An introduction to geometric analysis

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Contents

Introduction

These are notes for an introductory course in geometric analysis. We will focus on the analysis of ellliptic operators, whose propotype is the Laplacian of Riemannian manifolds, and on geometric applications to linear problems (spectrum) and nonlinear problems (Yamabe problem).

An important source of inspiration was the notes by Simon Donaldson [Don] which are a very good reference for these lectures.

The notes are not intended as self-contained: sometimes the proofs are omitted, short or left to the reader as exercises. The reader should complete these notes by referring to excellent textbooks like [Jos17, Li12]. A number of statements and arguments are borrowed to these books, as well as to [Don] and [LP87] on the Yamabe problem.

A few exercises are proposed in the text, some other ones at the end of the notes. Some of them are taken from the exercise classes of Thibault Lefeuvre the last years.

Chapter I

The scalar Laplacian on a Riemannian manifold

1 The Riemannian Laplacian

Let consider $\mathbb{R}^n$ with coordinates $(x^1, \dots, x^n)$. We denote the standard basis of vector fields by

$$\partial_i = \frac{\partial}{\partial x^i}.$$

The ***scalar Laplacian*** on $\mathbb{R}^n$ is the operator defined on functions of $\mathbb{R}^n$ by

$$\Delta f = -\sum_1^n \partial_i^2 f.$$

We can also write

$$\Delta f = d^* df,$$

where the operator d^* associates to a 1-form $\alpha = \alpha_i dx^i$ on $\mathbb{R}^n$ the function defined by

$$d^*\alpha = -\sum_1^n \partial_i \alpha_i.$$

We denote by $(\cdot,\cdot)$ the standard L^2 inner product on functions or forms on $\mathbb{R}^n$. By an integration by parts, one checks the following identities for compactly supported 1-form α and functions f, g:

$$(\alpha, df) = (d^*\alpha, f),$$
$$(\Delta f, g) = (df, dg) = (f, \Delta g).$$

The first identity says that d^* is the formal adjoint of d, and the second identity that Δ is formally selfadjoint.

Now we generalize this to any Riemannian manifold (M^n, g). Locally we can choose coordinates (x^i) in which we write the metric and the volume form as

$$g = g_{ij} dx^i dx^j, \quad \mathrm{vol} = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

We claim that a formal adjoint of the differential of functions is given by the following operator sending 1-forms on functions:

$$d^*(\alpha_i dx^i) = -\frac{1}{\sqrt{\det(g)}}\partial_i(\sqrt{\det(g)}g^{ij}\alpha_j). \tag{1.1}$$

This fact is easily checked by an integration by parts. There is also a more intrinsic way to see this: the metric gives an isomorphism $\sharp : T^*M \to TM$, given by

$$\alpha^\sharp = g^{ij}\alpha_j\partial_i, \tag{1.2}$$

and (1.1) can be rewritten as

$$d^*\alpha = -\frac{d(\alpha^\sharp \lrcorner \operatorname{vol})}{\operatorname{vol}}. \tag{1.3}$$

Then we can check that d^* is the formal adjoint of d:

$$\begin{aligned}\int_M d(f\alpha^\sharp \lrcorner \operatorname{vol}) &= \int_M df \wedge (\alpha^\sharp \lrcorner \operatorname{vol}) + fd(\alpha^\sharp \lrcorner \operatorname{vol}) \\ &= \int_M \big(\langle df, \alpha\rangle - fd^*\alpha\big)\operatorname{vol}.\end{aligned}$$

In the case M has no boundary, the LHS vanishes and we have proved

$$(df, \alpha) = (f, d^*\alpha). \tag{1.4}$$

Now we can define the scalar Laplacian of (M, g) to be the operator $\Delta = d^*d$: from (1.1) it is given by the explicit formula

$$\Delta f = -\frac{1}{\sqrt{\det(g)}}\partial_i(\sqrt{\det(g)}g^{ij}\partial_j f). \tag{1.5}$$

Formula (1.4) implies, for any functions f, g with compact support:

$$(\Delta f, g) = (df, dg) = (f, \Delta g). \tag{1.6}$$

The case where M has a boundary is also of interest: let $\vec{n}$ be the unit exterior normal vector, then one has $\alpha^\sharp \lrcorner \operatorname{vol} = \alpha_{\vec{n}} \operatorname{vol}_{\partial M}$, therefore Stokes theorem gives

$$\int_M \big(\langle df, \alpha\rangle - fd^*\alpha\big)\operatorname{vol} = \int_{\partial M} \alpha_{\vec{n}} \operatorname{vol}_{\partial M}. \tag{1.7}$$

Applying this formula twice, we obtain the analog of (1.6) with boundary:

$$\int_M \Delta f\, g \operatorname{vol} = \int_M f\Delta g \operatorname{vol} + \int_{\partial M} \big((\vec{n}\cdot g)f - (\vec{n}\cdot f)g\big)\operatorname{vol}_{\partial M}. \tag{1.8}$$

2 Main result

We now suppose that (M, g) is a compact connected manifold (without boundary). First observe from (1.6) that if $\Delta f = 0$ then $(\Delta f, f) = \|df\|^2 = 0$ and therefore f is constant. Therefore $\ker \Delta = \mathbb{R}$.

Still from (1.6) we have for any f the identity $(1, \Delta f) = 0$ that is $\int_M \Delta f \operatorname{vol} = 0$. It turns out that this is the only constraint:

Theorem 2.1. *If* $\varrho \in C^\infty(M)$ *and* $\int_M \varrho \,\mathrm{vol} = 0$, *then there exists a solution* $f \in C^\infty(M)$ *of the equation* $\Delta f = \varrho$, *which is unique up to an additive constant.*

Example 2.2 (Torus). On a torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ we can write the Laplacian through the decomposition in Fourier series:

$$f(x) = \sum_{\xi\in\mathbb{Z}^n} \hat{f}(\xi)e^{i\xi\cdot x}, \quad \Delta f(x) = \sum_{\xi\in\mathbb{Z}^n} |\xi|^2 e^{i\xi\cdot x}. \tag{2.1}$$

The theorem is then explicit: one can solve $\Delta f = \varrho$ if $\hat{\varrho}(0) = 0$. Then the solution f is given by $\hat{f}(\xi) = \frac{1}{|\xi|^2}\hat{\varrho}(\xi)$.

Example 2.3 (Green function). We now consider the case of the flat space $\mathbb{R}^n$. Of course it is not covered by the theorem since it is noncompact. There is a useful explicit solution of $\Delta f = \varrho$ which is obtained in the following way.

We define the Green function G as a radial function on $\mathbb{R}^n$ given by:

$$G(r) = \begin{cases} \frac{1}{(n-2)V_{n-1}}\frac{1}{r^{n-2}}, & n > 2, \\ -\frac{1}{2\pi}\log r, & n = 2. \end{cases} \tag{2.2}$$

Here V_{n-1} is the total volume of the sphere $S^{n-1} \subset \mathbb{R}^n$.

Then a solution of $\Delta f = \varrho$ is given by

$$f(x) = \int_{\mathbb{R}^n} \varrho(y)G(x-y)|dy|^n. \tag{2.3}$$

Of course we need the integral to converge, for simplicity we suppose that ϱ has compact support. Note that we do not require ϱ to have zero integral; indeed the integration by parts (1.8) with $g = 1$ has a nonzero boundary term on a large domain of $\mathbb{R}^n$.

Proof of (2.3). On radial functions the Laplacian on $\mathbb{R}^n$ writes as

$$\Delta f(r) = -r^{-(n-1)}\partial_r r^{n-1}\partial_r f(r)$$

so it is immediate to check that $\Delta G = 0$ outside the origin. Applying (1.8) for a function f with compact support, we obtain

$$\int_{\mathbb{R}^n\setminus B_\varepsilon} \Delta f \, G \,\mathrm{vol} = \int_{S_\varepsilon} (G\partial_r f - f\partial_r G) r^{n-1} \,\mathrm{vol}_{S^{n-1}}.$$

Only the term $f\partial_r G$ gives a nonzero limit when $\varepsilon \to 0$, and since $\partial_r G = -r^{-(n-1)}/V_{n-1}$ we obtain exactly

$$\int_{\mathbb{R}^n} \Delta f \, G \,\mathrm{vol} = f(0). \tag{2.4}$$

This actually means that $\Delta G = \delta_0$ (the Dirac function) in the sense of distributions. Translating (2.4) we obtain $f(x) = \int_{\mathbb{R}^n} \Delta f(y)G(x-y)|dy|^n$, that is

$$\Delta f * G = f. \tag{2.5}$$

Given a smooth function ϱ with compact support, and writing the convolution as $\varrho * G(x) = \int \varrho(x-y)G(y)|dy|^n$, we can commute differentiation and integration to obtain:

$$\Delta(\varrho * G) = (\Delta\varrho) * G = \varrho$$

by (2.5). Therefore we can take as solution of $\Delta f = \varrho$ the function $f := \varrho * G$. □

3 Proof of the theorem

The proof of Theorem 2.1 is difficult. There will be several steps.

Weak solutions

If $\Delta f = \varrho$ then from (1.6) we have $(df, dg) = (\varrho, g)$ for any smooth function g. Actually the converse is true: if f is smooth and $(df, dg) = (\varrho, g)$ for any smooth function g, then by integration by parts $(\Delta f, g) = (\varrho, g)$ for all functions g and therefore $\Delta f = \varrho$.

We define the Sobolev space H^1 by

$$H^1(M) = \{f \in L^2(M), df \in L^2(M)\}, \tag{3.1}$$

equipped with the Hilbert norm $\|f\|^2_{H^1} = \|f\|^2_{L^2} + \|df\|^2_{L^2}$. Actually $H^1(M)$ can be defined as the completion of $C^\infty(M)$ for this norm.

A ***weak solution*** of the equation $\Delta f = \varrho$ is a function $f \in H^1(M)$ such that

$$(df, dg) = (\varrho, g) \quad \text{for all functions } g \in C^\infty(M). \tag{3.2}$$

Observe that by density of $C^\infty \subset H^1$ it is equivalent to require this property for all functions $g \in H^1(M)$.

From the previous considerations it follows that if f is a weak solution of the equation and f is smooth, then f is a genuine solution: $\Delta f = \varrho$. It turns out that the smoothness of f is automatic:

Theorem 3.1 (Regularity). *If $\Delta f = \varrho$ in the weak sense with $\varrho \in C^\infty(M)$, then $f \in C^\infty(M)$.*

It follows that the resolution of the equation $\Delta f = \varrho$ is reduced to finding a weak solution of the equation.

Theorem 3.1 is a difficult theorem which we will not prove now: it is a special case of elliptic regularity, which will be explained later in these lectures, see corollary 18.2.

Variational principle

We consider the energy

$$E(f) = \int_M \Big(\frac{1}{2}|df|^2 - f\varrho\Big) \operatorname{vol}. \tag{3.3}$$

This gives a functional $E : C^\infty(M) \to \mathbb{R}$ which extends to a well-defined functional $E : H^1(M) \to \mathbb{R}$. It is differentiable with differential

$$d_f E(\dot f) = \int_M \langle df, d\dot f\rangle - \varrho \dot f. \tag{3.4}$$

Comparing with (3.2), we see that $d_f E = 0$ exactly when f is a weak solution of the equation $\Delta f = \varrho$. By theorem 3.1 this is equivalent to having a smooth solution of the equation.

Therefore solving the equation $\Delta f = \varrho$ is now reduced to finding a critical point of the functional E on H^1. We first state the following fundamental result, which will be proved in the next section.

Theorem 3.2 (Poincaré inequality). *If* (M^n, g) *is compact, then there exists* $c > 0$ *such that for any function* $f \in C^\infty(M)$ *with* $\int_M f \operatorname{vol} = 0$ *one has*

$$\int_M |df|^2 \operatorname{vol} \geq c \int_M f^2 \operatorname{vol}. \tag{3.5}$$

End of proof of theorem 2.1. Actually E is unchanged if we add a constant to f, so it is equivalent to restrict to functions with $\int_M f \operatorname{vol} = 0$.

Step 1. We use the Poincaré inequality and the Cauchy-Schwartz inequality $f\varrho \leq \frac{1}{2}(\frac{c}{2}f^2 + \frac{2}{c}\varrho^2)$ to obtain the lower bound

$$E(f) \geq \int_M \left(\frac{c}{4}|df|^2 - \frac{1}{c}\varrho^2\right) \operatorname{vol} \tag{3.6}$$

It follows that E has a lower bound; moreover if E is bounded then $\|df\|_{L^2}$ (and therefore $\|f\|_{H^1}$) is bounded.

Step 2. We choose a minimizing sequence $f_i \in H^1(M)$ for E with $\int_M f_i \operatorname{vol} = 0$. Since it is bounded in H^1, we can extract a subsequence which converges weakly in H^1 to a limit $f \in H^1$. Since the integral against any (smooth) function is a continuous linear form on $H^1(M)$, we also have at the limit $\int_M f \operatorname{vol} = 0$, and

$$\|df\|_{L^2} \leq \liminf \|df_i\|_{L^2}, \quad \int_M \varrho f_i \operatorname{vol} \to \int_M \varrho f \operatorname{vol}.$$

It follows that $E(f) \leq \liminf E(f_i) = \inf E$, so E attains a minimum at f. Therefore f is a critical point of f: this gives the required solution. □

Remark 3.3. There is a shorter proof of the Theorem applying Riesz representation theorem to the weak equation $(df, dg) = (\varrho, g)$ for all g. Poincaré inequality says that $\|df\|^2_{L^2}$ is an equivalent scalar product on H^1 (restricting to functions with zero integral). Our proof therefore more or less amounts to the proof of the Riesz theorem; the variational approach is very general and can be used in many situations where Riesz theorem does not apply.

4 Poincaré inequality

We begin by the local version, that is the version in $\mathbb{R}^n$:

Proposition 4.1. *Let* $\Omega \subset \mathbb{R}^n$ *be convex and bounded,* f *a function defined on an open set containing* $\overline{\Omega}$*, then*

$$\int_\Omega |df|^2 \geq c(\Omega) \int_\Omega |f - \bar{f}|^2 \tag{4.1}$$

where $\bar{f} = \frac{1}{V(\Omega)}\int_\Omega f$ *and* $V(\Omega)$ *is the volume of* Ω.

Proof. First we prove the following estimate: there is a constant $c_1(\Omega)$ such that for any $x \in \Omega$ one has

$$|f(x) - \bar{f}| \le c_1(\Omega)\int_\Omega \frac{|df(y)|}{|x-y|^{n-1}}|dy|^n. \tag{4.2}$$

To prove this inequality, translating Ω if necessary, we can suppose $0 \in \Omega$. The two sides (4.2) do not change if we add a constant to f, so we can also suppose $f(0) = 0$. We consider the function f in radial coordinates (r, u) where $u \in S^{n-1}$. Then, if $R(u) = \sup\{r, ru \in \Omega\}$, we have

$$\begin{aligned}
\bar{f} &= \frac{1}{V(\Omega)}\int_{S^{n-1}} d\,\mathrm{vol}^{S^{n-1}}(u)\int_0^{R(u)} f(r,u)r^{n-1}dr \\
&= \frac{1}{V(\Omega)}\int_{S^{n-1}} d\,\mathrm{vol}^{S^{n-1}}(u)\int_0^{R(u)} \Big(\int_0^r \partial_\rho f(\rho,u)d\rho\Big)r^{n-1}dr \\
&= \frac{1}{V(\Omega)}\int_{S^{n-1}} d\,\mathrm{vol}^{S^{n-1}}(u)\int_0^{R(u)} \frac{R(u)^n - \rho^n}{n}\partial_\rho f(\rho,u)d\rho.
\end{aligned}$$

Using $R(u)^n - \rho^n \le R(u)^n \le \mathrm{Diam}(\Omega)^n$ we obtain

$$|\bar{f}| \le \frac{\mathrm{Diam}(\Omega)^n}{nV(\Omega)}\int_\Omega \frac{|df|}{\rho^{n-1}}|dx|^n$$

which proves (4.2) with $c_1(\Omega) = \frac{\mathrm{Diam}(\Omega)^n}{nV(\Omega)}$.

We can rewrite (4.2) as

$$|f - \bar{f}| \le c_1(\Omega) K * g,$$

where g and K are functions on $\mathbb{R}^n$ defined by

$$g(x) = \begin{cases} |df(x)|, & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases} \qquad K(x) = \begin{cases} \frac{1}{|x|^{n-1}}, & |x| \le \mathrm{Diam}\,\Omega, \\ 0, & |x| > \mathrm{Diam}\,\Omega. \end{cases}$$

Using the general inequality $\|\phi * \psi\|_{L^p} \le \|\phi\|_{L^p}\|\psi\|_{L^1}$ for functions on $\mathbb{R}^n$, we obtain

$$\|f - \bar{f}\|_{L^2} \le c_1(\Omega)\|K\|_{L^1}\|g\|_{L^2} \le c_2(\Omega)\|df\|_{L^2}.$$

□

Proof of Theorem 3.2. We will actually prove the following: if ϱ is a function on M such that $\int_M \varrho\,\mathrm{vol} = 0$, then for any function f on M one has

$$\left|\int_M f\varrho\,\mathrm{vol}\right| \le c\|\varrho\|_{L^2}\|df\|_{L^2}. \tag{4.3}$$

One deduces the Poincaré inequality by taking $\varrho = f$.

The idea is to reduce to the local version (4.1) via a partition of unity. We write the proof in the case where M is covered by two open sets: $M = U_1 \cup U_2$ with U_i convex bounded set in $\mathbb{R}^n$ and (χ_1, χ_2) a corresponding partition of unity, then we define

$$\varrho_i = \chi_i \varrho - \left(\int_M \chi_i \varrho \,\mathrm{vol}\right)\sigma,$$

where σ is a function on M such that $\mathrm{Supp}\,\sigma \subset U_1 \cap U_2$ and $\int_M \sigma \,\mathrm{vol} = 1$. Therefore we have $\varrho = \varrho_1 + \varrho_2$ with

$$\mathrm{Supp}\,\varrho_i \subset U_i, \qquad \int_M \varrho_i \,\mathrm{vol} = 0, \qquad \|\varrho_i\|_{L^2} \le c\|\varrho\|_{L^2}.$$

Then it is sufficient to prove (4.3) for each ϱ_i, so we are back on the bounded convex set $U_i \subset \mathbb{R}^n$. Here we distinguish the metric g of M and the standard Euclidean metric g_0 of $\mathbb{R}^n$. Let us denote $\bar{f}_i$ the mean value of f on U_i for g_0. We write

$$\int_{U_i} \varrho_i f \,\mathrm{vol} = \int_{U_i} \varrho_i (f - \bar{f}_i)\,\mathrm{vol}\,.$$

Therefore

$$\begin{aligned}\left|\int_{U_i} \varrho_i f \,\mathrm{vol}\right| &\le \|\varrho_i\|_{L^2(U_i,g)} \|f - \bar{f}_i\|_{L^2(U_i,g)} \\ &\le c\|\varrho_i\|_{L^2(U_i,g)} \|f - \bar{f}_i\|_{L^2(U_i,g_0)} \\ &\le c'\|\varrho_i\|_{L^2(U_i,g)} \|df\|_{L^2(U_i,g_0)} \\ &\le c''\|\varrho_i\|_{L^2(U_i,g)} \|df\|_{L^2(U_i,g)}\end{aligned}$$

which proves (4.3). □

Exercise 4.2 (Manifolds with boundary). Let (M, g) be a smooth connected manifold with boundary. Let ν be the outward pointing unit vector field on the boundary ∂M. Recall that if X is a vector field on M, its divergence $\mathrm{div}(X)$ is defined such that $\mathcal{L}_X \mathrm{vol}_g = \mathrm{div}(X)\,\mathrm{vol}_g$.

1) Show that for all vector fields $X \in C^\infty(M, TM)$,

$$\int_M \mathrm{div}(X)\,\mathrm{vol}_g = \int_{\partial M} X \cdot \nu \,\mathrm{vol}_{g|_{\partial M}}\,.$$

2) More generally, show that for all $u \in C^\infty(M), X \in C^\infty(M, TM)$,

$$\int_M u\,\mathrm{div}(X)\,\mathrm{vol}_g = -\int_M \nabla u \cdot X \,\mathrm{vol}_g + \int_{\partial M} u\, X \cdot \nu \,\mathrm{vol}_{g|_{\partial M}}$$

3) Deduce that for all $u, v \in C^\infty(M)$,

$$\int_M u\,\Delta v \,\mathrm{vol}_g = \int_M \nabla u \cdot \nabla v \,\mathrm{vol}_g - \int_{\partial M} u\,\nabla v \cdot \nu \,\mathrm{vol}_{g|_{\partial M}}\,.$$

4) The space $H^1_0(M)$ is defined as the completion of $C^\infty_c(M)$ with respect to the H^1-norm. Show that for all $u \in H^1_0(M), X \in C^\infty(M, TM)$,

$$\int_M u \operatorname{div}(X) \operatorname{vol}_g = -\int_M \nabla u \cdot X \operatorname{vol}_g .$$

The *double* of M is the manifold M^{double} obtained by gluing two copies of M along a cylinder $\partial M \times [-1,1]$. More precisely,

$$M^{\text{double}} := M \times \{-1\} \sqcup \partial M \times [-1,1] \sqcup M \times \{1\} / \sim,$$

where $(x, \pm 1)_{M \times \{\pm 1\}} \sim (x, \pm 1)_{\partial M \times [-1,1]}$ for all $x \in \partial M$. We will admit that M^{double} has the structure of a nice smooth closed manifold. Observe that there is a natural embedding $M \hookrightarrow M^{\text{double}}$. We extend the metric g to a smooth metric g^{double} such that $g^{\text{double}} = g$ on $M \times \{\pm 1\}$ and g^{double} is arbitrary on $\partial M \times [-1,1]$.

5) Draw a picture of M^{double}.

6) In this doubling process, it is key to guarantee that $g^{\text{double}} = g$ on $M \times \{\pm 1\}$. Why do we need to glue a cylinder then?

7) Given $f \in H^1_0(M)$, we define Ef as the function on M^{double} such that $Ef = f$ on $M \times \{-1\}$ and $Ef = 0$ on $\partial M \times [-1,1]$ and $Ef = -f$ on $M \times \{1\}$. Compute ∇Ef and show that $E : H^1_0(M) \to H^1(M)$ is continuous.

