AUSLANDER-REITEN THEORY FOR FINITE DIMENSIONAL ALGEBRAS

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3. Irreducible morphisms and almost split sequences

A algebra, L, M, N modules in $\operatorname{mod} A$

A homomorphism $f : L \to M$ in $\operatorname{mod} A$ is called **left minimal** if every $h \in \operatorname{End}_A(M)$ s.t. $hf = f$ is an isomorphism.

A homomorphism $g : M \to N$ in $\operatorname{mod} A$ is called **right minimal** if every $e \in \operatorname{End}_A(M)$ s.t. $ge = g$ is an isomorphism.

A homomorphism $f : L \to M$ in $\operatorname{mod} A$ is called **left almost split** if:

- f is not a section
- for every homomorphism $u : L \to U$ which is not a section there exists $u' : M \to U$ s.t. $u'f = u$.

A homomorphism $g : M \to N$ in $\operatorname{mod} A$ is called **right almost split** if:

- f is not a retraction
- for every homomorphism $v : V \to N$ which is not a retraction there exists $v' : V \to M$ s.t. $gv' = v$.

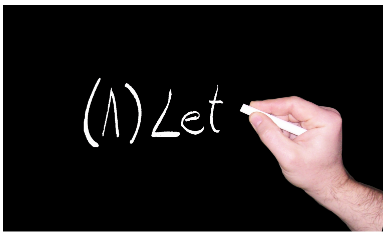
A homomorphism $f : L \to M$ in $\operatorname{mod} A$ is called **left minimal almost split** if it is both left minimal and left almost split.

A homomorphism $g : M \to N$ in $\operatorname{mod} A$ is called **right minimal almost split** if it is both right minimal and right almost split.

Proposition 3.1.

(1) Let $f : L \to M$ and $f' : L \to M'$ be left minimal almost split homomorphisms in $\operatorname{mod} A$*. Then there exists an isomorphism $h : M \to M'$ in* $\operatorname{mod} A$ *s.t. $f' = hf$.*

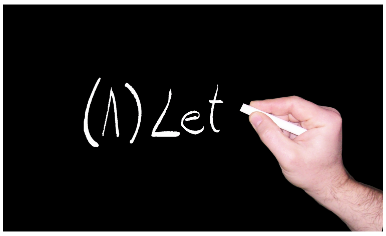
(2) Let $g : M \to N$ and $g' : M' \to N$ be right minimal almost split homomorphisms in $\operatorname{mod} A$*. Then there exists an isomorphism $e : M \to M'$ in* $\operatorname{mod} A$ *s.t. $g = g'e$.*

Proof. 

Lemma 3.2.

(1) *Let $f : L \to M$ be a left almost split homomorphism in* $\operatorname{mod} A$. *Then the module L is indecomposable.*

(2) *Let $g : M \to N$ be a right almost split homomorphism in* $\operatorname{mod} A$. *Then the module N is indecomposable.*

Proof. 

A homomorphism $f : X \to Y$ in $\operatorname{mod} A$ is said to be **irreducible** provided:

(1) f is neither a section nor a retraction

(2) if $f = f_1 f_2$, then either f_1 is a retraction or f_2 is a section.

Lemma 3.3. *Let $f : X \to Y$ be an irreducible homomorphism in* $\operatorname{mod} A$. *Then f is either a proper monomorphism or a proper epimorphism.*

Lemma 3.4. (Auslander-Reiten) *Let P be an indecomposable projective module in* $\operatorname{mod} A$*, and $u : \operatorname{rad} P \hookrightarrow P$ be the inclusion homomorphism. Then*

(1) u is right minimal almost split in $\operatorname{mod} A$

(2) u is irreducible in $\operatorname{mod} A$.

Lemma 3.5. (Auslander-Reiten) *Let I be an indecomposable injective module in* $\operatorname{mod} A$*, and $v : I \twoheadrightarrow I/\operatorname{soc} I$ the canonical epimorphism. Then*

(1) v is left minimal almost split in $\operatorname{mod} A$

(2) v is irreducible in $\operatorname{mod} A$.

Lemma 3.6. (Bautista) *Let X, Y be indecomposable modules in* $\operatorname{mod} A$*, and $f \in \operatorname{Hom}_A(X, Y)$. Then f is an irreducible homomorphism iff $f \in \operatorname{rad}_A(X, Y) \setminus \operatorname{rad}_A^2(X, Y)$.*

Lemma 3.7. *Let* $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ *be a nonsplit short exact sequence in* $\operatorname{mod} A$.

(1) *The homomorphism* f *is irreducible iff for every homomorphism* $v : V \to N$, *there exists* $v_1 : V \to M$ *s.t.* $v = gv_1$ *or* $v_2 : M \to V$ *s.t.* $g = vv_2$.

(2) *The homomorphism* g *is irreducible iff for every homomorphism* $u : L \to U$, *there exists* $u_1 : M \to U$ *s.t.* $u = u_1 f$ *or* $u_2 : U \to M$ *s.t.* $f = u_2 u$.

Corollary 3.8.

(1) *Let* $f : L \to M$ *be an irreducible monomorphism in* $\operatorname{mod} A$. *Then* $N = \operatorname{Coker} f$ *is indecomposable.*

(2) *Let* $g : M \to N$ *be an irreducible epimorphism in* $\operatorname{mod} A$. *Then* $L = \operatorname{Ker} g$ *is indecomposable.*

Theorem 3.9.

(1) *Let* $f : L \to M$ *be a nonzero left minimal almost split homomorphism in* $\operatorname{mod} A$*. Then*

(a) f *is irreducible in* $\operatorname{mod} A$*.*

(b) *A homomorphism* $f' : L \to M'$ *in* $\operatorname{mod} A$ *is irreducible iff* $M' \neq 0$ *and there exists a direct sum decomposition* $M \cong M' \oplus M''$ *and a homomorphism* $f'' : L \to M''$ *s.t.* $\begin{bmatrix} f' \\ f'' \end{bmatrix} : L \to M' \oplus M''$ *is left minimal almost split homomorphism in* $\operatorname{mod} A$*.*

(2) *Let* $g : M \to N$ *be a nonzero right minimal almost split homomorphism in* $\operatorname{mod} A$*. Then*

(a) g *is irreducible in* $\operatorname{mod} A$*.*

(b) *A homomorphism* $g' : M' \to N$ *in* $\operatorname{mod} A$ *is irreducible iff* $M' \neq 0$ *and there exists a direct sum decomposition* $M \cong M' \oplus M''$ *and a homomorphism* $g'' : M'' \to N$ *s.t.* $\begin{bmatrix} g' & g'' \end{bmatrix} : M' \oplus M'' \to N$ *is right minimal almost split homomorphism in* $\operatorname{mod} A$*.*

A short exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

in $\operatorname{mod} A$ is called an **almost split sequence (Auslander-Reiten sequence)** provided:

- f is left minimal almost split
- g is right minimal almost split

Remarks.

(1) An almost split sequence is never split (because f is not a section and g is not a retraction).

(2) The modules L and N are indecomposable (it follows from Lemma 3.2).

(3) L is not injective and N is not projective.

Lemma 3.10. *Let*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

and

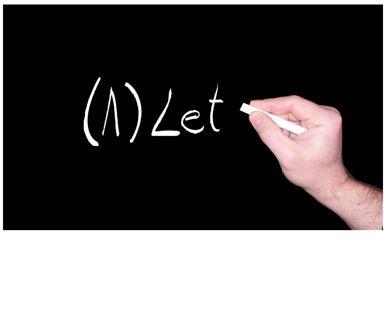
$$0 \longrightarrow L' \xrightarrow{f'} M' \xrightarrow{g'} N' \longrightarrow 0$$

be two almost split sequences in $\operatorname{mod} A$. *TFAE*

(1) The two sequences are isomorphic.

