Almost periodic patterns

Pierre-Antoine Guihéneuf and Yves Meyer

Abstract. Two new definitions of almost periodic patterns are discussed with applications to quasicrystals. This note complements *Quasicrystals, almost periodic patterns, mean periodic functions and irregular sampling,* African Diaspora Journal of Mathematics, Volume 13, Number 1 (2012) 1-45.

1. Introduction

Are quasicrystals almost periodic patterns? This issue was already addressed in [2]. Here, as it was the case in [2], quasicrystals are defined as *regular model sets.*

Almost periodic patterns will be defined by a property which is entirely different from the one used in [2]. As in [2] a discrete point set $\Lambda \subset \mathbb{R}^n$ is an almost periodic pattern if and only if the corresponding sum of Dirac masses $\mu = \sum_{\lambda\in\Lambda} \delta_\lambda$ is an almost periodic measure. But our new definitions of almost periodic measures (Definition 6, Section 2, and Definition 10, Section 6) differ from the one used in [2] and are based on Hermann Weyl's seminal work which is summarized in this introduction.

A continuous function f defined on $\mathbb{R}^n$ is almost periodic in the sense of Bohr if it is the limit for the L^∞ norm of a sequence $P_j = \sum_{\omega\in F_j} c(j,\omega)\exp(i\omega\cdot x)$ of generalized trigonometric polynomials. Hermann Weyl replaced the L^∞ norm by a suitable L^p norm, $p\in[1,\infty)$, to define new spaces of almost periodic functions. In what follows $B(x,R)\subset\mathbb{R}^n$ is the ball of radius R centered at x and $|B(x,R)| = c_n R^n$ denotes its volume.

Definition 1. *The Weyl norm $\|f\|_{w,p}$, $p\in[1,\infty)$, of a function $f\in L^p_{loc}(\mathbb{R}^n)$ is defined by*

$$\|f\|_{w,p} = \limsup_{R\to\infty} \Big[\sup_{x\in\mathbb{R}^n} \Big(\frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)|^p \, dy \Big)^{1/p} \Big] \tag{1.1}$$

whenever the right hand side of (1.1) *is finite.*

The space $\mathcal{W}_p$ of almost periodic functions in the sense of H. Weyl is defined as follows.

1991 *Mathematics Subject Classification.* Primary ; Secondary.
Key words and phrases. Quasicrystals, Almost periodic measures.
P-A. G. acknowledges support from CNRS, France.
YM acknowledges support from CNRS, France.

Definition 2. *If $p \in [1, \infty)$ a function $f \in L^p_{loc}(\mathbb{R}^n)$ belongs to $\mathcal{W}_p$ if it is the limit for the norm $\| \cdot \|_{w,p}$ of a sequence of generalized trigonometric polynomials.*

Besicovitch used a less demanding norm defined as

$$\|f\|_{b,p} = \limsup_{R\to\infty} \big(\frac{1}{|B(0,R)|}\int_{B(0,R)} |f(y)|^p\,dy\big)^{1/p} \tag{1.2}$$

Definition 3. *A function $f \in L^p_{loc}(\mathbb{R}^n)$ is almost periodic in the sense of Besicovitch if it is the limit for the norm $\|\cdot\|_{b,p}$ of a sequence of generalized trigonometric polynomials. We then write $f \in \mathcal{B}_p$.*

The Besicovitch space $\mathcal{B}_p$ is larger than the Weyl space $\mathcal{W}_p$. For example we have

Proposition 1. *If $1 \leq p < \infty$ the function $f(x)$ of the real variable x defined by*

$$f(x) = \sum_1^\infty \frac{1}{n}\sin(2^{-n}x) \tag{1.3}$$

belongs to $\mathcal{B}_p$ but not to $\mathcal{W}_p$.