8) Using the Poincaré inequality on M^{double}, deduce that there exists $C > 0$ such that for all $f \in H^1_0(M)$, $\|f\|_{L^2(M)} \leq C \|\nabla f\|_{L^2(M)}$.

9) Show that the inclusion $H^1_0(M) \hookrightarrow L^2(M)$ is compact.

10) Given $\rho \in C^\infty(M)$, solve $\Delta f = \rho$ with $f \in H^1_0(M)$ in the weak sense (that is, when applied to $\varphi \in C^\infty_{\text{comp}}(M^\circ)$).

11) Show that the solution is unique.

12) Show that $\rho \mapsto \Delta^{-1}\rho$ is defined on $L^2(M)$ and compact.

5 Spectral decomposition and first eigenvalue

We begin by the following compactness theorem for the Sobolev inclusion:

Theorem 5.1 (Compactness). *The injection* $H^1(M) \hookrightarrow L^2(M)$ *is compact.*

Proof. We need to prove that a bounded sequence (f_i) in H^1 has a convergent subsequence in L^2.

The first step is to prove the result in the case of a torus $\mathbb{T}^n$. Using the Fourier series (2.1), the H^1 norm of f_i is given by

$$\|f_i\|^2_{H^1} = \sum_{\xi \in \mathbb{Z}^n} (1 + |\xi|^2) |f_i(\xi)|^2.$$

If $\|f_i\|^2_{H^1} \leq c$, we can extract a subsequence, still denoted f_i, such that for any $\xi \in \mathbb{Z}^n$ we have a convergence $f_i(\xi) \to f_\infty(\xi)$ when $i \to \infty$. The limit f_∞ also satisfies $\|f_\infty\|_{H^1}^2 \leq c$. Then we write, for any $R > 0$,

$$\begin{aligned}\|f_i - f_\infty\|_{L^2}^2 &= \sum_{\xi\in\mathbb{Z}^n} |f_\infty(\xi) - f_i(\xi)|^2 \\ &\leq \sum_{|\xi|<R} |f_\infty(\xi) - f_i(\xi)|^2 + \frac{1}{R^2}\sum_{|\xi|\geq R} |\xi|^2 |f_\infty(\xi) - f_i(\xi)|^2 \\ &\leq \sum_{|\xi|<R} |f_\infty(\xi) - f_i(\xi)|^2 + \frac{2c}{R^2}.\end{aligned}$$

Taking $i \to \infty$, we obtain $\limsup \|f_i - f_\infty\|_{L^2}^2 \leq \frac{2c}{R^2}$ for any $R > 0$. Therefore $f_i \to f_\infty$ in L^2.

The general case follows by localising in coordinate balls, considered as balls in the torus: choose a covering of M by a finite number of open sets U_j which are diffeomorphic to balls of radius 1 in $\mathbb{R}^n$. We can consider these balls as embedded in $\mathbb{T}^n$ instead, and we consider a partition of unity (χ_j) subordinate to (U_j). Then we have an equivalence of norms

$$\|f\|_{L^2(M,g)}^2 \sim \sum_j \|\chi_j f\|_{L^2(\mathbb{T}^n,g_0)}^2, \qquad \|f\|_{H^1(M,g)}^2 \sim \sum_j \|\chi_j f\|_{H^1(\mathbb{T}^n,g_0)}^2.$$

Therefore the result follows by applying the torus result to each $\chi_j f$. □

We can now establish the spectral theory for the Laplacian. For each ϱ with $\int_M \varrho \operatorname{vol} = 0$, we have found a unique $f := G\varrho$ such that $\Delta f = \varrho$ and $\int_M f \operatorname{vol} = 0$. Moreover, by (1.6) and (3.5):

$$\|df\|_{L^2}^2 = (\varrho, f) \leq \|\varrho\|_{L^2}\|f\|_{L^2} \leq c^{-\frac{1}{2}}\|\varrho\|_{L^2}\|df\|_{L^2}. \tag{5.1}$$

Therefore $\|df\|_{L^2} \leq c^{-\frac{1}{2}}\|\varrho\|_{L^2}$ which means that G is continous as an operator $L^2(M) \to H^1(M)$. Composing by the compact injection $H^1(M) \hookrightarrow L^2(M)$ we deduce that the operator $G : L^2(M) \to L^2(M)$ is compact.

The spectral theory of compact selfadjoint operators applies: it follows that there exist a Hilbertian basis (ϕ_i) of $L^2(M)$ such that

$$G\phi_i = \mu_i\phi_i, \quad \mu_i \to 0.$$

Here $\mu_i > 0$ since $\Delta(\mu_i\phi_i) = \phi_i$ and therefore $\mu_i\|d\phi_i\|_{L^2}^2 = \|\phi_i\|_{L^2}^2$. Taking $\lambda_i = 1/\mu_i$ we get

$$\Delta\phi_i = \lambda_i\phi_i, \quad \lambda_i \to +\infty. \tag{5.2}$$

A priori we have restricted to functions with zero integral on M, but we can add the constant function $\phi_0 = \frac{1}{\sqrt{\mathrm{Vol}}}$ with $\lambda_0 = 0$. We can order the eigenvalues λ_i so that

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \tag{5.3}$$

and decomposing any function on this Hilbertian basis we obtain that the optimal constant in the Poincaré inequality (3.5) is exactly $c = \lambda_1$.

Exercise 5.2 (Laplace spectrum on the sphere). The purpose of this exercise is to compute the Laplace eigenvalues on the sphere (S^n, g_{can}) equipped with the round metric. Let $\mathbf{P}_m(\mathbb{R}^{n+1})$ be the space of homogeneous polynomials of degree $m \geq 0$ on $\mathbb{R}^{n+1}$, and

$$\mathbf{H}_m(\mathbb{R}^{n+1}) := \{u \in \mathbf{P}_m(\mathbb{R}^{n+1}), \ \mid \Delta u = 0\},$$

the space of harmonic homogeneous polynomials.

We introduce the operator

$$\partial : \mathbf{P}_m(\mathbb{R}^{n+1}) \to \mathrm{Diff}^m, \qquad \partial(\sum_{|\alpha|=m} c_\alpha x^\alpha) = \sum_{|\alpha|=m} c_\alpha \partial_1^{\alpha_1} ... \partial_n^{\alpha_n}.$$

and define the scalar product

$$\langle P, Q \rangle := \partial(P)\overline{Q}, \qquad \forall P, Q \in \mathbf{P}_m(\mathbb{R}^{n+1}). \tag{5.4}$$

1) Compute $\mathbf{H}_0(\mathbb{R}^{n+1}), \mathbf{H}_1(\mathbb{R}^{n+1})$.

2) Verify that (5.4) defines indeed a Hermitian scalar product (or Euclidean scalar product in restriction to real-valued polynomials).

3) Compute the dimension of $\mathbf{P}_m(\mathbb{R}^{n+1})$.

4) Let $R \in \mathbf{P}_k(\mathbb{R}^{n+1})$ and define

$$m_R : \mathbf{P}_m(\mathbb{R}^{n+1}) \to \mathbf{P}_{m+k}(\mathbb{R}^{n+1}), \qquad m_R(P) := PR.$$

Show that $m_R^* : \mathbf{P}_{m+k}(\mathbb{R}^{n+1}) \to \mathbf{P}_m(\mathbb{R}^{n+1})$ is given by $m_R^*(P) = \partial(\overline{R})P$. What do you get for $R := |x|^2$?

5) Show that for $m \geq 2$,

$$\mathbf{P}_m(\mathbb{R}^{n+1}) = \mathbf{H}_m(\mathbb{R}^{n+1}) \oplus |x|^2 \mathbf{P}_{m-2}(\mathbb{R}^{n+1}).$$

Deduce that $\mathbf{P}_m(\mathbb{R}^{n+1}) = \oplus_{k \geq 0} |x|^{2k} \mathbf{H}_{m-2k}(\mathbb{R}^{n+1})$ (with the convention that $\mathbf{H}_{m-2k} = \{0\}$ for $m - 2k < 0$).

6) Let $f \in C^\infty(\mathbb{R}^{n+1})$. Show that

$$\Delta_{S^n}(f|_{S^n}) = (\Delta f + n\partial_r f + \partial_r^2 f)|_{S^n}.$$

7) Show that if $f \in \mathbf{H}_m(\mathbb{R}^{n+1})$, then $P := f|_{S^{n+1}}$ satisfies $\Delta_{S^n} P = m(m + n - 1)P$.

8) Deduce the Laplace spectrum of (S^n, g_{can}).

Exercise 5.3 (Laplace spectrum on the torus). Let Λ be a lattice in $\mathbb{R}^n$ and set $\mathbb{T}^n := \mathbb{R}^n/\Lambda$. Define Λ^* the dual lattice, as the set of vectors $\lambda^* \in \mathbb{R}^n$ such that $\langle \lambda^*, \lambda \rangle \in \mathbb{Z}$ for all $\lambda \in \Lambda$.

1) Show that $f_{\lambda^*}(x) := e^{2i\pi\langle \lambda^*, x\rangle}$ is a well-defined function on $\mathbb{T}^n$ and a Laplace eigenfunction for the eigenvalue $4\pi^2|\lambda^*|^2$. 2) Show that it forms a basis of $L^2(\mathbb{T}^n)$.

On a closed Riemannian manifold (M, g), Weyl's law describes the asymptotic growth of Laplace eigenvalues. It was proved by Weyl in 1911. It shows that

$$\#\{\mu \leq T, \mu \text{ eigenvalue of } \Delta\} \sim (2\pi)^{-n} T^{n/2} \omega_n \operatorname{vol}_g(M),$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ (and the eigenvalues are counted with multiplicities).

3) Prove Weyl's law on the torus.

Chapter II

Elliptic operators and Hodge theory

6 Definition

If E is a vector bundle over M, we denote by $C^\infty(M,E)$, or sometimes $C^\infty(E)$, the space of smooth sections of E.

A linear operator $P : C^\infty(M,E) \to C^\infty(M,F)$ between sections of two bundles E and F is a ***differential operator of order*** d if, in any local trivialisation of E and F over a coordinate chart (x^i), one has

$$Pu(x) = \sum_{|\alpha|\leq d} a^\alpha(x)\partial_\alpha u(x),$$

where $\alpha = (\alpha_1,\dots,\alpha_k)$ is a multiindex with each $\alpha_i \in \{1\dots n\}$, $|\alpha| = k$, $\partial_\alpha = \partial_{\alpha_1}\dots\partial_{\alpha_n}$, and $a^\alpha(x)$ is a matrix representing an element of $\mathrm{Hom}(E_x,F_x)$.

The ***principal symbol*** of P is defined for $x \in M$ and $\xi \in T_x^*M$ by taking only the terms of order d in P:

$$\sigma_P(x,\xi) = \sum_{|\alpha|=d} a^\alpha(x)\xi_\alpha,$$

where $\xi_\alpha = \xi_{\alpha_1}\cdots\xi_{\alpha_d}$ if $\xi = \xi_i dx^i$. It is a degree d homogeneous polynomial in the variable ξ with values in $\mathrm{Hom}(E_x,F_x)$.

A priori, it is not clear from the formula in local coordinates that the principal symbol is intrinsically defined. But it is easy to check that one has the following more intrinsic definition: suppose $f \in C^\infty(M)$, $t \in \mathbb{R}$ and $u \in C^\infty(M,E)$, then

$$e^{-tf(x)}P\big(e^{tf(x)}u(x)\big)$$

is a polynomial of degree d in the variable t, whose monomial of degree d is a homogeneous polynomial of degree d in $df(x)$. It is actually

$$t^d\sigma_P(x,df(x))u(x).$$

The following property of the principal symbol is obvious.

Lemma 6.1. $\sigma_{P\circ Q}(x,\xi) = \sigma_P(x,\xi)\circ\sigma_Q(x,\xi)$.

Examples 6.2. 1) The principal symbol of the exterior derivative $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is

$$\sigma_d(x,\xi) = \xi\wedge . \tag{6.1}$$

Indeed $e^{-tf} d(e^{tf}\alpha) = tdf\wedge\alpha + d\alpha$. The same is true for the exterior derivative d^∇ on vector valued differential forms, see section 11.

2) If one has a connection $\nabla : C^\infty(E) \to \Omega^1(E)$ on a vector bundle E, then similarly $e^{-tf}\nabla(e^{tf}u) = tdf\otimes u + \nabla u$. Therefore

$$\sigma_\nabla(x,\xi) = \xi\otimes : E_x \to T^*_x M\otimes E_x. \tag{6.2}$$

We now add metrics. Suppose (M^n, g) is an oriented Riemannian manifold, and $E \to M$ a unitary bundle. Then on sections of E with compact support, one can define the L^2 scalar product and the L^2 norm:

$$(s,t) = \int_M \langle s,t\rangle_E \,\mathrm{vol}, \qquad \|s\|^2 = \int_M \langle s,s\rangle_E \,\mathrm{vol}. \tag{6.3}$$

If E and F are unitary bundles and $P : C^\infty(E) \to C^\infty(F)$ is a linear operator, then a *formal adjoint of* P is an operator $P^* : C^\infty(F) \to C^\infty(E)$ satisfying

$$(Ps,t)_E = (s,P^*t)_F \tag{6.4}$$

for all sections $s\in C^\infty_c(E)$ and $t\in C^\infty_c(F)$.

Lemma 6.3. *Any differential operator* $P : C^\infty(E)\to C^\infty(F)$ *of order d has a formal adjoint* P^*, *whose principal symbol is*

$$\sigma_{P^*}(x,\xi) = (-1)^d \sigma_P(x,\xi)^*.$$

Exercise 6.4. Prove the lemma in the following way. In local coordinates, write $\mathrm{vol} = v(x)dx^1\wedge\cdots\wedge dx^n$. Choose orthonormal trivialisations of E and F, and write $P = \sum a^\alpha(x)\partial_\alpha$. Then prove that

$$P^*t = \sum_{|\alpha|\le d} (-1)^{|\alpha|}\frac{1}{v(x)}\partial_\alpha\big(v(x)a^\alpha(x)^* t\big).$$

Definition 6.5. A differential operator $P : C^\infty(E)\to C^\infty(F)$ is an ***elliptic operator*** if for any $x\in M$ and $\xi\neq 0$ in T^*_xM, the principal symbol $\sigma_P(x,\xi) : E_x\to F_x$ is injective.

7 Main result

Here is our main theorem on elliptic operators. It will be proved in section 19.

Theorem 7.1. *Suppose* (M^n,g) *is a compact oriented Riemannian manifold, and* $P : C^\infty(E)\to C^\infty(F)$ *is an elliptic operator, with* $\operatorname{rank} E = \operatorname{rank} F$. *Then*

1. $\ker(P)$ *is finite dimensional;*

2. *there is a L^2 orthogonal sum*

$$C^\infty(M,F) = \ker(P^*) \oplus P\big(C^\infty(M,E)\big).$$

We can apply the result to the scalar Laplacian and obtain immediately Theorem 2.1 as a special case.

Remark that $\ker(P^*)$ is also finite dimensional, since P^* is elliptic if P is elliptic. The difference $\dim\ker P - \dim\ker P^*$ is the ***index*** of P, defined by

$$\operatorname{ind}(P) = \dim\ker P - \dim\operatorname{coker} P.$$

Operators with finite dimensional kernel and cokernel are called ***Fredholm operators***, and the index is invariant under continuous deformation among Fredholm operators. Since ellipticity depends only on the principal symbol, it follows immediately that the index of P depends only on σ_P. The fundamental index theorem of Atiyah-Singer gives a topological formula for the index, see the book [BGV04].

A useful special case is that of a formally selfadjoint elliptic operator. Its index is of course zero. The invariance of the index then implies that any elliptic operator with the same symbol (or whose symbol is a deformation through elliptic symbols) has also index zero.

8 The Hodge operator

Let V be a n-dimensional oriented Euclidean vector space (thought as a tangent space of an oriented Riemannian n-manifold). Therefore there is a canonical volume element $\mathrm{vol} \in \Lambda^n V^*$. The exterior product $\Lambda^p V^* \wedge \Lambda^{n-p} V^* \to \Lambda^n V^*$ is a non degenerate pairing. Therefore, for a form $\beta \in \Lambda^p V^*$, one can define $*\beta \in \Lambda^{n-p}V^*$ by its wedge product with p-forms:

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \,\mathrm{vol} \tag{8.1}$$

for all $\beta \in \Lambda^p V^*$. The operator $* : \Lambda^p V^* \to \Lambda^{n-p} V^*$ is called the ***Hodge $*$ operator***.

In more concrete terms, if $(e_i)_{i=1\dots n}$ is a direct orthonormal basis of V, then $(e^I)_{I\subset\{1,\dots,n\}}$ is an orthonormal basis of ΛV^*. One checks easily that

$$\begin{aligned} *1 &= \mathrm{vol}, & *e^1 &= e^2 \wedge e^3 \wedge \cdots \wedge e^n, \\ *\,\mathrm{vol} &= 1, & *e^i &= (-1)^{i-1} e^1 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots e^n. \end{aligned}$$

More generally,

$$*e^I = \varepsilon(I, \complement I) e^{\complement I}, \tag{8.2}$$

where $\varepsilon(I, \complement I)$ is the signature of the permutation $(1, \dots, n) \to (I, \complement I)$.

Exercise 8.1. Suppose that in the basis (e_i) the quadratic form is given by the matrix $g = (g_{ij})$, and write the inverse matrix $g^{-1} = (g^{ij})$. Prove that for a 1-form $\alpha = \alpha_i e^i$ one has

$$*\alpha = (-1)^{i-1} \sqrt{\det(g)} g^{ij} \alpha_j e^1 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge e^n. \tag{8.3}$$

Using $\alpha^\sharp$ defined in (1.2), this can also be written

$$*\alpha = \alpha^\sharp \lrcorner \,\mathrm{vol}\,. \tag{8.4}$$

Exercise 8.2. Prove that $*^2 = (-1)^{p(n-p)}$ on $\Lambda^p V^*$.

If n is even, then $* : \Lambda^{n/2}V^* \to \Lambda^{n/2}V^*$ satisfies $*^2 = (-1)^{n/2}$. Therefore, if $n/2$ is even, the eigenvalues of $*$ on $\Lambda^{n/2}V^*$ are ± 1, and $\Lambda^{n/2}V^*$ decomposes accordingly as

$$\Lambda^{n/2}V^* = \Lambda_+^{n/2} \oplus \Lambda_-^{n/2}. \tag{8.5}$$

The elements of $\Lambda_+^{n/2}$ are called *selfdual forms*, and the elements of $\Lambda_-^{n/2}$ *antiselfdual forms*. For example, if $n = 4$, then $\Lambda_\pm$ is generated by the forms

$$e^1 \wedge e^2 \pm e^3 \wedge e^4, \quad e^1 \wedge e^3 \mp e^2 \wedge e^4, \quad e^1 \wedge e^4 \pm e^2 \wedge e^3. \tag{8.6}$$

Exercise 8.3. If $n/2$ is even, prove that the decomposition (8.5) is orthogonal for the quadratic form $\Lambda^{n/2}V^* \wedge \Lambda^{n/2}V^* \to \Lambda^n V^* \simeq \mathbb{R}$, and

$$\alpha \wedge \alpha = \pm |\alpha|^2 \operatorname{vol} \quad \text{if } \alpha \in \Lambda_\pm. \tag{8.7}$$

Exercise 8.4. If u is an orientation-preserving isometry of V, that is $u \in SO(V)$, prove that u preserves the Hodge operator. This means the following: u induces an isometry of V^*, and an isometry $\Lambda^p u$ of $\Lambda^p V^*$ defined by $(\Lambda^p u)(x^1 \wedge \cdots \wedge x^p) = u(x^1) \wedge \cdots \wedge u(x^p)$. Then for any p-form $\alpha \in \Lambda^p V^*$ one has

$$*(\Lambda^p u)\alpha = (\Lambda^{n-p} u) * \alpha.$$

This illustrates the fact that an orientation-preserving isometry preserves every object canonically attached to a metric and an orientation.

9 Adjoint operator

We have already calculated the adjoint of the differential of functions, see equations (1.1)– (1.3). Given (8.4) it can also be written for any 1-form α as

$$d^*\alpha = - * d * \alpha. \tag{9.1}$$

This is a special case of the following general formula. Denote by $\Omega^p(M)$ the space of smooth differential p-forms on M, and $\Omega_c^p(M)$ the version with compact support.

Lemma 9.1. *The formal adjoint of the exterior derivative* $d : \Omega^p(M) \to \Omega^{p+1}(M)$ *is*

$$d^* = (-1)^{np+1} * d * . \tag{9.2}$$

Proof. For $\alpha \in \Omega_c^p(M)$ and $\beta \in \Omega_c^{p+1}(M)$ one has the equalities:

$$\begin{aligned}\int_M \langle d\alpha, \beta \rangle \operatorname{vol}^g &= \int_M du \wedge *v \\ &= \int_M d(u \wedge *v) - (-1)^p u \wedge d * v\end{aligned}$$

by Stokes theorem, and using exercice 8.2:

$$= (-1)^{p+1+p(n-p)} \int_M u \wedge * * d * v$$
$$= (-1)^{pn+1} \int_M \langle u, *d * v\rangle \operatorname{vol}^g .$$

□

Remarks 9.2. 1) If n is even then the formula simplifies to $d^* = - * d*$.

2) The same formula gives an adjoint for the exterior derivative $d^\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$ associated to a unitary connection ∇ on a bundle E.

3) By lemma 6.3 the principal symbol of d^* is $-\sigma_d(x,\xi)^*$. The adjoint of the exterior product by ξ is the internal product by $\xi^\sharp$, so we obtain

$$\sigma_{d^*}(x,\xi)\alpha = -\xi^\sharp \lrcorner \alpha. \tag{9.3}$$

10 Hodge theory

Definition 10.1. Let (M^n, g) be an oriented Riemannian manifold. The ***Hodge-De Rham Laplacian*** on p-forms is defined by

$$\Delta\alpha = (dd^* + d^*d)\alpha.$$

On functions we recover the scalar Laplacian that we have already seen.

Clearly, Δ is a formally selfadjoint operator. The definition is also valid for E-valued p-forms, using the exterior derivative d^∇, where E has a metric connection ∇.

Exercise 10.2. On p-forms in $\mathbb{R}^n$ prove that $\Delta(\alpha_I dx^I) = (\Delta\alpha_I)dx^I$.

Exercise 10.3. Prove that $*$ commutes with Δ.

