INSTABILITY FOR THE ROTATION SET OF DIFFEOMORPHISMS OF THE TORUS HOMOTOPIC TO THE IDENTITY

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Abstract. The aim of this short note is to explain how the arguments of the "closing lemma with time control" of F. Abdenur and S. Crovisier [AC12] can be used to answer Question 1 of the article "Instability for the rotation set of homeomorphisms of the torus homotopic to the identity" of S. Addas-Zanata [AZ04].

In this short note, we explain how to get a C^1 version of a perturbation result of the rotation set of homeomorphisms of the torus homotopic to the identity, obtained by S. Addas-Zanata in [AZ04]: consider some diffeomorphism f of the torus, isotopic to the identity, and suppose that some extreme point (t, ω) of the rotation set of f has at least one irrational coordinate. Then there exists a perturbation g of f, which is arbitrarily C^1-close to f, such that the rotation set of g contains some vector that was not in the rotation set of f.

We will use the notations of [AZ04]. Let us recall the most useful ones: we will denote $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ the flat torus. The space $D^1(\mathbf{T}^2)$ will be the set of C^1-diffeomorphism of the torus $\mathbf{T}^2$ homotopic to the identity, endowed with the classical C^1 topology on compact spaces; $D^1(\mathbf{R}^2)$ will be the set of lifts to the plane of elements of $D^1(\mathbf{T}^2)$. Given $\tilde f \in D^1(\mathbf{R}^2)$, its *rotation set* will be defied as

$$\rho(\tilde f) = \bigcap_{i=1}^{\infty} \overline{\bigcup_{n\ge i} \Big\{ \frac{\tilde f^n(\tilde x) - \tilde x}{n} \mid \tilde x \in \mathbf{R}^2 \Big\}}.$$

For $\tilde x \in \mathbf{R}^2$, we will denote

$$\rho(\tilde x, n, \tilde f) = \frac{\tilde f^n(\tilde x) - \tilde x}{n}$$

the rotation vector of the segment of orbit $\tilde x, \tilde f(\tilde x), \cdots, \tilde f^n(\tilde x)$, and when it is well defined (for example for a periodic point),

$$\rho(\tilde x, \tilde f) = \lim_{n\to+\infty} \rho(\tilde x, n, \tilde f).$$

We will also consider ω a volume or a symplectic form on $\mathbf{T}^2$, whose lift to $\mathbf{R}^2$ will also be denoted by ω.

We will prove the following result.

Theorem 1. *Let $\tilde f \in D^1(\mathbf{R}^2)$ be such that $\rho(\tilde f)$ has an extremal point $(t, \omega) \notin \mathbf{Q}^2$. Then there exists $\tilde g \in D^1(\mathbf{R}^2)$, arbitrarily C^1-close to f, such that $\rho(\tilde g) \cap \rho(\tilde f)^{\complement} \neq \emptyset$ (and in particular, $\rho(\tilde g) \neq \rho(\tilde f)$).*

Moreover, if $\tilde f$ preserves ω, then $\tilde g$ can be supposed to preserve it too.

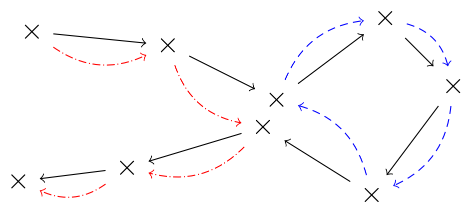

FIGURE 1. If the rotation vector of the initial orbit (in black) is in $\{L > 0\}$, then the rotation vector of one of the two pseudo orbits (in red and in blue) too.

We will prove this theorem by replacing the C^0 perturbation result of [AZ04] by a closing lemma in topology C^1, obtained by adapting the arguments of Theorem 6 of [AC12].