One easily checks that f is well defined and belongs to $\mathcal{C}^\infty$. Let us prove that f belongs to Besicovitch space $\mathcal{B}_p$. For simplifying the discussion let us assume $p = 2$. Let us consider $R_N(x) = f(x) - \sum_1^N \frac{1}{m}\sin(2^{-m}x)$ and

$$\epsilon_N = \limsup_{R\to\infty} R^{-1}\int_0^R |R_N(x)|^2\,dx \tag{1.4}$$

We shall prove that ϵ_N tends to 0 as N tends to infinity. Let us define an integer M by $R \in [2\pi.2^M, 2\pi.2^{M+1}]$ and split $[0, 2\pi.2^{M+1}]$ into $[0, 2\pi.2^N] \bigcup_{N+1}^M J_m$ where $J_m = [2\pi.2^m, 2\pi.2^{m+1}]$. It can be assumed that $M \geq N$ in what follows. We obviously have $R_N(x) = \sum_{N+1}^\infty \frac{1}{m}\sin(2^{-m}x)$ which together with $|\sin x| \leq |x|$ implies

$$|R_N(x)| \leq 2^{-N}\frac{|x|}{N} \tag{1.5}$$

It yields $\int_0^{2\pi.2^N} |R_N(x)|^2\,dx \leq N^{-2}4^N$ but this bound does not matter since it will be erased by the factor R^{-1} as R tends to infinity in (1.4). Let $\eta_N(m) = \int_{2\pi.2^m}^{2\pi.2^{m+1}} |R_N(x)|^2\,dx$. We then have

$$\int_{2\pi.2^N}^R |R_N(x)|^2\,dx \leq \eta_N(N) + \cdots + \eta_N(M) \tag{1.6}$$

If $x \in J_m$ we decompose $R_N(x) = \sum_{N+1}^\infty \frac{1}{j}\sin(2^{-j}x)$ into $R_N(x) = \sum_{N+1}^m(\cdot) + R_m(x)$. When $N+1 \leq j \leq m$ the sine functions $\sin(2^{-j}x)$ are orthogonal on the interval J_m. We also have by (1.5) $|R_m(x)| \leq C\frac{1}{m}$ if $x \in J_m$ which finally leads to $\eta_N(m) \simeq 2^m m^{-1}$. We have proved that $\sum_{N+1}^M \eta_N(m) \simeq 2^M(N+1)^{-1}$ which yields $\epsilon_N \simeq (N+1)^{-1}$ as announced.

For proving the second statement in Proposition 1 we observe that all the derivatives of f are almost periodic functions in the sense of Bohr. Let us argue by contradiction and assume that f belongs to $\mathcal{W}_p$. Then $\|f \star \phi\|_\infty \le C$ when ϕ is a compactly supported test function. If $\int \phi = 1$ then $\|f - f \star \phi\|_\infty \le \|\frac{d}{dx} f\|_\infty \le C$. Finally $f \in \mathcal{W}_p \Rightarrow f \in L^\infty$. To disprove this conclusion one computes $f(\frac{2\pi}{7} \cdot 2^q)$ for a large integer q. Taking in account that $\sin(\frac{2\pi}{7}) + \sin(\frac{4\pi}{7}) + \sin(\frac{8\pi}{7}) > 0$ one easily obtains $f(\frac{2\pi}{7} \cdot 2^q) \ge c \log q$ for a positive constant c.

A subset $M \subset \mathbb{R}^n$ is *relatively dense* if there exists a positive R such that each ball $B(x, R)$ with radius R (whatever be its center x) contains at least a point x in M. This definition was introduced by Besicovitch. If $f \in \mathcal{W}_p$ for every $\epsilon > 0$ the set $M(\epsilon)$ of $\tau \in \mathbb{R}^n$ such that $\|f(\cdot + \tau) - f(\cdot)\|_{w,p} \le \epsilon$ is relatively dense. However this property does not characterize the space $\mathcal{W}_p$ and the Heaviside function is a counter example.

