3 APPARENTLY UNRELATED PICTURES

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PICTURE 1

WHAT IS IT?

On the left: initial picture. On the right: picture obtained by rotating successively the first one 25 times with a consumer software, with angles chosen arbitrarily (the last rotation being performed so that the resulting image is in the correct orientation). We observe that these rotations induce a quite strong blur.

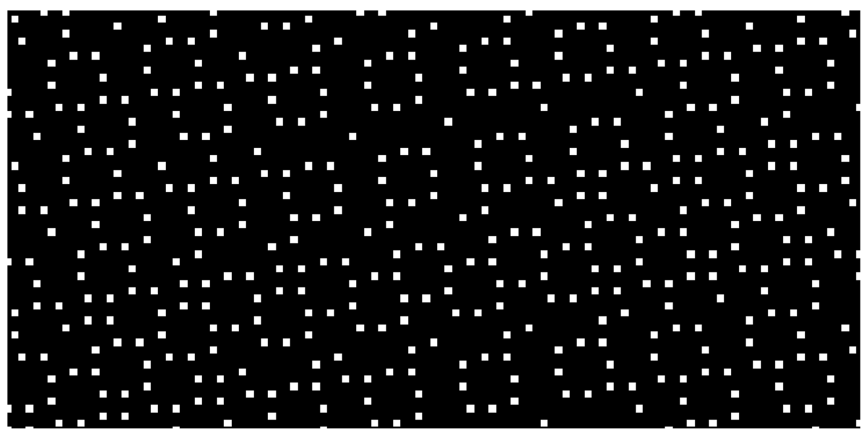
EXPLANATION

We rotate images made of pixels, thus we have to apply discretizations of the rotations.

The *discretization* $\widehat{A}$ of a linear map A is defined as

$$\widehat{A}: \begin{array}{ccc} \mathbf{Z}^n & \longrightarrow & \mathbf{Z}^n \\ x & \longmapsto & \pi(Ax), \end{array}$$

where $\pi : \mathbf{R}^n \to \mathbf{Z}^n$ is a projection on the nearest integer point.



Example of set $\widehat{A}(\mathbf{Z}^2)$.

Given a sequence $(A_k)_{k\geq 1}$ of matrices of $SL_n(\mathbf{R})$, we want to estimate the density of the sets $\Gamma_k = (\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^n)$, called the *rate of injectivity* (where $B_R := B(0,R)$):

$$\tau^k(A_1,\cdots,A_k) = \lim_{R\to+\infty} \frac{\operatorname{Card}\left(\Gamma_k \cap B_R\right)}{\operatorname{Card} \mathbf{Z}^n \cap B_R} \in]0,1].$$

We then have the following theorem:

Theorem 1 (2015, [1]). *For a generic sequence $(A_i)_i \in \ell^\infty(Sl_2(\mathbf{R}))$ (or of $\ell^\infty(O_2(\mathbf{R}))$), we have*

