What is the effect of spatial discretizations on dynamical systems?

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Abstract

We are interested in the dynamical behaviour of discretizations of a generic conservative homeomorphism of a compact manifold.

It turns out that the dynamics of the discretizations of such homeomorphims does not depend on the homeomorphim itself but rather on the order of the discretization.

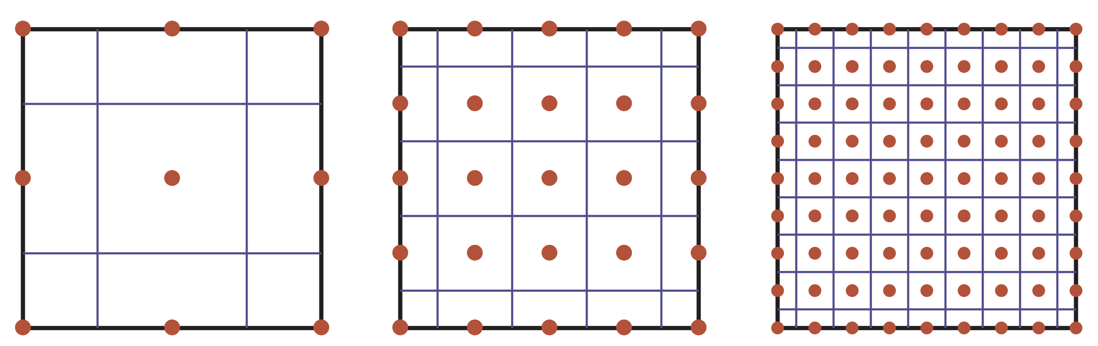
Problem

We study the link between the dynamical behaviour of a dynamical system and the dynamical behaviour of its numerical simulations: can some dynamical features be detected on numerical simulations?

A computer works with a finite number of decimal places. When one simulate a discrete-time dynamical system, numerical errors made at each iteration may add up, so that after a while the numerically calculated orbit of a point will have nothing in common with the actual one. Nevertheless, a numerically calculated orbit is close to an actual orbit at any time, thus one can hope that the collective behaviour of numerically calculated orbits provides informations about the collective behaviour of actual orbits.

The model

Spatial discretization is modeled by numerical truncation. Consider a discrete time dynamical system f whose phase space is the torus $\mathbf{T}^2$ (under reasonable assumptions, results remain true for an arbitrary compact manifold of dimension ≥ 2, possibly with boundary). A numerical simulation of the system with a precision 2^{-N} replaces the continuous phase space $\mathbf{T}^2$ by a discrete space E_N made of points of $\mathbf{T}^2$ whose coordinates are binary numbers with at most N decimal places, and replaces the map f by its discretization $f_N : E_N \to E_N$ that maps $x \in E_N$ to the point (or one of the points) of E_N nearest $f(x)$.



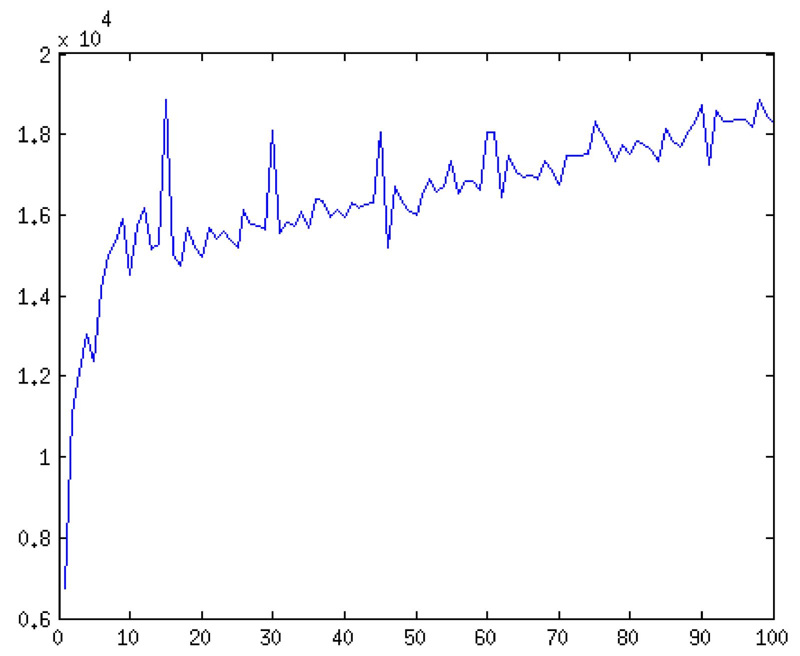
Uniform discretization grids of order 1, 2 and 3 on $\mathbf{T}^2$ and their associated cubes.