2. Almost periodic measures

Following Laurent Schwartz a Borel measure μ on $\mathbb{R}^n$ is almost periodic if and only if for every compactly supported continuous function g the convolution product $\mu \star g$ is an almost periodic function. Everything depends now on the definition of an almost periodic function which is adopted. If we consider the standard definition given by Bohr, the corresponding definition of almost periodic measures is too demanding and does not cover the case of quasicrystals. *Generalized almost periodic functions* where introduced in [2] to address this issue. A real valued function f is a generalized almost periodic function if for every positive ϵ there exist two standard almost periodic functions u_ϵ and v_ϵ such that $u_\epsilon \le f \le v_\epsilon$ and $\|v_\epsilon - u_\epsilon\|_{w,1} \le \epsilon$ and almost periodic measures are defined accordingly. In the last section of this note the space of generalized almost periodic functions will be replaced by the Weyl space $\mathcal{W}_1$ and almost periodic measures will be defined by $\mu \star g \in \mathcal{W}_1$ for every compactly supported continuous function g. The Weyl space is larger than the space of generalized almost periodic functions. A completely distinct definition of almost periodic measures is given now (Definition 5). This definition does not use any smoothing and is motivated by Robert Moody's seminal work [3].

Definition 4. *The Weyl norm of a Borel measure μ on $\mathbb{R}^n$ is defined by*

$$\|\mu\|_w = \limsup_{R\to\infty} \sup_{x\in\mathbb{R}^n} |B(x,R)|^{-1} |\mu|(B(x,R)) \tag{2.1}$$

whenever the right hand side of (2.1) *is finite.*

If $\mu = f\,dx$ where f is locally integrable then $\|\mu\|_w = \|f\|_{w,1}$.

For $\tau \in \mathbb{R}^n$ we denote by μ_τ the measure μ translated by τ. Almost periodic measures are defined as follows.

Definition 5. *A Borel measure μ is almost periodic in the sense of Weyl if for every positive ϵ there exists a relatively dense set $M(\epsilon) \subset \mathbb{R}^n$ such that*

$$\tau \in M(\epsilon) \Rightarrow \|\mu_\tau - \mu\|_w \le \epsilon \tag{2.2}$$

This definition of almost periodic measures differs from the one given in [2]. Here is a simple example. The Dirac comb σ on $\mathbb{Z}$ is almost periodic and the same holds for the Dirac comb σ' on $\sqrt{2}\mathbb{Z}$. Let $\mu = \sigma + \sigma'$. Then μ cannot be almost periodic in the sense given by Definition 5 since $\|\mu_\tau - \mu\|_w \ge 2$ if $\tau \ne 0$. However μ is an almost periodic measure in the strongest sense. For every compactly supported continuous function g the convolution product $\mu \star g$ is a standard almost periodic function ($\mu \star g$ being the sum of two periodic functions). It now seems that our Definition 5 is stronger than the standard one or the one given in [2]. It is not the case and here is a counter example. If $\lambda_k = k^2 + \sqrt{2}k,\ k \in \mathbb{Z}$, then the measure $\sigma = \sum_{-\infty}^{+\infty} \delta_{\lambda_k}$ is not a generalized almost periodic measure in the sense given in [2] (see Proposition 2.27 in [2]). However σ trivially satisfies the requirement of Definition 5 since $\|\sigma\|_w = \|\sigma_\tau\|_w = 0$.

Let us study another one dimensional example.

Proposition 2. *Let r_k, $k \in \mathbb{Z}$, be a sequence of real numbers tending to* 0. *Then the perturbed Dirac comb $\sigma = \sum_{k\in\mathbb{Z}} \delta_{k+r_k}$ is an almost periodic measure if and only if the upper density of the set $E = \{k;\, r_k \neq 0\}$ is* 0.