Lemma 2 (Closing lemma with rotation control)**.** *Let $\tilde f \in D^1(\mathbf{T}^2)$, $L : \mathbf{R}^2 \to \mathbf{R}$ a non-trivial affine form, and $\mathcal{V}$ a C^1-neighbourhood of f. Then, there exists $N \in N$ such that for every non-periodic point x of f, there exists a neighbourhood V of x such that if $n \geq N$ and $y \in V$ are such that $f^n(y) \in V$ and $L\big(\rho(\tilde y, n, \tilde f)\big) > 0$, then there exists $g \in \mathcal{V}$ such that y is a periodic point[1] of g satisfying $L\big(\rho(\tilde y, \tilde g)\big) > 0$. Moreover, if f preserves ω, then g can be supposed to preserve it too.*

The idea of the proof of this lemma is identical to that of Theorem 6 of [AC12], by replacing the dichotomy "ℓ divides / does not divide the length of the orbit" by the dichotomy "$L\big(\rho(\tilde x, n, \tilde f)\big) > 0$ / $L\big(\rho(\tilde x, n, \tilde f)\big) \leq 0$". More precisely, the proof of the connecting lemma of S. Hayashi [Hay97] builds a "closable" pseudo-orbit[2] from a recurrent orbit of f, by making *shortcuts* in this orbit; each time such a shortcut is performed there are two possibilities of creating a new pseudo-orbit (see Figure 1). If the initial orbit belongs to the set $\{L(\rho) > 0\}$, then at least one of these two new pseudo-orbits also belongs to the set $\{L(\rho) > 0\}$ (as the rotation vector of the initial orbit is a barycentre of the two new ones)[3].

Proof of Lemma 2. Simply remark that Proposition 4 of [AC12] still holds when condition

3. The length of the periodic pseudo-orbit $(y_1, \cdots, y_n = y_0)$ is not a multiple of ℓ.

is replaced by the condition

3. The periodic pseudo-orbit[4] $(y_1, \cdots, y_n = y_0)$ satisfies $L\big(\frac{\tilde y_n - \tilde y_0}{n}\big) > 0$.

The rest of the proof is identical to Section 3.3.1 of [AC12]. □

We now explain how this connecting lemma with rotation control can be applied to adapt the proof of Theorem 1 of [AZ04] to the C^1 case. Let us quickly recall the main arguments of the proof in the C^0 case. As the rotation set is convex [MZ89], there exists a supporting line of $\rho(\tilde f)$ at (t, ω), in other words an affine

[1]Note that in general, this period is different from n.

[2]A pseudo-orbit is called *closable* if Pugh's algebraic lemma (Lemma 4 of [AC12], see also [Pug67]) can be applied simultaneously to every jump of the pseudo-orbit, to make it become a real orbit.

[3]This corresponds to the initial argument of [AC12]: "If ℓ does not divide the length of the initial orbit, then it also does not divides the length of at least one of these two new pseudo-orbits".

[4]To be rigorous here, pseudo-orbits must be considered in the cover $\mathbf{R}^2$ and perturbations of diffeormorphisms performed in $\mathbf{T}^2$.

map $L : \mathbf{R}^2 \to \mathbf{R}$ such that $L(t,\omega) = 0$ and $L(v) \le 0$ for every $v \in \rho(\tilde f)$. Thus, if we build g close to f such that there exists $v \in \rho(\tilde g)$ satisfying $L(v) > 0$, then we are done.