(2) The modules L and L' are isomorphic.

(3) The modules N and N' are isomorphic.

Proof. 

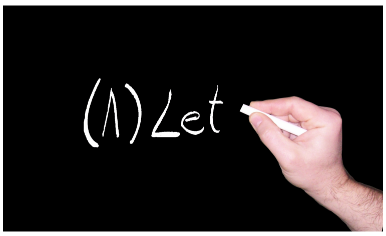
Lemma 3.11. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

be a commutative diagram in $\operatorname{mod} A$, *where the rows are exact and not split. Then*

(1) If L is indecomposable and w is an isomorphism, then u and hence v are isomorphisms.

(2) If N is indecomposable and u is an isomorphism, then w and hence v are isomorphisms.

Proof. 

Theorem 3.12. *Let*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be a short exact sequence in $\operatorname{mod} A$. *TFAE*

(1) *The given sequence is almost split sequence.*

(2) L *is indecomposable, and* g *is right almost split.*

(3) N *is indecomposable, and* f *is left almost split.*

(4) f *is left minimal almost split.*

(5) g *is right minimal almost split.*

(6) L *and* N *are indecomposable, and* f, g *are irreducible.*

Some implications of the proof.

- (1) $\Rightarrow$ (4), (1) $\Rightarrow$ (5) by definition of almost split sequence
- (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) by Lemma 3.2
- (1) $\Rightarrow$ (6) by Lemma 3.2 and Theorem 3.9.

4. The Auslander-Reiten theorems

A algebra

proj A	full subcategory of mod A consisting of all projective modules
inj A	full subcategory of mod A consisting of all injective modules

Consider the contravariant functor (following **Auslander** and **Bridger**)

$$(-)^t = \mathrm{Hom}_A(-, A) : \mathrm{mod}\, A \to \mathrm{mod}\, A^{\mathrm{op}}$$

then the functor $(-)^t$ induces the duality

$$\mathrm{proj}\, A \underset{(-)^t}{\overset{(-)^t}{\rightleftarrows}} \mathrm{proj}\, A^{\mathrm{op}}$$

Remarks. Let e be an idempotent in A.

(1) $(eA)^t = \mathsf{Hom}_A(eA, A) = Ae = eA^{\mathsf{op}}$.

(2) Every module in $\operatorname{proj} A$ is a direct sum of the modules of the form eA, where e is primitive.

(3) Every module in $\operatorname{proj} A^{\mathsf{op}}$ is a direct sum of the modules of the form $Ae = eA^{\mathsf{op}}$, where e is primitive.

Let $M \in \operatorname{mod} A$ and
$(\star)$ $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$ be a minimal projective presentation of M in $\operatorname{mod} A$.

For $(\star)$ we have in $\operatorname{mod} A^{\mathsf{op}}$ the induced exact sequence

$$0 \longrightarrow M^t \xrightarrow{p_0^t} P_0^t \xrightarrow{p_1^t} P_1^t \longrightarrow \operatorname{Coker} p_1^t \longrightarrow 0.$$

Then $\operatorname{Coker} p_1^t$ we denoted by $\operatorname{Tr} M$ and called **transpose** of M.

Remark. $\operatorname{Tr}(M)$ is uniquely determined by M, up to isomorphism.

Proposition 4.1. *Let M be an indecomposable module in* $\text{mod}\, A$*. Then*

(1) $\text{Tr}(M)$ *has no nonzero projective direct summands.*

(2) If M is not projective, then the sequence $P_0^t \xrightarrow{p_1^t} P_1^t \longrightarrow \text{Tr}\, M \longrightarrow 0$ *induced from* $(\star)$ *is a minimal projective presentation of* $\text{Tr}\, M$ *in* $\text{mod}\, A^{\text{op}}$.

(3) M is projective iff $\text{Tr}\, M = 0$.

(4) If M is not projective, then $\text{Tr}\, M$ *is indecomposable and* $\text{Tr}(\text{Tr}\, M) \cong M$.

(5) If M, N are indecomposable nonprojective, then $M \cong N$ iff $\text{Tr}\, M \cong \text{Tr}\, N$.

Remark. Tr does not define a duality.

For $M, N \in \operatorname{mod} A$ we define two ideals $\mathscr{P}_A$ and $\mathscr{I}_A$ of $\operatorname{mod} A$.

$$\mathscr{P}_A(M,N) = \left\{ f \in \mathrm{Hom}_A(M,N) \;\middle|\; \begin{array}{c} f = f_2 f_1, \\ f_1 \in \mathrm{Hom}_A(M,P), \\ f_2 \in \mathrm{Hom}_A(P,N), \\ P \in \operatorname{proj} A \end{array} \right\}$$

$$\mathscr{I}_A(M,N) = \left\{ g \in \mathrm{Hom}_A(M,N) \;\middle|\; \begin{array}{c} g = g_2 g_1, \\ g_1 \in \mathrm{Hom}_A(M,I), \\ g_2 \in \mathrm{Hom}_A(I,N), \\ I \in \operatorname{inj} A \end{array} \right\}$$

Now, we can define the **projectively stable category**: $\underline{\operatorname{mod}} A = \operatorname{mod} A / \mathscr{P}_A$

- objects of $\underline{\operatorname{mod}} A$ = objects of $\operatorname{mod} A$
- K-vector space of morphisms from M to N in $\underline{\operatorname{mod}} A$ is the quotient vector space $\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N)/\mathscr{P}_A(M,N)$
- the composition of morphisms in $\underline{\operatorname{mod}} A$ is induced from the composition of homomorphisms in $\operatorname{mod} A$

Similarly, we define the **injectively stable category**: $\overline{\text{mod}}A = \text{mod}\,A/\mathscr{I}_A$.

Proposition 4.2. *The transpose* Tr *induces a duality* $\underline{\text{mod}}A \underset{\text{Tr}}{\overset{\text{Tr}}{\rightleftarrows}} \underline{\text{mod}}A^{\text{op}}$

Recall that we have the standard duality
$D = \text{Hom}_K(-, K) : \text{mod}\,A \to \text{mod}\,A^{\text{op}}$

$$\begin{array}{ccc} \text{mod}\,A & \underset{D}{\overset{D}{\rightleftarrows}} & \text{mod}\,A^{\text{op}} \\ \cup & & \cup \\ \text{proj}\,A & \underset{D}{\overset{D}{\rightleftarrows}} & \text{inj}\,A^{\text{op}} \end{array}$$

So, D induces a duality between the stable categories $D : \underline{\text{mod}}A \to \overline{\text{mod}}A^{\text{op}}$

In particular, we have the equivalences of the categories

$\tau_A = D\,\text{Tr} : \underline{\text{mod}}A \to \overline{\text{mod}}A$,
$\tau_A^{-1} = \text{Tr}\,D : \overline{\text{mod}}A \to \underline{\text{mod}}A$

called the **Auslander-Reiten functors**.

For $M \in \operatorname{mod} A$ we have well-defined modules in $\operatorname{mod} A$:

$$\tau_A M = D\,\mathrm{Tr}(M) \quad \text{and} \quad \tau_A^{-1} M = \mathrm{Tr}\,D(M)$$

called the **Auslander-Reiten translations** of M.

Proposition 4.3. *Let M be a module in $\operatorname{mod} A$. Then*

(1) $\mathrm{pd}_A\, M \leq 1$ *iff* $\mathrm{Hom}_A(D(A), \tau_A M) = 0$.

(2) $\mathrm{id}_A\, M \leq 1$ *iff* $\mathrm{Hom}_A(\tau_A^{-1} M, A) = 0$.