$$\lim_{k\to+\infty} \tau^k(A_1,\cdots,A_k) = 0.$$

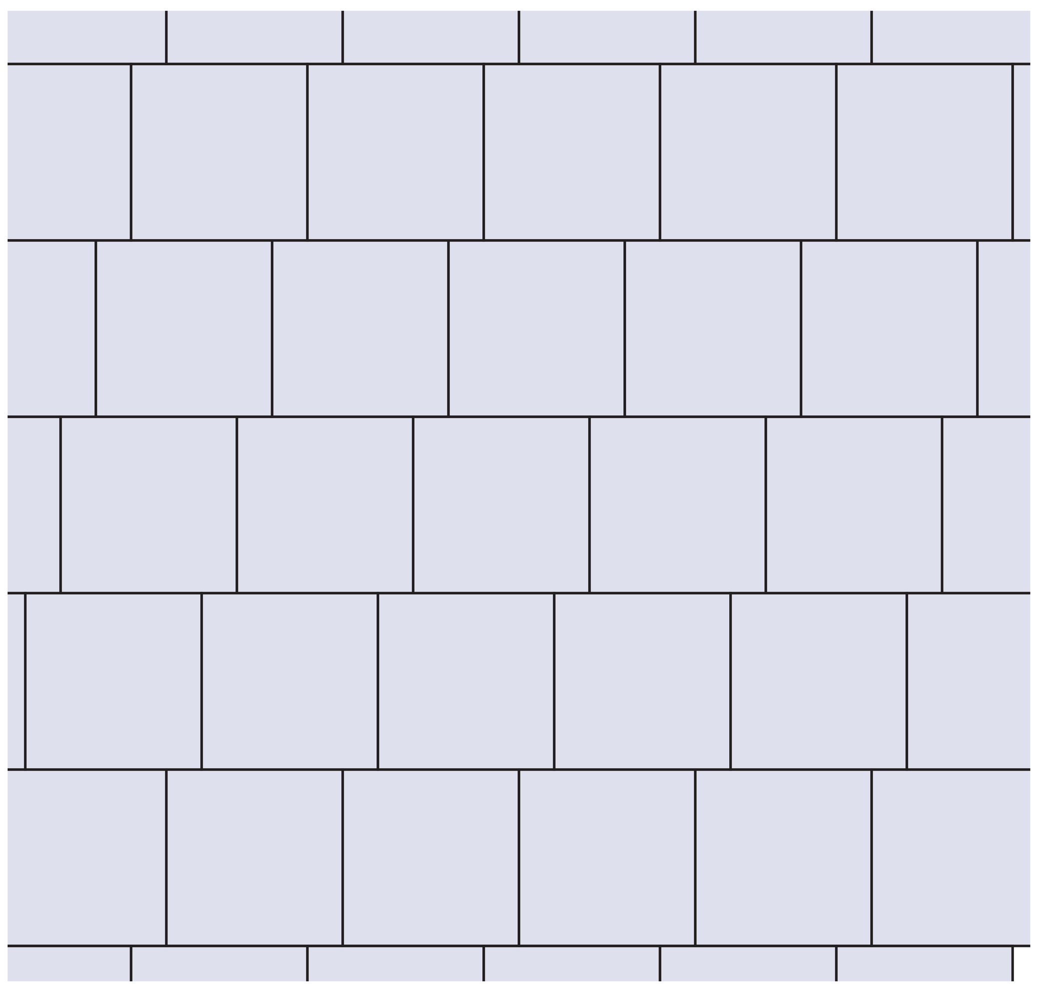
This expresses that in a certain sense, we can not avoid phenomenons like the blur of Picture 1 in the general case.

BIBLIOGRAPHY

[1] Pierre-Antoine Guihéneuf. *Discrétisations spatiales de systèmes dynamiques génériques*. PhD thesis, Université Paris-Sud, 2015.

[2] Sherman K. Stein and Sándor Szabó. *Algebra and tiling*, volume 25 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC, 1994.

PICTURE 2



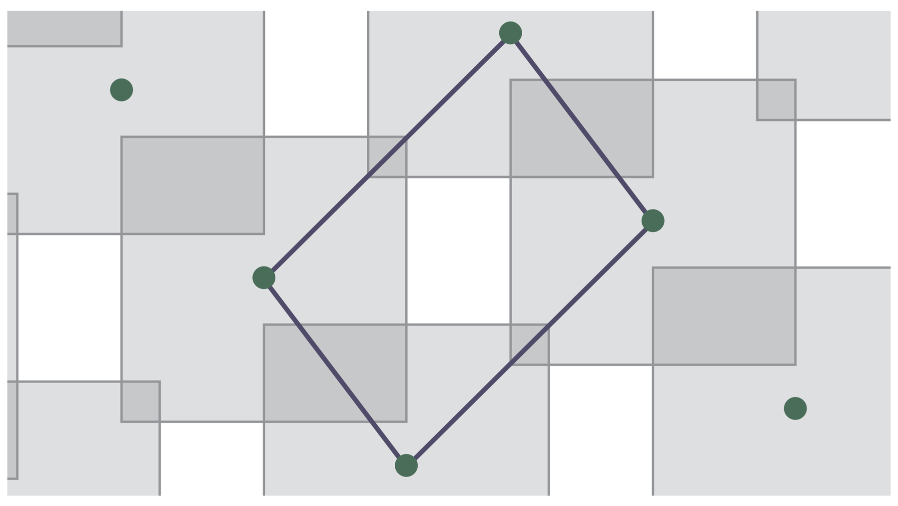
WHAT IS IT?

