DEGREE OF RECURRENCE OF GENERIC DIFFEOMORPHISMS

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Abstract. We study the spatial discretizations of dynamical systems: can we recover some dynamical features of a system from numerical simulations? Here, we tackle this issue for the simplest algorithm possible: we compute long segments of orbits with a fixed number of digits. We show that the dynamics of the discretizations of a C^1 generic conservative diffeomorphism of the torus is very different from that observed in the C^0 regularity. The proof of our results involves in particular a local-global formula for discretizations, as well as a study of the corresponding linear case, which uses ideas from the theory of quasicrystals.

Résumé. Le problème qui nous intéresse est celui de la discrétisation spatiale des systèmes dynamiques : est-il possible de retrouver certaines propriétés dynamiques d'un système à partir de simulations numériques ? Nous donnons ici des éléments de réponse à cette question dans le cas où les simulations se font avec l'algorithme le plus simple possible : on calcule des segments d'orbites très longs, à précision décimale fixée. Nous démontrons que le comportement dynamique des discrétisations d'un C^1-difféomorphisme conservatif générique du tore est complètement différent de celui observé dans le cas de la régularité C^0. La preuve des résultats passe en particulier par l'obtention d'une formule reliant les comportement local et global des discrétisations, ainsi qu'une étude du cas linéaire utilisant des concepts issus de la théorie des quasi-cristaux.

Contents

1. Introduction

This paper is concerned with the issue of numerical experiments on dynamical systems. More precisely, consider a discrete-time dynamical system f on the torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ endowed with Lebesgue measure [1], and define some spatial discretizations of this system in the following way: consider the collection $(E_N)_{N\in\mathbf{N}}$ of uniform grids on $\mathbf{T}^n$

$$E_N = \left\{ \left(\frac{i_1}{N}, \cdots, \frac{i_n}{N}\right) \in \mathbf{R}^n/\mathbf{Z}^n \,\middle|\, 1 \le i_1, \cdots, i_n \le N \right\},$$

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1. We will see in Appendix A a more general framework where our results remain true.

and define a Euclidean projection P_N on the nearest point of E_N; in other words $P_N(x)$ is (one of) the point(s) of E_N which is the closest from x. This projection allows to define the discretizations of f.

Definition 1. The *discretization* $f_N : E_N \to E_N$ of f on the grid E_N is the map $f_N = P_N \circ f_{|E_N}$.

Such discretizations f_N are supposed to reflect what happens when segments of orbits of the system f are computed with a computer: in particular, if we take $N = 10^k$, the discretization models computations made with k decimal places. The question we are interested in is then the following: can the dynamics of the system f be inferred from the dynamics of (some of) its discretizations f_N?

Here, we will focus on a number associated to the combinatorics of the discretizations, called the degree of recurrence. As every discretization is a finite map $f_N : E_N \to E_N$, each of its orbits is eventually periodic. Thus, the sequence of sets $(f_N^k(E_N))_{k\in\mathbf{N}}$ (the order of discretization N being fixed) is eventually constant, equal to a set $\Omega(f_N)$ called the *recurrent set*[2]. This set coincides with the union of the periodic orbits of f_N; it is the biggest subset of E_N on which the restriction of f_N is a bijection. Then, the *degree of recurrence* of f_N, denoted by $D(f_N)$, is the ratio between the cardinality of $\Omega(f_N)$ and the cardinality of the grid E_N.

Definition 2. Let E be a finite set and $\sigma : E \to E$ be a finite map on E. The *recurrent set* of σ is the union $\Omega(\sigma)$ of the periodic orbits of σ. The *degree of recurrence* of the finite map σ is the ratio

$$D(\sigma) = \frac{\operatorname{Card}\big(\Omega(\sigma)\big)}{\operatorname{Card}(E)}.$$

This degree of recurrence $D(f_N) \in [0,1]$ represents the loss of information induced by the iteration of the discretization f_N. For example if $D(f_N) = 1$, then f_N is a bijection. Also, a finite map with a degree of recurrence equal to 1 preserves the uniform measure on E_N, and thus can be considered as *conservative*.

The goal of this paper is to study the behaviour of the degree of recurrence $D(f_N)$ as N goes to infinity and for a *generic* dynamics f of the torus $\mathbf{T}^n$, $n \geq 2$. More precisely, on every Baire space $\mathcal{B}$ it is possible to define a good notion of genericity: a property on elements of $\mathcal{B}$ will be said *generic* if satisfied on at least a countable intersection of open and dense subsets of $\mathcal{B}$.

For our purpose, the spaces $\operatorname{Diff}^r(\mathbf{T}^n)$ and $\operatorname{Diff}^r(\mathbf{T}^n, \operatorname{Leb})$ of respectively C^r-diffeomorphisms and Lebesgue measure preserving C^r-diffeomorphisms of $\mathbf{T}^n$ are Baire spaces for every $r \in [0, +\infty]$, when endowed with the classical metric on C^r-diffeomorphisms[3]. Elements of $\operatorname{Diff}^r(\mathbf{T}^n)$ will be called *dissipative* and elements of $\operatorname{Diff}^r(\mathbf{T}^n, \operatorname{Leb})$ *conservative*. Also, the spaces $\operatorname{Diff}^0(\mathbf{T}^n)$ and $\operatorname{Diff}^0(\mathbf{T}^n, \operatorname{Leb})$ of homeomorphisms will be denoted by respectively $\operatorname{Homeo}(\mathbf{T}^n)$ and $\operatorname{Homeo}(\mathbf{T}^n, \operatorname{Leb})$.

We will also consider the case of generic expanding maps of the circle. Indeed, for any $r \in [1, +\infty]$, the space $\mathcal{D}^r(\mathbf{S}^1)$ of C^r expanding maps[4] of the circle $\mathbf{S}^1$, endowed with the classical C^r topology, is a Baire space.