Theorem 4.4. (Auslander-Reiten) *Let $M, N \in \operatorname{mod} A$. Then there exist isomorphisms of K-vector spaces*

$$D\underline{\mathrm{Hom}}_A(\tau_A^{-1} N, M) \cong \mathrm{Ext}_A^1(M, N) \cong D\overline{\mathrm{Hom}}_A(N, \tau_A M).$$

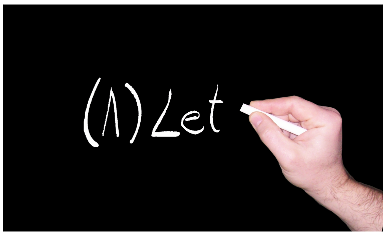
Corollary 4.5. *Let* $M, N \in \operatorname{mod} A$.

(1) *If* $\operatorname{pd}_A M \leq 1$, *then there exists a* K*-linear isomorphism*

$$\operatorname{Ext}_A^1(M, N) \cong D \operatorname{Hom}_A(N, \tau_A M).$$

(2) *If* $\operatorname{id}_A M \leq 1$ *then there exists a* K*-linear isomorphism*

$$\operatorname{Ext}_A^1(M, N) \cong D \operatorname{Hom}_A(\tau_A^{-1} N, M).$$

Proof. 

Theorem 4.6. (Auslander-Reiten)

(1) *For any indecomposable nonprojective module* $M \in \operatorname{mod} A$, *there exists an almost split sequence in* $\operatorname{mod} A$

$$0 \to \tau_A M \to E \to M \to 0.$$

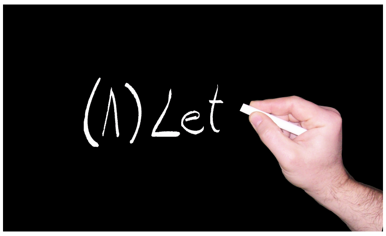
(2) *For any indecomposable noninjective module* $N \in \operatorname{mod} A$, *there exists an almost split sequence in* $\operatorname{mod} A$

$$0 \to N \to F \to \tau_A^{-1} N \to 0.$$

Proposition 4.7.

(1) *Let M be an indecomposable nonprojective module in* $\operatorname{mod} A$. *Then there exists an irreducible morphism $f : X \to M$ iff there exists an irreducible morphism $f' : \tau_A M \to X$.*

(2) *Let N be an indecomposable noninjective module in* $\operatorname{mod} A$. *Then there exists an irreducible morphism $g : N \to Y$ iff there exists an irreducible morphism $g' : Y \to \tau_A^{-1} N$.*

Proof. 

Corollary 4.8.

(1) *Let S be a simple projective noninjective module in* $\operatorname{mod} A$. *If $f : S \to M$ is irreducible, then M is projective. In particular, we have in* $\operatorname{mod} A$ *the almost split sequence*

$$0 \to S \to P \to \tau_A^{-1} S \to 0,$$

where P is projective.

(2) *Let S be a simple injective nonprojective module in* $\operatorname{mod} A$. *If $g : M \to S$ is irreducible, then M is injective. In particular, we have in* $\operatorname{mod} A$ *the almost split sequence*

$$0 \to \tau_A S \to I \to S \to 0,$$

where I is injective.

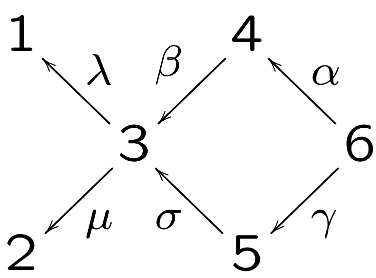
Proposition 4.9. *Let P be a nonsimple indecomposable projective-injective module. Then the sequence*

$$0 \longrightarrow \operatorname{rad} P \xrightarrow{[q,u]^T} (\operatorname{rad} P/\operatorname{soc} P) \oplus P \xrightarrow{[-j,v]} P/\operatorname{soc} P \longrightarrow 0,$$

where u, j are the inclusion homomorphisms and q, v are the canonical epimorphisms, is an almost split sequence in $\operatorname{mod} A$*.*

Example 4.10. Let $A = KQ/I$, where

Q :



and I is the ideal of KQ generated by $\beta\lambda$, $\sigma\mu$, $\alpha\beta - \gamma\sigma$.

S_1 - simple projective noninjective summand of $\operatorname{rad} P_3$

S_2 - simple projective noninjective summand of $\operatorname{rad} P_3$

• by Corollary 4.8 (1) we have almost split sequences

$$0 \to S_1 \to P_3 \to P_3/S_1 \to 0, \quad P_3/S_1 = \tau_A^{-1} S_1$$

$$0 \to S_2 \to P_3 \to P_3/S_2 \to 0, \quad P_3/S_2 = \tau_A^{-1} S_2$$

S_6 - simple injective nonprojective

• by Corollary 4.8 (2) we have an almost split sequence

$$0 \to P_6/S_3 \to I_4 \oplus I_5 \to S_6 \to 0, \; P_6/S_3 = \tau_A S_6$$

$P_6 = I_3$ - projective-injective

• by Proposition 4.9 we have an almost split sequence

$$0 \to \operatorname{rad} P_6 \to S_4 \oplus S_5 \oplus P_6 \to P_6/S_3 \to 0,$$

where $S_4 \oplus S_5 = \operatorname{rad} P_6 / \operatorname{soc} P_6$

5. The Auslander-Reiten quiver of an algebra

A finite dimensional K-algebra over a field K

Z indecomposable module in $\operatorname{mod} A$

$\operatorname{End}_A(Z)$ local K-algebra

$$\begin{aligned} F_Z &= \operatorname{End}_A(Z)/\operatorname{rad}\operatorname{End}_A(Z) \\ &= \operatorname{End}_A(Z)/\operatorname{rad}_A(Z,Z) \\ &\quad \text{division } K\text{-algebra} \end{aligned}$$

X, Y indecomposable modules in $\operatorname{mod} A$

$$\operatorname{irr}_A(X,Y) = \operatorname{rad}_A(X,Y)/\operatorname{rad}_A^2(X,Y)$$

the **space of irreducible homomorphisms** from X to Y

$\operatorname{irr}_A(X,Y)$ is an F_Y-F_X-bimodule by

$$(h+\operatorname{rad}_A(Y,Y))(f+\operatorname{rad}_A^2(X,Y)) = hf+\operatorname{rad}_A^2(X,Y)$$

$$(f+\operatorname{rad}_A^2(X,Y))(g+\operatorname{rad}_A(X,X)) = fg+\operatorname{rad}_A^2(X,Y)$$

for $f \in \operatorname{rad}_A(X,Y)$, $g \in \operatorname{End}_A(X)$, $h \in \operatorname{End}_A(Y)$

$$d_{XY} = \dim_{F_Y} \operatorname{irr}_A(X,Y)$$

$$d'_{XY} = \dim_{F_X} \operatorname{irr}_A(X,Y)$$

Γ_A **Auslander Reiten quiver** of A
valued translation quiver defined as follows:

- The vertices of Γ_A are the isoclasses $\{X\}$ of indecomposable modules X in $\operatorname{mod} A$
- For two vertices $\{X\}$ and $\{Y\}$, there exists an arrow $\{X\} \longrightarrow \{Y\}$ iff $\operatorname{irr}_A(X, Y) \neq 0$. Then we have in Γ_A the valued arrow

$$\{X\} \xrightarrow{(d_{XY}, d'_{XY})} \{Y\}$$

- τ_A translation of Γ_A defined on each nonprojective vertex $\{X\}$ of Γ_A by

$$\tau_A\{X\} = \{\tau_A X\} = \{D \operatorname{Tr} X\}$$

- τ_A^{-1} translation of Γ_A defined on each noninjective vertex $\{X\}$ of Γ_A by

$$\tau_A^{-1}\{X\} = \{\tau_A^{-1} X\} = \{\operatorname{Tr} DX\}$$

Tr - **transpose**, D - **standard duality**

We identify a vertex $\{X\}$ of Γ_A with the indecomposable module X and write

$X \xrightarrow{(d_{XY}, d'_{XY})} Y$ instead of $\{X\} \xrightarrow{(d_{XY}, d'_{XY})} \{Y\}$

and $X \longrightarrow Y$ instead of $X \xrightarrow{(1,1)} Y$

X, Y indecomposable modules in $\operatorname{mod} A$ (vertices of Γ_A)

$d_{XY} =$ multiplicity of Y in the codomain of a minimal left almost split homomorphism in $\operatorname{mod} A$ with the domain X

$$X \xrightarrow{f} M = Y^{d_{XY}} \oplus M'$$

M' without direct summand isomorphic to Y

Remarks.