Lemma 10.4. *The principal symbol of the Hodge-De Rham Laplacian is*

$$\sigma_\Delta(x,\xi) = -|\xi|^2.$$

In particular Δ is an elliptic operator.

Proof. By (9.3)

$$\sigma_\Delta(x,\xi) = -(\xi^\sharp\lrcorner)(\xi\wedge) - (\xi\wedge)(\xi^\sharp\lrcorner).$$

It is sufficient to calculate for $\xi = e^1$ where (e_i) is an orthonormal basis of TM, and the result is then immediate. □

Let (M^n, g) be a closed Riemannian oriented manifold. Consider the De Rham complex

$$0 \to C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0.$$

Remind that the De Rham cohomology in degree p is defined by

$$H^p = \{\alpha \in \Omega^p(M), d\alpha = 0\}/d\Omega^{p-1}(M).$$

Definition 10.5. A ***harmonic form*** is a C^∞ form such that $\Delta\alpha = 0$.

Lemma 10.6. *If $\alpha \in C_c^\infty(M, \Omega^p)$, then α is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$. In particular, on a compact connected manifold, any harmonic function is constant.*

Proof. It is clear that if $d\alpha = 0$ and $d^*\alpha = 0$, then $\Delta\alpha = d^*d\alpha + dd^*\alpha = 0$. Conversely, if $\Delta\alpha = 0$, because

$$(\Delta\alpha, \alpha) = (d^*d\alpha, \alpha) + (dd^*\alpha, \alpha) = \|d\alpha\|^2 + \|d^*\alpha\|^2,$$

we deduce that $d\alpha = 0$ and $d^*\alpha = 0$. □

Remark 10.7. The lemma remains valid on complete manifolds, for L^2 forms α such that $d\alpha$ and $d^*\alpha$ are also L^2. This is proved by taking cut-off functions χ_j, such that $\chi_j^{-1}(1)$ are compact domains which exhaust M, and $|d\chi_j|$ remains bounded by a fixed constant C. Then

$$\begin{aligned}\int_M \langle \Delta\alpha, \chi_j\alpha\rangle \,\mathrm{vol} &= \int_M \big(\langle d\alpha, d(\chi_j\alpha)\rangle + \langle d^*\alpha, d^*(\chi_j\alpha)\rangle\big)\,\mathrm{vol} \\ &= \int_M \big(\chi_j(|d\alpha|^2 + |d^*\alpha|^2) + \langle d\alpha, d\chi_j \wedge \alpha\rangle - \langle d^*\alpha, \nabla\chi_j \lrcorner \alpha\rangle\big)\,\mathrm{vol}\end{aligned}$$

Using $|d\chi_j| \le C$ and taking j to infinity, one obtains $(\Delta\alpha, \alpha) = \|d\alpha\|^2 + \|d^*\alpha\|^2$.

Note $\mathbf{H}^p$ the space of harmonic p-forms on M. Theorem 7.1 on elliptic operators can be applied to the Hodge-De Rham Laplacian to give:

Theorem 10.8. *Let (M^n, g) be a compact closed oriented Riemannian manifold. Then:*

1. *$\mathbf{H}^p$ is finite dimensional;*
2. *one has a decomposition $\Omega^p(M) = \mathbf{H}^p \oplus \Delta(\Omega^p(M))$, which is orthogonal for the L^2 scalar product.*

We now derive some immediate consequences.

Corollary 10.9. *Same hypothesis. One has the orthogonal decomposition*

$$\Omega^p(M) = \mathbf{H}^p \oplus d\big(\Omega^{p-1}(M)\big) \oplus d^*\big(\Omega^{p+1}(M)\big),$$

where

$$\ker d = \mathbf{H}^p \oplus d\big(\Omega^{p-1}(M)\big), \tag{10.1}$$

$$\ker d^* = \mathbf{H}^p \oplus d^*\big(\Omega^{p+1}(M)\big). \tag{10.2}$$

Note that since harmonic forms are closed, there is a natural map $\mathbf{H}^p \to H^p$. The equality (10.1) implies immediately:

Corollary 10.10. *Same hypothesis. The map $\mathbf{H}^p \to H^p$ is an isomorphism.*

Using exercise 10.3, we obtain:

Corollary 10.11 (Poincaré duality). *Same hypothesis. The Hodge $*$ operator induces an isomorphism $* : \mathbf{H}^p \to \mathbf{H}^{n-p}$. In particular the corresponding Betti numbers are equal, $b_p = b_{n-p}$.*

Remark 10.12. As an immediate consequence, if M is connected then $\mathrm{H}^n = \mathbb{R}$ since $\mathrm{H}^0 = \mathbb{R}$. Since $*1 = \mathrm{vol}^g$ and $\int_{\mathrm{M}} \mathrm{vol}^g > 0$, an identification with $\mathbb{R}$ is just given by integration of n-forms on M.

Remark 10.13. In Kähler geometry there is a decomposition of harmonic forms using the (p,q) type of forms, $\mathbf{H}^k \otimes \mathbb{C} = \oplus_0^k \mathbf{H}^{p,k-p}$, and corollary 10.11 can then be refined as an isomorphism $* : \mathbf{H}^{p,q} \to \mathbf{H}^{m-q,m-p}$, where $n = 2m$.

Remark 10.14. Suppose that n is a multiple of 4. Then by exercises 8.3 and 10.3, one has an orthogonal decomposition

$$\mathbf{H}^{n/2} = \mathbf{H}_+ \oplus \mathbf{H}_-. \tag{10.3}$$

Under the wedge product, the decomposition is orthogonal, $\mathbf{H}_+$ is positive and $\mathbf{H}_-$ is negative, therefore the signature of the manifold is (p,q) with $p = \dim \mathbf{H}_+$ and $q = \dim \mathbf{H}_-$.

Exercise 10.15. Suppose again that n is a multiple of 4. Note $d_\pm : \Omega^{n/2-1}(\mathrm{M}) \to \Omega_\pm(\mathrm{M})$ the projection of d on selfdual or antiselfdual forms. Prove that on $(n/2-1)$-forms, one has $d_+^* d_+ = d_-^* d_-$. Deduce that the cohomology of the complex

$$0 \to C^\infty(\mathrm{M}) \xrightarrow{d} \Omega^1(\mathrm{M}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n/2-1}(\mathrm{M}) \xrightarrow{d_+} \Omega_+(\mathrm{M}) \to 0 \tag{10.4}$$

is $\mathbf{H}^0, \mathbf{H}^1, \ldots, \mathbf{H}^{n/2-1}, \mathbf{H}_+$.

Exercise 10.16. Using exercise 10.2, calculate the harmonic forms and the cohomology of a flat torus $\mathbb{R}^n/\mathbb{Z}^n$.

Exercise 10.17. Let (M, g) be a compact oriented Riemannian manifold.

1) If γ is an orientation-preserving isometry of (M, g) and α a harmonic form, prove that $\gamma^*\alpha$ is harmonic.

2) (requires some knowledge of Lie groups) Prove that if a *connected* Lie group G acts on M, then the action of G on $\mathrm{H}^\bullet(\mathrm{M}, \mathbb{R})$ given by $\alpha \to \gamma^*\alpha$ is trivial[1].

3) Deduce that harmonic forms are invariant under $\mathrm{Isom}(\mathrm{M}, g)^o$, the connected component of the identity in the isometry group of M. Apply this observation to give a proof that the cohomology of the n-sphere vanishes in degrees $k = 1, \ldots, n-1$ (prove that there is no $\mathrm{SO}(n+1)$-invariant k-form on S^n using the fact that the representation of $\mathrm{SO}(n)$ on $\Omega^k\mathbb{R}^n$ is irreducible and therefore has no fixed nonzero vector).

[1]If ξ belongs to the Lie algebra of G and X_ξ is the associated vector field on M given by the infinitesimal action of G (that is defined by $\mathrm{X}_\xi(x) = \frac{d}{dt} e^{t\xi} x|_{t=0}$), then one has $\frac{d}{dt}(e^{t\xi})^*\alpha|_{t=0} = \mathcal{L}_{\mathrm{X}_\xi}\alpha = i_{\mathrm{X}_\xi} d\alpha + d i_{\mathrm{X}_\xi}\alpha$. Deduce that if α is closed, then the infinitesimal action of G on $\mathrm{H}^\bullet(\mathrm{M}, \mathbb{R})$ is trivial.

Chapter III

Bochner formula and applications

11 More on connections

We recall without proof some basic properties of connections on vector bundles.

Let $E \to M$ be a vector bundle with a connection $\nabla : C^\infty(E) \to \Omega^1(E)$. In a local trivialization of E and in local coordinates (x^i) one can write locally

$$\nabla s = ds + dx^i \otimes a_i s,$$

where s is a section of E (seen as a map to $\mathbb{R}^k$ or $\mathbb{C}^k$ in the trivialization) and a_i are local maps from M to End E, that is to $k \times k$ matrices. We can evaluate on the basic vector field ∂_i and write

$$\nabla_i s = \partial_i s + a_i s.$$

We can also define the local connection 1-form $a = a_i dx^i$: it is a 1-form with values in End E; then we can write in a compact form the connection as

$$\nabla s = ds + as.$$

If ∇ is a unitary connection (that is preserves a scalar or Hermitian product h on E), then in an orthonormal trivialization of E it turns out that the matrices a_i are antisymmetric (real case) or anti-Hermitian (complex case), that is take value in the Lie algebras $\mathfrak{o}(E)$ or $\mathfrak{u}(E)$.

A connection on a bundle E induces a connection on all associated bundles: E^*, $\Lambda^p E$, etc. The principle is that algebraic operations are invariant under the connection, for example the connection ∇^{E^*} is deduced from ∇^E by

$$d\langle s^*, s\rangle = \langle \nabla^{E^*} s^*, s\rangle + \langle s^*, \nabla^E s\rangle.$$

Taking a trivialization, it follows quickly that

$$a_i^{E^*} = -(a_i^E)^t.$$

Another important case is that of the bundle $\mathrm{End}\,E = E^* \otimes E$. Again, invariance of the evaluation of an endomorphism of E on a section of E gives the rule

$$\nabla^E(u(s)) = (\nabla^{\mathrm{End}\,E}u)(s) + u(\nabla^E s),$$

from which one deduces in a trivialization the equality

$$a_i^{\mathrm{End}\,E}(u) = [a_i^E, u].$$

The curvature of ∇ is a 2-form with values in End E defined by

$$F_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{11.1}$$

A direct calculation from $\nabla = d + a$ gives

$$F_{X,Y} = X \cdot a_Y - Y \cdot a_X - a_{[X,Y]} + [a_X, a_Y] = (da + a \wedge a)_{X,Y}, \tag{11.2}$$

that is $F = da + a \wedge a$. In coordinates we can write $F = \sum_{i<j} F_{ij} dx^i \wedge dx^j$ with

$$F_{ij} = \partial_i a_j - \partial_j a_i + [a_i, a_j]. \tag{11.3}$$

Note that for a unitary connection this is still with values in $\mathfrak{o}(E)$ or $\mathfrak{u}(E)$.

12 The Bianchi identity

We denote by $\Omega^p(M, E) := C^\infty(M, \Lambda^p T^*M \otimes E)$ the space of p-forms with values in E. For example the curvature F is an element of $\Omega^2(\mathrm{End}\,E)$. One can extend ∇ uniquely to an exterior differential on E-valued differential forms:

$$d^\nabla : \Omega^p(E) \longrightarrow \Omega^{p+1}(E) \tag{12.1}$$

satisfying the Leibniz identity, for α a differential form and σ an E-valued differential form:

$$d^\nabla(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^{|\alpha|} \alpha \wedge d^\nabla \sigma. \tag{12.2}$$

This extension can be defined by the local formula $d^\nabla \sigma = d\sigma + a \wedge \sigma$ in a trivialization of E in which $\nabla = d + a$ as above. It is equivalent to the formula:

$$\begin{aligned}(d^\nabla \sigma)_{X_0,\dots,X_p} = &\sum_0^p (-1)^i \nabla_{X_i}(\sigma_{X_0,\dots,\widehat{X_i},\dots,X_k}) \\ &+ \sum_{0\le i<j\le p} (-1)^{i+j} \sigma_{[X_i,X_j],X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_k}.\end{aligned} \tag{12.3}$$

This extension leads to a nice interpretation of the curvature: recall that the exterior differential satisfies $d \circ d = 0$. The curvature is precisely the defect for $d^\nabla \circ d^\nabla$ to vanish:

Lemma 12.1. *1° The curvature* F^∇*, seen as an operator* $C^\infty(E) \to \Omega^2(E)$*, is* $F^\nabla = d^\nabla \circ d^\nabla$.
2° As an operator $\Omega^p(E) \to \Omega^{p+2}(E)$*, one has* $F^\nabla = d^\nabla \circ d^\nabla$.

Proof. Let us choose a local trivialization of E, and write the connection $\nabla = d+a = d + a_i dx^i$, where each a_i is End(E)-valued. Then, for a section s of E, we have $d^\nabla s = ds + as$ and

$$\begin{aligned} d^\nabla(d^\nabla s) &= (d+a)(d+a)s \\ &= d(as) + a \wedge ds + a \wedge as \\ &= (da + a \wedge a)s. \end{aligned}$$

This proves the first statement. The proof of the second one is similar. □

The lemma implies that if (E, ∇) is a flat bundle ($\mathrm{F}^\nabla = 0$), then we have an associated complex

$$0 \to \mathrm{C}^\infty(\mathrm{M},\mathrm{E}) \xrightarrow{d^\nabla} \Omega^1(\mathrm{E}) \xrightarrow{d^\nabla} \cdots \xrightarrow{d^\nabla} \Omega^n(\mathrm{E}) \to 0$$

and we can define the De Rham cohomology with values in E as in the usual case. The Hodge-De Rham Laplacian $d^\nabla(d^\nabla)^* + (d^\nabla)^* d^\nabla$ has still symbol $-|\xi|^2$ and therefore all the results of Hodge theory extend to this situation.

We deduce from the lemma the following important identity:

Proposition 12.2 (***differential Bianchi identity***). *The curvature of a connection satisfies the identity*

$$d^\nabla \mathrm{F}^\nabla = 0.$$

Remark that $\mathrm{F}^\nabla \in \Omega^2(\mathrm{End}\,\mathrm{E})$ so d^∇ is the exterior derivative associated to the connection ∇ on End E, and $d^\nabla \mathrm{F}^\nabla \in \Omega^3(\mathrm{End}\,\mathrm{E})$.

Proof. We give two proofs. The first proof is abstract: let us distinguish ∇ on E and $\bar\nabla$ on End E. Recall that, as a linear operator on E, for $u \in \mathrm{C}^\infty(\mathrm{End}\,\mathrm{E})$ one has $\bar\nabla u = \nabla \circ u - u \circ \nabla$. Then the reader will check that, as operators $\mathrm{C}^\infty(\mathrm{E}) \to \Omega^3(\mathrm{E})$, one has $d^{\bar\nabla} \mathrm{F}^\nabla = d^\nabla \circ \mathrm{F}^\nabla - \mathrm{F}^\nabla \circ d^\nabla$. But since $\mathrm{F}^\nabla = d^\nabla \circ d^\nabla$, we obtain

$$d^{\bar\nabla} \mathrm{F}^\nabla = d^\nabla \circ d^\nabla \circ d^\nabla - d^\nabla \circ d^\nabla \circ d^\nabla = 0.$$

The second proof is a calculation: in a trivialization where $\nabla = d + a$ we have $\mathrm{F}^\nabla = da + a \wedge a$ and therefore

$$d^\nabla \mathrm{F}^\nabla = d(da + a \wedge a) + [a, da + a \wedge a]$$

since the connection form acts by bracket on End E. But $d(da) = 0$, $[a, da] = a \wedge da - da \wedge a = -d(a \wedge a)$ and finally $[a, a \wedge a] = 0$ by the Jacobi identity. So $d^\nabla \mathrm{F}^\nabla = 0$. □

Suppose now that ∇ is a unitary connection on E and (M, g) is a Riemannian manifold. The operator d^∇ has an adjoint $(d^\nabla)^*$ which is again given by formula (9.2).

Combining ∇ and the Levi-Civita connection of M gives a connection on $\Lambda^p \mathrm{T}^*\mathrm{M} \otimes \mathrm{E}$, that is on the bundle of E-valued p-forms. This connection can be used to give an alternative expression of $(d^\nabla)^*$: for 1-forms one obtains the following lemma.

Lemma 12.3. *Let* E *be a vector bundle with unitary connection* ∇*, then the formal adjoint of* $\nabla : C^\infty(M, E) \to \Omega^1(M, E)$ *is*

$$\nabla^* \alpha = -\operatorname{Tr}^g(\nabla u) = -\sum_1^n (\nabla_{e_i}\alpha)(e_i).$$

Proof. Take a local orthonormal basis (e_i) of TM, and consider an E-valued 1-form $\alpha = \alpha_i e^i$. We have $*\alpha = (-1)^{i-1}\alpha_i e^1 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge e^n$. One can suppose that just at the point p one has $\nabla e_i(p) = 0$, therefore $de^i(p) = 0$ and, still at the point p,

$$d^\nabla * \alpha = \sum_1^n (\nabla_i \alpha_i) e^1 \wedge \cdots \wedge e^n.$$

Finally $\nabla^*\alpha(p) = -\sum_1^n (\nabla_i \alpha_i)(p)$. □

Remark 12.4. Actually the same formula is also valid for p-forms. Indeed, $d^\nabla : \Omega^p(M) \to \Omega^{p+1}(M)$ can be deduced from the covariant derivative $\nabla : \Omega^p(M) \to \Omega^1(M, \Lambda^p T^* M)$ by the formula[1]

$$d^\nabla = (p+1)\mathbf{a} \circ \nabla,$$

where $\mathbf{a}$ is the antisymmetrization of a $(p+1)$-tensor. Also observe that if $\alpha \in \Lambda^p V^* \subset \otimes^p V^*$, its norm as a p-form differs from its norm as a p-tensor by

$$|\alpha|^2_{\Lambda^p V^*} = p! |\alpha|^2_{\otimes^p V^*}.$$

Putting together this two facts, one can calculate that $(d^\nabla)^*$ is the restriction of ∇^* to antisymmetric tensors in $T^*M \otimes \Lambda^p T^* M$. We get the formula

$$(d^\nabla)^* \alpha = -\sum_1^n e_i \lrcorner \nabla_i \alpha. \tag{12.4}$$

13 The Ricci tensor

If R is the Riemannian curvature tensor of (M, g), the ***Ricci tensor*** Ric is defined by the formula

$$\operatorname{Ric}(X, Y) = \operatorname{Tr}(Z \to R_{Z,X} Y).$$

In an orthonormal basis (e_i) of the tangent bundle, one has

$$\operatorname{Ric}(X, Y) = \sum \langle R_{e_i, X} Y, e_i \rangle, \tag{13.1}$$

which we can write $\operatorname{Ric}_{jk} = R_{ijk}{}^i$. From the symmetries of the curvature tensor $R_{ijk}{}^i = R_{kii}{}^j = R_{ikj}{}^i$ that is

$$\operatorname{Ric}(X, Y) = \operatorname{Ric}(Y, X), \tag{13.2}$$

so the Ricci tensor is a symmetric 2-tensor.

[1] This formula is true as soon as ∇ is a torsion free connection on M.

The ***scalar curvature*** of the metric is the function defined by

$$\mathrm{Scal} = \mathrm{Tr}(g^{-1}\,\mathrm{Ric}) = \sum_1^n \mathrm{Ric}(e_i, e_i).$$

For example, in dimension 2, in an orthonormal basis (e_1, e_2), if $K = R_{122}{}^1$ is the (Gauss) curvature, then one obtains immediately

$$\mathrm{Ric} = Kg, \quad \mathrm{Scal} = 2K.$$

For the sphere S^n one has $\mathrm{Ric} = (n-1)g$ and $\mathrm{Scal} = n(n-1)$.
For the hyperbolic space H^n one has $\mathrm{Ric} = -(n-1)$ and $\mathrm{Scal} = -n(n-1)$.

Proposition 13.1 (***differential Bianchi identity***)**.** *The Riemannian curvature satisfies the identity*

$$(\nabla_X R)_{Y,Z} + (\nabla_Y R)_{Z,X} + (\nabla_Z R)_{X,Y} = 0.$$

Proof. This is just a way of writing the Bianchi identity 12.2, using formula (12.3) with the help of the connection induced on $\Omega^2 \otimes \mathfrak{o}(TM)$. □

Proposition 13.2 (***Bianchi identity***)**.** *One has*

$$\delta\,\mathrm{Ric} = -\frac{1}{2} d\,\mathrm{Scal},$$

where the **divergence** $\delta\phi$ *of a 2-tensor* ϕ *is the 1-form defined by* $(\delta\phi)_X = -\sum_1^n (\nabla_{e_i}\phi)(e_i, X)$.

Proof. We choose an orthonormal basis (e_i) of TM such that just at the point x one has $\nabla e_i(x) = 0$, and we calculate only at the point x. We can also suppose that $\nabla X(x) = 0$, then we have

$$(d\,\mathrm{Scal})_X(x) = \mathcal{L}_X \sum_{i,j=1}^n \langle R_{e_i,e_j} e_j, e_i\rangle = \sum_{i,j=1}^n \langle \nabla_X R_{e_i,e_j} e_j, e_i\rangle.$$

Then, using the differential Bianchi identity,

$$\begin{aligned}(\delta\,\mathrm{Ric})_X(x) &= -\sum_1^n \nabla_{e_i} \mathrm{Ric}(e_i, X) = -\sum_{i,j=1}^n \nabla_{e_i} \langle R_{e_j,X} e_i, e_j\rangle \\ &= \sum_{i,j=1}^n \langle \nabla_{e_j} R_{X,e_i} e_i + \nabla_X R_{e_i,e_j} e_i, e_j\rangle \\ &= -(\delta\,\mathrm{Ric})_X + (d\,\mathrm{Scal})_X.\end{aligned}$$

□

From the definition, if f is a function then $\delta(fg) = -df$, so the Bianchi identity can also be written

$$\delta\Big(\mathrm{Ric} - \frac{\mathrm{Scal}}{2}\Big) = 0. \tag{13.3}$$

An ***Einstein metric*** is a Riemannian metric g which satisfies

$$\mathrm{Ric} = \Lambda g. \tag{13.4}$$

The constant Λ is called the ***cosmological constant*** in physics.