The study of generic dynamics is motivated by the phenomenon of resonance that appears on some very specific examples like that of the linear automorphism of the torus $(x, y) \mapsto (2x + y, x + y)$: as noticed by É. Ghys in [Ghy94],the fact hat this map is linear with integer coefficients forces the discretizations on the uniform grids

2. Note that a priori, $(f_N)^k \neq (f^k)_N$.
3. For example, $d_{C^1}(f, g) = \sup_{x\in\mathbf{T}^n} d(f(x), g(x)) + \sup_{x\in\mathbf{T}^n} \|Df_x - Dg_x\|$.
4. Recall that a C^1 map f is *expanding* if for every $x \in \mathbf{S}^1$, $|f'(x)| > 1$.

to be bijections with a (very) small order (see also [DF92] or the introduction of [Gui15c]). One can hope that this behaviour is exceptional; to avoid them, É. Ghys proposes to study the case of generic maps.

To begin with, let us recall what happens for generic homeomorphisms. The following result can be found in Section 5.2 of [Gui15d].

Theorem 3 (Guihéneuf)**.** *Let $f \in \mathrm{Homeo}(\mathbf{T}^n)$ be a generic dissipative homeomorphism. Then $D(f_N) \underset{N\to+\infty}{\longrightarrow} 0$.*

This theorem expresses that there is a total loss of information when a discretization of a generic homeomorphism is iterated. This is not surprising, as a generic dissipative homeomorphism has an "attractor dynamics" (see for example [AA13]). The following result is the Corollary 5.24 of [Gui15c] (see also Section 4.3 of [Gui15d] and Proposition 2.2.2 of [Mie05]).

Theorem 4 (Guihéneuf, Miernowski)**.** *Let $f \in \mathrm{Homeo}(\mathbf{T}^n, \mathrm{Leb})$ be a generic conservative homeomorphism. Then the sequence $(D(f_N))_{N\in\mathbf{N}}$ accumulates on the whole segment $[0,1]$.*

Thus, the degree of recurrence of a generic *conservative* homeomorphism accumulates on the biggest set on which it can a priori accumulate. Then, the behaviour of this combinatorial quantity depends a lot on the order N of the discretization and not at all on the dynamics of the homeomorphism f. Moreover, as there exists a subsequence $(N_k)_{k\in\mathbf{N}}$ such that $D(f_N) \underset{N\to+\infty}{\longrightarrow} 0$, there exists "a lot" of discretizations that do not reflect the conservative character of the homeomorphism.

In this paper, we study the asymptotic behaviour of the degree of recurrence in higher regularity. Our results can be summarized in the following theorem (see Corollary 39, Theorem 41 and Corollary 49).

Theorem A. *Let f be either*
- *a generic dissipative C^1-diffeomorphism of $\mathbf{T}^n$;*
- *a generic conservative C^1-diffeomorphism of $\mathbf{T}^n$;*
- *a generic C^r expanding map[5] of the circle $\mathbf{S}^1$, for any $1 \le r \le \infty$;*

Then

$$D(f_N) \underset{N\to+\infty}{\longrightarrow} 0.$$

We must admit that the degree of recurrence gives only little information about the dynamics of the discretizations. Theorem A becomes interesting when compared to the corresponding case in C^0 regularity: in the conservative setting, it indicates that the bad behaviours observed for generic conservative homeomorphisms should disappear in higher regularity: the behaviour of the rate of injectivity is less irregular for generic conservative C^1 diffeomorphisms than for generic conservative homeomorphisms(compare Theorems 4 and A). This big difference suggests that the wild behaviours of the global dynamics of discretizations of generic conservative homeomorphisms observed in [Gui15d] may not appear for higher regularities. For example, we can hope that — contrary to what happens for generic conservative homeomorphisms — it is possible to recover the physical measures of a generic C^1 conservative diffeomorphism by looking at some well-chosen invariant measures of the discretizations, as suggested by some numerical experiments (see also [Gui15e]).

5. This point was previously announced by P.P. Flockerman and O. E. Lanford, but remained unpublished, see [Flo02].

Despite this, Theorem A shows that when we iterate the discretizations of a generic conservative diffeomorphism, we lose a great amount of information. Moreover, although f is conservative, its discretizations tend to behave like dissipative maps. This can be compared with the work of P. Lax [Lax71]: for any conservative homeomorphism f, there is a bijective finite map arbitrarily close to f. Theorem A states that for a generic conservative C^1 diffeomorphism, the discretizations never possess this property.

But the main interest of Theorem A lies in the techniques used to prove it: we will link the global and local behaviours of the discretizations, thus reduce the proof to that of a linear statement.

Firstly, as it can be obtained as the decreasing limit of finite time quantities, the degree of recurrence is maybe the easiest combinatorial invariant to study: we will deduce its behaviour from that of the rate of injectivity.

Definition 5. Let $n \geq 1$, $f : \mathbf{T}^n \to \mathbf{T}^n$ an endomorphism of the torus and $t \in \mathbf{N}$. The *rate of injectivity* in time t and for the order N is the quantity

$$\tau^t(f_N) = \frac{\operatorname{Card}\big((f_N)^t(E_N)\big)}{\operatorname{Card}(E_N)}.$$

Then, the *upper rate of injectivity* of f in time t is defined as

$$\tau^t(f) = \limsup_{N\to+\infty} \tau^t(f_N), \tag{1}$$

and the *asymptotic rate of injectivity* of f is

$$\tau^\infty(f) = \lim_{t\to+\infty} \tau^t(f)$$

(as the sequence $(\tau^t(f))_t$ is decreasing, the limit is well defined).