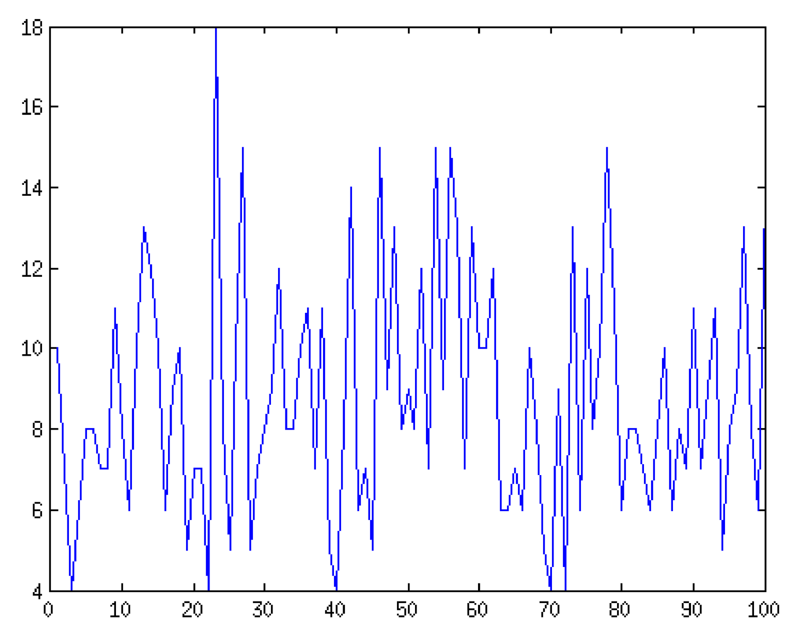
Let λ be Lebesgue measure (resusts can be generalized to arbitrary *good measures*) and consider *generic* homeomorphisms of $\mathrm{Homeo}(\mathbf{T}^2, \lambda)$ (homeomorphisms of $\mathbf{T}^2$ which preserve λ): a property is said to be generic if it is true on a dense G_δ subset of $\mathrm{Homeo}(\mathbf{T}^2, \lambda)$; as genericity is stable under countable intersections one can talk (abusively) about generic homeomorphisms.

Finite maps

The dynamics of any finite map $\sigma : E_N \to E_N$ is quite simple: given $x \in E_N$, the orbit $(\sigma^k(x))_k$ is preperiodic. To study the dynamics of σ one can focus on simple quantities such as the cardinal of the union of periodic orbits of σ, the number of periodic orbits of σ, their lengths...

Some simulations





Number of periodic orbits depending on N for a perturbation of identity (left) and a perturbation of standard linear Anosov map (right), on grids of size $N = 128k$, $k = 1, \ldots, 100$ (more simulations in [1]).