This is a tiling of the plane by squares. It can be easily shown that for every such tiling of the plane, any square has a common edge with another square. More generally, in any dimension:

Theorem 2 (Hajós, 1941). *Let Λ be a lattice of $\mathbf{R}^n$. Then the collection $\{B_\infty(\lambda, 1/2)\}_{\lambda\in\Lambda}$ of unit hypercubes centred on points of Λ tiles the plane if and only if in a canonical basis of $\mathbf{R}^n$ (that is, permuting coordinates if necessary), Λ admits a generating matrix which is upper triangular with ones on the diagonal.*

The proof of this theorem involves fine results of group theory (see [2]). Surprisingly, Hajós' theorem becomes wrong if we do not suppose that the centres of the cubes form a lattice of $\mathbf{R}^n$, as soon as $n \geq 8$ (but it remains true for $n \leq 6$, the case $n = 7$ is still open).

WHAT LINK WITH PICTURE 1?



It can be proved that if the matrix A is totally irrational (meaning that $A\mathbf{Z}^n$ is equidistributed mod $\mathbf{Z}^n$, which is a generic condition), then $\tau^1(A) = D\left(\bigcup_{\gamma\in A\mathbf{Z}^n} B_\infty(\gamma,1/2)\right)$ (where D denotes the density). So Hajós' theorem tells us when the equality $\tau^1(A) = 1$ holds.

The same construction holds for arbitrary times k: if we set the matrix of $M_{nk}(\mathbf{R})$

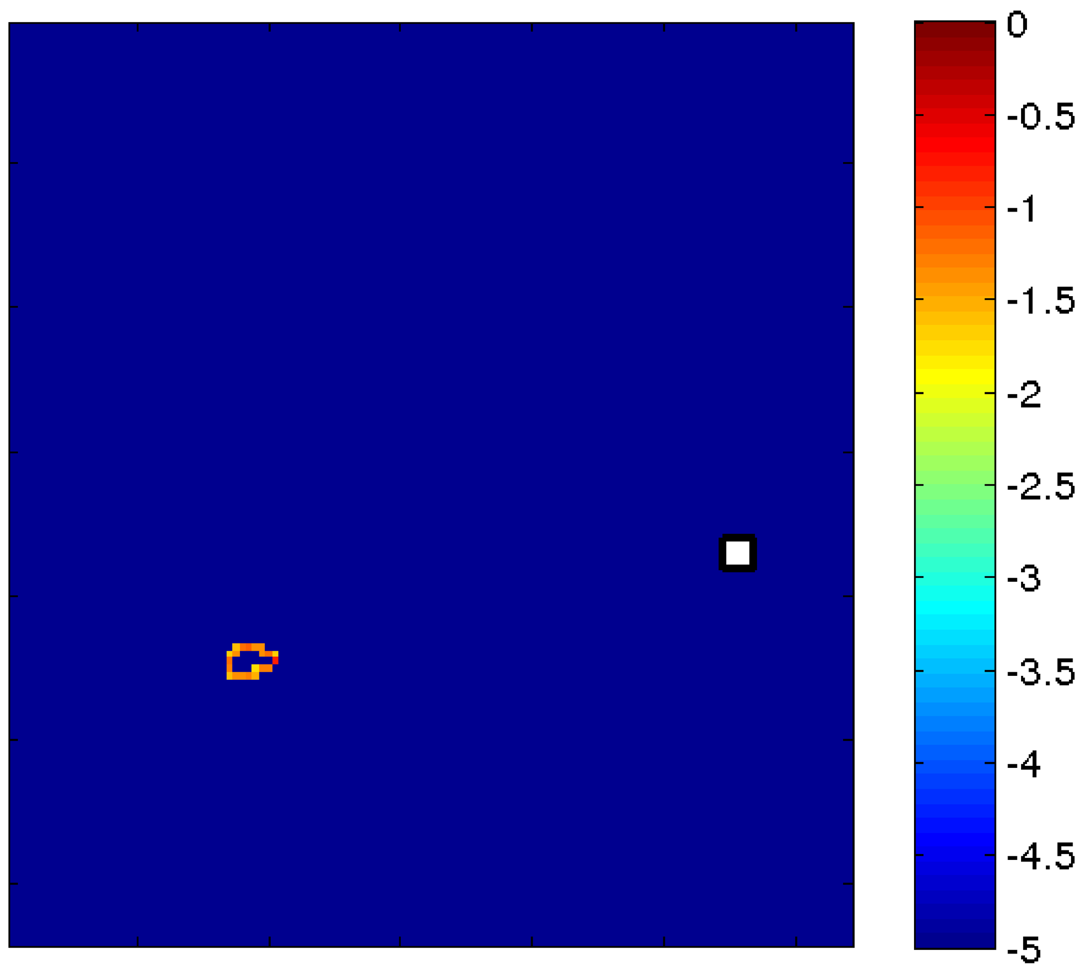
$$M_{A_1,\cdots,A_k} = \begin{pmatrix} A_1 & -\mathrm{Id} & & \\ & A_2 & \ddots & \\ & & \ddots & -\mathrm{Id} \\ & & & A_k \end{pmatrix},$$