14 Bochner formula

Let (E, ∇) be a bundle equipped with a unitary connection over an oriented Riemannian manifold (M^n, g). Then $\nabla : C^\infty(E) \to \Omega^1(E)$ and we can define the ***rough Laplacian*** $\nabla^*\nabla$ acting on sections of E. Using a local orthonormal basis (e_i) of TM, from lemma 12.3 it follows that

$$\nabla^*\nabla s = \sum_1^n -\nabla_{e_i}\nabla_{e_i} s + \nabla_{\nabla_{e_i} e_i} s. \tag{14.1}$$

If we calculate just at a point p and we choose a basis (e_i) which is parallel at p, then the second term vanishes.

In particular, using the Levi-Civita connection, we get a Laplacian $\nabla^*\nabla$ acting on p-forms. It is not equal to the Hodge-De Rham Laplacian, as follows from:

Lemma 14.1 (Bochner formula). *Let* (M^n, g) *be an oriented Riemannian manifold. Then for any* 1*-form* α *on* M *one has*

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}(\alpha). \tag{14.2}$$

Remark 14.2. There is a similar formula (Weitzenböck formula) on p-forms: the difference $\Delta\alpha - \nabla^*\nabla\alpha$ is a zero-th order term involving the curvature of M.

Proof of the lemma. We have $d\alpha_{X,Y} = (\nabla_X\alpha)_Y - (\nabla_Y\alpha)_X$, therefore

$$d^*d\alpha_X = -\sum_1^n (\nabla_{e_i} d\alpha)_{e_i,X} = \sum_1^n -(\nabla_{e_i}\nabla_{e_i}\alpha)_X + (\nabla_{e_i}\nabla_X\alpha)_{e_i},$$

where in the last equality we calculate only at a point p, and we have chosen the vector fields (e_i) and X parallel at p.

Similarly, $d^*\alpha = -\sum_1^n (\nabla_{e_i}\alpha)_{e_i}$, therefore

$$dd^*\alpha_X = -\sum_1^n \nabla_X((\nabla_{e_i}\alpha)_{e_i}) = -\sum_1^n (\nabla_X\nabla_{e_i}\alpha)_{e_i}.$$

Therefore, still at the point p, comparing with (14.1),

$$(\Delta\alpha)_X = (\nabla^*\nabla\alpha)_X + \sum_1^n (R_{e_i,X}\alpha)_{e_i} = (\nabla^*\nabla\alpha)_X + \mathrm{Ric}(\alpha)_X. \tag{14.3}$$

□

Remark 14.3. There is a similar formula if the exterior derivative is coupled with a bundle E equipped with a connection ∇. The formula for the Laplacian $\Delta = (d^\nabla)^* d^\nabla + d^\nabla (d^\nabla)^*$ becomes

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}(\alpha) + \mathcal{R}^\nabla(\alpha), \tag{14.4}$$

where the additional last term involves the curvature of ∇,

$$\mathcal{R}^\nabla(\alpha)_X = \sum_1^n R^\nabla_{e_i,X}\alpha(e_i). \tag{14.5}$$

The proof is exactly the same as above, a difference arises just in the last equality of (14.3), when one analyses the curvature term: the curvature acting on α is that of $\Omega^1 \otimes E$, so equals $R \otimes 1 + 1 \otimes R^\nabla$, from which:

$$\sum_1^n (R_{e_i,X}\alpha)_{e_i} = \mathrm{Ric}(\alpha)_X + \sum_1^n R^\nabla_{e_i,X}\alpha(e_i).$$

Now let us see an application of the Bochner formula. Suppose M is compact. By Hodge theory, an element of $H^1(M)$ is represented by a harmonic 1-form α. By the Bochner formula, we deduce $\nabla^*\nabla\alpha + \mathrm{Ric}(\alpha) = 0$. Taking the scalar product with α, one obtains

$$\|\nabla\alpha\|^2 + (\mathrm{Ric}(\alpha), \alpha) = 0. \tag{14.6}$$

If $\mathrm{Ric} \geq 0$, this equality implies $\nabla\alpha = 0$ and $\mathrm{Ric}(\alpha) = 0$. If $\mathrm{Ric} > 0$, then $\alpha = 0$; if $\mathrm{Ric} \geq 0$ we get only that α is parallel, therefore the cohomology is represented by parallel forms. Suppose that M is connected, then a parallel form is determined by its values at one point p, so we get an injection

$$\mathbf{H}^1 \hookrightarrow T_p^*M.$$

Therefore $\dim \mathbf{H}^1 \leq n$, with equality if and only if M has a basis of parallel 1-forms. This implies that M is flat, and by Bieberbach's theorem that M is a torus. Therefore we deduce:

Corollary 14.4. *If* (M^n, g) *is a compact connected oriented Riemannian manifold, then:*

- *if* $\mathrm{Ric} > 0$*, then* $b_1(M) = 0$*;*
- *if* $\mathrm{Ric} \geq 0$*, then* $b_1(M) \leq n$*, with equality if and only if* (M, g) *is a flat torus.*

This corollary is a typical example of application of Hodge theory to prove vanishing theorems for the cohomology: one uses Hodge theory to represent cohomology classes by harmonic forms, and then a Weitzenböck formula to prove that the harmonic forms must vanish or be special under some curvature assumption.

15 Positive Ricci and the first eigenvalue

We have seen that Bochner formula (14.2) contrains the topology when the Ricci tensor is nonnegative. It also contrains the lowest eigenvalue, as we shall see in the two following results.

Theorem 15.1. *Suppose* (M^n, g) *is a compact connected oriented Riemannian manifold with* $\mathrm{Ric} \geq \varrho > 0$*. Then we have the following lower bound on the first eigenvalue of* (M, g)*:*

$$\lambda_1 \geq \frac{n}{n-1}\varrho. \tag{15.1}$$

The case $\mathrm{Ric} \geq 0$ is more subtle, and the following result is rather recent (Li-Yau 1980, Zhong-Yang 1984):

Theorem 15.2. *Under the same hypotheses, with only* $\mathrm{Ric} \geq 0$*, one has*

$$\lambda_1 \geq \frac{\pi^2}{\mathrm{Diam}(M,g)^2}. \tag{15.2}$$

We will actually prove only a weaker inequality below.

Both results are optimal, since the lower bound is obtained:

- in theorem 15.1, for the round sphere S^n ($n > 1$) since $\lambda_1(S^n) = n$;
- in theorem 15.2, for the circle S^1 since $\lambda_1(S^1) = 1$.

Proof of theorem 15.1. By Bochner formula (14.2), we have for any function f the equality

$$\Delta df = \nabla^* \nabla df + \mathrm{Ric}(df).$$

We have $\Delta df = (d^*d + dd^*)df = dd^*df = d\Delta f$. Therefore when we integrate against df, we obtain

$$\|\Delta f\|^2 = \|\nabla df\|^2 + \int_M \mathrm{Ric}(df, df)\,\mathrm{vol}\,.$$

Since $\Delta f = -\mathrm{Tr}(\nabla df)$ we have $|\nabla df|^2 \geq \frac{1}{n}|\Delta f|^2$. Injecting this in the previous equality and using the hypothesis on Ricci, we obtain

$$\big(1 - \frac{1}{n}\big)\|\Delta f\|^2 \geq \rho\|df\|^2 = \rho(\Delta f, f).$$

The theorem follows by applying to an eigenfunction f for the eigenvalue λ_1. □

The rest of the section is now devoted to the proof of a weaker form of theorem 15.2, namely, under the same hypothesis, the Li-Yau estimate

$$\lambda_1 \geq \frac{\pi^2}{2\,\mathrm{Diam}(M,g)^2}. \tag{15.3}$$

We follow [Jos17, §4.6]. We need the two following ingredients:

Proposition 15.3. *For any 1-form* α *one has*

$$\frac{1}{2}\Delta|\alpha|^2 = \langle \Delta\alpha, \alpha\rangle - |\nabla\alpha|^2 - \mathrm{Ric}(\alpha,\alpha). \tag{15.4}$$

Proof. One has $\frac{1}{2}\Delta|\alpha|^2 = \nabla^*\langle\alpha, \nabla\alpha\rangle = \langle\alpha, \nabla^*\nabla\alpha\rangle - |\nabla\alpha|^2$. The result then follows from the Bochner formula (14.2). □

Proposition 15.4 (Weak maximum principle). *If* f *is a function on a Riemannian manifold* (M,g) *which attains a local maximum at a point* x*, then* $(\Delta f)(x) \geq 0$*.*

Proof. From the explicit form (1.5) of the Laplacian, since $df(x) = 0$ we have $(\Delta f)(x) = -g^{ij}(x)\partial^2_{ij} f(x) \geq 0$. □

Proof of (15.3). We take an eigenfunction u for the eigenvalue λ_1. By multiplying u by a constant, we can normalize so that

$$1 = \sup u > \inf u = -k \geq -1$$

for some $k > 0$. We will prove the following estimate: if $\mathrm{Ric} \geq 0$ then

$$|du|^2 \leq \frac{2\lambda_1}{1+k}(1-u)(k+u). \tag{15.5}$$

We show how the estimate (15.3) can be deduced from (15.5): take two points $x, y \in \mathrm{M}$ with $u(x) = -k$ and $u(y) = 1$ and choose a minimizing geodesic $(\gamma(t))_{t\in[0,d]}$ joining x to y. The idea here is that if we have some control on $|du|$, then u cannot vary too quickly along γ, so γ has to be long enough:

$$\pi = \int_{-k}^{1} \frac{du}{\sqrt{(1-u)(k+u)}} = \int_0^d \frac{du(\dot\gamma(t))dt}{\sqrt{(1-u(\gamma(t)))(k+u(\gamma(t)))}} \leq \sqrt{\frac{2\lambda_1}{1+k}}d.$$

Therefore $\pi \leq \sqrt{2\lambda_1}d \leq \sqrt{2\lambda_1}\,\mathrm{Diam}(\mathrm{M}, g)$, which is (15.3).

So the proof is now reduced to proving the estimate (15.5), which is the most difficult part. We center u by considering

$$v = \frac{u - \frac{1-k}{2}}{\frac{1+k}{2}}$$

which now satisfies $-1 \leq v \leq 1$ and $\Delta v = \lambda(v + c)$ with $c = \frac{1-k}{1+k}$. We consider the function

$$\mathrm{F} = \frac{|dv|^2}{1-v^2}.$$

Then the estimate (15.5) is equivalent to proving

$$\mathrm{F} \leq \lambda(1+c). \tag{15.6}$$

Of course F is not well defined since $1 - v^2$ vanishes at some points: the reader can check that the proof below applies by replacing v by $v/(1+\varepsilon)$, then the result is obtained by making $\varepsilon \to 0$, so for simplicity we will ignore this issue.

Take a point x at which F attains its maximum, so $d\mathrm{F}(x) = 0$ and $\Delta\mathrm{F}(x) \geq 0$. Write $\mathrm{F} = f/g$, then $d\mathrm{F} = df/g - f dg/g^2$. Because $d\mathrm{F}(x) = 0$ the formula for $\Delta\mathrm{F}(x)$ simplifies into $\Delta\mathrm{F} = \frac{\Delta f}{g} - \frac{f\Delta g}{g^2}$ (at the point x). But $\Delta g = \Delta(1-v^2) = -2v\Delta v + 2|dv|^2$, so we obtain, still at the point x:

$$\frac{1}{2}\Delta|dv|^2 \geq \frac{|dv|^2}{1-v^2}(|dv|^2 - v\Delta v). \tag{15.7}$$

On the other hand, by (15.4) and the hypothesis on Ricci we have

$$\frac{1}{2}\Delta|dv|^2 = \langle \Delta dv, dv\rangle - |\nabla dv|^2 - \mathrm{Ric}(dv, dv) \leq \lambda|dv|^2 - |\nabla dv|^2. \tag{15.8}$$

Since $\frac{1}{2}d\mathrm{F} = \frac{\langle dv, \nabla dv\rangle}{1-v^2} + \frac{v|dv|^2 dv}{(1-v^2)^2}$ vanishes at x, we obtain, still at the point x:

$$\frac{v|dv|^3}{1-v^2} \leq |dv||\nabla dv|.$$

Injecting in (15.8) gives

$$\frac{1}{2}\Delta|dv|^2 \leq \lambda|dv|^2 - \frac{v^2|dv|^4}{(1-v^2)^2}. \tag{15.9}$$

Comparing with (15.7), using $\Delta v = \lambda(v+c)$, one obtains $\frac{|dv|^2}{1-v^2} \leq \lambda(v+c)$ at x. □

Exercise 15.5 (Cheeger's inequality). The purpose of this exercise is to establish Cheeger's inequality, proved by Cheeger in 1970. Set

$$h := \inf_{\Sigma} \frac{\mathrm{vol}_{n-1}(\Sigma)}{\min(\mathrm{vol}_n(\mathrm{M}_1), \mathrm{vol}_n(\mathrm{M}_2))},$$

where Σ runs over all codimension 1 submanifolds disconnecting M, that is $\mathrm{M}\setminus\Sigma = \mathrm{M}_1 \sqcup \mathrm{M}_2$. Then:

$$\lambda_1(\mathrm{M}, g) \geq h^2/4. \tag{15.10}$$

Let f be a real-valued eigenfunction for the eigenvalue λ_1, and further assume that 0 is a regular value of f. Set $\mathrm{M}_\pm := \{\pm f \geq 0\}, \mathrm{M}_0 = \{f = 0\}$. Up to switching the role of f and $-f$, we further assume $\mathrm{vol}(\mathrm{M}_+) \leq \mathrm{vol}(\mathrm{M}_-)$.

1) Give a heuristic argument to explain why it is reasonable that h is a good measure of λ_1.

2) Show that

$$\int_{\mathrm{M}_+} |\nabla f|^2 \,\mathrm{vol}_g = \lambda_1 \int_{\mathrm{M}_+} f^2 \,\mathrm{vol}_g\,.$$

3) Deduce that

$$\lambda_1 \geq \frac{1}{4}\left(\frac{\int_{\mathrm{M}_+} |\nabla(f^2)|\,\mathrm{vol}_g}{\int_{\mathrm{M}_+} f^2\,\mathrm{vol}_g}\right)^2.$$

4) Using the coarea formula (21.11), show that

$$\int_{\mathrm{M}_+} |\nabla(f^2)|\,\mathrm{vol}_g \geq h \int_{\mathrm{M}_+} f^2\,\mathrm{vol}_g,$$

and conclude.

5) Treat the general case where 0 is not necessarily a regular value.

Cheeger's inequality is known to be sharp in some cases. An upper bound for λ_1 involving h was also proved by Buser in 1982. The Cheeger constant also plays an important role in graph theory.

16 Negative Ricci and Killing fields

Let (M^n, g) be a Riemannian manifold, X a vector field on M and (ϕ_t) the associated flow of diffeomorphisms of M.

Lemma and Definition 16.1. *We say that* X *is a* **Killing field** *is one of the two following equivalent conditions is satisfied:*

1. *the flow of* X *is a flow of isometries of* (M, g)*;*
2. *the covariant derivative of* X *satisfies, for all tangent vectors* Y *and* Z*:*

$$\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0. \tag{16.1}$$

The space of Killing fields is the Lie algebra of the group of isometries of (M, g), which is known to be a Lie group (and it is compact if M is compact).

Proof. As usual the flow (ϕ_t) generated by X preserves g if and only if the Lie derivative $\mathcal{L}_X g := \frac{d}{dt}|_{t=0}\phi_t^* g = 0$. We now identify $\mathcal{L}_X g(Y, Z)$ with the LHS of (16.1), which will prove the lemma:

$$\mathcal{L}_X g(Y, Z) = X \cdot g(Y, Z) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z)$$

and as $\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X$ we obtain

$$\mathcal{L}_X g(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X).$$

□

Let α be the 1-form dual to X via the metric g, that is $\alpha^\sharp = X$. Then equation (16.1) says that X is a Killing field if and only if $\nabla\alpha$ is an antisymmetric 2-tensor, that is a 2-form. Since the antisymmetric part of $\nabla\alpha$ is just $\frac{1}{2}d\alpha$, we obtain that X is a Killing field if and only if

$$\nabla\alpha = \frac{1}{2}d\alpha. \tag{16.2}$$

This implies that $d^*d\alpha = 2\nabla^*\nabla\alpha$. On the other hand, we have

$$d^*\alpha = -\operatorname{Tr}(\nabla\alpha) = 0$$

since $\nabla\alpha$ is antisymmetric. Therefore, if X is a Killing field,

$$\Delta\alpha = d^*d\alpha = 2\nabla^*\nabla\alpha. \tag{16.3}$$

Comparing with the Bochner formula (14.2), we have proved:

Proposition 16.2. *If* X *is a Killing field, then the dual 1-form* α *satisfies*

$$\nabla^*\nabla\alpha = \operatorname{Ric}(\alpha) \tag{16.4}$$

□

Remark 16.3. One can prove the more general Kostant formula which gives all the 2nd order derivatives of α in terms of the Riemannian curvature:

$$\nabla\nabla\alpha(Y, Z, T) = -\langle R_{X,Y}Z, T\rangle.$$

Corollary 16.4. *If* (M, g) *is compact connected and* $\mathrm{Ric} \leq 0$ *everywhere, and* $\mathrm{Ric} < 0$ *at least at one point, then* (M, g) *does not admit any Killing field. In particular the group of isometries of* (M, g) *is finite.*

Proof. From (16.4) we deduce by integration by parts

$$\|\nabla\alpha\|^2 = (\mathrm{Ric}\,\alpha, \alpha) \leq 0.$$

Therefore $\nabla\alpha = 0$ and $\langle \mathrm{Ric}\,\alpha, \alpha\rangle = 0$ everywhere. Since $\mathrm{Ric} < 0$ at least at one point, at this point we have $\alpha = 0$ and therefore $\alpha = 0$ everywhere since $\nabla\alpha = 0$. □

Chapter IV

General theory of elliptic operators

17 Sobolev spaces

The first step is to introduce the Sobolev space $H^s(\mathbb{R}^n)$ of tempered distributions f on $\mathbb{R}^n$ such that the Fourier transform satisfies

$$\|f\|^2_{H^s} := \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s |d\xi|^n < +\infty. \tag{17.1}$$

Equivalently, $H^s(\mathbb{R}^n)$ is the space of functions $f \in L^2(\mathbb{R}^n)$ which admit s derivatives in distribution sense[1] in L^2, and

$$\|f\|^2_{H^s} \sim \sum_{|\alpha|\leq s} \|\partial_\alpha s\|^2_{L^2}. \tag{17.2}$$

(But observe that the definition (17.1) is valid also for any real s).

If M is a compact manifold and E a vector bundle over M, then one can define the space $C^k(M, E)$ of sections of E whose coefficients are of class C^k in any trivialisation of E, and $H^s(M, E)$ the space of sections of E whose coefficients in any trivialisation and any coordinate chart are functions of class H^s in $\mathbb{R}^n$. If M is covered by a finite number of charts (U_j) with trivialisations of $E|_{U_j}$ by a basis of sections $(e_{j,a})_{a=1,\dots,r}$, choose a partition of unity (χ_j) subordinate to (U_j), then a section u of E can be written $u = \sum \chi_j u_{j,\alpha} e_{j,\alpha}$ with $\chi_j u_{j,\alpha}$ a function with compact support in $U_j \subset \mathbb{R}^n$, therefore

$$\|u\|_{C^k} = \sup_{j,\alpha} \|\chi_j u_{j,\alpha}\|_{C^k(\mathbb{R}^n)}, \quad \|u\|^2_{H^s} = \sum \|\chi_j u_{j,\alpha}\|^2_{H^s(\mathbb{R}^n)}. \tag{17.3}$$

Up to equivalence of norms, the result is independent of the choice of coordinate charts and trivialisations of E.

[1] Weak derivative: $g = \partial_\alpha f$ if for any $\phi \in C^\infty_c(\mathbb{R}^n)$ one has $\int_{\mathbb{R}^n} (\partial_\alpha \phi) f = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \phi g$.

There is another approach to define C^k and H^s norms for sections of E. Suppose that M^n has a Riemannian metric, and E is equipped with a unitary connection ∇. Then one can define

$$\|u\|_{C^k} = \sup_{j\leq k}\sup_M |\nabla^j u|, \quad \|u\|^2_{H^s} = \sum_0^s \int_M |\nabla^j u|^2 \operatorname{vol}^g . \tag{17.4}$$

Remark 17.1. On a noncompact manifold, the definition (17.3) does not give a well defined class of equivalent norms when one changes the trivialisations. On the contrary, definition (17.4), valid only for integral s, can be useful if (M, g) is non compact; the norms depend on the geometry at infinity of g and ∇.

Example 17.2. If M is a torus $\mathbb{T}^n$, then the regularity can be seen on the Fourier series: $f \in H^s(\mathbb{T}^n)$ if and only if

$$\|f\|^2_{H^s} = \sum_{\xi\in\mathbb{Z}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 < +\infty.$$

From the inverse formula $f(x) = \sum_\xi \hat{f}(\xi)\exp^{i\xi\cdot x}$, by the Cauchy-Schwartz inequality,

$$|f(x)| \leq \sum_{\xi\in\mathbb{Z}^n} |\hat{f}(\xi)| \leq \|f\|_{H^s}\Big(\sum_\xi (1+|\xi|^2)^{-s}\Big)^{1/2} < +\infty \quad \text{if } s > \frac{n}{2}.$$

It follows that there is a continuous inclusion $H^s \subset C^0$ is $s > \frac{n}{2}$. Similarly it follows that $H^s \subset C^k$ if $s > k + \frac{n}{2}$.

Of course the same results are true on $\mathbb{R}^n$ using Fourier transform, and one obtains the following lemma.

Lemma 17.3 (Sobolev). *If* M^n *is compact,* $k \in \mathbb{N}$ *and* $s > k + \frac{n}{2}$*, then there is a continuous and compact injection* $H^s \subset C^k$.

The fact that the inclusion is compact follows from the following lemma (which is obvious on a torus, and the general case follows):

Lemma 17.4 (Rellich). *If* M^n *is compact and* $s > t$*, then the inclusion* $H^s \subset H^t$ *is compact.*

In particular

$$\cap_{s\geq 0} H^s(M, E) = C^\infty(M, E), \qquad \cup_{s\leq 0} H^s(M, E) = \mathcal{D}'(M, E), \tag{17.5}$$

where $\mathcal{D}'(M, E)$ is the space of all E-valued distributions.