The link between the degree of recurrence and the rates of injectivity is made by the trivial formula:

$$D(f_N) = \lim_{t\to+\infty} \tau^t(f_N).$$

Furthermore, when N is fixed, the sequence $(\tau^t(f_N))_t$ is decreasing in t, so $D(f_N) \leq \tau^t(f_N)$ for every $t \in \mathbf{N}$. Taking the upper limit in N, we get

$$\limsup_{N\to+\infty} D(f_N) \leq \tau^t(f)$$

for every $t \in \mathbf{N}$, so considering the limit $t \to +\infty$, we get

$$\limsup_{N\to+\infty} D(f_N) \leq \lim_{t\to+\infty} \tau^t(f) = \tau^\infty(f). \tag{2}$$

In particular, if we have an upper bound on $\tau^\infty(f)$, this will give a bound on $\limsup_{N\to+\infty} D(f_N)$. This reduces the proof of Theorem A to the study of the asymptotic rate of injectivity $\tau^\infty(f)$.

For generic dissipative C^1-diffeomorphisms, the fact that the sequence $(D(f_N))_N$ converges to 0 (Theorem A) is an easy consequence of Equation 2 and of a theorem of A. Avila and J. Bochi (Theorem 37, see also [AB06]).

For conservative diffeomorphisms and expanding maps, the study of the rates of injectivity will be the opportunity to understand the local behaviour of the discretizations: we will "linearize" the problem and reduce it to a statement about generic sequences of linear maps.

Let us first define the corresponding quantities for linear maps: as in general a linear map does not send $\mathbf{Z}^n$ into $\mathbf{Z}^n$, we will approach it by a discretization. For any $A \in GL_n(\mathbf{R})$ and any $x \in \mathbf{Z}^n$, $\widehat{A}(x)$ is defined as the point of $\mathbf{Z}^n$ which is the closest from $A(x)$. Then, for any sequence $(A_k)_k$ of linear maps, the *rate of injectivity* $\tau^k(A_1,\cdots,A_k)$ is defined as the density of the set $(\widehat{A}_k\circ\cdots\circ\widehat{A}_1)(\mathbf{Z}^n)$ (see Definition 7):

$$\tau^k(A_1,\cdots,A_k)=\limsup_{R\to+\infty}\frac{\operatorname{Card}\big((\widehat{A_k}\circ\cdots\circ\widehat{A_1})(\mathbf{Z}^n)\cap B(0,R)\big)}{\operatorname{Card}\big(\mathbf{Z}^n\cap B(0,R)\big)}\in]0,1],$$

and the *asymptotic rate of injectivity* $\tau^\infty((A_k)_k)$ is the limit of $\tau^k(A_1,\cdots,A_k)$ as k tends to infinity. These definitions are made to mimic the corresponding definitions for diffeomorphisms. Then, the following statement asserts that that the rates of injectivity of a generic map are obtained by averaging the corresponding quantities for the differentials of the diffeomorphism, and thus makes the link between local and global behaviours of the discretizations (Theorem 26).

Theorem B. *Let $r\in[1,+\infty]$, and $f\in\operatorname{Diff}^r(\mathbf{T}^n)$ (or $f\in\operatorname{Diff}^r(\mathbf{T}^n,\operatorname{Leb})$) be a generic diffeomorphism. Then $\tau^k(f)$ is well defined (that is, the limit superior in (1) is a limit) and satisfies:*

$$\tau^k(f)=\int_{\mathbf{T}^n}\tau^k\left(Df_x,\cdots,Df_{f^{k-1}(x)}\right)\mathrm{d}\operatorname{Leb}(x).$$

The same kind of result holds for generic expanding maps (see Theorem 31).

For conservative diffeomorphisms, the core of the proof of Theorem A is the study of the rate of injectivity of generic sequences of matrices with determinant 1, that we will conduct in Section 2. Indeed, applying Theorem B wich links the local and global behaviours of the diffeomorphism, together with Rokhlin tower theorem and a statement of local linearization of diffeomorphisms (Lemma 42 combined with the smoothing result [Avi10] of A. Avila), we reduce the proof of Theorem A to the main result of Section 2 (Theorem 9).

Theorem C. *For a generic sequence of matrices $(A_k)_{k\geq1}$ of $\ell^\infty(SL_n(\mathbf{R}))$, we have*

$$\tau^\infty\big((A_k)_{k\geq1}\big)=0.$$

This linear statement has also nice applications to image processing. For example — the result remains true for a generic sequence of *isometries* —, it says that if we apply a naive algorithm, the quality of a numerical image will be necessarily deteriorated by rotating this image many times by a generic sequence of angles (see [Gui15b]). Theorem C is also the central technical result of [Gui15e], wich study the physical measures of discretizations of generic conservative C^1-diffeomorphisms.

The proof of Theorem C (which is the most difficult and most original part of the proof of Theorem A) uses the nice formalism of model sets, developed initially for the study of quasicrystals. Using an equidistribution property, this allows to get a geometric formula for the computation of the rate of injectivity of a generic sequence of matrices (Proposition 12): the rate of injectivity of a generic sequence $A_1,\cdots,A_k$ of matrices of $SL_n(\mathbf{R})$ can be expressed in terms of areas of intersections of cubes in $\mathbf{R}^{nk}$. This formula, by averaging what happens in the image sets $(\widehat{A}_k\circ\cdots\circ\widehat{A}_1)(\mathbf{Z}^n)$, reflects the global behaviour of these sets. Also, it transforms the iteration into a passage in high dimension. These considerations allow to prove Theorem C, by geometric considerations about the volume of intersections of cubes, without having to make "clever" perturbations of the sequence of matrices (that is, the perturbations made a each iteration are chosen independently from that made in the past or in the future).

Note that the fact that the local-global formula is true for C^r-generic diffeomorphisms does not help to conclude about the degree of recurrence of such maps: a priori, we need to perturb the derivative of such maps on a large subset of the torus. However, it is likely that Theorem 41 remains true for these higher regularities, at least on some generic sets of open subsets of $\text{Diff}^r(\mathbf{T}^n, \text{Leb})$.

