Periodic points for generic conservative homeomorphisms

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The goal of this short note is to describe the set of periodic points for a generic conservative homeomorphism of a compact topological manifold.

We consider a smooth manifold X of dimension $n \geq 2$, equipped with a *good* (in the sense of Oxtoby-Ulam [OU41]) measure μ. We denote by $\mathrm{Homeo}(X,\mu)$ the space of homeomorphisms of X preserving the measure μ (that we will call *conservative* homeomorphisms. For a more precise description of the setting, see [Gui12].

Proposition 1. *For $f \in \mathrm{Homeo}(X)$, denote*

$$\mathrm{Per}_\tau(f) = \{x \in X \mid f^\tau(x) = x\}$$

the set of periodic points of period τ for f. Then, generically in $\mathrm{Homeo}(X,\mu)$, $\bigcup_{\tau\geq 0} \mathrm{Per}_\tau(f)$ is dense in X and has zero μ-measure. Moreover,

1. *for any $\tau \geq 1$, the set $Per_\tau(f)$ is either empty, either perfect (it has no isolated point);*
2. *if $\mathrm{Per}_\tau(f) \neq \emptyset$, then for any $\ell > 1$, the set $\mathrm{Per}_{\ell\tau}(f) \setminus \mathrm{Per}_\tau(f)$ is nonempty and contains $\mathrm{Per}_\tau(f)$ in its closure.*

Proof. The fact that $\bigcup_{\tau\geq 0} \mathrm{Per}_\tau(f)$ is dense in X is proved in [Gui12, Section 3.2] using the local modification technique (see for example [Gui15] for an English version of this method).

To prove that the set of periodic points is generically of zero measure, it suffices to use Birkhoff ergodic theorem together with the fact that a generic element of $\mathrm{Homeo}(X,\mu)$ is ergodic (see [OU41]).

We now prove point 1. We will only treat the case of fixed points ($\tau = 1$): generically in $\mathrm{Homeo}(X,\mu)$, there exists a residual set $R \subset \mathrm{Homeo}(X,\mu)$ on which either $\mathrm{Per}_1(f) = \mathrm{Fix}(f)$ is empty, or it is a Cantor set of zero Hausdorff dimension. The proof in the general case $\tau > 1$ is identical.

Let F be the set of $f \in \mathrm{Homeo}(X,\mu)$ having at least one fixed point. By an easy compactness argument, the set F is closed in $\mathrm{Homeo}(X,\mu)$. Let $f \in F$. Then, $\mathrm{Fix}(f)$ is a nonempty compact set.

Let us now show that generically, $\mathrm{Fix}(f)$ is perfect (*i.e.* it has no isolated point). Let $(U_k)_{k\in\mathbf{N}}$ be a basis of the topology of X. For $M \in \mathbf{N}$, let E_M the set of $f \in \mathrm{Homeo}(X,\mu)$ satisfying the two following properties:

(i) $\exists m \geq M \mid \mathrm{Card}(\mathrm{Fix}(f) \cap U_m) \geq 2$;

(ii) $\forall m \leq M$, $\mathrm{Card}(\mathrm{Fix}(f) \cap U_m) \geq 1 \implies \mathrm{Card}(\mathrm{Fix}(f) \cap U_m) \geq 2$.

Let us show that the interior of E_M is dense in F for any $M \geq 0$. Let $f \in F, m \in \mathbf{N}$ and $\delta > 0$. As $f \in F$, it possesses at least one fixed point $p \in X$. As $(U_k)_{k \in \mathbf{N}}$ is a basis of the topology, there exists $m \geq M$ such that $p \in U_m$. By local perturbation (Théorème 3.2 of [Gui12]), it is possible to perturb f so that it coincides with $\pm\mathrm{Id}$ in a neighbourhood of p. We then make another perturbation to make two of the created fixed points persistent (Définition 3.6.of [Gui12]); this gives $g \in \mathrm{Homeo}(X, \mu)$ which is δ-close to f and with two fixed points in U_m. This treats the case of (i). To treat (ii), if suffices to remark that this construction can be made each time $\mathrm{Fix}(f) \cap U_m \neq \emptyset$.

Let $E = \bigcap_{M \in \mathbf{N}} \mathrm{int}(E_M)$. Then E is a G_δ dense subset of F. One can easily prove that any $f \in E$ has an infinite number of fixed points and that none of them are isolated. This shows that generically, the set $\mathrm{Fix}(f)$ is either empty or perfect.

For point 2., consider the set

$$E = \bigcap_{\tau, \ell, m \in \mathbf{N}^*} \mathrm{int}(E_{\tau,\ell,m}).$$

with

$$E_{\tau,\ell,m} = \big\{ f \in \mathrm{Homeo}(X, \mu) \mid \mathrm{Per}_\tau(f) \cap U_m \neq \emptyset \implies (\mathrm{Per}_{\ell\tau}(f) \setminus \mathrm{Per}_\tau(f)) \cap U_m \neq \emptyset \big\}.$$

The set E is clearly a G_δ set, any element of which satisfying the conclusion of 2. It remains to prove that it is dense.