18 Introduction to pseudodifferential operators

Suppose we have a differential operator $P = \sum_{|\alpha|\leq d} a^\alpha(x)\partial_\alpha$ on $\mathbb{R}^n$, then using Fourier transform we can write, for all tempered distributions $f \in \mathcal{D}(\mathbb{R}^n)$

$$Pf(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \sum_{|\alpha|\leq d} a^\alpha(x)(i\xi)^\alpha \hat{f}(\xi) e^{i\xi\cdot x} |d\xi|^n.$$

Using the ***total symbol***

$$\sigma(x,\xi) = \sum_{|\alpha|\leq d} a^{\alpha}(x)(i\xi)^{\alpha} \tag{18.1}$$

we see that $P = \mathrm{Op}_{\sigma}$, where for suitable functions $\sigma(x,\xi)$ the operator Op_{σ} is defined for $f \in \mathcal{D}(\mathbb{R}^n)$ by the formula

$$\mathrm{Op}_{\sigma} f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \sigma(x,\xi)\hat{f}(\xi)e^{i\xi\cdot x}|d\xi|^n. \tag{18.2}$$

A standard class of symbols σ on $\mathbb{R}^n$ is the class S^d of smooth functions $\sigma(x,\xi)$ on $\mathbb{R}^n \times (\mathbb{R}^n)^*$ which satisfy, for any $\alpha,\beta \in \mathbb{N}$ and any compact $K \subset \mathbb{R}^n$

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta}\sigma(x,\xi)| \leq c_{\alpha,\beta,K}(1+|\xi|)^{d-|\beta|} \qquad \text{for all } x \in K, \xi \in \mathbb{R}^n. \tag{18.3}$$

In that case formula (18.2) defines a function $\mathrm{Op}_{\sigma} f$ since $\hat{f}$ is a fast decaying function: this is the definition of a ***pseudodifferential operator*** on $\mathbb{R}^n$.

This definition extends to an operator between sections on two bundles E and F on a manifold M. We choose a covering (U_j) by open sets of trivializations of E and F, and smooth functions χ_j such that (χ_j^2) is a partition of unity. We say that a symbol $\sigma \in C^{\infty}(T^*M, \mathrm{Hom}(E,F))$ is in the class $S^d(M,E,F)$ if it satisfies (18.3) in each trivialization. Then using formula (18.2) in each trivialization, we define for a section $f \in C^{\infty}(M,E)$ the ***pseudodifferential operator*** on M

$$\mathrm{Op}_{\sigma} f = \sum \chi_j \,\mathrm{Op}_{\sigma}(\chi_j f). \tag{18.4}$$

Observe that in this definition Op_{σ} depends not only on σ but also on the choice of the trivializations and the partition of unity (χ_j^2).

We need the following properties of pseudodifferential operators (see [Ala23]):

1. *Extension to Sobolev spaces.* If $\sigma \in S^d(M,E,F)$ then Op_{σ} extends as a continuous operator

$$\mathrm{Op}_{\sigma} : H^s(M,E) \longrightarrow H^{s-d}(M,F). \tag{18.5}$$

   A special case is that of a symbol $\sigma \in S^{-\infty}(M,E,F) = \cap_d S^d(M,E,F)$. In that case Op_{σ} sends continuously $\mathcal{D}'(M,E,F)$ into $C^{\infty}(M,E,F)$: an operator with this property is called a ***regularizing operator***. The regularizing operators are exactly the operators R given by a C^{∞} kernel $K(x,y) \in \mathrm{Hom}(E_y,F_x)$, such that

$$(Rf)(x) = \int_M K(x,y)f(y)\,\mathrm{vol}(y). \tag{18.6}$$

   One works usually on pseudodifferential operators modulo the space $\mathcal{R}$ of regularizing operators, that is considering operators of the type $\mathrm{Op}_{\sigma} + R$ with $R \in \mathcal{R}$.

2. *Composition.* If $\sigma \in S^d(M,E,F)$ and $\sigma' \in S^{d'}(M,F,G)$ then there exists a symbol $\sigma'\diamond\sigma \in S^{d+d'}(M,E,G)$ such that

$$\mathrm{Op}_{\sigma'} \circ \mathrm{Op}_{\sigma} = \mathrm{Op}_{\sigma'\diamond\sigma} \mod \mathcal{R}. \tag{18.7}$$

   Moreover one has $\sigma'\diamond\sigma - \sigma'\circ\sigma \in S^{d+d'-1}(M,E,G)$.

3. *Differential operators.* We have seen that a differential operator P of order d on $\mathbb{R}^n$ is also a pseudodifferential operator with symbol given by (18.1). In that case the total symbol is polynomial in ξ, and the part of degree d is exactly the principal symbol of P, up to a factor i^d. In general, a differential operator P of order d on a manifold M is also a pseudodifferential operator, that is there exists a symbol $\sigma \in \mathrm{S}^d$ such that

$$\mathrm{P} = \mathrm{Op}_\sigma \mod \mathcal{R}. \tag{18.8}$$

Moreover the dominant term of σ is given by the principal symbol of P: one has $\sigma(x,\xi) - i^d \sigma_{\mathrm{P}}(x,\xi) \in \mathrm{S}^{d-1}$.

One says that Op_σ is an ***elliptic pseudodifferential operator*** of order d if $\sigma \in \mathrm{S}^d(\mathrm{M},\mathrm{E},\mathrm{F})$ satisfies: for all (x,ξ) such that $|\xi| > \mathrm{A}$ for some A large enough, one has

$$|\sigma(x,\xi)u| \geq c|\xi|^d|u|. \tag{18.9}$$

In particular, if $\operatorname{rk}\mathrm{E} = \operatorname{rk}\mathrm{F}$ this implies that $\sigma(x,\xi)$ is invertible for $|\xi| > \mathrm{A}$. Of course an elliptic differential operator of order d is also elliptic in the sense of pseudodifferential operators.

Theorem 18.1 (Gårding inequality). *Suppose* P *is an elliptic pseudodifferential operator of order d. If* $f \in \mathrm{L}^2(\mathrm{M},\mathrm{E})$ *satisfies* $\mathrm{P}f \in \mathrm{H}^s(\mathrm{M},\mathrm{F})$ *then* $f \in \mathrm{H}^{s+d}(\mathrm{M},\mathrm{E})$ *and one has the following* **elliptic estimate***:*

$$\|f\|_{\mathrm{H}^{s+d}} \leq c_s\big(\|\mathrm{P}f\|_{\mathrm{H}^s} + \|f\|_{\mathrm{L}^2}\big). \tag{18.10}$$

Sketch of proof. Limit to the case $\operatorname{rk}\mathrm{E} = \operatorname{rk}\mathrm{F}$. Suppose that $\varsigma(x,\xi)$ is an elliptic symbol: then one can define a symbol $\varsigma'(x,\xi) = \chi(\xi)\varsigma(x,\xi)^{-1} \in \mathrm{S}^{-d}$, where χ is a cutoff function such that $\chi(\xi) = 0$ for $|\xi| < \mathrm{A}$ and $\chi(\xi) = 1$ for $|\xi| > 2\mathrm{A}$. Since $\varsigma'(x,\xi)\circ \varsigma(x,\xi) = \mathrm{Id}$ for $|\xi| > 2\mathrm{A}$, we have $\mathrm{Op}_{\varsigma'\circ\varsigma} = \mathrm{Id} \mod \mathcal{R}$ and since $\mathrm{Op}_{\varsigma'}\circ\mathrm{Op}_\varsigma = \mathrm{Op}_{\varsigma'\diamond\varsigma} \mod \mathcal{R}$ we get

$$\mathrm{Op}_{\varsigma'}\circ\mathrm{Op}_\varsigma = \mathrm{Id} + \mathrm{Op}_\rho, \qquad \rho \in \mathrm{S}^{-1}.$$

Inverting formally $(\mathrm{Id}+\mathrm{Op}_\rho)^{-1} = \mathrm{Id} - \mathrm{Op}_\rho + \mathrm{Op}_\rho^2 + \cdots$, we can construct a symbol $\tau \sim 1 - \rho + \rho^{\diamond 2} + \cdots$ such that

$$(\mathrm{Id}+\mathrm{Op}_\rho)^{-1} = \mathrm{Op}_\tau \mod \mathcal{R}.$$

It follows that

$$\mathrm{Op}_{\tau\diamond\varsigma'}\circ\mathrm{Op}_\varsigma = \mathrm{Id} \mod \mathcal{R}.$$

In summary, if P is an elliptic operator of order d, there exists a symbol $\sigma \in \mathrm{S}^{-d}$ and a regularizing operator $\mathrm{R} \in \mathcal{R}$ such that

$$\mathrm{Op}_\sigma \circ \mathrm{P} = \mathrm{Id} + \mathrm{R}. \tag{18.11}$$

In particular $f = \mathrm{Op}_\sigma(\mathrm{P}f) + \mathrm{R}f$. If $\mathrm{P}f \in \mathrm{H}^s$ then $\mathrm{Op}_\sigma(\mathrm{P}f) \in \mathrm{H}^{s+d}$ and $\mathrm{R}f \in \mathrm{C}^\infty$. So $f \in \mathrm{H}^{s+d}$ and the estimate (18.10) also follows. □

From the elliptic estimate and the fact that $\cap_s \mathrm{H}^s = \mathrm{C}^\infty$, we obtain:

Corollary 18.2. *If* P *is elliptic and* $\mathrm{P}f$ *is* C^∞ *then* f *is* C^∞.

Exercise 18.3. Prove (18.10) for operators with constant coefficients on the torus.

19 Proof of the main theorem

We now prove theorem 7.1. Actually we prove the statement in the general setting of elliptic pseudodifferential operators.

First let us prove the first statement: the kernel of P is finite dimensional. By the elliptic estimate (18.10), for $u \in \ker(P)$ one has

$$\|u\|_{H^{s+d}} \leq C\|u\|_{L^2}.$$

Therefore the first identity map in the following diagram is continuous:

$$(\ker P, L^2) \longrightarrow (\ker P, H^{s+d}) \longrightarrow (\ker P, L^2).$$

The second inclusion is compact by lemma 17.4. The composite map is the identity of ker P equipped with the L^2 scalar product, it is therefore a compact map. This implies that the closed unit ball of ker(P) is compact, therefore ker(P) is a finite dimensional vector space.

Now let us prove the theorem in Sobolev spaces. We consider P as an operator

$$P : H^{s+d}(M, E) \longrightarrow H^s(M, F), \tag{19.1}$$

and in these spaces we want to prove

$$H^s(M, F) = \ker(P^*) \oplus \operatorname{im}(P). \tag{19.2}$$

We claim that for any $\epsilon > 0$, there exists an L^2 orthonormal family $(v_1, \dots, v_N)$ in H^{s+d}, such that

$$\|u\|_{L^2} \leq \epsilon\|u\|_{H^{s+d}} + \Big(\sum_1^N |(v_j, u)|^2\Big)^{1/2}. \tag{19.3}$$

Suppose for the moment that the claim is true. Then combining with the elliptic estimate (18.10), we deduce

$$(1 - C\epsilon)\|u\|_{H^{s+d}} \leq C\|Pu\|_{H^s} + C\Big(\sum_1^N |(v_j, u)|^2\Big)^{1/2}.$$

Choose $\epsilon = \frac{1}{2C}$, and let T be the subspace of sections in $H^{s+d}(M, E)$ which are L^2 orthogonal to the $(v_i)_{i=1\dots N}$. Then we obtain

$$2\|u\|_{H^{s+d}} \leq C\|Pu\|_{H^s} \quad \text{for } u \in T.$$

It follows that P(T) is closed in $H^s(M, F)$. But im(P) is the sum of P(T) and the image of the finite dimensional space generated by the $(v_i)_{i=1\dots N}$, so im(P) is closed as well in $H^s(M, F)$.

Finally the statement (19.1) in the Sobolev spaces H^s implies the statement for the space C^∞, which finishes the proof of the theorem. Indeed, suppose that $v \in C^\infty(M, F)$ is L^2 orthogonal to $\ker(P^*)$. Fix any $s \geq 0$ and apply (19.2) in H^s: therefore there exists $u \in H^{s+d}(M, E)$ such that $Pu = v$. Then u is C^∞ by corollary 18.2.

It remains to prove the claim (19.3). Choose a Hilbertian basis (v_j) of L^2, and suppose that the claim is not true. Then there exists a sequence of $(u_N) \in H^{s+d}(M, E)$ such that

1. $\|u_N\|_{L^2} = 1$,
2. $\epsilon\|u_N\|_{H^{s+d}} + \left(\sum_1^N |(v_j, u_N)|^2\right)^{1/2} < 1$.

From the second condition we deduce that (u_N) is bounded in $H^{s+d}(E)$, therefore there is a weakly convergent subsequence in $H^{s+d}(E)$, and the limit satisfies

$$\epsilon\|u\|_{H^{s+d}} + \|u\|_{L^2} \leq 1.$$

By the compact inclusion $H^{s+d} \subset L^2$ this subsequence is strongly convergent in $L^2(E)$ so by the first condition, the limit u satisfies

$$\|u\|_{L^2} = 1,$$

which is a contradiction.

20 Green operator

Theorem 20.1. *Suppose that* $E = F$ *and* P *is a selfadjoint elliptic operator of order* d *acting on sections of* E. *Denote by* $\Pi_{ker P}$ *the* L^2*-orthogonal projection on* $\ker P$. *Then there exists a pseudodifferential operator* G *of order* $-d$ *such that on* $C^\infty(M, E)$ *one has*

$$\mathrm{Id} = P \circ G + \Pi_{\ker P} = G \circ P + \Pi_{\ker P}. \tag{20.1}$$

Proof. In the decomposition $C^\infty(M, E) = (\ker P)^\perp \oplus \ker P$ the operator $G = P^{-1} \oplus 0$ gives (20.1). There remains to prove that G is a pseudodifferential operator. We know that there exists a pseudodifferential operator Q of order $-d$ such that $QP = \mathrm{Id} + R$ with $R \in \mathcal{R}$. Now

$$\begin{aligned} Q &= Q(PG + \Pi) \\ &= (\mathrm{Id} + R)G + Q\Pi \\ &= G + RG + Q\Pi. \end{aligned}$$

But RG and $Q\Pi$ are regularizing operators, so it follows that $G = Q \mod \mathcal{R}$ is a pseudodifferential operator of order $-d$. □

Remark 20.2. This result implies the existence of a spectral decomposition of P as in section 5.

21 Other functional spaces

Given a vector bundle (E, h) over a compact Riemannian manifold (M, g), we have seen the Sobolev spaces $H^s(M, E)$, but other functional spaces are also useful:

- ***Sobolev spaces*** $W^{k,p}(M, E)$ for $1 \leq p < \infty$: this is the space of L^p sections with k derivatives in L^p, that is the completion of $C^\infty(M, E)$ for the norm

$$\|f\|^p_{W^{k,p}} = \int_M \left(|f|^p + |\nabla f|^p + \cdots + |\nabla^k f|^p\right) \mathrm{vol}. \tag{21.1}$$

- ***Hölder spaces*** $C^{k,\alpha}(M, E)$ for some $k \in \mathbb{N}$ and $0 < \alpha < 1$: for a function one defines a kind of norm of the "α-derivative" of f by
$$|f|_\alpha = \sup_{d(x,y)<\rho} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}. \tag{21.2}$$
This is a priori defined for a function; for a section of a vector bundle, we choose trivializations and take the sup over the various charts. We then define the norms
$$\|f\|_{C^\alpha} = \|f\|_{C^0} + |f|_\alpha, \qquad \|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + |\nabla^k f|_\alpha. \tag{21.3}$$
The derivative $\nabla^k f$ can be taken to be the standard derivative in each trivialization, or we can choose a connection ∇ on E. The various choices (charts, trivializations, etc.) give equivalent norms.

Proposition 21.1 (Sobolev injections). *One has the following continuous inclusions:*

- *If* $k - \frac{n}{p} \geq k' - \frac{n}{p'}$ *then* $W^{k,p} \subset W^{k',p'}$.
- *If* $k - \frac{n}{p} \geq k' + \alpha$ *then* $W^{k,p} \subset C^{k',\alpha}$.

Moreover, these inclusions are compact in case the inequality is strict.

We will not prove the inclusions in general, but offer some comments. We can reduce to the case $k' = 0$.

First if we are on $\mathbb{R}^n$ rather than a compact manifold, we have the following inequality. Take q such that $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$, then there is a constant C such that for any compactly supported function f on $\mathbb{R}^n$ one has
$$\|f\|_{L^q} \leq C\|\nabla^k f\|_{L^p}. \tag{21.4}$$
The exponent q is the only possible exponent in this inequality, because of the invariance by homothety: if we consider $f_\varepsilon(x) = f(\varepsilon x)$ then the LHS is multiplied by $\varepsilon^{-n/q}$ and the RHS by $\varepsilon^{k-n/p}$: the numbers $-\frac{n}{q}$ and $k - \frac{n}{p}$ are the conformal weights of these norms, and this explains why they enter in the hypothesis of the proposition.

From (21.4) one deduces the first Sobolev inclusion on a compact manifold, reducing to the case of $\mathbb{R}^n$ via a partition of unity. The exponent q can then be taken smaller than the one in (21.4).

The hypothesis for the second Sobolev inclusion can also be explained using the conformal weights of the two norms. Again we do not give any proof, except for the inequality $W^{1,p} \subset C^0$ on $\mathbb{R}^n$ if $p > n$ which follows from
$$f(0) = \frac{1}{V_{n-1}} \int_{\mathbb{R}^n} \frac{1}{r^{n-1}} \frac{\partial f}{\partial r} r^{n-1} dr \,\mathrm{vol}_{S^{n-1}}. \tag{21.5}$$
If $p > n$ then the dual exponent s such that $\frac{1}{p} + \frac{1}{s} = 1$ satisfies $s < \frac{n}{n-1}$ which is exactly the condition to have $\frac{1}{r^{n-1}} \in L^s$, so Hölder inequality gives
$$|f(0)| \leq \frac{1}{V_{n-1}} \left\| \frac{1}{r^{n-1}} \right\|_{L^s} \|df\|_{L^p}.$$

In the limit case $p = n$, the inclusion is not true, as one can see in $\mathbb{R}^n$ with the function

$$f(x) = \ln|\ln(e|x|)| \tag{21.6}$$

which vanishes for $|x| = 1$ and is extended by zero outside B_1. Then $df \in L^n$ but f is unbounded at $x = 0$.

Remark 21.2. Formula (21.5) almost gives (21.4): it can be rewritten as $|f| \leq \frac{1}{r^{n-1}} * |df|$. One has the well-known inequality $\|f * g\|_{L^q} \leq \|f\|_{L^p}\|g\|_{L^s}$ if $\frac{1}{p} + \frac{1}{s} = 1 + \frac{1}{q}$. To obtain $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ we need $s = \frac{n}{n-1}$ but this is the limit exponent since $\frac{1}{r^{n-1}} \in L^s$ only for $s < \frac{n}{n-1}$.

Remark 21.3. The case $p = 1$ gives the inclusion $W^{1,1} \subset L^{\frac{n}{n-1}}$ which is related to the ***isoperimetric inequality*** in $\mathbb{R}^n$: for a bounded domain $\Omega \subset \mathbb{R}^n$ one has

$$\mathrm{Vol}(\Omega)^{\frac{n}{n-1}} \leq c_n \,\mathrm{Vol}(\partial\Omega). \tag{21.7}$$

See exercises.

Finally, the following theorem says basically that the elliptic theory that we have seen on the Sobolev spaces H^s extends to the spaces $W^{k,p}$ and $C^{k,\alpha}$.

Theorem 21.4 (Elliptic estimates). *If* P *is an elliptic operator of order d on the compact manifold* M*, then one has:*

1. *If* $Pf \in W^{k,p}$ *then* $f \in W^{k+d,p}$ *and one has the Calderon-Zygmund estimate*

$$\|f\|_{W^{k+d,p}} \leq c\big(\|Pf\|_{W^{k,p}} + \|f\|_{L^p}\big). \tag{21.8}$$

2. *If* $Pf \in C^{k,\alpha}$ *then* $f \in C^{k+d,\alpha}$ *and one has the Schauder estimate*

$$\|f\|_{C^{k+d,\alpha}} \leq c\big(\|Pf\|_{C^{k,\alpha}} + \|f\|_{C^0}\big). \tag{21.9}$$

Moreover one has the L^2*-orthogonal decompositions*

$$W^{k,p} = P(W^{k+d,p}) \oplus \ker P^*, \qquad C^{k+\alpha} = P(C^{k+d,\alpha}) \oplus \ker P^*. \tag{21.10}$$

Remark 21.5. It is important to note that this theorem is not true for C^k spaces (hence the use of $C^{k,\alpha}$ spaces). For example the function $f(x,y) = (x^2-y^2)\sqrt{|\ln|x||}$ on $\mathbb{R}^2$ is not C^2 but $\Delta f \in C^0$.

Hölder spaces are particularly useful in certain nonlinear problems, because the $C^{k,\alpha}$ spaces, like the C^k spaces, are algebras. It turns out that $W^{k,p}$ is an algebra only when one has the inclusion $W^{k,p} \subset C^0$.

Exercise 21.6 (Coarea formula). Let (M,g) be a closed manifold. For $f \in C^\infty(M)$ show that:

1) If f is positive, then:

$$\int_M f\,\mathrm{vol}_g = \int_0^{+\infty} \mathrm{vol}(f^{-1}(t,\infty))dt.$$

2) If f is a Morse function, then:

$$\int_M |\nabla f| \ \mathrm{vol}_g = \int_0^{+\infty} \mathrm{vol}_{n-1}(f^{-1}(t))dt. \tag{21.11}$$

Formula (21.11) is called the ***coarea formula***. It is actually valid for all $f \in C^\infty(M)$ (admitted).

Exercise 21.7 (Isoperimetric inequality and sharp Sobolev inequalities). Let (M, g) be a complete Riemannian manifold. We say that $C > 0$ is an isoperimetric constant in (M, g) if for all relatively compact domain $\Omega \subset M$ with smooth boundary,

$$C \leq \frac{\mathrm{vol}_{n-1}(\partial\Omega)^n}{\mathrm{vol}_n(\Omega)^{n-1}}$$

We say that the Sobolev embedding $W^{1,1}(M) \hookrightarrow L^{n/(n-1)}(M)$ is satisfied with constant $C > 0$ if for all $u \in C^\infty_{\mathrm{comp}}(M)$,

$$\|u\|_{L^{n/(n-1)}(M)} \leq C\|\nabla u\|_{L^1(M)}.$$

The purpose of this exercise is to investigate the relationship between these two constants.