The end of this paper is devoted to the results of the simulations we have conducted about the degree of recurrence of C^1-diffeomorphisms and expanding maps; it shows that in practice, the degree of recurrence tends to 0, at least for the examples of diffeomorphisms we have tested.

Recall that we will see in Appendix A that the quite restrictive framework of the torus $\mathbf{T}^n$ equipped with the uniform grids can be generalized to arbitrary manifolds, provided that the discretizations grids behave locally (and almost everywhere) like the canonical grids on the torus.

To finish, we state some questions related to Theorem A that remain open.

- What is the behaviour of the degree of recurrence of discretizations of generic C^r-expanding maps of the torus $\mathbf{T}^n$, for $n \geq 2$ and $r > 1$? In the view of Theorem A, we can conjecture that this degree of recurrence tends to 0. To prove it we would need a generalization of Lemma 45 to bigger dimensions.
- What is the behaviour of the degree of recurrence of discretizations of generic C^r-diffeomorphisms of the torus $\mathbf{T}^n$ for $r > 1$? These questions seem to be quite hard, as it may require some perturbation results in the C^r topology for $r > 1$.

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2. The linear case

We begin by the study of the linear case, corresponding to the "local behaviour" of C^1 maps. We first define the linear counterpart of the discretization.

Definition 6. The map $P : \mathbf{R} \to \mathbf{Z}$ is defined as a projection from $\mathbf{R}$ onto $\mathbf{Z}$. More precisely, for $x \in \mathbf{R}$, $P(x)$ is the unique[6] integer $k \in \mathbf{Z}$ such that $k - 1/2 < x \leq k + 1/2$. This projection induces the map

$$\pi : \begin{array}{ccc} \mathbf{R}^n & \longmapsto & \mathbf{Z}^n \\ (x_i)_{1\leq i\leq n} & \longmapsto & \big(P(x_i)\big)_{1\leq i\leq n} \end{array}$$

which is an Euclidean projection on the lattice $\mathbf{Z}^n$. For $A \in M_n(\mathbf{R})$, we denote by $\widehat{A}$ the *discretization* of A, defined by

$$\widehat{A} : \begin{array}{ccc} \mathbf{Z}^n & \longrightarrow & \mathbf{Z}^n \\ x & \longmapsto & \pi(Ax). \end{array}$$

This definition allows us to define the rate of injectivity for sequences of linear maps.

6. Remark that the choice of where the inequality is strict and where it is not is arbitrary.

FIGURE 1. Successive images of $\mathbf{Z}^2$ by discretizations of random matrices in $SL_2(\mathbf{R})$, a point is black if it belongs to $(\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^2)$. The A_i are chosen randomly, using the singular value decomposition: they are chosen among the matrices of the form $R_\theta D_t R_{\theta'}$, with R_θ the rotation of angle θ and D_t the diagonal matrix $\mathrm{Diag}(e^t, e^{-t})$, the θ, θ' being chosen uniformly in $[0, 2\pi]$ and t uniformly in $[-1/2, 1/2]$. From left to right and top to bottom, $k = 1$, 3, 20.

Definition 7. Let $A_1, \cdots, A_k \in GL_n(\mathbf{R})$. The *rate of injectivity* of $A_1, \cdots, A_k$ is the quantity [7] [8]

$$\tau^k(A_1, \cdots, A_k) = \limsup_{R \to +\infty} \frac{\mathrm{Card}\left((\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^n) \cap B_R)\right)}{\mathrm{Card}\left(\mathbf{Z}^n \cap B_R\right)} \in]0, 1],$$

and for an infinite sequence $(A_k)_{k \geq 1}$ of invertible matrices, as the previous quantity is decreasing in k, we can define the *asymptotic rate of injectivity*

$$\tau^\infty\left((A_k)_{k\geq 1}\right) = \lim_{k \to +\infty} \tau^k(A_1, \cdots, A_k) \in [0, 1].$$

For a typical example of the sets $(\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^2)$, see Figure 1. Finally, we define a topology on the set of sequences of linear maps.

Definition 8. We fix once for all a norm $\|\cdot\|$ on $M_n(\mathbf{R})$. For a bounded sequence $(A_k)_{k\geq 1}$ of matrices of $SL_n(\mathbf{R})$, we set

$$\|(A_k)_k\|_\infty = \sup_{k \geq 1} \|A_k\|.$$

In other words, we consider the space $\ell^\infty(SL_n(\mathbf{R}))$ of uniformly bounded sequences of matrices of determinant 1 endowed with this natural metric.

We can now state the main result of this section.

Theorem 9. *For a generic sequence of matrices $(A_k)_{k\geq 1}$ of $\ell^\infty(SL_n(\mathbf{R}))$, we have*

$$\tau^\infty\left((A_k)_{k\geq 1}\right) = 0.$$

Moreover, for every $\varepsilon > 0$, the set of$(A_k)_{k\geq 1} \in \ell^\infty(SL_n(\mathbf{R}))$ such that $\tau^\infty\left((A_k)_{k\geq 1}\right) < \varepsilon$ is open and dense.

Remark 10. The second part of this statement is easily deduced from the first by applying the continuity of τ^k on a generic subset (Remark 14).

Remark 11. The same statement holds for generic sequences of isometries (see also [Gui15b]).

7. By definition, $B_R = B_\infty(0, R)$.

8. In the sequel we will see that the lim sup is in fact a limit for a generic sequence of matrices. It can also easily shown that it is a limit in the general case.

We take advantage of the rational independence between the matrices of a generic sequence to obtain geometric formulas for the computation of the rate of injectivity. The tool used to do that is the formalism of *model sets*[9] (see for example [Moo00] or [Mey12] for surveys about model sets, see also [Gui15a] for the application to the specific case of discretizations of linear maps).