Combinatorial results

Theorem 1 (Miernowski). *For a generic homeomorphism $f \in \mathrm{Homeo}(\mathbf{T}^2, \lambda)$, for infinitely many integers N, f_N is a cyclic permutation.*

Theorem 2. *For a generic homeomorphism $f \in \mathrm{Homeo}(\mathbf{T}^2, \lambda)$, f has a periodic point, and for infinitely many integers N, f_N has a unique periodic orbit, whose period equals to the smallest period of periodic points of f; moreover one can suppose that E_N is covered by a single (pre-periodic) orbit of f_N.*

Theorem 3. *Let $\vartheta : \mathbf{N} \to \mathbf{R}$ such that $\vartheta(N) = o(\mathrm{Card}(E_N))$. For a generic homeomorphism $f \in \mathrm{Homeo}(\mathbf{T}^2, \lambda)$ and for infinitely many integers N, the discretization f_N is a permutation and has at least $\vartheta(N)$ cycles which are pairwise conjugated.*

Lesson: The dynamics of a single discretization of a generic homeomorphism has in general nothing to do with the dynamics of the initial homeomorphism.

Sketches of proofs

We set that every time homeomorphisms are well approximated by a certain type of discrete maps, this type of discrete maps appears infinitely many times on the discretizations of a generic homeomorphism.

A *type of approximation* $\mathcal{T} = (\mathcal{T}_N)_{N\in\mathbf{N}}$ is a sequence of subsets of the set $\mathcal{F}(E_N, E_N)$ of applications from E_N into itself.

A type of approximation $\mathcal{T}$ is *dense* if for all $f \in \mathrm{Homeo}(\mathbf{T}^2, \lambda)$, all $\varepsilon > 0$ and all $N_0 \in \mathbf{N}$, there exists $N \geq N_0$ and $\sigma_N \in \mathcal{T}_N$ such that $d_n(f, \sigma_N) < \varepsilon$ (d_N is the uniform distance restricted to E_N).

Theorem 4. *Let $\mathcal{T}$ be a dense type of approximation. Then for a generic $f \in \mathrm{Homeo}(\mathbf{T}^2, \lambda)$ and for all $N_0 \in \mathbf{N}$, there exists $N \geq N_0$ such that $f_N \in \mathcal{T}_N$.*

So to prove theorems 1, 2 and 3, we just have to prove that the corresponding types of approximations are dense.

Theorem 5 (Lax, Alpern). *The set of cyclic permutations of the grids E_N is a dense type of approximation (wonderful combinatorial proof based on marriage lemma).*

Proposition 6 (Variations of Lax's theorem).
For all $\varepsilon > 0$, the set of applications of E_N into itself which have a single orbit and whose periodic orbit covers a proportion smaller than ε of E_N is a dense type of approximation.
Let $\vartheta : \mathbf{N} \to \mathbf{R}$ such that $\vartheta(N) = o(\mathrm{Card}(E_N))$. Then the set of permutations of E_N which have at least $\vartheta(N)$ periodic orbits with the same length is a dense type of approximation.

Average behaviour

The previous theorems express that the dynamics of a single discretization does not reflect the actual dynamics of the homeomorphism. However, one might reasonably expect that the properties of the homeomorphism are transmitted to many discretizations. It is not so, for instance:

Theorem 7. *For a generic conservative homeomorphism f, the proportion of integers N between 1 and M such that f_N is a cyclic permutation accumulates on both 0 and 1 when M goes to infinity.*

In fact, for most of the properties considered in the previous paragraph, the frequency they appear on discretizations of orders smaller than M accumulates on both 0 and 1 when M goes to infinity.

Lesson: A dynamical property of a generic homeomorphism can not be deduced from the frequency it appears on discretizations.

And after...

- One can prove that it is possible to recover some important dynamical features of a generic homeomorphism by looking at the corresponding dynamical features of *all* the discretizations; for example one can retrieve the set of all invariant compact sets or the set of all invariant measures. However it is impossible to recover physical measures of generic conservative homeomorphisms (see [1]).
- The dissipative case (without assumption of preservation of a given measure) is much more simple: the results express that generically, the dynamics of the discretizations tends to that of the initial homeomorphism (see [1]).
- So far, only little is known about discretizations of generic C^1-diffeomorphisms. However, one can hope that the behaviour of discretizations reflects better the dynamics of the initial dynamical system.

Bibliography

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[3] Tomasz Miernowski. *Dynamique discrétisée et stochastique, géométrie conforme*. PhD thesis, ENS Lyon, 2005.