Let us prove Proposition 2. If $\tau \notin \mathbb{Z}$ then $k + r_k \neq j + r_j + \tau$ for $|k| \geq k_0$ which implies $\|\sigma_\tau - \sigma\|_w \geq 2$. Then $\|\sigma_\tau - \sigma\|_w \leq 1$ implies $\tau \in \mathbb{Z}$. Finally the conclusion follows immediately from Definition 5.

Definition 6. *A Borel measure μ is uniformly almost periodic if for every positive ϵ there exists a relatively dense set $M(\epsilon)$ and a positive number $R(\epsilon)$ such that*

$$\tau \in M(\epsilon),\ x \in \mathbb{R}^n,\ R \geq R(\epsilon) \Rightarrow |B(x,R)|^{-1} \int_{B(x,R)} d\,|\mu_\tau - \mu| \leq \epsilon \tag{2.3}$$

The sum $\sigma = \sum_{k=1}^{\infty} \delta_k$ is the right half of the Dirac comb. It is an example of an almost periodic measure which is not uniformly almost periodic. Here $M(\epsilon) = \mathbb{Z}$ and $\sigma_\tau - \sigma = -\sum_{k=1}^{\tau} \delta_k$ if $\tau \geq 1$. When $\tau \leq -1$ we have $\sigma_\tau - \sigma = \sum_{k=\tau+1}^{0} \delta_k$. Therefore

$$\|\sigma_\tau - \sigma\|_w = 0 \quad (\forall \tau \in \mathbb{Z}) \tag{2.4}$$

Therefore σ is almost periodic in the sense of Weyl. It is not uniformly almost periodic. Indeed (2.3) implies $R \geq |\tau|/\epsilon$. Uniformity with respect to τ makes the difference between (2.2) and (2.3).

3. Almost periodic patterns

A subset Λ of $\mathbb{R}^n$ is a Delone set if there exist two radii $R_2 > R_1 > 0$ such that

(a) each ball with radius R_1, whatever be its location, shall contain at most one point in Λ
(b) each ball with radius R_2, whatever be its location, shall contain at least one point in Λ.

Almost periodic patterns are defined as follows.

Definition 7. *A Delone set Λ is an almost periodic pattern if the sum of Dirac masses $\mu = \sum_{\lambda\in\Lambda} \delta_\lambda$ is a uniformly almost periodic measure (in the sense given by Definition* 6*).*

Definition 8. *If Λ is a discrete set, $R > 0$, and if*

$$D_R^+(\Lambda) = \sup_{x\in\mathbb{R}^n} |B(x,R)|^{-1} \#[(B(x,R)) \cap \Lambda] \tag{3.1}$$

The uniform upper density of Λ *is defined by*

$$D^+(\Lambda) = \limsup_{R\to\infty} D_R^+(\Lambda) \tag{3.2}$$

We then have

Lemma 1. *Let* Λ *be a Delone set and let* μ *be the sum of Dirac masses* $\mu = \sum_{\lambda\in\Lambda} \delta_\lambda$. *Then* $\|\mu\|_w = D^+(\Lambda)$.

The symmetric difference between $A, B \subset \mathbb{R}^n$ is denoted by $A \bigtriangleup B$. An equivalent definition of an almost periodic pattern is given by the following lemma.

Lemma 2. *A Delone set* Λ *is an almost periodic pattern if and only if for every positive* ϵ *there exists a* $R(\epsilon) > 0$ *and a relatively dense set* $M(\epsilon)$ *such that*

$$R \geq R(\epsilon),\ \tau \in M(\epsilon) \Rightarrow D_R^+[(\Lambda+\tau) \bigtriangleup \Lambda] \leq \epsilon \tag{3.3}$$