Let us summarize the different notations we will use throughout this section. We will denote by 0^k the origin of the space $\mathbf{R}^k$, and $W^k =]-1/2, 1/2]^{nk}$ (unless otherwise stated). In this section, we will denote $B_R = B_\infty(0, R)$ and $D_c(E)$ the density of a "continuous" set $E \subset \mathbf{R}^n$, defined as (when the limit exists)

$$D_c(E) = \lim_{R \to +\infty} \frac{\operatorname{Leb}(B_R \cap E)}{\operatorname{Leb}(B_R)},$$

while for a discrete set $E \subset \mathbf{R}^n$, the notation $D_d(E)$ will indicate the discrete density of E, defined as (when the limit exists)

$$D_d(E) = \lim_{R \to +\infty} \frac{\operatorname{Card}(B_R \cap E)}{\operatorname{Card}(B_R \cap \mathbf{Z}^n)},$$

We will consider $(A_k)_{k \geq 1}$ a sequence of matrices of $SL_n(\mathbf{R})$, and denote

$$\Gamma_k = (\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^n).$$

Also, Λ_k will be the lattice $M_{A_1,\cdots,A_k} \mathbf{Z}^{n(k+1)}$, with

$$M_{A_1,\cdots,A_k} = \begin{pmatrix} A_1 & -\operatorname{Id} & & & \\ & A_2 & -\operatorname{Id} & & \\ & & \ddots & \ddots & \\ & & & A_k & -\operatorname{Id} \\ & & & & \operatorname{Id} \end{pmatrix} \in M_{n(k+1)}(\mathbf{R}), \tag{3}$$

and $\widetilde{\Lambda}_k$ will be the lattice $\widetilde{M}_{A_1,\cdots,A_k} \mathbf{Z}^{nk}$, with

$$\widetilde{M}_{A_1,\cdots,A_k} = \begin{pmatrix} A_1 & -\operatorname{Id} & & & \\ & A_2 & -\operatorname{Id} & & \\ & & \ddots & \ddots & \\ & & & A_{k-1} & -\operatorname{Id} \\ & & & & A_k \end{pmatrix} \in M_{nk}(\mathbf{R}).$$

Finally, we will denote

$$\overline{\tau}^k(A_1, \cdots, A_k) = D_c\Big(W^{k+1} + \Lambda_k\Big)$$

the *mean rate of injectivity in time* k of $A_1, \cdots, A_k$.

2.1. **A geometric viewpoint to compute the rate of injectivity in arbitrary times.** We begin by motivating the introduction of model sets by giving an alternative construction of the image sets $(\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^n)$ using this formalism.

Let $A_1, \cdots, A_k \in GL_n(\mathbf{R})$, then

$$\begin{aligned} \Gamma_k &= (\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^n) \\ &= \big\{ p_2(\lambda) \mid \lambda_k \in \Lambda_k,\, p_1(\lambda) \in W^k \big\} \\ &= p_2\Big(\Lambda \cap \big(p_1^{-1}(W^k)\big)\Big), \end{aligned} \tag{4}$$

with p_1 the projection on the nk first coordinates and p_2 the projection on the n last coordinates. This allows us to see the set Γ_k as a model set.

9. Also called *cut-and-project* sets.

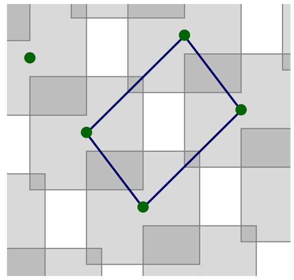

FIGURE 2. Geometric construction to compute the rate of injectivity: the green points are the elements of Λ, the blue parallelogram is a fundamental domain of Λ and the grey squares are centred on the points of Λ and have radii $1/2$. The rate of injectivity is equal to the area of the intersection between the union of the grey squares and the blue parallelogram.

Here, we suppose that the set $p_1(\Lambda_k)$ is dense (thus, equidistributed) in the image set $\operatorname{im} p_1$ (note that this condition is generic among the sequences of invertible linear maps). In particular, the set $\{p_2(\gamma) \mid \gamma \in \Lambda_k\}$ is equidistributed in the window W^k.

The following property makes the link between the density of Γ_k — that is, the rate of injectivity of $A_1, \cdots, A_k$ — and the density of the union of unit cubes centred on the points of the lattice Λ_k (see Figure 2). This formula seems to be very specific to the model sets defined by the matrix $M_{A_1,\cdots,A_k}$ and the window W^k, it is unlikely that it can be generalized to other model sets.

Proposition 12. *For a generic sequence of matrices $(A_k)_k$ of $SL_n(\mathbf{R})$, we have*

$$D_d(\Gamma_k) = D_c\left(W^k + \widetilde{\Lambda}_k\right) = \overline{\tau}^k(A_1, \cdots, A_k).$$

Remark 13. The density on the left of the equality is the density of a discrete set (that is, with respect to counting measure), whereas the density on the right of the equality is that of a continuous set (that is, with respect to Lebesgue measure). The two notions coincide when we consider discrete sets as sums of Dirac masses.

Remark 14. Proposition 12 asserts that for a generic sequence of matrices, the rate of injectivity τ^k in time k coincides with the mean rate of injectivity $\overline{\tau}^k$, which is continuous and piecewise polynomial of degree $\leq nk$ in the coefficients of the matrix.

Remark 15. The formula of Proposition 12 could be used to compute numerically the mean rate of injectivity in time k of a sequence of matrices: it is much faster to compute the volume of a finite number of intersections of cubes (in fact, a small number) than to compute the cardinalities of the images of a big set $[-R, R]^n \cap \mathbf{Z}^n$.