(1) We know that there exists a minimal left almost split homomorphism $X \to M$ in $\operatorname{mod} A$

- if X is injective, then by Lemma 3.5 we have $X \to X/\operatorname{soc} X = M$
- if X is not injective, then by Theorem 3.12 (4) and Theorem 4.6 (2) we have almost split sequence

$$0 \longrightarrow X \xrightarrow{f} M \longrightarrow \tau_A^{-1} X \longrightarrow 0,$$

where f is minimal left almost split.

(2) By Theorem 3.9, for any irreducible homomorphism $X \to Y^m$ we get $m \leq d_{XY}$.

$d'_{XY} =$ multiplicity of X in the domain of a minimal right almost split homomorphism in $\operatorname{mod} A$ with the codomain Y

$$N' \oplus X^{d'_{XY}} = N \xrightarrow{g} Y$$

N' without direct summand isomorphic to X

Remarks.

(1) We know that there exists a minimal right almost split homomorphism $N \to Y$ in $\operatorname{mod} A$

- if Y is projective, then by Lemma 3.4 we have $N = \operatorname{rad} Y \to Y$
- if Y is not projective, then by Theorem 3.12 (5) and Theorem 4.6 (1) we have almost split sequence

$$0 \longrightarrow \tau_A Y \xrightarrow{f} N \longrightarrow Y \longrightarrow 0,$$

where f is minimal left almost split.

(2) By Theorem 3.9, for any irreducible homomorphism $X^n \to Y$ we get $n \le d'_{XY}$.

Proposition 5.1. *Let X, Y be indecomposable modules in* $\operatorname{mod} A$*, and assume that there exists an irreducible homomorphism from X to Y. Then*

(1) If Y is nonprojective, then $d'_{\tau_A Y, X} = d_{XY}$.

(2) If X is noninjective, then $d_{Y, \tau_A^{-1} X} = d'_{XY}$.

Remarks. Assume $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ is an arrow in Γ_A

(1) If A is an algebra of finite representation type, then

$$d_{XY} = 1 \text{ or } d'_{XY} = 1.$$

(2) If A is an algebra over an algebraically closed field K, then

$$d_{XY} = d'_{XY} \text{ (because } F_X \cong K \cong F_Y\text{).}$$

In particular, $d_{XY} = d'_{XY} = 1$ if A is of finite representation type.

Component of Γ_A = connected component of the quiver Γ_A

Shapes of components of Γ_A give important information on A and $\operatorname{mod} A$

Δ locally finite valued quiver without loops and multiple arrows

Δ_0 set of vertices of Δ

Δ_1 set of arrows of Δ

$d, d' : \Delta_1 \to \Delta_0$ the valuation maps

$$x \xrightarrow{(d_{xy}, d'_{xy})} y$$

$\mathbb{Z}\Delta$ valued translation quiver

$(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0 = \left\{(i,x) \,\middle|\, i \in \mathbb{Z}, x \in \Delta_0\right\}$ set of vertices of $\mathbb{Z}\Delta$.

$(\mathbb{Z}\Delta)_1$ set of arrows of $\mathbb{Z}\Delta$ consists of the valued arrows

$$(i,x) \xrightarrow{(d_{xy}, d'_{xy})} (i,y), \quad (i+1,y) \xrightarrow{(d'_{xy}, d_{xy})} (i,x),$$

$i \in \mathbb{Z}$, for all arrows $x \xrightarrow{(d_{xy}, d'_{xy})} y$ in Δ_1.

The translation $\tau : \mathbb{Z}\Delta_0 \to \mathbb{Z}\Delta_0$ is defined by

$$\tau(i,x) = (i+1,x) \text{ for all } i \in \mathbb{Z},\ x \in \Delta_0.$$

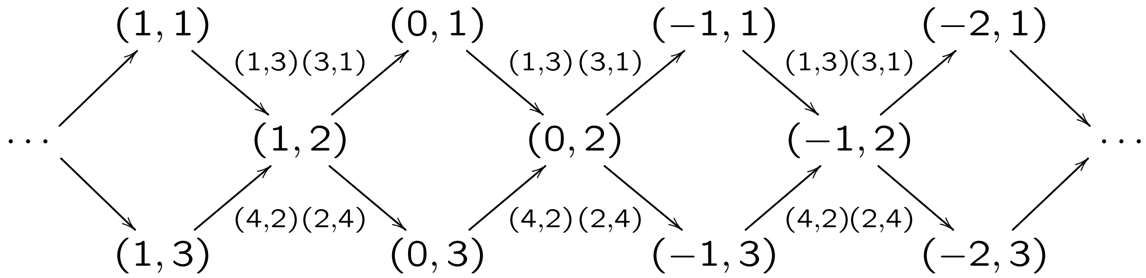
$\mathbb{Z}\Delta$ **stable valued translation quiver**

For a subset I of $\mathbb{Z}$, $I\Delta$ is the full translation subquiver of $\mathbb{Z}\Delta$ given by the set of vertices $(I\Delta)_0 = I \times \Delta_0$.

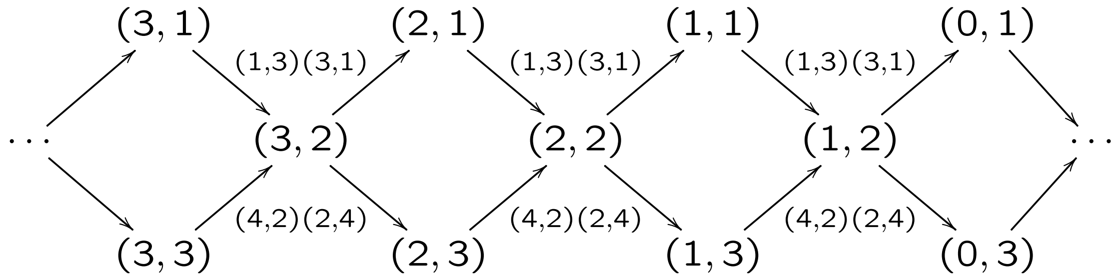
In particular, we have the valued translation subquivers $\mathbb{N}\Delta$ and $(-\mathbb{N})\Delta$ of $\mathbb{Z}\Delta$.