1) Let (N, g) be a smooth Riemannian manifold with boundary, and ν be the inward pointing unit vector field on ∂N. Show that

$$\phi : [0, \epsilon) \times \partial N \to N, \qquad (t, x) \mapsto \exp_x(t\nu(x))$$

is a diffeomorphism to a local neighborhood of ∂N for $\epsilon > 0$ small enough.

2) Show that $g = dt^2 + h_t$, where h_t is a smooth metric on ∂N.

3) Show that the Sobolev inequality with constant C implies the isoperimetric inequality with constant C^{-n}.

Conversely, we want to show that the isoperimetric inequality with constant C implies the Soblev inequality with constant $C^{-1/n}$.

4) Show that it suffices to show the Sobolev inequality for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ with $f \geq 0$.

5) Assume $f \geq 0$, and $f \in C^\infty_{\mathrm{comp}}(M)$. Show that

$$\int_M |f|^{\frac{n}{n-1}} \mathrm{vol} = \frac{n}{n-1} \int_0^{+\infty} \mathrm{vol}_n(f \geq t) t^{\frac{1}{n-1}} dt,$$

$$\int_M |\nabla f| \mathrm{vol} \geq C \int_0^{+\infty} \mathrm{vol}_n(f \geq t)^{\frac{n-1}{n}} dt.$$

6) Deduce the Sobolev inequality.

It is a classical theorem that the isoperimetric inequality is saturated in $\mathbb{R}^n$ for Ω being the unit ball. However, proving similar isoperimetric inequalities in general Riemannian manifolds is still an open question.

Chapter V

The scalar curvature

22 Gauss curvature on surfaces

We consider a Riemannian surface (S, g). As we have seen in section 13 the curvature tensor reduces to the Gauss curvature $K^g = {R_{122}}^1$, and $\mathrm{Ric}^g = Kg$, $\mathrm{Scal}^g = 2K$.

The ***Gauss-Bonnet formula*** relates the curvature and the topology: if (S, g) is compact connected oriented then

$$\frac{1}{2\pi}\int_S K^g \,\mathrm{vol}^g = \chi(S) = 2 - 2g(S), \tag{22.1}$$

where $\chi(S)$ is the Euler characteristic of S and $g(S)$ its genus.

In particular, if (S, g) has constant curvature, which we can assume to be $K = \pm 1$ or 0, then there is the following constraint on the genus:

- $g = 0$ if $K = 1$;
- $g = 1$ if $K = 0$;
- $g \geq 2$ if $K = -1$.

Conversely we can try to construct metrics of constant curvature on any surface. The standard metrics on S^2 and $\mathbb{T}^2$ give the answers for genus 0 and 1. In higher genus we start from (S, g_0) and we are looking for a metric of the form $g = e^u g_0$. Such a metric is said to be conformal to g_0 and the space $\{g = e^u g_0, u \in C^\infty(S)\}$ is called the ***conformal class*** of g_0.

Theorem 22.1. *Let* (S, g_0) *be a compact connected oriented surface with* $\chi(S) < 0$. *Then the conformal class of* g_0 *contains a unique metric* g *with* $K^g \equiv -1$.

Remark 22.2. This theorem establishes a bijection between the two first following spaces:

1. the space of metrics with $K \equiv -1$ (hyperbolic metrics) on S, modulo $\mathrm{Diff}_0(S)$;
2. the space of conformal classes on S, modulo $\mathrm{Diff}_0(S)$;

3. the space of Riemann surface structures on S, modulo $\mathrm{Diff}_0(S)$.

These are three descriptions of the ***Teichmüller space***, which is diffeomorphic to $\mathbb{R}^{6g-6}$ and plays an important role in geometry.

We now establish the equation to solve to prove the theorem.

Lemma 22.3. *If $g = e^{2f} g_0$ then*

$$K^g = e^{-2f}(K^{g_0} + \Delta^{g_0} f). \tag{22.2}$$

Remark 22.4. Since $\mathrm{vol}^g = e^{2f}\,\mathrm{vol}^{g_0}$ this formula implies that $\int_S K^g \mathrm{vol}^g$ remains constant in the conformal class of g_0, as follows from (22.1).

Proof. We claim that the Levi-Civita connection of g is given by

$$\nabla^g_X Y = \nabla^{g_0}_X Y + (X\cdot f)Y + (Y\cdot f)X - g_0(X,Y)\nabla^{g_0} f. \tag{22.3}$$

The reader can check that indeed this formula defines a connection which is torsion free and preserves the metric g. We rewrite the formula more compactly as

$$\nabla^g_X = \nabla^{g_0}_X + df(X) + df\wedge X, \tag{22.4}$$

where $A\wedge B \in \mathfrak{so}_2$ is defined by $A\wedge B(U) = g_0(A,U)B - g_0(B,U)A$. A metric connection in rank 2 has the local form $\nabla = d + a$ with a a $\mathfrak{so}_2$-valued 1-form; since $\mathfrak{so}_2$ is abelian, it follows that $F^\nabla = da$ is linear in a. In particular, from (22.4) it follows that

$$R^g = R^{g_0} + d(df\wedge\cdot).$$

Calculating at a point x, we can suppose that $\nabla X(x) = \nabla Y(x) = 0$ and therefore

$$d(df\wedge\cdot)_{X,Y} = \nabla_X(df\wedge Y) - \nabla_Y(df\wedge X) = (\nabla_X df)\wedge Y - (\nabla_Y df)\wedge X.$$

It follows that, for a g_0-orthonormal basis (e_1,e_2),

$$g_0(R^g_{e_1,e_2}e_2,e_1) = K^{g_0} - \nabla_{e_1}df(e_1) - \nabla_{e_2}df(e_2) = K^{g_0} + \Delta^{g_0}df.$$

The formula for $K^g = g(R^g_{e^{-f}e_1,e^{-f}e_2}e^{-f}e_2, e^{-f}e_1)$ follows. □

Proof of theorem 22.1. Now the theorem amounts to solving the equation $e^{-2f}(K^{g_0} + \Delta f) = -1$, that is

$$\Delta f + e^{2f} = -K^{g_0}. \tag{22.5}$$

First we reduce to the case where $K^{g_0} < 0$ everywhere. It is sufficient to replace g_0 by a metric $e^{2f}g_0$ so that $K^{e^{2f}g_0} < 0$. Note $\bar{K}_0$ the mean value of K^{g_0} on S, then $\bar{K}_0 < 0$ by (22.1). From theorem 2.1 there exists a solution f to the equation $K^{g_0} + \Delta f = \bar{K}_0$, and it follows by (22.2) that $K^{e^{2f}g_0} = e^{-2f}\bar{K}_0 < 0$.

So we can now suppose $K^{g_0} < 0$ everywhere and taking the function $\lambda = -K^{g_0} > 0$ we want to solve the equation

$$\Delta f + e^{2f} = \lambda. \tag{22.6}$$

We first prove uniqueness. If we have two solutions f and g of (22.6), then taking the difference we obtain

$$\Delta(f-g)+e^{2f}-e^{2g}=0.$$

By the maximum principle, at a maximum of $f-g$ we have $\Delta(f-g)\geq 0$ and therefore $e^{2f}-e^{2g}\leq 0$, which proves that $f-g\leq 0$ everywhere. Inverting the role of f and g we obtain that $f=g$.

To solve (22.6) we use a ***continuity method***: for each $t\in[0,1]$ we are looking for a solution of the equation

$$\Delta f+e^{2f}=1-t+t\lambda. \tag{22.7}$$

For $t=0$ we have the obvious solution $f=0$, and for $t=1$ this is our equation (22.6). We consider

$$\mathrm{I}=\{t\in[0,1],\text{there is a solution of (22.7)}\}.$$

To prove that $\mathrm{I}=[0,1]$, we prove that it is open and closed (it is not empty since $0\in\mathrm{I}$).

We begin by the openness: we consider the operator $\Theta(f)=\Delta f+e^{2f}$, we claim that it is a well-defined operator $\mathrm{H}^2\to\mathrm{L}^2$. This comes from the Sobolev inclusion $\mathrm{H}^2\subset\mathrm{C}^0$ in dimension 2, so e^{2f} is C^0 if $f\in\mathrm{H}^2$. Moreover Θ is differentiable (as the reader can check), with differential

$$d_f\Theta(\dot f)=\Delta\dot f+2e^{2f}\dot f.$$

So the linearization $d_f\Theta:\mathrm{H}^2\to\mathrm{L}^2$ is $\Delta+2e^{2f}$ which is a second order selfadjoint elliptic operator. Moreover it has no kernel, for if $d_f\Theta(\dot f)=0$, then integrating by parts we obtain $(d_f\Theta(\dot f),\dot f)=\int|d\dot f|^2+2e^{2f}|\dot f|^2=0$ and therefore $\dot f=0$. Therefore by theorem 7.1 $d_f\Theta:\mathrm{H}^2\to\mathrm{L}^2$ is invertible. By the implicit function theorem applied to $\Theta:\mathrm{H}^2\to\mathrm{L}^2$, if we have a solution of (22.7) for some t, then we obtain a solution in H^2 for nearby values of t. The solution is actually C^∞ (see the regularity result at the end of the proof), therefore I is open.

We now prove the closedness. Suppose f is a solution of (22.7). We first prove a priori C^0 bounds on f, using the weak maximum principle (proposition 15.4):

- at a maximum of f, by the maximum principle $\Delta f\geq 0$ so $e^{2f}\leq 1-t+t\lambda\leq \mathrm{C}=1+\sup\lambda$;
- at a minimum of f, again $\Delta f\leq 0$ so $e^{2f}\geq 1-t+t\lambda\geq\varepsilon>0$ since $\lambda>0$.

Finally we obtain $\frac{1}{2}\ln\varepsilon\leq f\leq\frac{1}{2}\ln\mathrm{C}$, that is we have a uniform C^0 bound on f.

We can now finish the proof of closedness: suppose we have a sequence of solutions f_i of (22.7) for $t_i\to\tau$. The uniform C^0 bound on f_i implies a uniform C^0 bound on Δf_i by (22.7), in particular a uniform L^p bound on Δf_i for any p. From the elliptic estimate (21.8) we deduce a uniform $\mathrm{W}^{2,p}$ bound on f_i. Taking $p>2$ we have a compact inclusion $\mathrm{W}^{2,p}\subset\mathrm{H}^2$ and therefore we can extract a sequence $f_i\to f$ converging strongly in H^2. This implies that f is a solution of (22.7) for $t=\tau$. We need to prove additionally that f is C^∞.

This is a general regularity statement: suppose we have a solution $f \in H^2$ of equation (22.7), then f is C^∞. Observe indeed that if $f \in H^2$ then $e^{2f} \in C^0$ and it follows quickly that $e^{2f} \in H^2$. From (22.7) we obtain that $\Delta f \in H^2$ and therefore $f \in H^4$. Bootstrapping we obtain that $f \in H^6$, $f \in H^8$, etc. that is $f \in \cap_{s>0} H^s = C^\infty$. □

23 The Yamabe problem

We now consider the constant scalar curvature problem for a general Riemannian manifold (M^n, g). To simplify notation we use the usual notation R for the scalar curvature. We consider a metric $\tilde{g} = e^{2f} g$ in the conformal class of g, and we want to solve the equation

$$\tilde{R} = \text{cst}. \tag{23.1}$$

This problem can be seen as a variational problem in the following way. We define

$$I(\tilde{g}) = \frac{n-2}{4(n-1)} \int_M \tilde{R} \operatorname{vol}^{\tilde{g}} . \tag{23.2}$$

There is an obvious nonuniqueness for solutions of (23.1), since one can multiply $\tilde{g}$ by a constant. So it is natural to kill this ambiguity by imposing the constraint

$$\operatorname{Vol}(\tilde{g}) = 1. \tag{23.3}$$

Proposition 23.1. *The Euler-Lagrange equation for* $I(\tilde{g})$ *under the constraint (23.3) is* $\tilde{R} \equiv$ *cst.*

This is called the ***Yamabe problem***.

Theorem 23.2 (Yamabe, Trudinger, Aubin, Schoen,...). *In the conformal class of* g *there exists a metric* $\tilde{g}$ *which minimizes* $I(\tilde{g})$ *among metrics with* $\operatorname{Vol}(\tilde{g}) = 1$. *In particular* $\tilde{R}$ *is constant.*

The aim of this chapter is to prove this difficult theorem in most cases.

Proof of proposition 23.1. We begin by setting up the problem. First we need the formula for $\tilde{R}$: in the same way we obtained (22.2), one can prove

$$\tilde{R} = e^{-2f} \left(R + 2(n-1)\Delta f + (n-1)(n-2)|df|^2 \right). \tag{23.4}$$

The following formalism is more convenient for this problem if $n > 2$: write $\tilde{g} = e^{2f} g = u^{\frac{4}{n-2}} g$ then

$$\tilde{R} = u^{-\frac{n+2}{n-2}} \left(4\frac{n-1}{n-2} \Delta u + R u \right) \tag{23.5}$$

and

$$\operatorname{Vol}(\tilde{g}) = \int_M u^{\frac{2n}{n-2}} \operatorname{vol}^g . \tag{23.6}$$

Finally, with an integration by parts:

$$\mathrm{I}(\tilde{g}) = \int_{\mathrm{M}} \left(u\Delta u + \frac{n-2}{4(n-1)} \mathrm{R} u^2 \right) \mathrm{vol}^g \tag{23.7}$$

$$= \int_{\mathrm{M}} \left(|du|^2 + \frac{n-2}{4(n-1)} \mathrm{R} u^2 \right) \mathrm{vol}^g \tag{23.8}$$

to minimize under the constraint $\int_{\mathrm{M}} u^p \,\mathrm{vol}^g = 1$, where $p = \frac{2n}{n-2}$ is the Sobolev exponent which appears in the Sobolev embedding $\mathrm{H}^1 \subset \mathrm{L}^p$, that is $\frac{1}{p} = \frac{1}{2} - \frac{1}{n}$.
Under the form (23.8) the Euler-Lagrange equation is clearly

$$\Delta u + \frac{n-2}{4(n-1)} \mathrm{R} u = \lambda u^{p-1} \tag{23.9}$$

for some constant λ. Since $p - 1 = \frac{n+2}{n-2}$ this can be rewritten as

$$\frac{n-2}{4(n-1)} \tilde{\mathrm{R}} = \lambda. \tag{23.10}$$

This proves proposition 23.1. □

Observe that (23.10) implies $\lambda = \mathrm{I}(\tilde{g})$. Since we are looking for a minimum of I, it is natural to define:

Definition 23.3. The ***Yamabe constant*** of the conformal class of g is

$$\lambda(\mathrm{M}, g) = \inf\{\mathrm{I}(\tilde{g}), \tilde{g} \text{ conformal to } g, \mathrm{Vol}(\tilde{g}) = 1\}. \tag{23.11}$$

It makes no difference to suppose a priori that $\mathrm{Vol}(g) = 1$. Since $p > 2$ we have $\|u\|_{\mathrm{L}^2(\mathrm{M},g)} \leq \|u\|_{\mathrm{L}^p(\mathrm{M},g)} = 1$ and therefore we see from (23.8) that $\mathrm{I}(\tilde{g})$ is bounded below if $\mathrm{Vol}(\tilde{g}) = 1$, therefore $\lambda(\mathrm{M}, g) > -\infty$.
We now rewrite the Yamabe problem in the following final setup:

- $p = \frac{2n}{n-2}$ is the Sobolev exponent of the inclusion $\mathrm{H}^1 \subset \mathrm{L}^p$; it is useful to keep in mind that $p - 1 = \frac{n+2}{n-2}$ and $p - 2 = \frac{4}{n-2}$;
- we define a differential operator

$$\mathrm{L}_g u = \Delta u + \frac{n-2}{4(n-1)} \mathrm{R} u \tag{23.12}$$

so that if

$$\tilde{g} = u^{p-2} g \tag{23.13}$$

then

$$\tilde{\mathrm{R}} = 4\frac{n-1}{n-2} u^{-p+1} \mathrm{L}_g u; \tag{23.14}$$

- we want to solve the equation

$$\mathrm{L}_g u = \lambda u^{p-1} \tag{23.15}$$

where $\lambda = \lambda(\mathrm{M}, g)$ is the Yamabe constant; this equation implies $\tilde{\mathrm{R}} = 4\frac{n-1}{n-2}\lambda$;

- this equation has a variational formulation: it is satisfied by a minimum of the functional $I(u)$ defined for $u > 0$ by (23.8), under the constraint $\|u\|_{L^p} = 1$.

The operator L_g is called the ***conformal Laplacian*** because it enjoys the following invariance property under conformal changes:

Proposition 23.4. *For any positive function* Ω *one has*

$$L_{\Omega^2 g} u = \Omega^{-\frac{n+2}{2}} L_g(\Omega^{\frac{n-2}{2}} u). \tag{23.16}$$

Proof. It is sufficient to prove the identity (23.16) in the case $u > 0$. Write $g_1 = u_1^{\frac{4}{n-2}} g$ and $g_2 = (u_2 u_1)^{\frac{4}{n-2}} g$, then

$$R_2 = 4\frac{n-1}{n-2}(u_1 u_2)^{-p+1} L_g(u_1 u_2) = 4\frac{n-1}{n-2} u_2^{-p+1} L_{g_1} u_2.$$

Therefore $L_{g_1} u_2 = u_1^{-p+1} L_g(u_1 u_2)$ which proves the proposition by taking $\Omega^2 = u_1^{\frac{4}{n-2}}$. □

24 Non critical case

Fix p' so that $2 \leq p' < p$. We consider the problem of minimizing $I(u)$ under the constraint $\|u\|_{L^{p'}} = 1$. The derivation of the Euler-Lagrange equation is similar to what was done in the previous section, and we find that a minimizer for this problem should satisfy the equation

$$L_g u = \lambda_{p'} u^{p'-1} \tag{24.1}$$

which is the equation to solve.

Theorem 24.1. *The equation (24.1) always admits a smooth positive solution, which is a minimizer of* $I(u)$ *under the constraint* $\|u\|_{L^{p'}} = 1$.

Proof. Observe that the functional $I(u) = \int_M |du|^2 + \frac{n-2}{4(n-1)} R u^2$ is well-defined on the Sobolev space H^1. Moreover if $u \in H^1$ then $|u| \in H^1$ and $I(u) = I(|u|)$. So for any $u \in H^1$ we have that $I(u) = \lim_{\varepsilon \to 0} I((\varepsilon + |u|)/\|\varepsilon + |u|\|_{L^{p'}})$ and it follows that the infimum of I on positive functions is the same as the infimum of I on all functions of H^1 such that $\|u\|_{L^{p'}} = 1$.

The natural method to prove the theorem is to take a minimizing sequence $(u_i > 0)$ of I under the constraint $\|u_i\|_{L^{p'}} = 1$, and to prove some convergence. We have

$$\left| \int_M R u^2 \operatorname{vol} \right| \leq (\sup |R|) \|u\|^2_{L^{p'}} \leq \sup |R|$$

so $\int_M |du_i|^2$ is bounded. It follows that $\|u_i\|_{H^1}$ is bounded. Since $H^1 \subset L^{p'}$ is compact (this is where we use $p' < p$), we deduce that there exist a limit u such that

$$u_i \rightharpoonup u \text{ in } H^1, \qquad u_i \to u \text{ in } L^{p'}.$$

In particular $\|u\|_{L^{p'}} = 1$ so $u \neq 0$ (we will see that this fails for $p' = p$).

Summarizing, we obtain a function $u \in H^1$ satisfying:

- $u \geq 0$
- u is a minimizer of I on H^1 with the constraint $\|u\|_{L^{p'}} = 1$: this implies that it is a weak solution of the equation (24.1), that is it satisfies for any function φ

$$\int_M \big(\langle du, d\varphi\rangle + \frac{n-2}{4(n-1)} Ru\varphi\big)\,\mathrm{vol} = \lambda_{p'} \int_M u^{p'-1}\varphi\,\mathrm{vol}\,.$$

Lemma 24.2. *A weak solution $u \geq 0$ in H^1 of equation (24.1) is actually C^∞ and positive everywhere if $u \not\equiv 0$.*

Applying this lemma to the previous function u solves our problem: the function $u > 0$ is smooth and minimizes $I(u)$ under the constraint $\|u\|_{L^{p'}} = 1$; it is in particular a solution of equation (24.1). □

Proof of lemma 24.2. Since L_g is an elliptic operator, we shall use equation (24.1) to obtain more regularity on u. Suppose $u \in L^r$, then $u^{p'-1} \in L^{\frac{r}{p'-1}}$, so by equation (24.1) $u \in W^{2,\frac{r}{p'-1}} \subset L^{r'}$ where r' is given by the Sobolev inclusion (proposition 21.1):

$$\frac{1}{r'} = \frac{p'-1}{r} - \frac{2}{n} = \frac{1}{r} + \big(\frac{p'-2}{r} - \frac{2}{n}\big).$$

Using $p' < p$ and supposing $r \geq p'$, we have

$$\frac{p'-2}{r} - \frac{2}{n} < \frac{p-2}{p} - \frac{2}{n} = 0$$

so it follows that

$$\frac{1}{r'} < \frac{1}{r} - \varepsilon$$

for some fixed $\varepsilon > 0$. So starting from $u \in L^{r_0}$ with $r_0 = p'$, we obtain $u \in L^{r_1}$ for $1/r_1 < 1/r_0 - \varepsilon$; iterating we obtain $u \in L^{r_j}$ with $1/r_j < 1/p' - j\varepsilon$. So we can obtain $u \in W^{2,r}$ for r as large as we want, which implies that $u \in C^{1,\alpha}$ by proposition 21.1. Then $u^{p'-1} \in C^{1,\alpha}$ so again elliptic regularity for equation (24.1) gives $u \in C^{3,\alpha}$, and iterating this we obtain $u \in C^\infty$.

There remains to prove that if $u \not\equiv 0$ then $u > 0$ everywhere. From (24.1) we obtain

$$(\Delta + m)u = \big(m - \frac{n-2}{4(n-1)} + \lambda u^{p'-2}\big)u \geq 0$$

for $m > 0$ large enough. The result is then a consequence of the strong maximum principle below. □

Theorem 24.3 (Strong Maximum Principle). *Suppose $h \geq 0$ is a smooth function on* M. *Suppose $u \geq 0$ is a C^2 function on* M *satisfying the differential inequality*

$$\Delta u + hu \geq 0.$$

If u vanishes at some point on M, *then $u \equiv 0$.*

Observe that the weak maximum principle in proposition 15.4 gives that at a zero of u (which is a minimum since $u \geq 0$) one has $\Delta u \leq 0$. The strict inequality would contradict $\Delta u + hu \geq 0$, but this is not given by the weak maximum principle. The strong maximum principle gives an answer in that case.