This implies the weaker property

$$D^+[(\Lambda+\tau) \bigtriangleup \Lambda] \leq \epsilon \tag{3.4}$$

Robert Moody introduced an even weaker property in [3] where the uniform upper density is replaced by

$$d(\Lambda) = \limsup_{R\to\infty}\ |B(0,R)|^{-1} \#[(B(0,R)) \cap \Lambda] \tag{3.5}$$

and (3.4) by

$$d[(\Lambda+\tau) \bigtriangleup \Lambda] \leq \epsilon \tag{3.6}$$

Here is an example of a set of integers which is almost periodic if the definition given by Robert Moody is used but is not an almost periodic pattern. Let $E \subset \mathbb{Z}$ be the union of the intervals $[2^j, 2^j + j],\ j \geq 1$, and let $\Lambda = \mathbb{Z} \setminus E$. For every τ we have $d[(\Lambda+\tau) \bigtriangleup \Lambda] = 0$. However Λ is not an almost periodic pattern. We argue by contradiction and assume that for every $\epsilon > 0$ there exists a relatively dense set $M(\epsilon)$ of almost periods of Λ. Therefore for every j there exists a $\tau \in M(\epsilon)$ such that $|\tau - j| \leq C(\epsilon)$. Then $[2^j, 2^j + j]$ is disjoint from $[2^j, 2^j + j] + \tau$ up to an interval of length less than $2C(\epsilon)$. It implies $D^+[(\Lambda+\tau) \bigtriangleup \Lambda] \geq 1 - 2C(\epsilon)/j$ and the expected contradiction will be reached when $1 - 2C(\epsilon)/j > \epsilon$.

4. Regular model sets are almost periodic patterns

A regular model set Λ is defined by the cut and projection method. Let us assume that $\Gamma \subset \mathbb{R}^{n+m}$ is a lattice. Then $p_1 : \mathbb{R}^{n+m} \mapsto \mathbb{R}^n$ is defined by $p_1(x, y) = x$ when $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and similarly $p_2(x, y) = y$. We are assuming that $p_1 : \Gamma \mapsto \mathbb{R}^n$ is injective and that $p_2 : \Gamma \mapsto \mathbb{R}^m$ has a dense range. Let $W \subset \mathbb{R}^m$ be a Riemann integrable compact set.

Definition 9. *With these notations the regular model set* Λ *is defined by*

$$\Lambda = \{\lambda = p_1(\gamma);\ \ \gamma \in \Gamma,\ \ p_2(\gamma) \in W\} \tag{4.1}$$

We know that Λ has a uniform density given by $\operatorname{dens}\Lambda = D^+(\Lambda) = c|W|$ where $c = c(\Gamma)$ and $|W|$ is the Lebesgue measure of W. Keeping in mind Definitions 6 and 7 we have

THEOREM 1. *Regular model sets are almost periodic patterns.*

In other terms for every positive ϵ there exists a Delone set $M(\epsilon)$ such that (3.3) holds for every $\tau \in M(\epsilon)$ and for every $R \geq R(\epsilon)$. Most of the points in the model set Λ also belong to $\tau + \Lambda$ (this set of "good points" in Λ depends on τ).

The proof is not difficult. Let $N(\eta)$ the model set defined by

$$N(\eta) = \{x = p_1(\gamma);\ \ \gamma \in \Gamma,\ \ |p_2(\gamma)| \leq \eta\} \tag{4.2}$$

Then we shall prove that $N(\eta)$ is the $M(\epsilon)$ we are looking for if η is small enough. More precisely we have

LEMMA 3. *There exists a regular model set $Q(\epsilon)$ such that $D^+(Q(\epsilon)) \leq \epsilon$ and*

$$\tau \in N(\eta) \Rightarrow (\Lambda + \tau) \bigtriangleup \Lambda \subset Q(\epsilon) \tag{4.3}$$

Lemma 3 obviously implies Theorem 1.