Example 5.2. Let $\Delta :\quad 1 \xrightarrow{(1,3)} 2 \xleftarrow{(4,2)} 3$

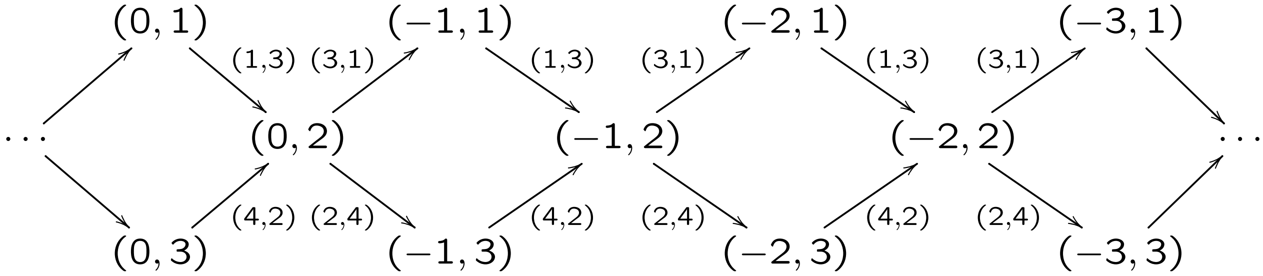
$\mathbb{Z}\Delta$ of the form



$\mathbb{N}\Delta$ of the form

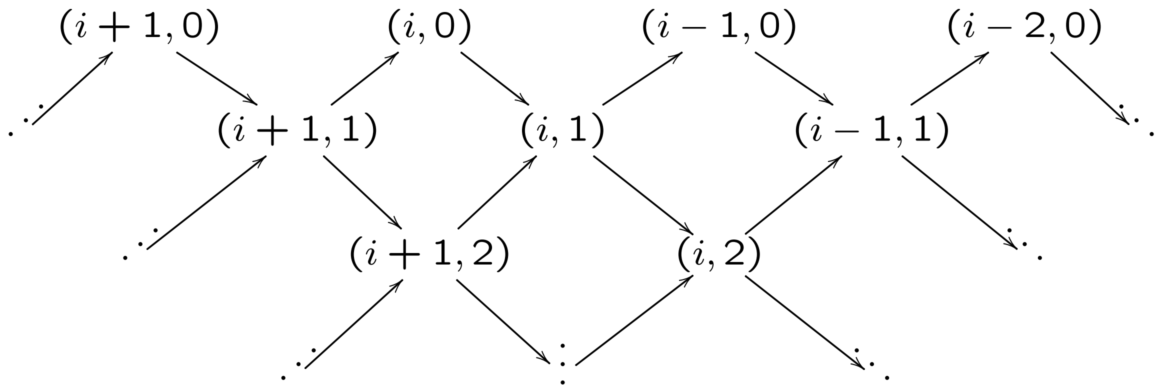


$(-\mathbb{N})\Delta$ of the form



$$\mathbb{A}_\infty : \qquad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \dots$$

$\mathbb{Z}\mathbb{A}_\infty$ is the translation quiver



$\tau(i,j) = (i+1, j)$ for all $i \in \mathbb{Z}$, $j \in \mathbb{N}$.

For $r \geq 1$, we may consider the translation quiver

$$\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$$

obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each vertex x with $\tau^r x$ and each arrow $x \to y$ with $\tau^r x \to \tau^r y$.

$\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ **stable tube of rank** r.

A algebra

$\mathscr{C}$ component of Γ_A is **regular** if $\mathscr{C}$ contains neither a projective module nor an injective module (equivalently, τ_A and τ_A^{-1} are defined on all vertices of $\mathscr{C}$)

Theorem 5.3. **(Liu, Zhang)** *Let $\mathscr{C}$ be a regular component of Γ_A. The following equivalences hold.*

(1) $\mathscr{C}$ contains an oriented cycle iff $\mathscr{C}$ is a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some $r \geq 1$.

(2) $\mathscr{C}$ is acyclic iff $\mathscr{C}$ is of the form $\mathbb{Z}\Delta$ for a connected, locally finite, acyclic, valued quiver Δ.

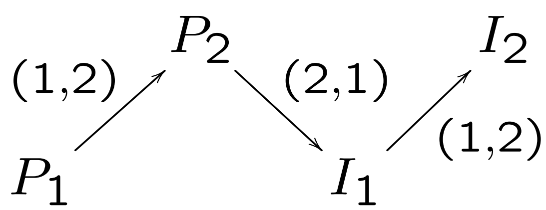
A component $\mathscr{C}$ of Γ_A is **postprojective** if $\mathscr{C}$ is acyclic and each module in $\mathscr{C}$ is of the form $\tau_A^{-m}P$ for a projective module P in $\mathscr{C}$ and some $m \geq 0$.

A component $\mathscr{C}$ of Γ_A is **preinjective** if $\mathscr{C}$ is acyclic and each module in $\mathscr{C}$ is of the form $\tau_A^{m}I$ for an injective module I in $\mathscr{C}$ and some $m \geq 0$.

Example 5.4. Let

$$A = \begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{C} & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ c & b \end{bmatrix} \in \mathbb{M}_2(\mathbb{C}) \mid a \in \mathbb{R}, b, c \in \mathbb{C} \right\}$$

Then A is 5-dimensional $\mathbb{R}$-algebra of finite representation type and Γ_A is of the form



where $e_1 = \begin{bmatrix} 1_{\mathbb{R}} & 0 \\ 0 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathbb{C}} \end{bmatrix}$,

$P_1 = e_1 A$, $P_2 = e_2 A$, $I_1 = D(Ae_1) = \mathrm{Hom}_{\mathbb{R}}(Ae_1, \mathbb{R})$,

$I_2 = D(Ae_2) = \mathrm{Hom}_{\mathbb{R}}(Ae_2, \mathbb{R})$,

$$P_1 = \begin{bmatrix} \mathbb{R} & 0 \\ 0 & 0 \end{bmatrix} = S_1 = e_1 A / e_1 \operatorname{rad} A$$

simple projective

$(\operatorname{rad} A = \begin{bmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{bmatrix}$, $A / \operatorname{rad} A \cong \mathbb{R} \times \mathbb{C})$

$$P_2 = \begin{bmatrix} 0 & 0 \\ \mathbb{C} & \mathbb{C} \end{bmatrix}$$

$$S_2 = e_2A/e_2\,\mathrm{rad}\,A = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{C} \end{bmatrix} \cong \mathbb{C}$$

- we have the valued arrow $P_1 \xrightarrow{(d_{P_1P_2}, d'_{P_1P_2})} P_2$

- $d_{P_1P_2}$ multiplicity of P_2 in a minimal left almost split homomorphism $P_1 \to M$

One can show that we have an almost split sequence $0 \to P_1 \to P_2 \to I_1 \to 0$.

Therefore, $d_{P_1P_2} = 1$.

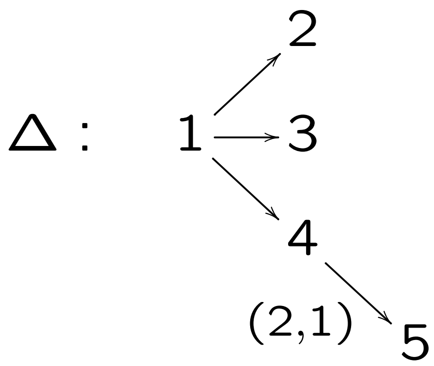
- $d'_{P_1P_2}$ multiplicity of P_1 in a minimal right almost split homomorphism $N \to P_2$

$$N = \mathrm{rad}\,P_2 = \begin{bmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mathbb{R} \oplus \mathbb{R}i & 0 \end{bmatrix} \cong P_1 \oplus P_1$$

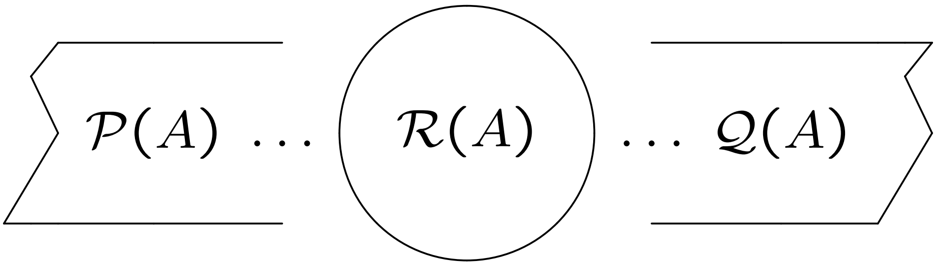
Therefore, $d'_{P_1P_2} = 2$.