Proof. Using normal coordinates we can suppose that we are in $\mathbb{R}^n$ with $u(0) = 0$. In polar coordinates we note $u(r, \sigma)$ with $\sigma \in S^{n-1}$. The metric is $g = dr^2 + r^2(g_{S^{n-1}} + O(r^2))$, the volume form is $\mathrm{vol} = \phi r^{n-1} dr \wedge \mathrm{vol}^{S^{n-1}}$ with $\phi = 1 + O(r^2)$. We consider the function

$$I(r) = \int_{B_r} u \,\mathrm{vol} = \int_{B_r} u(\rho, \sigma)\phi(\rho, \sigma)\rho^{n-1} d\rho |d\sigma|^{n-1}.$$

It follows that

$$\partial_r I = \int_{S_r} u(r, \sigma)\phi(r, \sigma) r^{n-1} |d\sigma|^{n-1}$$

$$r^{n-1}\partial_r(r^{-n-1}\partial_r I) = \int_{S_r} r^{n-1}\big(\partial_r u + u\frac{\partial_r \phi}{\phi}\big)\phi |d\sigma|^{n-1}.$$

Integrating the inequality $\Delta u + hu \geq 0$ on B_r gives

$$\int_{S_r} -\partial_r u \,\phi r^{n-1} |d\sigma|^{n-1} + \int_{B_r} hu \,\mathrm{vol} \geq 0.$$

Therefore we obtain

$$r^{n-1}\partial_r(r^{-n-1}\partial_r I) \leq \int_{B_r} hu \,\mathrm{vol} + \int_{S_r} u\frac{\partial_r \phi}{\phi}\phi r^{n-1} |d\sigma|^{n-1}.$$

Using $h \geq 0$ and $|\partial_r \phi / \phi| = O(r)$ we obtain finally, for some constants $a, b > 0$,

$$r^{n-1}\partial_r(r^{-n-1}\partial_r I) \leq aI + br\partial_r I.$$

Also from its definition, note that $I = O(r^{n+2})$ and $r\partial_r I = O(r^{n+2})$.

This kind of differential inequality is often studied by finding a suitable function which satisfies an opposite inequality: here we choose $J(r) = r^{n+1}$ so that

$$r^{n-1}\partial_r r^{-n-1}\partial_r J = (n+1)r^{n-1} \geq aJ + br\partial_r J$$

for $r \leq r_0$ small enough. Therefore, still for $r \leq r_0$, one has

$$r^{n-1}\partial_r r^{-n-1}\partial_r(I - \epsilon J) \leq a(I - \epsilon J) + br\partial_r(I - \epsilon J). \tag{24.2}$$

Suppose $I(r_0) > 0$. Then we can choose $\epsilon > 0$ small enough so that $(I - \epsilon J)(r_0) \geq 0$. But when $r \to 0$ we have

$$(I - \epsilon J)(r) \sim -\epsilon r^{n+1}.$$

It follows that $I - \epsilon J$ has a negative maximum in the interval $(0, r_0)$. This contradicts (24.2). Therefore one must have $I(r_0) = 0$, which implies that $u \equiv 0$ on B_{r_0}. □

25 The conformal class of the sphere

We use the description of the standard sphere S^n via stereographic projection. In these coordinates the metric of the sphere is conformal to the Euclidean metric, and we can write it in the form (23.13) as

$$g_{S^n} = \frac{4g_{\mathbb{R}^n}}{(1+|x|^2)^2} = 4u_1^{p-2} g_{\mathbb{R}^n} \quad \text{with } u_1 = \frac{1}{(1+|x|^2)^{\frac{n-2}{2}}}. \tag{25.1}$$

It is known that the group of conformal diffeomorphisms of S^n is generated by:

- rotations (these are isometries of S^n);
- translations $\tau_v(x) = x + v$ in $\mathbb{R}^n$;
- dilations $\delta_\alpha(x) = \alpha^{-1}x$ in $\mathbb{R}^n$.

In particular we obtain in the conformal class of S^n the following family of metrics which are also isometric to S^n:

$$\delta_\alpha^* g_{S^n} = 4u_\alpha^{p-2} g_{\mathbb{R}^n} \quad \text{with } u_\alpha(x) = \left(\frac{\alpha}{\alpha^2 + |x|^2}\right)^{\frac{n-2}{2}}. \tag{25.2}$$

In $\mathbb{R}^n$ we have the Sobolev injection $H^1 \subset L^p$, and one defines the ***Sobolev constant*** by

$$\mu_0 = \inf_{H^1} \frac{\|du\|_{L^2}^2}{\|u\|_{L^p}^2}. \tag{25.3}$$

Consider the Yamabe problem on S^n: starting from the metric $g_{\mathbb{R}^n}$ in the conformal class, we see that the functional I of (23.8) reduces to

$$I(u) = \int_{\mathbb{R}^n} |du|^2 |dx|^n \tag{25.4}$$

and the variational formulation is to minimize I under the constraint $\|u\|_{L^p(\mathbb{R}^n)} = 1$. Therefore the Yamabe constant is

$$\lambda(S^n) = \inf_{\|u\|_{L^p}=1} I(u) = \mu_0. \tag{25.5}$$

Of course we are not exactly in the setting of section 23 since $\mathbb{R}^n$ is S^n minus a point and $g_{\mathbb{R}^n}$ is conformal to g_{S^n} outside this point. The reader can check that the infimum of I can be checked on H^1 functions of $\mathbb{R}^n$ only.

26 The Yamabe problem on the sphere

Theorem 26.1. *The infimum of the Yamabe functional in the conformal class of* (S^n, g_{S^n}) *is realized exactly on the metrics on* S^n *obtained from* g_{S^n} *by a conformal diffeomorphism.*

Moreover any metric on S^n *conformal to* g_{S^n} *and with constant scalar curvature, is obtained from* g_{S^n} *by a conformal diffeomorphism.*

The first part of the theorem is difficult and will not be proved in these notes.

Remarks 26.2. 1) It follows that the functions u_α realize the minimum of I. But observe that $u_\alpha \to 0$ when $\alpha \to 0$ so contrarily to the equation for $p' < p$ (section 24) a minimizing sequence can converge to 0.

2) Also one has $u_\alpha \to \infty$ when $\alpha \to \infty$ so there is no a priori C^0 bound for the solutions of the equation.

3) Since a minimizer of the Yamabe functional has constant scalar curvature, the second part of the theorem implies the first part, if one knows a priori the infimum of I is realized by a metric in the conformal class.

Proof. We only prove the second statement. We need the following formula for the Ricci tensor under a conformal change: if $g = \phi^2 g_0$ then

$$\mathrm{Ric}(g_0) = \mathrm{Ric}(g) + \phi^{-1}\Big((n-2)\nabla d\phi - (n-1)\frac{|d\phi|^2}{\phi} g - (\Delta\phi)g\Big) \tag{26.1}$$

where all operators are with respect to g. Taking the trace free part,

$$\mathrm{Ric}_0(g_0) = \mathrm{Ric}_0(g) + (n-2)\phi^{-1}\big(\nabla d\phi + \frac{1}{n}(\Delta\phi)g\big). \tag{26.2}$$

Apply this with $g = \phi^2 g_{S^n}$ with g having constant scalar curvature, we obtain

$$\mathrm{Ric}_0(g) = -(n-2)\phi^{-1}\big(\nabla d\phi + \frac{1}{n}(\Delta\phi)g\big). \tag{26.3}$$

Therefore

$$\begin{aligned}\int_{S^n} \phi|\,\mathrm{Ric}_0(g)|^2 \,\mathrm{vol}^g &= \int_{S^n} \langle \mathrm{Ric}_0(g), -(n-2)(\nabla d\phi + \frac{1}{n}(\Delta\phi)g\rangle \,\mathrm{vol}^g \\ &= -(n-2)\int_{S^n} \langle \delta\,\mathrm{Ric}_0(g), d\phi\rangle \,\mathrm{vol}^g \\ &= 0\end{aligned}$$

since by the Bianchi identity $\delta\,\mathrm{Ric}_0(g) = 0$.

It follows that $\mathrm{Ric}_0(g) = 0$, that is g is an Einstein metric. The result is now a consequence of the following:

Fact 26.3. *If* $\mathrm{Ric}(g) = \Lambda g$ *and g is conformal to* g_{S^n}*, then g has constant sectional curvature, which implies that there is a diffeomorphism Φ of* S^n *such that* $\Phi^* g = g_{S^n}$.

Since g and g_{S^n} are conformal, Φ has to be a conformal diffeomorphism. □

We will not give the proof of the fact 26.3, but here is the main idea. If $n = 3$ then Ric contains the information of the whole curvature tensor, so an Einstein metric has constant sectional curvature. If $n \geq 4$, there is a component of the Riemannian curvature called the ***Weyl tensor***, noted W, which detects when a metric is locally conformal to the standard sphere S^n. The data of both W and Ric gives the whole curvature tensor, see for example [Bes87]. So if g is conformal to g_{S^n} and is Einstein, then its Weyl tensor $W = 0$ and $\mathrm{Ric} = \Lambda$: since these determine completely the Riemannian curvature, the sectional curvature is constant.

27 Epsilon regularity

We have seen that in the case $p' < p$ a weak solution of the Yamabe equation is controled in C^0 thanks to the Sobolev inequality (see the proof of lemma 24.2. The proof fails in the case $p' = p$ and indeed we have seen in section 26 that a sequence of solutions can diverge.

The following is an example of ε-regularity: it says that if some energy is locally small enough, then a solution is C^0 controled as if the problem was linear. Similar statements exist in various nonlinear geometric problems. In our case the energy is just the L^p norm:

Theorem 27.1. *Let h be a function on (M, g). There exist constants $\varepsilon_0, r_0, c > 0$ such that: if $u > 0$ is a smooth solution of the equation $\Delta u + hu = u^{p-1}$, then for any ball B of radius $r \leq r_0$ one has:*

$$\textit{if} \quad \int_B u^p \,\mathrm{vol} = \varepsilon < \varepsilon_0 \quad \textit{then} \quad u(0) \leq c\frac{\varepsilon^{\frac{1}{p}}}{r^{\frac{n}{p}}}. \tag{27.1}$$

We need some tools to prove the theorem.

Proposition 27.2. *Suppose (M^n, g) is a compact Riemannian manifold. Then there are constants $C, r_0 > 0$ such that if we have a function $f \geq 0$ on a geodesic ball B_r ($r \leq r_0$) satisfying $\Delta f \leq 2nk$ then*

$$f(0) \leq C \fint_{B_r} f + k|x|^2. \tag{27.2}$$

Proof. We have $\Delta r^2 = -2n + O(r^2)$ so for $r \leq r_0$ (with $r_0 > 0$ sufficiently small) we have $\Delta(f + k(1+\epsilon)r^2) \leq 0$. By considering $f + k(1+\epsilon)r^2$, and up to increasing slightly the constant C we are reduced to the case $k = 0$.

Therefore we can suppose $\Delta f \leq 0$. We use this inequality by integrating on geodesic balls: with notations similar to that of the proof of theorem 24.3,

$$0 \geq \int_{B_r} \Delta f \,\mathrm{vol} = \int_{S_r} -r^{n-1}\phi \partial_r f |d\sigma|^{n-1}.$$

Define $m(r) = \fint_{S_r} f\phi|d\sigma|^{n-1}$, then we obtain

$$\begin{aligned} \partial_r m &\geq \int_{S_r} f \partial_r \phi |d\sigma|^{n-1} \\ &\geq -cr \int_{S_r} f\phi|d\sigma|^{n-1} \\ &\geq -crm \end{aligned}$$

It follows that $m(r) \geq m(0)e^{-cr^2/2} = f(0)e^{-cr^2/2}$ and therefore $f(0) \leq C \fint_{S_r} f$. The proposition follows. □

Remark 27.3. If we are on the flat space $\mathbb{R}^n$ then in the proposition one can take $C = 1$, and the hypothesis $f \geq 0$ is not needed. This follows from the proof of the proposition, observing that in $\mathbb{R}^n$ we have exactly $\Delta r^2 = -2n$ and $\phi = 1$.

Actually one can see by the same technique that if f is a harmonic function in $\mathbb{R}^n$, then one has the well-known ***mean formula***

$$f(0) = \fint_{B_r} f. \tag{27.3}$$

From this formula one can deduce the Liouville theorem: a bounded harmonic function on $\mathbb{R}^n$ is constant.

Proof of theorem 27.1. We simplify the proof by taking $h = 0$. The reader can check that the addition of h gives additional terms which do not perturb the following arguments.

We first limit ourselves to the special case when u has a maximum M at a point x_0, and we consider the balls centered at x_0. The equation is $\Delta u = u^{p-1} \leq M^{p-1}$ which gives via (27.2) an estimate $M = u(x_0) \leq C \fint_{B_r} u + M^{p-1} r^2$. From Hölder inequality

$$\int_{B_r} u \leq \left(\int_{B_r} u^p \right)^{\frac{1}{p}} \mathrm{Vol}(B_r)^{\frac{p-1}{p}}$$

it follows that

$$M \leq c \left(r^{-\frac{n}{p}} \varepsilon^{\frac{1}{p}} + r^2 M^{p-1} \right).$$

Taking $\lambda = r^{n/p} M$ we can rewrite this as

$$\lambda \leq c\big(\varepsilon^{\frac{1}{p}} + \lambda^{p-1}\big). \tag{27.4}$$

When r is small then λ is small. Since $p > 2$, if $\varepsilon > 0$ is small enough, say $\varepsilon < \varepsilon_0$, the first zero of the function $c(\varepsilon^{1/p} + \lambda^{p-1}) - \lambda$ is approximately $\lambda_1 \sim c\varepsilon^{1/p}$, so the inequality (27.4) imposes the restriction $\lambda \leq \lambda_1 \leq 2c\varepsilon^{1/p}$, which we rewrite as

$$M \leq 2c \frac{\varepsilon^{\frac{1}{p}}}{r^{\frac{n}{p}}}. \tag{27.5}$$

This proves the theorem for the special case of a point x_0 at which u attains a maximum.

The case of a general point is similar: we restrict to a ball B_r but u can reach its maximum at the boundary of the ball. To avoid this we introduce the distance d to the boundary of the ball, and we take

$$M = \max_{B_r} u(x) d(x)^{\frac{n}{p}}. \tag{27.6}$$

This is attained at a point x_0 in the interior of the ball, and we take a smaller ball $B_\rho(x_0)$ now centered at x_0 so that $\rho < \frac{1}{2} d(x_0)$, so the ball remains at some distance of the boundary of B_r. On $B_\rho(x_0)$ we now have $u \leq 2^{n/p}M$. We can then proceed as in the first case to prove the estimate (27.5) for some larger constant c':

$$u(x_0) \leq c' \frac{\varepsilon^{\frac{1}{p}}}{\rho^{\frac{n}{p}}}.$$

But we want to estimate $u(0)$:

$$u(0) \leq \frac{\rho^{\frac{n}{p}}}{r^{\frac{n}{p}}} u(x_0) \leq c' \frac{\varepsilon^{\frac{1}{p}}}{r^{\frac{n}{p}}}.$$

This proves the result. Note that the choice of the exponent $\frac{n}{p}$ in (27.6) is used only in this last step. □

Theorem 27.1 reduces the C^0-estimate to having small L^p norm of u on small balls. Note that this typically fails in the case of the sphere S^n for the functions u_α defined by (25.2), since the L^p norm is more and more concentrated near the origin as $\alpha \to 0$. This case is excluded in the following proposition, which gives a simple criterion to control local L^p norms:

Proposition 27.4. *If the Yamabe constant of* (M, g) *satisfies* $\lambda(M, g) < \mu_0 = \lambda(S^n)$, *then for any* $\varepsilon > 0$ *there exists* $r > 0$ *such that for any solution* u *of the Yamabe equation (23.15) with* $\|u\|_{L^p} = 1$ *and any ball* B_r *of* (M, g) *one has*

$$\int_{B_r} u^p < \varepsilon.$$

Proof. Take $x_0 \in M$ and for simplicity assume that g is flat near x_0.
Take χ a cutoff function near x_0. Since g is flat near x_0, the Yamabe equation is $\Delta u = \lambda u^{p-1}$ near x_0. By integration by parts,

$$(d(\chi^2 u), du) = \lambda \int \chi^2 u^p.$$

Observing that

$$\langle d(\chi u), d(\chi u)\rangle = |d\chi|^2 u^2 + 2\langle \chi du, u d\chi\rangle + \chi^2 |du|^2 = |d\chi|^2 u^2 + \langle d(\chi^2 u), du\rangle,$$

we obtain

$$\|d(\chi u)\|_2^2 = \lambda \int \chi^2 u^p + \int |d\chi|^2 u^2.$$

We have the following Hölder inequalities:

$$\int \chi^2 u^p \leq \Big(\int (\chi^2 u^2)^{p/2}\Big)^{2/p} \big((u^{p-2})^{p/(p-2)}\big)^{(p-2)/p} = \|\chi u\|_{L^p}^2 \|u\|_{L^p}^{p-2},$$

$$\int |d\chi|^2 u^2 \leq \Big(\int (u^2)^{p/2}\Big)^{2/p} \Big(\int |d\chi|^{2p/(p-2)}\Big)^{(p-2)/p} = \|u\|_{L^p}^2 \|d\chi\|_{L^n}^2.$$

Applying these inequalities and the Sobolev inequality in $\mathbb{R}^n$, we obtain

$$\mu_0 \|\chi u\|_{L^p}^2 \leq \|d(\chi u)\|_2^2 \leq \lambda \|\chi u\|_{L^p}^2 \|u\|_{L^p}^{p-2} + \|u\|_{L^p}^2 \|d\chi\|_{L^n}^2.$$

Since $\|u\|_{L^p} = 1$ it follows that

$$(\mu_0 - \lambda)\|\chi u\|^2_{L^p} \leq \|d\chi\|^2_{L^n}. \tag{27.7}$$

The fact that the Sobolev embedding $W^{1,n} \subset C^0$ fails says that one can find cutoff functions χ with $\|d\chi\|_{L^n}$ as small as we want. To be more specific we start from the function f in example (21.6) and we define a cutoff function χ_ρ with support in B_r by

$$\chi_\rho(x) = \begin{cases} \frac{f(rx)}{f(r\rho)} & \text{if } |x| \geq \rho, \\ 1 & \text{if } |x| \leq \rho. \end{cases}$$

Then $\|d\chi_\rho\|_{L^n} \to 0$ when $\rho \to 0$, so for ρ small enough (27.7) implies

$$\int_{B_\rho} u^p < \varepsilon.$$

This concludes the proof in the case g is flat around x_0. In general one argues that on a small enough ball the metric is close to the flat metric. The main difference is the term Ru in $L_g u$ which gives an additional term $\|\chi u\|_2^2$ in the integration by parts. But this term is negligible before $\|\chi u\|^2_{L^p}$ on small enough balls. □

28 Resolution for Yamabe constant less than that of the sphere

The C^0 estimate from theorem 27.1 do not apply to general functions, but only to solutions of the Yamabe problem. Therefore we need to use to solutions of equations. We will use a solution $u_{p'}$ ($2 \leq p' < p$) of the equation (24.1), attaining the inf in

$$\lambda_{p'} = \inf_{\|u\|_{L^{p'}}=1} (L_g u, u) = \inf \frac{(L_g u, u)}{\|u\|^2_{L^{p'}}}.$$

Lemma 28.1. *Suppose* $\mathrm{Vol}(g) = 1$. *Then* $|\lambda_{p'}|$ *is a decreasing function of* p'. *Moreover:*

- *if* $\lambda(M, g) < 0$ *then* $\lambda_{p'} < 0$ *for all* $p' \in [2, p)$*;*
- *if* $\lambda(M, g) \geq 0$ *then* $\lambda_{p'}$ *is left continuous.*

Proof. We have

$$\lambda_{p''} = \inf \frac{\|u\|^2_{L^{p'}}}{\|u\|^2_{L^{p''}}} \frac{(L_g u, u)}{\|u\|^2_{L^{p'}}}.$$

If $p' \leq p''$ then $\|u\|^2_{L^{p'}} \leq \|u\|^2_{L^{p''}}$ and it follows that $|\lambda_{p''}| \leq |\lambda_{p'}|$.
If $\lambda(M, g) < 0$ then there exists u such that $(L_g u, u) < 0$ and it follows that $\lambda_{p'} < 0$ for any $p' \in [2, p]$.
If $\lambda(M, g) \geq 0$ then similarly $\lambda_{p'} \geq 0$ for any $p' \in [2, p]$. Given $\varepsilon > 0$ and p' there exists u such that $\frac{(L_g u, u)}{\|u\|_{L^{p'}}} < \lambda_{p'} + \varepsilon$. Therefore for $p'' < p'$ close enough to p' we have

$$\lambda_{p'} \leq \lambda_{p''} \leq \frac{(L_g u, u)}{\|u\|_{L^{p''}}} < \lambda_{p'} + 2\varepsilon.$$

This prove the left continuity of $\lambda_{p'}$. □

Theorem 28.2. *If* $\lambda(M,g) < \lambda(S^n)$ *a family* $(u_{p'})_{p'<p}$ *of solutions of (24.1) with* $\|u_{p'}\|_{L^{p'}} = 1$ *admits a convergent subsequence to a function* $u > 0$*, smooth, which realizes the minimum of the Yamabe functional. In particular* $u^{p-2}g$ *has constant scalar curvature.*

Proof. We have

$$L_g u_{p'} = \lambda_{p'} u_{p'}^{p'-1}. \tag{28.1}$$

It is not difficult to check that the C^0 estimate resulting from theorem 27.1 and proposition 27.4 extends to (28.1), and from the hypothesis $\lambda(M,g) < \lambda(S^n)$ it is actually uniform for $p' < p$.

From equation (28.1) we then obtain that $L_g u_{p'}$ is also uniformly bounded in C^0, so by elliptic regularity $u_{p'}$ is bounded in $C^{1,\alpha}$. Bootstrapping we obtain that $u_{p'}$ is bounded in $C^{k,\alpha}$ for any k. In particular, by Ascoli's theorem, we can extract a subsequence which converges strongly in C^2 to a limit u when $p' \to p$. We have $u \geq 0$ and $\|u\|_{L^p} = 1$.