Proof of Proposition 12. We want to determine the density of Γ_k. By Equation (4), we have

$$x \in \Gamma_k \iff x \in \mathbf{Z}^n \text{ and } \exists \lambda \in \Lambda_k : x = p_2(\lambda),\, p_1(\lambda) \in W^k.$$

But if $x = p_2(\lambda)$, then we can write $\lambda = (\widetilde{\lambda}, 0^n) + (0^{(k-1)n}, -x, x)$ with $\widetilde{\lambda} \in \widetilde{\Lambda}_k$. Thus,

$$\begin{aligned} x \in \Gamma_k &\iff x \in \mathbf{Z}^n \text{ and } \exists \widetilde{\lambda} \in \widetilde{\Lambda}_k : (0^{(k-1)n}, -x) - \widetilde{\lambda} \in W^k \\ &\iff x \in \mathbf{Z}^n \text{ and } (0^{(k-1)n}, x) \in \bigcup_{\widetilde{\lambda} \in \widetilde{\Lambda}_k} \widetilde{\lambda} - W^k. \end{aligned}$$

Thus, $x \in \Gamma_k$ if and only if the projection of $(0^{(k-1)n}, x)$ on $\mathbf{R}^{nk}/\widetilde{\Lambda}_k$ belongs to $\bigcup_{\widetilde{\lambda}\in\widetilde{\Lambda}_k} \widetilde{\lambda} - W^k$. Then, the proposition follows directly from the fact that the points of the form $(0^{(k-1)n}, x)$, with $x \in \mathbf{Z}^n$, are equidistributed in $\mathbf{R}^{nk}/\widetilde{\Lambda}_k$.

To prove this equidistribution, we compute the inverse matrix of $\widetilde{M}_{A_1,\cdots,A_k}$:

$$\widetilde{M}^{-1}_{A_1,\cdots,A_k} = \begin{pmatrix} A_1^{-1} & A_1^{-1}A_2^{-1} & A_1^{-1}A_2^{-1}A_3^{-1} & \cdots & A_1^{-1}\cdots A_k^{-1} \\ & A_2^{-1} & A_2^{-1}A_3^{-1} & \cdots & A_2^{-1}\cdots A_k^{-1} \\ & & \ddots & \ddots & \vdots \\ & & & A_{k-1}^{-1} & A_{k-1}^{-1}A_k^{-1} \\ & & & & A_k^{-1} \end{pmatrix}.$$

Thus, the set of points of the form $(0^{(k-1)n}, x)$ in $\mathbf{R}^{nk}/\widetilde{\Lambda}_k$ corresponds to the image of the action

$$\mathbf{Z}^n \ni x \longmapsto \begin{pmatrix} A_1^{-1}\cdots A_k^{-1} \\ A_2^{-1}\cdots A_k^{-1} \\ \vdots \\ A_{k-1}^{-1}A_k^{-1} \\ A_k^{-1} \end{pmatrix} x$$

of $\mathbf{Z}^n$ on the canonical torus $\mathbf{R}^{nk}/\mathbf{Z}^{nk}$. But this action is ergodic (even in restriction to the first coordinate) when the sequence of matrices is generic. ☐

Recall the problem raised by Theorem 9: we want to make τ^k tend to 0 as k tends to infinity. By an argument of equidistribution stated by Proposition 12, generically, it is equivalent to make the mean rate of injectivity $\overline{\tau}^k$ tend to 0 when k goes to infinity, by perturbing every matrix in $SL_n(\mathbf{R})$ of at most $\delta > 0$ (fixed once for all). The conclusion of Theorem 9 is motivated by the phenomenon of concentration of the measure on a neighbourhood of the boundary of the cubes in high dimension.

Proposition 16. *Let W^k be the infinite ball of radius $1/2$ in $\mathbf{R}^k$ and v^k the vector $(1, \cdots, 1) \in \mathbf{R}^k$. Then, for every $\varepsilon, \delta > 0$, there exists $k_0 \in \mathbf{N}^*$ such that for every $k \geq k_0$, we have* $\operatorname{Leb}\big(W^k \cap (W^k + \delta v^k)\big) < \varepsilon$.

The case of equality $\overline{\tau}^k = 1$ is given by Hajós theorem.

Theorem 17 (Hajós, [Haj41])**.** *Let Λ be a lattice of $\mathbf{R}^n$. Then the collection of squares $\{B_\infty(\lambda, 1/2)\}_{\lambda\in\Lambda}$ tiles $\mathbf{R}^n$ if and only if in a canonical basis of $\mathbf{R}^n$ (that is, permuting coordinates if necessary), Λ admits a generating matrix which is upper triangular with ones on the diagonal.*

Remark 18. The kind of questions addressed by Hajós theorem are in general quite delicate. For example, we can wonder what happens if we do not suppose that the centres of the cubes form a lattice of $\mathbf{R}^n$. O. H. Keller conjectured in [Kel30] that the conclusion of Hajós theorem is still true under this weaker hypothesis. This conjecture was proven to be true for $n \leq 6$ by O. Perron in [Per40a, Per40b], but remained open in higher dimension until 1992, when J. C. Lagarias and P. W. Shor proved in [LS92] that Keller's conjecture is false for $n \geq 10$ (this result was later improved by [Mac02] which shows that it is false as soon as $n \geq 8$; the case $n = 7$ is to our knowledge still open).

Combining Hajós theorem with Proposition 12, we obtain that the equality $\overline{\tau}^k = 1$ occurs if and only if the lattice given by the matrix $M_{A_1,\cdots,A_k}$ satisfies the conclusions of Hajós theorem [10]. The heuristic suggested by the phenomenon of concentration

10. Of course, this property can be obtained directly by saying that the density is equal to 1 if and only if the rate of injectivity of every matrix of the sequence is equal to 1.

of the measure is that if we perturb “randomly” any sequence of matrices, we will go “far away” from the lattices satisfying Hajós theorem and then the rate of injectivity will be close to 0.