Let us treat the set $\Lambda + \tau \setminus \Lambda$ when $\tau \in N(\eta)$. The treatment of $\Lambda \setminus \Lambda + \tau$ will be similar. If $x \in \Lambda + \tau$ we have $x = p_1(\gamma) + p_1(\gamma_0) = p_1(\gamma + \gamma_0)$ where $\gamma,\, \gamma_0 \in \Gamma,\ p_2(\gamma) \in W$ and $|p_2(\gamma_0| \leq \eta$. If $x \notin \Lambda$ we have $p_2(\gamma + \gamma_0) \notin W$. It implies that $p_2(\gamma + \gamma_0) \in W_\eta$ where $W_\eta \subset W$ is defined as the set of points $y \notin W$ such that the distance from y to the boundary of W does not exceed η. We have proved the following

$$\Lambda + \tau \setminus \Lambda \subset Q_\eta = \{\lambda = p_1(y);\ \ y \in \Gamma,\ \ p_2(y) \in W_\eta\} \tag{4.4}$$

Let us stress that the model set Q_η does not depend on τ. We now observe that $|W_\eta|$ tends to 0 as η tends to 0. The uniform density of the model set Q_η defined by the window W_η does not exceed ϵ if η is small enough and we then set $Q(\epsilon) = Q_\eta$. As it was said the treatment of $\Lambda \setminus \Lambda + \tau$ is similar, W_η being replaced by the set W^η of points $y \in W$ such that the distance from y to the boundary of W does not exceed η. This ends the proof.

Is the converse implication true ? Let us assume that a Delone set Λ is an almost periodic pattern. Is $\Lambda - \Lambda$ a Delone set ? Here is a one dimensional counter example.

LEMMA 4. *Let $\Lambda = \cup_0^\infty \Lambda_j$ where $\Lambda_j = 2^j + r_j + 2^{j+1}\mathbb{Z}$. If $r_j \in (0, 1/3]$ then Λ is an almost periodic pattern. If $r_j \in (0, 1/3]$ tends to 0 as j tends to infinity then $\Lambda - \Lambda$ cannot be a Delone set.*

Let us observe that $2^j + 2^{j+1}\mathbb{Z} = 2^j\mathbb{Z} \setminus 2^{j+1}\mathbb{Z}$ which implies that Λ is a Delone set. Moreover Λ is an almost periodic pattern since $\Lambda_0 \cup \ldots \cup \Lambda_{j-1}$ is 2^j-periodic and the uniform upper density of $\Lambda_j \cup \ldots$ is 2^{-j}. Let us directly check that $\Lambda - \Lambda$ is not a Delone set. We have $2^j \in 2\mathbb{Z} = \Lambda_0 - \Lambda_0$. Moreover $2^j + r_j + 2^k \in \Lambda_j$ if $k \geq j + 1$. Finally $r_k + 2^k \in \Lambda_k$ which implies $2^j + r_j - r_k \in \Lambda - \Lambda$. But 2^j also belongs to $\Lambda - \Lambda$. Therefore $\Lambda - \Lambda$ cannot be a Delone set since $r_j - r_k$, $k \geq j + 1$, tends to 0 as j tends to infinity. Let us observe that Λ is also an almost periodic pattern if the definition given in [2] is adopted.

5. Large patches of model sets

Let $B(x, R) \subset \mathbb{R}^n$ be the ball centered at x with radius R. Patches of Λ are defined as $\mathcal{P}(x, R) = \Lambda \cap B(x, R),\ x \in \mathbb{R}^n,\ R > 0$. Finally $\#E$ denotes the number of elements of E. Then we have

THEOREM 2. *Let Λ be an almost periodic pattern. For every positive ϵ there exists a positive number $R(\epsilon)$ such that the following property holds:*

$\forall x,\, \forall y \in \mathbb{R}^n$ *there exists a translation* $\tau \in \mathbb{R}^n,\ \tau = \tau(x, y, \epsilon)$, *such that:*

$$\forall R \geq R(\epsilon), \quad \#[\mathcal{P}(x,R) \triangle (\mathcal{P}(y,R) - \tau)] \leq \epsilon R^n \tag{5.1}$$

This property was discovered by the first author. We do not know if (5.1) characterizes almost periodic patterns. The proof of Theorem 2 gives more. Indeed there exists a relatively dense Delone set $M(\epsilon)$ such that one can impose $\tau = \tau(x, y, \epsilon) \in M(\epsilon)$ in (5.1). This improved statement is then a characterization of almost periodic patterns.