- we get the valued arrow $P_1 \xrightarrow{(1,2)} P_2$

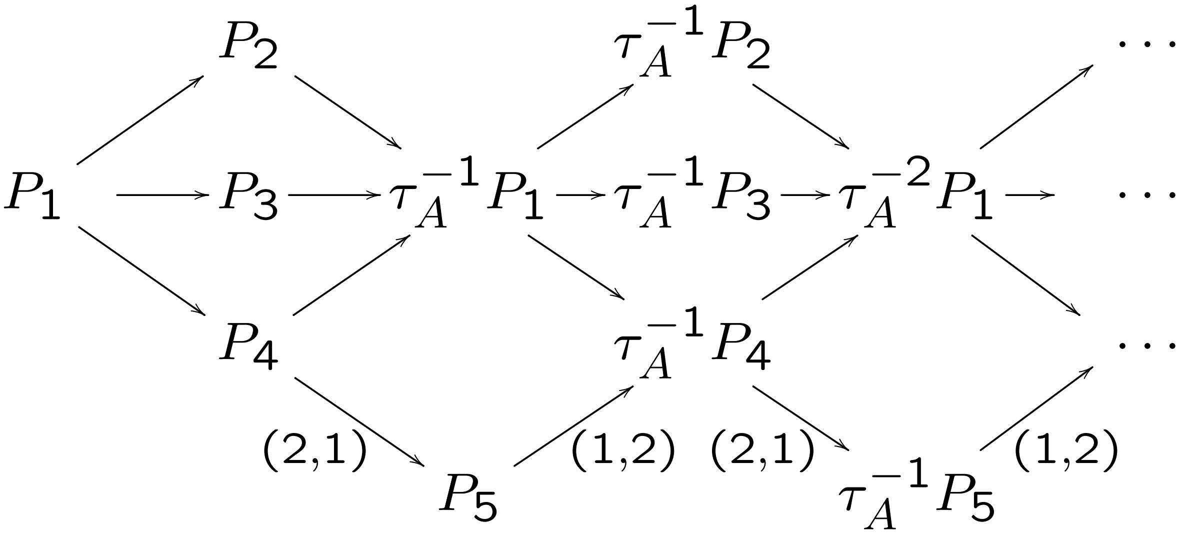
Example 5.5. Let $A = \begin{bmatrix} \mathbb{C} & 0 & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & 0 & \mathbb{C} & 0 & 0 \\ \mathbb{C} & 0 & 0 & \mathbb{C} & 0 \\ \mathbb{C} & 0 & 0 & \mathbb{C} & \mathbb{R} \end{bmatrix}$



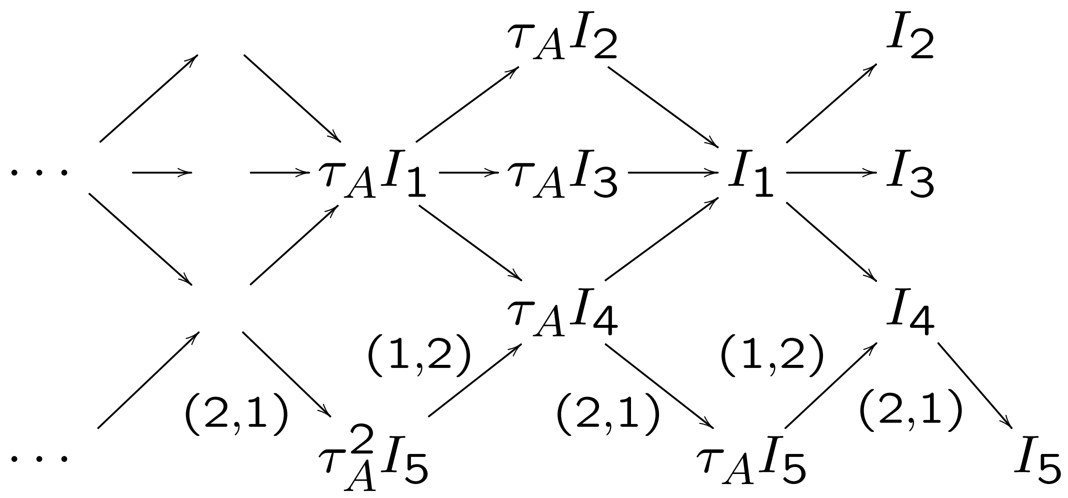
Then Γ_A:



where $\mathcal{P}(A) = (-\mathbb{N})\Delta$ is a postprojective component containing all indecomposable projective A-modules of the form



$\mathcal{Q}(A) = \mathbb{N}\Delta$ is a preinjective component containing all indecomposable injective A-modules of the form



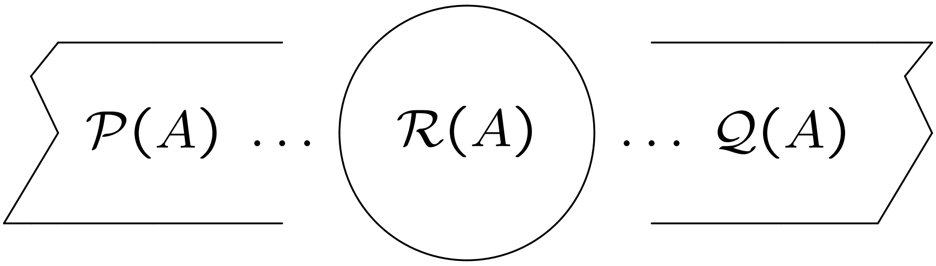
$\mathcal{R}(A)$ is a family of all regular components containing

- one stable tube of rank 3
- one stable tube of rank 2
- infinitely many stable tubes of ranks 1

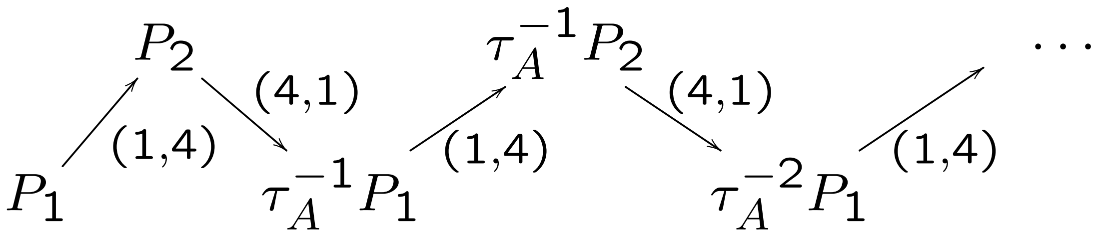
Example 5.6. Let $A = \begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{H} \end{bmatrix}$.

$$\Delta : \qquad 1 \xrightarrow{(1,4)} 2$$

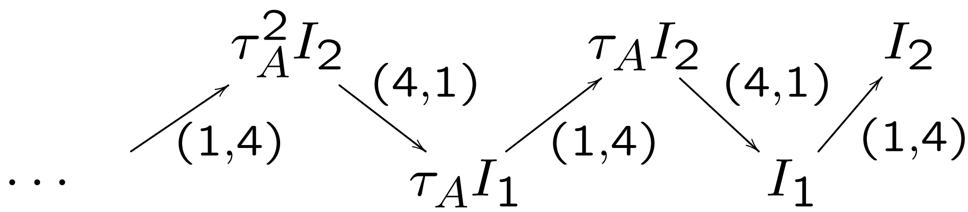
Then Γ_A:



$\mathcal{P}(A) = (-\mathbb{N})\Delta$ is a postprojective comp. containing all indecomp. proj. A-modules of the form



$\mathcal{Q}(A) = \mathbb{N}\Delta$ is a preinjective comp. containing all indecomp. inj. A-modules of the form



$\mathcal{R}(A)$ is a family of all regular components containing infinitely many stable tubes of ranks 1.