The ergodic theorem implies the existence of a point $x_0 \in \mathbf{T}^2$ which is recurrent for f and such that $\rho(\tilde x_0, \tilde f) = (t, \omega)$. At this point there are two possibilities. Either there exists n arbitrarily large such that $f^n(x_0)$ is close to x_0 and $L\big(\rho(\tilde x_0, n, \tilde f)\big) > 0$; in this case it suffices to apply a C^0 closing lemma to x_0 and $f^n(x_0)$ to get the theorem. Or for every n large enough such that $f^n(x_0)$ is close to x, we have $L\big(\rho(\tilde x_0, n, \tilde f)\big) \le 0$. This case is a bit more complicated: we begin by proving that in this case, it is possible to suppose that $L\big(\rho(\tilde x_0, n, \tilde f)\big) < 0$ (Lemma 3 of [AZ04]). Let n_0 be such a number (large enough); a theorem of recurrence of G. Atkinson [Atk76] implies the existence of a time $n_1 \gg n_0$ such that $L\big(n_1\rho(\tilde x_0, n_1, \tilde f)\big)$ is arbitrarily close to 0. A calculation shows that in this case, $L\big(\rho(\tilde f^{n_0}(\tilde x_0), n_1 - n_0, \tilde f)\big) > 0$: the rotation vector of the segment of orbit between $\tilde f^{n_0}(\tilde x_0)$ and $\tilde f^{n_1}(\tilde x_0)$ belongs to $\{L > 0\}$. It then suffices to apply the C^0 closing lemma to $\tilde f^{n_0}(\tilde x_0)$ and $\tilde f^{n_1}(\tilde x_0)$.

Proof of Theorem 1. Let $\tilde f \in D^1(\mathbf{R}^2)$ be such that $\rho(\tilde f)$ has an extremal point $(t,\omega) \notin \mathbf{Q}^2$, and $\mathcal{V}$ a C^1-neighbourhood of f. We fix once for all a lift $\tilde f$ of f, and choose $L : \mathbf{R}^2 \to \mathbf{R}$ an affine form such that $L(t,\omega) = 0$ and $L(v) \le 0$ for every $v \in \rho(\tilde f)$. Let $x_0 \in \mathbf{T}^2$ be a recurrent point of f such that $\rho(\tilde x_0, \tilde f) = (t,\omega)$. Lemma 2 gives us a number $N \in \mathbf{N}$ and a neighbourhood V of x_0. The proof of Theorem 1 of [AZ04] summarized in the previous discussion gives us a point $y = f^{n_0}(x_0)$ (with n_0 possibly equal to 0) and a time $n_1 \ge N$ such that $\tilde f^{n_1}(\tilde y) \in V$ and $L\big(\rho(\tilde y, n_1, \tilde f)\big) > 0$. Applying Lemma 2, we get $g \in \mathcal{V}$ such that y is a periodic point of g satisfying $L\big(\rho(\tilde y, \tilde g)\big) > 0$. This proves the theorem.

Moreover, if f preserves ω, then g can be supposed to preserve it too. □

Remark 3. Theorem 1 of [AZ04] is also true in the C^0 measure-preserving case. To see it, it suffices to replace the C^0 closing lemma by the measure-preserving one (see for example Lemma 13 of [OU41]).

References

[AC12] Flavio Abdenur and Sylvain Crovisier, *Transitivity and topological mixing for C^1 diffeomorphisms*, Essays in mathematics and its applications, Springer, Heidelberg, 2012, pp. 1–16. MR 2975581

[Atk76] Giles Atkinson, *Recurrence of co-cycles and random walks*, J. London Math. Soc. (2) **13** (1976), no. 3, 486–488. MR 0419727

[AZ04] Salvador Addas-Zanata, *Instability for the rotation set of homeomorphisms of the torus homotopic to the identity*, Ergodic Theory Dynam. Systems **24** (2004), no. 2, 319–328. MR 2054045 (2005c:37074)

[Hay97] Shuhei Hayashi, *Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows*, Ann. of Math. (2) **145** (1997), no. 1, 81–137. MR 1432037

[MZ89] Michał Misiurewicz and Krystyna Ziemian, *Rotation sets for maps of tori*, J. London Math. Soc. (2) **40** (1989), no. 3, 490–506. MR 1053617 (91f:58052)

[OU41] John Oxtoby and Stanislaw Ulam, *Measure-preserving homeomorphisms and metrical transitivity*, Ann. of Math. **42** (1941), no. 2, 874–920.

[Pug67] Charles Pugh, *The closing lemma*, Amer. J. Math. **89** (1967), 956–1009. MR 0226669 (37 #2256)

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