If $\lambda(M) \geq 0$ then $\lim \lambda_{p'} = \lambda_p$ by the previous lemma, so $I(u) = \lambda_p$ and u is a solution of the equation $L_g u_p = \lambda_p u^{p-1}$. It has no zero by the strong maximum principle (theorem 24.3).

If $\lambda(M) < 0$ then $\lambda_{p'}$ is an increasing function of p' so has a limit $\lambda = \lim_{p'\to p} \lambda_{p'} \leq \lambda_p$. Then $I(u) = \lambda \geq \lambda_p = \inf I$ so $\lambda = \lambda_p$ and the same applies. □

29 Non conformally flat case

Theorem 28.2 provides a solution of the Yamabe problem, provided we have $\lambda(M,g) < \lambda(S^n)$. The following theorem says that this is always the case:

Theorem 29.1. *One has always* $\lambda(M,g) < \lambda(S^n)$*, except if* (M^n, g) *is conformal to* (S^n, g_{S^n}).

With the case of the sphere (theorem 26.1), this concludes the resolution of the Yamabe problem.

Theorem 29.1 is difficult and was proved in increasing generality. The first observation is:

Proposition 29.2 (Aubin). *One has always* $\lambda(M,g) \leq \mu_0$.

Proof. We need to construct functions φ so that $I(\varphi)$ is as close as we want to μ_0. The idea is to use the family (25.2) of radial functions (u_α) on $\mathbb{R}^n$ which are minimizers on $\mathbb{R}^n$: they satisfy $\|\nabla u_\alpha\|_2^2 = \mu_0 \|u_\alpha\|_{L^p}^2$. A small neighbourhood of a point in (M,g) looks more and more like a small ball in $\mathbb{R}^n$, and when $\alpha \to 0$ the function u_α has its energy more and more concentrated around 0, so the idea is to graft it on (M,g) to obtain the wanted test function φ.

First we need to cut u_α so that it has compact support. Let $\chi = \chi(r)$ be a cuf-off function on $\mathbb{R}^n$ so that

$$\chi(r) = \begin{cases} 1 & \text{if } r \le \varepsilon, \\ 0 & \text{if } r \ge 2\varepsilon. \end{cases}$$

The function $\varphi = \chi u_\alpha$ satisfies

$$\begin{aligned} \int_{\mathbb{R}^n} |d\varphi|^2 &= \int_{B_{2\varepsilon}} \chi^2 |du_\alpha|^2 + 2\chi u_\alpha \langle d\chi, du_\alpha\rangle + u_\alpha^2 |d\chi|^2 \\ &\le \int_{\mathbb{R}^n} |\partial_r u_\alpha|^2 + C \int_{B_{2\varepsilon}\setminus B_\varepsilon} u_\alpha |\partial_r u_\alpha| + u_\alpha^2. \end{aligned} \tag{29.1}$$

One has $\partial_r u_\alpha = -(n-2)\frac{r}{\alpha}(\frac{\alpha}{\alpha^2+r^2})^{n/2}$ therefore we have

$$u_\alpha \le \frac{\alpha^{\frac{n-2}{2}}}{r^{n-2}}, \quad |\partial_r u_\alpha| \le (n-2)\frac{\alpha^{\frac{n-2}{2}}}{r^{n-1}}.$$

Now fix ε and make $\alpha \to 0$: the second term in (29.1) satisfies

$$\int_{B_{2\varepsilon}\setminus B_\varepsilon} u_\alpha |\partial_r u_\alpha| + u_\alpha^2 = O(\alpha^{n-2}).$$

Now analyze the first term of (29.1):

$$\begin{aligned} \int_{\mathbb{R}^n} |\partial_r u_\alpha|^2 &= \mu_0 \left(\int_{B_\varepsilon} u_\alpha^p + \int_{\mathbb{R}^n\setminus B_\varepsilon} u_\alpha^p \right)^{\frac{2}{p}} \\ &\le \mu_0 \left(\int_{B_{2\varepsilon}} \varphi^p + \int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{\alpha^n}{r^{2n}} \right)^{\frac{2}{p}} \\ &\le \mu_0 \left(\int_{B_{2\varepsilon}} \varphi^p \right)^{\frac{2}{p}} + O(\alpha^n). \end{aligned}$$

Putting all this together, (29.1) finally gives for some constant C

$$\frac{\|d\varphi\|_2^2}{\|\varphi\|_{L^p}^2} \le \mu_0 + C\alpha^{n-2}. \tag{29.2}$$

So cutting off u_α gives an error term of size controled by α^{n-2}.

We are now ready to pass to the manifold (M, g): near a point p we consider normal coordinates (x^i) and we consider the function φ in these coordinates (taking ε small enough). Then we still have $|d\varphi|^2 = |\partial_r \varphi|^2$, but we have an error term in the volume form:

$$\mathrm{vol}^g = (1 + O(r^2))dx^1 \wedge \cdots \wedge dx^n.$$

We have

$$\begin{aligned} I(\varphi) &= \int_{B_{2\varepsilon}} \left(|d\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2\right) \mathrm{vol}^g \\ &\le (1 + C\varepsilon^2)\left(\mu_0 \|\varphi\|_{L^p}^2 + C\alpha^{n-2} + C \int_{(0,2\varepsilon)\times S_r} u_\alpha^2 r^{n-1} dr |d\sigma|^{n-1} \right). \end{aligned} \tag{29.3}$$

We fix $\varepsilon > 0$ small enough, and then $\alpha > 0$ small enough. By formula (29.4) below, the last term in the RHS is $O(\alpha)$ so we obtain finally

$$\frac{I(\varphi)}{\|\varphi\|^2_{L^p}} \leq (1 + C\varepsilon^2)(\mu_0 + C\alpha).$$

It follows that $\lambda(M, g) \leq \mu_0$. □

In the proof, we have used the following fact, which is left to the reader. Note $f \approxeq g$ is there is a constant C such that $C^{-1}g \leq f \leq Cg$.

Fact 29.3. *Fix* $\varepsilon > 0$. *Let* $F(\alpha) = \int_0^\varepsilon r^k u_\alpha^2 r^{n-1} dr$. *Then one has when* $\alpha \to 0$*:*

$$F(\alpha) \approxeq \begin{cases} \alpha^{k+2} & n > k+4 \\ \alpha^{k+2}|\ln \alpha| & n = k+4 \\ \alpha^{n-2} & n < k+4. \end{cases} \tag{29.4}$$

We can now prove the following result, which is the first step towards theorem 29.1. We recall that (M, g) is locally conformally flat if any point in M has a neighbourhood which is conformally equivalent to an open set in $\mathbb{R}^n$.

Theorem 29.4. *If* $n \geq 6$ *and* (M, g) *is not locally conformally flat, then* $\lambda(M, g) < \mu_0$.

The proof of the theorem relies on ***normal conformal coordinates***. Before stating the theorem, it is useful to recall a geometric interpretation of the scalar curvature: in normal coordinates around a point p, there is an expansion for the determinant of the metric given by:

$$\det(g_{ij}) = 1 - \frac{1}{3}\operatorname{Ric}(p)_{ij}x^i x^j - \frac{1}{6}\nabla_k \operatorname{Ric}(p)_{ij}x^i x^j x^k + \cdots$$

and in particular we obtain for the volume form:

$$\sqrt{\det(g_{ij})} = 1 - \frac{1}{6}\operatorname{Ric}(p)_{ij}x^i x^j + \cdots$$

Integrating on a sphere of radius r, we can compare with the volumes for the standard Euclidean metric g_0:

$$\begin{aligned} \int_{S_r} \operatorname{vol}^g_{S^r} &= \int_{S_r} \big(1 - \frac{1}{6}\operatorname{Ric}(p)_{ii}(x^i)^2 + O(r^3)\big)\operatorname{vol}^{g_0}_{S_r} \\ &= V_{n-1}r^{n-1}\big(1 - \frac{R(p)}{6n}r^2 + O(r^3)\big). \end{aligned} \tag{29.5}$$

Therefore the scalar curvature measures the distortion of volumes of small spheres (or balls) for g with that for g_0.

Theorem 29.5 (Günther 1993). *Given* (M^n, g_0) *and a point* $p \in M$, *there exists a metric* g *in the conformal class of* g_0 *such that in* g*-normal coordinates near* p *one has*

$$\det(g_{ij}) = 1.$$

For $n \geq 5$ *one then has* $R(g) = O(r^2)$ *and* $\Delta R(p) = \frac{1}{6}|W(p)|^2$.

Note that in normal conformal coordinates one has always $R(p) = 0$ by (29.5). The theorem says that for dimension at least 5 one has a second order vanishing.

Proving that one can obtain $\det(g) = 1$ up to any finite order near p is a classical result, as is the consequence on R and ΔR. Proving that one can achieve $\det(g) \equiv 1$ near p is much more difficult. In our proof we need this to be true only to any finite order, but the arguments become more clear with the full strength of the theorem, so this is the point of view that we shall use.

Proof of theorem 29.4. We refine the technique of proposition 29.2. Restarting from (29.2), we now use normal conformal coordinates on (M, g), that is we can suppose that the volume form is exactly that of $\mathbb{R}^n$, that is we have exactly $\mathrm{vol}^g = dx^1 \wedge \cdots \wedge dx^n$. It follows that the estimate for the first term of $I(g)$ in (29.3) is now enhanced into

$$\int_{B_{2\varepsilon}} |d\varphi|^2 \,\mathrm{vol}^g \leq \mu_0 \|\varphi\|^2_{L^p} + C\alpha^{n-2}.$$

For the second term of $I(g)$, we now write

$$\int_{B_{2\varepsilon}} R\varphi^2 |dx|^n \leq \int_{B_\varepsilon} R u_\alpha^2 |dx|^n + c \int_{B_{2\varepsilon}\setminus B_\varepsilon} u_\alpha^2 |dx|^n.$$

The first term is

$$\begin{aligned}\int_{B_\varepsilon} R u_\alpha^2 |dx|^n &= \int_{B_\varepsilon} \big(\partial^2_{ij} R(p) \frac{x^i x^j}{2} + O(r^3)\big) u_\alpha^2 r^{n-1} dr |d\sigma|^{n-1} \\ &= V_{n-1} \int_0^\varepsilon \big(-\Delta R(p) \frac{r^2}{2n} + O(r^3)\big) u_\alpha^2 r^{n-1} dr \\ &= V_{n-1} \int_0^\varepsilon \big(-|W(p)|^2 \frac{r^2}{12n} + O(r^3)\big) u_\alpha^2 r^{n-1} dr.\end{aligned}$$

Putting everything together and using the estimate (29.4) we obtain

$$I(\varphi) = \int_{B_{2\varepsilon}} |d\varphi|^2 \,\mathrm{vol}^g \leq \begin{cases} \mu_0 \|\varphi\|^2_{L^p} - C|W(p)|^2 \alpha^4 + o(\alpha^4), & n > 6, \\ \mu_0 \|\varphi\|^2_{L^p} - C|W(p)|^2 \alpha^4 |\ln \alpha| + o(\alpha^4), & n = 6. \end{cases}$$

Since (M, g) is not locally conformally flat, we can choose p such that $W(p) \neq 0$. Taking α small enough we obtain

$$I(\varphi) < \mu_0 \|\varphi\|^2_{L^p}$$

so $\lambda(M, g) < \mu_0$. □

30 Conformally flat case and positive mass

The remaining cases are that of dimensions smaller than 6, or conformally flat metrics. Since we can solve the problem when $\lambda(M, g) < \lambda(S^n)$, we can suppose at least $\lambda(M, g) > 0$. Since $\lambda(M, g) = \inf(L_g u, u)/\|u\|^2_{L^p}$, this implies that $\ker L_g = 0$. Since $\lambda_{p'} > 0$ as well in that case, by theorem 24.1 we can solve the equation

$L_g u = \lambda_{p'} u^{p'-1}$ for some $p' < p$ and the corresponding metric $\tilde{g} = u^{p'-2} g$ has scalar curvature $\tilde{R} = 4\frac{n-1}{n-2} u^{-p+1} L_g u = 4\frac{n-1}{n-2}\lambda_{p'} u^{p'-p} > 0$. Replacing g by $\tilde{g}$ if necessary, we can suppose that g has positive scalar curvature.

The Green function

In stereographic coordinates on the sphere we can write

$$g_{\mathbb{R}^n} = \frac{(1+|x|^2)^2}{4} g_{S^n} = G^{p-2} g_{S^n}, \quad G(x) = \left(\frac{1+|x|^2}{2}\right)^{\frac{2}{p-2}}. \tag{30.1}$$

We can think of G as the conformal change from S^n minus the north pole N to $\mathbb{R}^n$. Since coordinates on S^n near the north pole are obtained by the inversion $y = \frac{x}{|x|^2}$, up to multiplying G by a constant we have $G \sim \frac{1}{r^{n-2}}$ where r is the distance to N in S^n. Since the scalar curvature of $\mathbb{R}^n$ is zero and $p - 2 = 4/(n-2)$, the function G also satisfies $L^{g_{S^n}} G = 0$. We summarize this by saying that G is a solution on S^n of the system:

$$\begin{aligned} L_{g_{S^n}} G &= 0, \\ G &\underset{r\to 0}{\sim} \frac{1}{r^{n-2}}. \end{aligned} \tag{30.2}$$

If we fix a point p in (M, g) the problem (30.2) still makes sense. A solution is called a ***Green function*** of (M, g). The existence is given by:

Proposition 30.1. *If $\lambda(M, g) > 0$ and $p \in M$ then a solution G to the system (30.2) exists and $G > 0$ everywhere. Moreover, if $n = 3, 4, 5$ or g is conformally flat near p, then near p one has for some constant A*

$$G = \frac{1}{r^{n-2}} + A + a \tag{30.3}$$

where $a = O(r)$ satisfies $\partial^k a = O(r^{1-k})$ for $k \leq 2$.

Proof. Let us begin to prove that a solution G has to be positive. We can suppose that $R > 0$. Then the strong maximum principle (theorem 24.3) implies that a minimum of a solution of $L_g u = 0$ has to be positive. Therefore $u > 0$.

Now pass to the existence of G. In normal conformal coordinates for g the function $G_0 = \frac{1}{r^{n-2}}$ satisfies

$$\Delta G_0 = \Delta_{\mathbb{R}^n} G_0 = 0.$$

Therefore

$$L_g G_0 = \frac{n-2}{4(n-1)} \frac{R}{r^{n-2}}.$$

Take a cutoff function χ, the idea is now to find the solution G under the form

$$G = \chi G_0 + \psi$$

where ψ is more regular than G near p and satisfies

$$L_g \psi = -L_g(\chi G_0). \tag{30.4}$$

If g is conformally flat near p, then actually $L_g G_0 = 0$ near p, so $-L_g(\chi G_0) = 0$ near p, so this is a smooth function on M. Since L_g is a positive elliptic operator, it follows that there exists a unique solution ψ of (30.4), this solution is smoth, and the proposition is proved.

In the cases $n = 3, 4, 5$, from Theorem 29.5 we obtain that R cancels to order 1 ($n = 3, 4$) or 2 ($n = 6$). Therefore

$$\frac{R}{r^{n-2}} = \begin{cases} O(1), & n = 3, \\ O(\frac{1}{r}), & n = 4, 5. \end{cases}$$

Therefore $L_g(\chi G_0) \in L^{n-\varepsilon}$ and we can solve (30.4) with $\psi \in W^{2,n-\varepsilon} \subset C^\alpha$ for any $\alpha < 1$. Outside p the function ψ is smooth, but this proves only that $\psi = A + O(r^\alpha)$ near p for any $\alpha < 1$. The full result actually requires to refine the approximate solution G_0 near p before solving (30.4). The details are left to the reader. □

The small dimension or conformally flat case

Limit ourselves now to the remaining cases of the Yamabe problem, that is $n = 3, 4, 5$ or $n \geq 6$ and g is conformally flat. By proposition 30.1 the metric $\hat{g} = G^{p-2} g$ has $\hat{R} = 0$. Geometrically, passing from (M, g) to $(M \setminus \{p\}, \hat{g})$ looks like passing from $S^n \setminus \{N\}$ to $\mathbb{R}^n$. This reflects in the following proposition.

Proposition 30.2. *Take coordinates ($x^j = y^j/|y|^2$) obtained by inversion from the normal conformal coordinates (y^j) on* M *near p. The metric $\hat{g} = G^{p-2} g$ on $M \setminus \{p\}$ satisfies near p (so when $r := |x| \to \infty$)*

$$\hat{g} = \gamma^{p-2}(x)\Big(\sum (dx^j)^2 + O(\tfrac{1}{r^2})\Big) \tag{30.5}$$

with

$$\gamma = 1 + \frac{A}{r^{n-2}} + O(\frac{1}{r^{n-1}}). \tag{30.6}$$

In the conformally flat case the term $O(\frac{1}{r^2})$ in (30.5) is not needed.

Proof. Left to the reader. □

Using this proposition, we now work with the metric $\hat{g}$: outside a large compact set K, we have $(M \setminus \{p\}) \setminus K \approx \mathbb{R}^n \setminus B_R$. The advantage of $\hat{g}$ is that $\hat{R} = 0$, so that the Yamabe functional reduces to $\|d\phi\|_2^2$, as is the case on $\mathbb{R}^n$. Consider again our functions

$$u_\alpha(x) = \Big(\frac{\alpha}{\alpha^2 + r^2}\Big)^{\frac{n-2}{2}}.$$

When α is large u_α becomes almost constant on a large ball. So it is a good approximation of u_α on $M \setminus \{p\}$ to extend u_α outside $\mathbb{R}^n \setminus B_R$ by a constant, defining for large ρ and α the function

$$\phi = \begin{cases} u_\alpha(x) & \text{on } \mathbb{R}^n \setminus B_\rho \\ u_\alpha(\rho) & \text{on } r \leq \rho \end{cases} \tag{30.7}$$

Proposition 30.3. *With this choice of* ϕ *we have when* $\alpha \to +\infty$*, for some constant* $c > 0$*,*

$$\|d\phi\|_2^2 \le \mu_0 \|\phi\|_{L^p}^2 - c\frac{A}{\alpha^{n-2}} + O(\frac{1}{\alpha^{n-1}}). \tag{30.8}$$

In particular, if $A > 0$ *then* $\lambda(M, g) < \mu_0 = \lambda(S^n)$.

Proof. To simplify the proof, we limit to the conformally flat case. Since $|dr|^2 = \gamma^{-(p-2)}$,

$$\begin{aligned}
\|d\phi\|_2^2 &= \int_{\mathbb{R}^n \setminus B_\rho} |du_\alpha|^2 \gamma^{n(p-2)} |dx|^n \\
&= \int_{\mathbb{R}^n \setminus B_\rho} |\partial_r u_\alpha|^2 \gamma^2 |dx|^n \\
&= \int_{\mathbb{R}^n \setminus B_\rho} (u_\alpha \Delta_{\mathbb{R}^n} u_\alpha \gamma^2 - u_\alpha (\partial_r u_\alpha)(\partial_r \gamma)) |dx|^n + \int_{S_\rho} u_\alpha (\partial_r u_\alpha) \gamma^2 \rho^{n-1} |d\sigma|^{n-1}.
\end{aligned} \tag{30.9}$$

Observe that

$$\partial_r u_\alpha = -(n-2)\alpha^{\frac{n-2}{2}} \frac{r}{(\alpha^2 + r^2)^{\frac{n}{2}}}, \quad \Delta_{\mathbb{R}^n} u_\alpha = n(n-2) u_\alpha^{p-1},$$

Therefore the third term in (30.9) is $O(\frac{1}{\alpha^n})$. The first term is controled by

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_\rho} u_\alpha \Delta_{\mathbb{R}^n} u_\alpha \gamma^2 |dx|^n &= n(n-2) \int_{\mathbb{R}^n \setminus B_\rho} u_\alpha^{p-2} (u_\alpha \gamma)^2 |dx|^n \\
&\le n(n-2) \left(\int_{\mathbb{R}^n \setminus B_\rho} (u_\alpha \gamma)^p |dx|^n \right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^n \setminus B_\rho} u_\alpha^p |dx|^n \right)^{1-\frac{2}{p}} \\
&\le n(n-2) \|u_\alpha\|_{L^p(\mathbb{R}^n)}^{p-2} \|\phi\|_{L^p} \\
&\le \mu_0 \|\phi\|_{L^p}^2.
\end{aligned}$$

The crucial term is the second term in (30.9), which is equal to

$$-(n-2)^2 V_{n-1} \int_\rho^\infty \alpha^{-1} r \left(\frac{\alpha}{\alpha^2 + r^2} \right)^{n-1} \left(\frac{A}{r^{n-1}} + O(\frac{1}{r^n}) \right) r^{n-1} dr.$$

As in (29.4) one checks that when $\alpha \to +\infty$, for some $C > 0$,

$$\int_\rho^\infty \left(\frac{\alpha}{\alpha^2 + r^2} \right)^{n-1} r dr = \frac{C}{\alpha^{n-3}} + O(\frac{1}{\alpha^{n-2}}),$$

so the second term in (30.9) now becomes

$$-c\frac{A}{\alpha^{n-2}} + O(\frac{1}{\alpha^{n-1}})$$

for some $c > 0$. The proposition is proved. □

So the proof of the Yamabe problem is reduced to proving that the coefficient A in the development of the corresponding Green function (30.3) is positive. This is true:

Theorem 30.4. *Under the previous hypothesis, one has* $\mathrm{A} \geq 0$*, with* $\mathrm{A} = 0$ *if and only if* $(\mathrm{M} \setminus \{p\}, \hat{g}) = (\mathbb{R}^n, \sum (dx^j)^2)$.

Remark 30.5. There is a more general statement for any 'asymptotically flat' metric g on $\mathrm{M} \setminus \{p\}$ with $\mathrm{R}^g \geq 0$. Here asymptotically flat of order $\tau > 0$ means

$$g = \sum (dx^j)^2 + g_1, \qquad |\partial^k g_1| = \mathrm{O}(\frac{1}{r^{\tau+k}}).$$

The result requires $\tau > \frac{n}{2} - 1$.

The proof of the theorem reduces to the celebrated positive mass theorem originating in physics. It was proved by Schoen and Yau using minimal surfaces, and also by Witten in the spin case with a nice argument involving spinors and the Dirac operator, and a Weitzenböck formula which is similar to the Bochner formula that we used. See more details in [LP87].

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