6. Hereditary algebras

A finite dimensional K-algebra over a field K

A is **right hereditary** if any right ideal of A is a projective right A-module

A is **left hereditary** if any left ideal of A is a projective left A-module

Theorem 6.1. *TFAE*

(1) $\operatorname{gl.dim} A \leq 1$.

(2) A is right hereditary.

(3) A is left hereditary.

(4) Every right A-submodule of a projective module in $\operatorname{mod} A$ *is projective.*

(5) Every factor module of an injective module in $\operatorname{mod} A$ *is injective.*

(6) The radical $\operatorname{rad} P$ *of any indecomposable projective module P in* $\operatorname{mod} A$ *is projective.*

(7) The socle factor $I/\operatorname{soc} I$ *of any indecomposable injective module I in* $\operatorname{mod} A$ *is injective.*

A is **hereditary** if A is left and right hereditary

Q_A **valued quiver** of A

$1, 2, \dots, n$ vertices of Q_A

there is an arrow $i \longrightarrow j$ in Q_A if $\dim_K \mathrm{Ext}^1_A(S_i, S_j) \neq 0$, and has the valuation

$$(\dim_{\mathrm{End}_A(S_j)} \mathrm{Ext}^1_A(S_i, S_j), \dim_{\mathrm{End}_A(S_i)} \mathrm{Ext}^1_A(S_i, S_j))$$

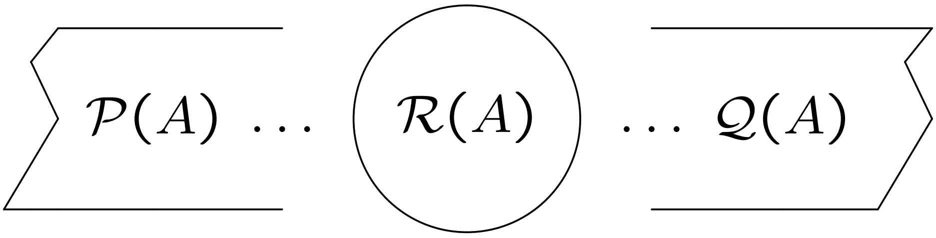
$\mathrm{End}_A(S_1), \mathrm{End}_A(S_2), \dots, \mathrm{End}_A(S_n)$ are **division** K-algebras

$G_A = \bar{Q}_A$ (underlying graph of Q_A)
valued graph of A

A hereditary K-algebra

- A is of **Dynkin type** if G_A is a Dynkin graph
- A is of **Euclidean type** if G_A is an Euclidean graph
- A is of **wild type** if G_A is neither a Dynkin nor Euclidean graph

Theorem 6.2. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K, and Q_A the valued quiver of A. Then Γ_A has the following shape*



- *$\mathcal{P}(A)$ is the postprojective component containing all indecomposable projective A-modules*
- *$Q(A)$ is the preinjective component containing all indecomposable injective A-modules*
- *$\mathcal{R}(A)$ is the family of all regular components*

Moreover,

(1) If A is of Dynkin type, then $\mathcal{P}(A) = \mathcal{Q}(A)$ is finite and $\mathcal{R}(A)$ is empty.

(2) If A is of Euclidean type, then $\mathcal{P}(A) \cong (-\mathbb{N})Q_A^{\text{op}}$, $\mathcal{Q}(A) \cong \mathbb{N}Q_A^{\text{op}}$ and $\mathcal{R}(A)$ is an infinite family of stable tubes, all but finitely many of them of rank one.

(3) If A is of wild type, then $\mathcal{P}(A) \cong (-\mathbb{N})Q_A^{\text{op}}$, $\mathcal{Q}(A) \cong \mathbb{N}Q_A^{\text{op}}$, and $\mathcal{R}(A)$ is an infinite family of components of type $\mathbb{Z}\mathbb{A}_\infty$.

7. The number of terms in the middle of almost split sequences

A algebra, $M \in \operatorname{mod} A$

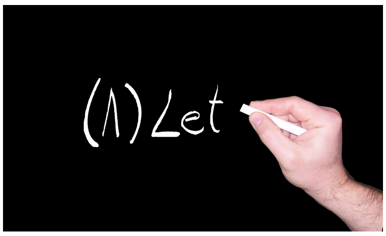
$\ell(M)$ length of M
(length of a composition series
$0 = M_0 \subset M_1 \subset \ldots \subset M_{\ell(M)} = M$,
M_{i+1}/M_i simple, $i \in \{0, \ldots, \ell(M) - 1\}$)

A path $X_1 \to X_2 \to \cdots \to X_{t-1} \to X_t$ in Γ_A is called **sectional** if for each $i \in \{1, \ldots, t-2\}$ we have $X_i \not\cong \tau_A X_{i+2}$.

Lemma 7.1. *Let*

$$0 \longrightarrow X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \longrightarrow 0$$

be an almost split sequence in $\operatorname{mod} A$, $Y_1, \ldots, Y_r$ *indecomposable modules, and* $\ell(Y_i) < \ell(X)$ *for any* $i \in \{1, \ldots, r\}$. *Then any sectional path in* Γ_A *ending with* Z *does not contain projective module.*

Proof. 

Proposition 7.2. *Let $f : X \to \bigoplus_{i=1}^{4} Y_i$ be an irreducible homomorphism in* $\operatorname{mod} A$, *X indecomposable, and $Y_1, \ldots, Y_4$ indecomposable nonprojectives. Let $\sum_{i=1}^{4} \ell(Y_i) \leq 2\ell(X)$. Then X has no projective predecessor in Γ_A.*

Corollary 7.3. *Let $f : X \to \bigoplus_{i=1}^{4} Y_i$ be an irreducible epimorphism in* $\operatorname{mod} A$, *X indecomposable, and $Y_1, \ldots, Y_4$ indecomposable nonprojectives. Then X has no projective predecessor in Γ_A.*

We have dual facts ...

Lemma 7.4. *Let*

$$0 \longrightarrow X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \longrightarrow 0$$

be an almost split sequence in $\operatorname{mod} A$, $Y_1, \ldots, Y_r$ *indecomposable modules, and* $\ell(Y_i) < \ell(Z)$ *for any* $i \in \{1, \ldots, r\}$. *Then any sectional path in* Γ_A *starting at* X *does not contain injective module.*

Proposition 7.5. *Let* $g : \bigoplus_{i=1}^{4} Y_i \to Z$ *be an irreducible homomorphism in* $\operatorname{mod} A$, Z *indecomposable, and* $Y_1, \ldots, Y_4$ *indecomposable noninjectives. Let* $\sum_{i=1}^{4} \ell(Y_i) \leq 2\ell(Z)$. *Then* Z *has no injective successor in* Γ_A.

Corollary 7.6. *Let* $g : \bigoplus_{i=1}^{4} Y_i \to Z$ *be an irreducible monomorphism in* $\operatorname{mod} A$, Z *indecomposable, and* $Y_1, \ldots, Y_4$ *indecomposable noninjectives. Then* Z *has no injective successor in* Γ_A.

