Symmetric spaces” (Pure and applied mathematics **8**), Marcel Dekker (1972).


Cmat, Ura 169 du Cnrs, Ecole Polytechnique, F-91128 PALAISEAU CEDEX