2.2. **A first step: proof that the asymptotic rate of injectivity is generically smaller than 1/2.**

As a first step, we prove that rate of injectivity of a generic sequence of $\ell^\infty(SL_n(\mathbf{R}))$ is smaller than 1/2.

Proposition 19. *Let $(A_k)_{k\ge 1}$ be a generic sequence of matrices of $SL_n(\mathbf{R})$. Then there exists a parameter $\lambda \in]0,1[$ such that for every $k\ge 1$, we have $\tau^k(A_1,\cdots,A_k)\le (\lambda^k+1)/2$. In particular, $\tau^\infty((A_k)_k)\le 1/2$.*

To begin with, we give a lemma estimating the sizes of intersections of cubes when the mean rate of injectivity $\overline{\tau}^k$ is bigger than 1/2.

Lemma 20. *Let $W^k =]-1/2,1/2]^k$ and $\Lambda\subset\mathbf{R}^k$ be a lattice with covolume 1 such that $D_c(W^k+\Lambda)\ge 1/2$. Then, for every $v\in\mathbf{R}^k$, we have*

$$D_c\big((W^k+\Lambda+v)\cap(W^k+\Lambda)\big)\ge 2D_c(W^k+\Lambda)-1.$$

Proof of Lemma 20. We first remark that $D_c(\Lambda+W^k)$ is equal to the volume of the projection of W^k on the quotient space $\mathbf{R}^k/\Lambda$. For every $v\in\mathbf{R}^k$, the projection of W^k+v on $\mathbf{R}^k/\Lambda$ has the same volume; as this volume is greater than 1/2, and as the covolume of Λ is 1, the projections of W^k and W^k+v overlap, and the volume of the intersection is bigger than $2D_c(W^k+\Lambda)-1$. Returning to the whole space $\mathbf{R}^k$, we get the conclusion of the lemma. □

A simple counting argument leads to the proof of the following lemma.

Lemma 21. *Let Λ_1 be a subgroup of $\mathbf{R}^m$, Λ_2 be such that $\Lambda_1\oplus\Lambda_2$ is a lattice of covolume 1 of $\mathbf{R}^m$, and C be a compact subset of $\mathbf{R}^m$. Let C_1 be the projection of C on the quotient $\mathbf{R}^m/\Lambda_1$, and C_2 the projection of C on the quotient $\mathbf{R}^m/(\Lambda_1\oplus\Lambda_2)$. We denote by*

$$a_i=\operatorname{Leb}\big\{x\in C_1 \mid \operatorname{Card}\{\lambda_2\in\Lambda_2 \mid x\in C_1+\lambda_2\}=i\big\}$$

(in particular, $\sum_{i\ge1}a_i=\operatorname{Leb}(C_1)$). Then,

$$\operatorname{Leb}(C_2)=\sum_{i\ge 1}\frac{a_i}{i}.$$

In particular, the area of C_2 (the projection on the quotient by $\Lambda_1\oplus\Lambda_2$) is smaller than (or equal to) that of C_1 (the projection on the quotient by Λ_1). The loss of area is given by the following corollary.

Corollary 22. *With the same notations as for Lemma 21, if we denote by*

$$D_1=\operatorname{Leb}\big\{x\in C_1 \mid \operatorname{Card}\{\lambda_2\in\Lambda_2 \mid x\in C_1+\lambda_2\}\ge 2\big\},$$

then,

$$\operatorname{Leb}(C_2)\le\operatorname{Leb}(C_1)-\frac{D_1}{2}.$$

Proof of Proposition 19. Let $(A_k)_{k\ge1}$ be a bounded sequence of matrices of $SL_n(\mathbf{R})$ and $\delta>0$. We proceed by induction on k and suppose that the proposition is proved for a rank $k\in\mathbf{N}^*$. Let $\widetilde{\Lambda}_k$ be the lattice spanned by the matrix $\widetilde{M}_{B_1,\cdots,B_k}$ and $W^k =]-1/2,1/2]^{nk}$ be the window corresponding to the model set Γ_k modelled on Λ_k (the lattice spanned by the matrix $M_{B_1,\cdots,B_k}$, see Equation (3)). By Proposition 12, if the sequence $(A_k)_k$ is generic, then we have

$$\tau^k(B_1,\cdots,B_k)=D_c\left(W^k+\widetilde{\Lambda}_k\right).$$

How to read these figures : The top of the figure represents the set $W^k + \widetilde{\Lambda}_k$ by the 1-dimensional set $[-1/2, 1/2] + \nu\mathbf{Z}$ (in dark blue), for a number $\nu > 1$. The bottom of the figure represents the set $W^{k+1} + \widetilde{\Lambda}_{k+1}$ by the set $[-1/2, 1/2]^2 + \Lambda$, where Λ is the lattice spanned by the vectors $(0, \nu)$ and $(1, 1-\varepsilon)$ for a parameter $\varepsilon > 0$ close to 0. The dark blue cubes represent the "old" cubes, that is, the thickening in dimension 2 of the set $W^k + \widetilde{\Lambda}_k$, and the light blue cubes represent the "added" cubes, that is, the rest of the set $W^{k+1} + \widetilde{\Lambda}_{k+1}$.

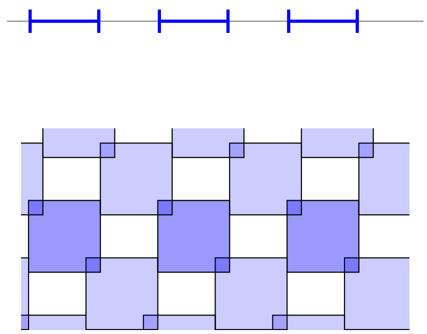

FIGURE 3. In the case where the rate is bigger than 1/2, some intersections of cubes appear automatically between times k and $k+1$.