Theorem 7.7. (Liu) *Let*

$$0 \longrightarrow X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \longrightarrow 0$$

be an almost split sequence in $\operatorname{mod} A$*, and* $Y_1, \dots, Y_r$ *indecomposable modules. Assume that* X *has a projective predecessor in* Γ_A *and* Z *has an injective successor in* Γ_A*. Then* $r \leq 4$*, and* $r = 4$ *implies that* Y_i *is projective-injective for some* $i \in \{1, \dots, 4\}$ *and* Y_j *is not projective-injective for any* $j \in \{1, \dots, 4\} \setminus \{i\}$*.*

Corollary 7.8. (Bautista-Brenner) *Let* A *an algebra of finite representation type, and*

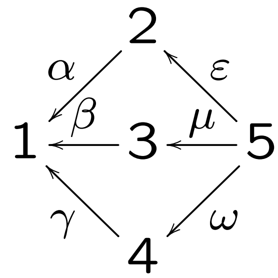
$$0 \longrightarrow X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \longrightarrow 0$$

be an almost split sequence in $\operatorname{mod} A$*, where* Y_i *is indecomposable for any* $i \in \{1, \dots, r\}$*. Then* $r \leq 4$*, and* $r = 4$ *implies that one of the* Y_i *is projective-injective.*

Remark. If A is of finite representation type, then any indecomposable module has a projective predecessor and an injective successor in Γ_A.

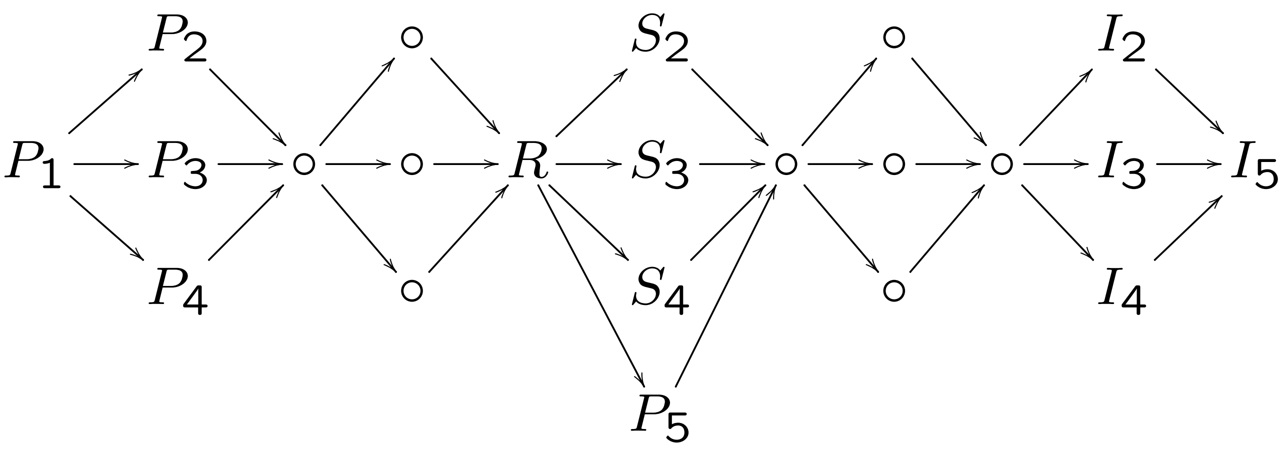
Example 7.9. Let $A = KQ/I$, where

Q :



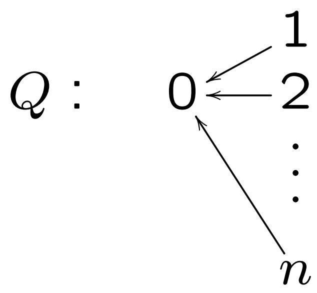
and I is the ideal of KQ generated by $\varepsilon\alpha - \mu\beta$, $\mu\beta - \omega\gamma$.

Then A is of finite representation type and Γ_A is of the form

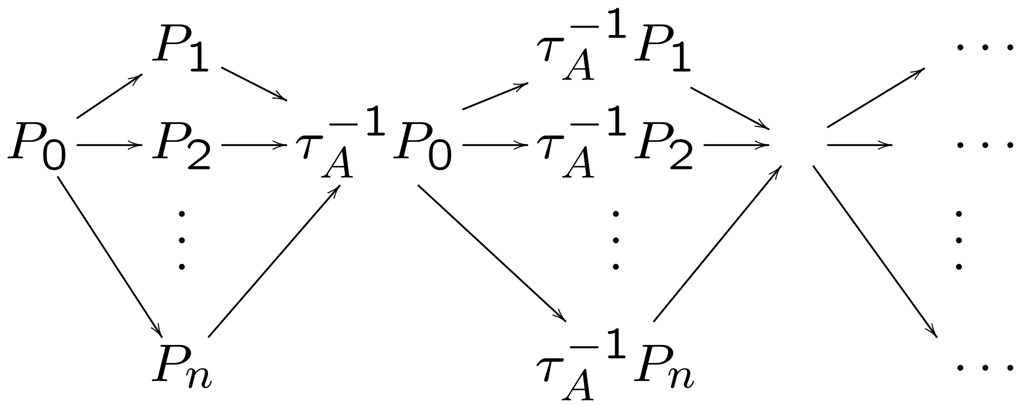


where $P_1 = S_1$, $R = \operatorname{rad} P_5$, $P_5 = I_1$, $I_5 = S_5$.

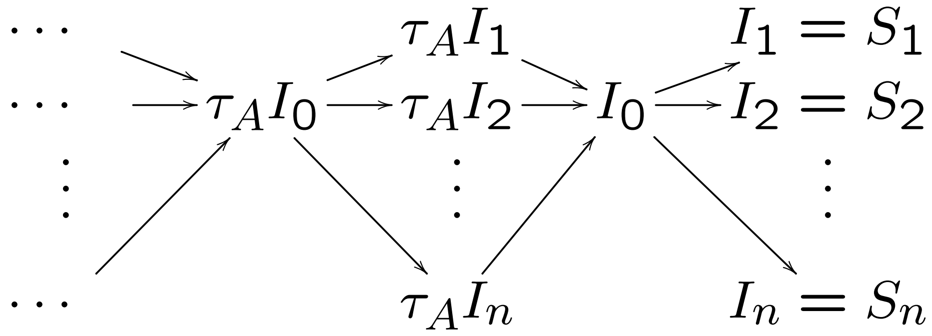
Example 7.10. Let $A = KQ$, where



Then Γ_A contains a component $\mathcal{P}(A)$ of the form



and a component $\mathcal{Q}(A)$ of the form



Conjecture 7.11. (Brenner-Butler) *Let A be a tame finite dimensional K-algebra over an algebraically closed field K,*

$$0 \longrightarrow X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \longrightarrow 0$$

an almost split sequence in $\operatorname{mod} A$, *and X indecomposable nonprojective module. Then $r \leq 5$.*

We give the affirmative answer in the case of cycle-finite algebras.

Let A be a finite dimensional K-algebra.

A **cycle** of indecomposable modules in $\operatorname{mod} A$ is a sequence

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{r-1} \xrightarrow{f_r} X_r = X_0$$

of nonzero nonisomorphisms in $\operatorname{mod} A$, where X_i is indecomposable for $i \in \{1, \ldots, r\}$, and such a cycle is said to be **finite** if the homomorphisms $f_1, \ldots, f_r$ do not belong to $\operatorname{rad}_A^{\infty}$.

An algebra A is said to be **cycle-finite** if all cycles between indecomposable modules in $\operatorname{mod} A$ are finite.

Theorem 7.12. (M.-de la Peña-Skowroński) *Let A be a cycle-finite K-algebra,*

$$0 \longrightarrow X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \longrightarrow 0$$

an almost split sequence between indecomposable modules in $\operatorname{mod} A$, and X nonprojective module. Then $r \leq 5$, and $r = 5$ implies that Y_i is projective-injective for some $i \in \{1, \dots, 5\}$ and Y_j is not projective-injective for any $j \in \{1, \dots, 5\} \setminus \{i\}$.

Remark. For finite dimensional cycle-finite algebras over an algebraically closed field Theorem 7.12 has been proved by **de la Peña and Takane**, by application of spectral properties of Coxeter transformations of algebras and **Liu's** results.