then for a generic sequence of matrices $(A_k)_k$, we have

$$\tau^k(A_1,\cdots,A_k) = D\left(\bigcup_{\gamma\in M_{A_1,\cdots,A_k}\mathbf{Z}^{nk}} B_\infty(\gamma,1/2)\right).$$

This replaces the iteration by a passage in higher dimension. Then, Theorem 1 can be seen as a statement of concentration of the measure around the faces of a cube in high dimensions.

PICTURE 3



WHAT IS IT?

This picture represents the density of a computed invariant measure of a conservative C^1-diffeomorphism f of the torus $\mathbf{T}^2$, C^1-close to Id. This density is represented in logarithmic scale (a yellow pixel has measure $\simeq 10^{-2}$). The measure represented is of the form

$$\mu_x^{f_N} = \lim_{M\to+\infty} \frac{1}{M}\sum_{m=0}^{M-1} (f_N^m)^*\delta_x$$

with $N = 23$. Here f_N denotes the map iterated when the computer works with N decimal places (in other words, the computations have been made with 23 binary digits), and x is the point located on the small black and white box.

WHAT LINK WITH PICTURE 1?

We define *grids* on the torus $\mathbf{T}^n$:

$$E_N = \left\{ \left(i_k/2^N\right)_{1\leq k\leq n} \;\middle|\; 0 \leq i_k \leq 2^N - 1 \right\},$$

and a projection $P_N : \mathbf{T}^n \to E_N$ on the nearest point of the grid. The *discretization* of $f \in \operatorname{Diff}^1(\mathbf{T}^n)$ is then the map $f_N = P_N \circ f : E_N \to E_N$.

We want to compare the invariant measures of f with that of f_N for large N. The invariant measures of f_N are supported by its periodic points (f_N is a finite map). For the diffeomorphism itself, it is conjectured that a C^1-generic conservative diffeomorphism is ergodic; thus we can expect to observe mainly Lebesgue measure among the invariant measures of f_N. This is not what we observe in practice (see Picture 3), nor in theory:

Theorem 3 (2015, [1]). *For any point $x \in \mathbf{T}^n$, for a generic $f \in \operatorname{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, for every f-invariant probability measure μ, there exists a subsequence of discretizations $(f_{N_k})_k$ such that,*

$$\mu_x^{f_{N_k}} \underset{k\to+\infty}{\longrightarrow} \mu.$$

By applying some results of C^1-generic dynamics (Abdenur, Avila, Bonatti, Crovisier, Mañé, Wilkinson…), we can reduce the proof of Theorem 3 to that of a statement similar to Theorem 1: we first approximate the measure μ by a periodic measure ω, and then merge the positive orbits of x and ω under f_N by perturbing the sequence of derivatives on ω such that it has a small rate of injectivity.