Fix $f \in \mathrm{Homeo}(X, \mu)$, $\delta > 0$ and $\tau, \ell, m \in \mathbf{N}^*$, we want to find $g \in \mathrm{Homeo}(X, \mu)$ with $d(f, g) < \delta$ and $g \in \mathrm{int}(E_{\tau,\ell,m})$.

There are two cases. First, if for any $g \in B(f, \delta)$, we have $\mathrm{Per}_\tau(g) \cap U_m = \emptyset$, then $f \in \mathrm{int}(E_{\tau,\ell,m})$ and there is nothing to prove. Otherwise, there exists $g \in B(f, \delta)$ and $p \in \mathrm{Per}_\tau(g) \cap U_m$. Consider $\rho > 0$ such that:

- $B(p, \rho) \subset U_m$;
- $B(p, \rho)$ is disjoint from the orbit $g(p), \dots, g^{\tau-1}(p)$;
- $\rho + d(f, g) < \delta$;

and $\varepsilon > 0$ such that $B(p, \varepsilon) \cup g^\tau(B(p, \varepsilon)) \subset B(p, \rho)$. Let $y_1, \dots, y_\ell \in B(p, \varepsilon)$ be ℓ different points, and for $i \in \mathbf{Z}/\ell\mathbf{Z}$, let $x_i = g^\tau(y_i)$. Then, applying the proposition of extension of finite maps (Proposition 2.7 of [Gui12]), one can find $\varphi \in \mathrm{Homeo}(X, \mu)$, supported in $B(p, \rho)$, such that for any $i \in \mathbf{Z}/\ell\mathbf{Z}$, we have $\varphi(x_i) = y_{i+1}$. Then, the map $h = \varphi \circ g$ has y_0 as a periodic point of period $\ell\tau$.

Note that to find such a periodic point, we could be tempted to apply the local modification technique to replace locally g around p by a rotation of order ℓ, however this does not work when f reverses the orientation.

Now, by the local modification technique [Gui12, Section 3.1], we make this periodic point y_0 stable; the resulting homeomorphism belongs to $\mathrm{int}(E_{\tau,\ell,m})$. $\square$

Proposition 2. *We suppose that $n = \dim(X) \neq 4$. Then, for a generic element in* $\mathrm{Homeo}(X, \mu)$ *which is isotopic to a diffeomorphism, for any $\tau \geq 1$, the set $Per_\tau(f)$ is either empty, either a Cantor set of zero Hausdorff dimension.*

Remark 3. It may happen that an adaptation of the arguments of [AHK03] or of Thom's transversality theorem [Tho54] leads to a similar statement without the hypotheses on the dimension and the isotopy class of a diffeomorphism.

Proof. As before, we will only treat the case of fixed points ($\tau = 1$): generically in $\mathrm{Homeo}(X, \mu)$, there exists a residual set $R \subset \mathrm{Homeo}(X, \mu)$ on which either $\mathrm{Per}_1(f) = \mathrm{Fix}(f)$ is empty, or it is a Cantor set of zero Hausdorff dimension. The proof in the general case $\tau > 1$ is identical.

As we have proved in Proposition 1 that for a generic element in $\mathrm{Homeo}(X, \mu)$, for any $\tau \geq 1$, the set $Per_\tau(f)$ is either empty, either perfect, it remains to prove that for a generic $f \in F$, the set $\mathrm{Fix}(f)$ has zero Hausdorff dimension and is totally disconnected. For $\varepsilon, s \in (0, 1)$, let $F_{\varepsilon,s}$ be the set of $f \in \mathrm{Homeo}(X, \mu)$ such that

$$\begin{cases} \exists B_1, \dots, B_k \text{ disjoint balls of diameters } \varepsilon' < \varepsilon \\ \text{such that } \ \mathrm{Fix}(f) \subset \bigcup_{1 \leq i \leq k} B_i \text{ and } k\varepsilon'^s < 1. \end{cases}$$

We now show that the sets $F_{\varepsilon,s}$ are open and dense in $\mathrm{Homeo}(X, \mu)$. Let $f \in \mathrm{Homeo}(X, \mu)$ and $\delta, \varepsilon, s \in (0, 1)$. We want to find $g \in \mathrm{int}(F_{\varepsilon,s})$ with $d(f, g) < \delta$.

This is where we use the hypothesis on the dimension: in [MÏ4], S. Müller shows the following theorem: If $n \leq 3$, then any conservative homeomorphism can be uniformly approximated by conservative diffeomorphisms. If $n \geq 5$, a conservative homeomorphismc an be uniformly approximated by conservative diffeomorphisms iff it is isotopic to a diffeomorphism.

Hence, it suffices first to approach f by a conservative diffeomorphism, and then to approach this diffeomorphism by a Kupka-Smale conservative diffeomorphism g: such a diffeomorphism has a finite number of fixed points (see [Pei67]) and is far from identity out of the (disjoint) balls of diameter $\varepsilon' < \varepsilon$ and centered on these fixed points, hence lies in $\mathrm{int}(F_{\varepsilon,s})$.

The set $\bigcap_{i,j>1} F_{1/i,1/j}$ is then a G_δ dense subset of $\mathrm{Homeo}(X, \mu)$, and any of its elements f is such that $\mathrm{Fix}(f)$ has all its connected components of diameter 0, and has Hausdorff dimension 0.

□

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