Property (5.1) is obvious if $|x - y| \leq R_0$ and $R \geq R_0(C_n\epsilon)^{-1}$. Then (5.1) holds with $\tau = 0$. Indeed we then have $|B(y,R) \triangle B(x,R)| \leq C_n|x-y|R^{n-1} \leq \epsilon R^n$. It yields (5.1) since Λ is a Delone set. It is the trivial case. Property (5.1) is only relevant if the distance between x and y is extremely large.

Theorem 2 in its full generality follows easily from the trivial case and from Lemma 2. We obviously have

$$B(y,R) \cap [(\Lambda + \tau) \triangle \Lambda] = \mathcal{P}(y,R) \triangle (\mathcal{P}(y-\tau,R) + \tau) \tag{5.2}$$

Therefore Λ is an almost periodic pattern if and only if there exists a relatively dense Delone set $M(\epsilon)$ such that

$$y \in \mathbb{R}^n,\ R \geq R(\epsilon),\ \tau \in M(\epsilon) \Rightarrow \frac{\#[\mathcal{P}(y,R) \triangle (\mathcal{P}(y-\tau,R)+\tau)]}{|B(y,R)|} \leq \epsilon \tag{5.3}$$

This is a restatement of Lemma 2 and it settles the case of the two patches $\mathcal{P}(y,R)$ and $\mathcal{P}(y-\tau,R)$. To compare $\mathcal{P}(y,R)$ to $\mathcal{P}(x,R)$ it suffices to observe that there exists a $\tau \in M(\epsilon)$ such that $|y - x - \tau| \leq C(\epsilon)$. We are finally led to compare $\mathcal{P}(x,R)$ to $\mathcal{P}(y-\tau,R)$ which is the trivial case.

6. Returning to Weyl spaces

The definition of an almost periodic measure is now modified.

DEFINITION 10. *A Borel measure μ is almost periodic if for for every compactly supported continuous function g the convolution product $\mu \star g$ belongs to the Weyl space $\mathcal{W}_1$.*

The definition of an almost periodic pattern is modified accordingly by demanding that the sum of Dirac masses $\mu = \sum_{\lambda\in\Lambda} \delta_\lambda$ be a uniformly almost periodic measure in the new sense. The set Λ constructed in Lemma 4 is an almost periodic pattern. Other examples are provided by the following result.

PROPOSITION 3. *Regular model sets are almost periodic patterns in the sense given by Definition 10.*

As it was said in the Introduction the space of generalized almost periodic functions is contained in the larger space $\mathcal{W}_1$. Therefore Proposition 3 is implied by Theorem 3.3 in [2].

References

[1] Jeffrey C. Lagarias *Geometric Models for Quasicrystals I. Delone Sets of Finite Type*. Discrete & Computational Geometry 21 (1999) 161-191.

[2] Yves Meyer *Quasicrystals, almost periodic patterns, mean periodic functions and irregular sampling.* African Diaspora Journal of Mathematics Volume 13, Number 1, (2012) 1-45.

[3] Robert V. Moody *Mathematical Quasicrystals: a tale of two topologies.* Proceedings of the International Congress of Mathematical Physics, Lisbon (2003).

Département de Mathématique, Université Paris-Sud, 91405 Orsay, France
E-mail address: Pierre-Antoine.Guiheneuf@math.u-psud.fr

CMLA, ENS-Cachan, 94235 Cachan Cedex, France
E-mail address: yves.meyer@cmla.ens-cachan.fr