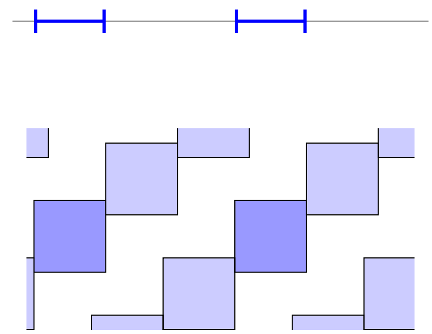

FIGURE 4. In the case where the rate is smaller than 1/2, there is not necessarily new intersections between times k and $k+1$.

We now choose a matrix B_{k+1} satisfying $\|A_{k+1} - B_{k+1}\| \le \delta$, such that there exists $x_1 \in \mathbf{Z}^n \setminus \{0\}$ such that $\|B_{k+1}x_1\|_\infty \le 1 - \varepsilon$, with $\varepsilon > 0$ depending only on δ and $\|(A_k)_k\|$ (and n): indeed, for every matrix $B \in SL_n(\mathbf{R})$, Minkowski theorem implies that there exists $x_1 \in \mathbf{Z}^n \setminus \{0\}$ such that $\|Bx_1\|_\infty \le 1$; it then suffices to modify slightly B to decrease $\|Bx_1\|_\infty$. We can also suppose that the sequence of matrices $B_1, \cdots, B_{k+1}$ is generic. Again, Proposition 12 reduces the calculation of the rate on injectivity $\tau^{k+1}(B_1, \cdots, B_{k+1})$ to that of the density of $W^{k+1} + \widetilde{\Lambda}_{k+1}$. By the form of the matrix $\widetilde{M}_{B_1,\cdots,B_k}$, this set can be decomposed into

$$W^{k+1} + \widetilde{\Lambda}_{k+1} = W^{k+1} + \begin{pmatrix} \widetilde{\Lambda}_k \\ 0^n \end{pmatrix} + \begin{pmatrix} 0^{n(k-1)} \\ -\operatorname{Id} \\ B_{k+1} \end{pmatrix} \mathbf{Z}^n.$$

In particular, as $|\det(B_{k+1})| = 1$, this easily implies that $D_c\left(W^{k+1} + \widetilde{\Lambda}_{k+1}\right) \le D_c\left(W^k + \widetilde{\Lambda}_k\right)$.

What we need is a more precise bound. We apply Corollary 22 to

$$\Lambda_1 = \big(\widetilde{\Lambda}_k, 0^n\big), \qquad \Lambda_2 = \begin{pmatrix} 0^{n(k-1)} \\ -\operatorname{Id} \\ B_{k+1} \end{pmatrix} \mathbf{Z}^n \qquad \text{and} \qquad C = W^{k+1}.$$

Then, the decreasing of the rate of injectivity between times k and $k+1$ is bigger than the D_1 defined in Corollary 22: using Lemma 20, we have

$$D_c\left(\left(W^k + \widetilde{\Lambda}_k\right) \cap \left(W^k + \widetilde{\Lambda}_k + \big(0^{n(k-1)}, -x_1\big)\right)\right) \ge 2D_d(\Gamma_k) - 1;$$

thus, as $\|x_1\|_\infty < 1 - \varepsilon$ (see Figure 3),

$$D_1 \ge \varepsilon^n \big(2D_d(\Gamma_k) - 1\big).$$

From Corollary 22 we deduce that

$$\begin{aligned} D_d(\Gamma_{k+1}) &= D_c\left(W^{k+1} + \widetilde{\Lambda}_{k+1}\right) \\ &\le D_c\left(W^k + \widetilde{\Lambda}_k\right) - \frac{1}{2} D_1 \\ &\le D_d(\Gamma_k) - \frac{1}{2}\varepsilon^n\big(2D_d(\Gamma_k) - 1\big). \end{aligned}$$

This proves the theorem for the rank $k+1$. □

2.3. **Proof of Theorem 9: generically, the asymptotic rate is zero.** We now come to the proof of Theorem 9. The strategy of proof is identical to that we used in the previous section to state that generically, the asymptotic rate is smaller than 1/2 (Proposition 19): we will use an induction to decrease the rate step by step. Recall that $\overline{\tau}^k(A_1, \cdots, A_k)$ indicates the density of the set $W^{k+1} + \Lambda_k$.

Unfortunately, if the density of $W^k + \widetilde{\Lambda}_k$ — which is generically equal to the density of the k-th image $\big(\widehat{A}_k \circ \cdots \circ \widehat{A}_1\big)(\mathbf{Z}^n)$ — is smaller than 1/2, then we can not apply exactly the strategy of proof of the previous section (see Figure 4). For example, if we take

$$A_1 = \operatorname{diag}(100, 1/100) \quad \text{and} \quad A_2 = \operatorname{diag}(1/10, 10),$$

then $\big(\widehat{A_2} \circ \widehat{A_1}\big)(\mathbf{Z}^2) = (2\mathbf{Z})^2$, and for every B_3 close to the identity, we have $\tau^3(A_1, A_2, B_3) = \tau^2(A_1, A_2) = 1/100$.

Moreover, if we set $A_k = \mathrm{Id}$ for every $k \ge 2$, then we can set $B_1 = A_1$, $B_2 = A_2$ and for each $k \ge 2$ perturb each matrix A_k into the matrix $B_k = \operatorname{diag}(1+\delta, 1/(1+\delta))$, with $\delta > 0$ small. In this case, there exists a time k_0 (minimal) such that $\tau^{k_0}(B_1, \cdots, B_{k_0}) < 1/100$. But if instead of $A_{k_0} = \mathrm{Id}$, we have $A_{k_0} = \operatorname{diag}(1/5, 5)$, this construction does not work anymore: we should have set $B_k = \operatorname{diag}(1/(1+\delta), 1+\delta)$. This suggests that we should take into account the next terms of the sequence $(A_k)_k$ to perform the perturbations. And things seems even more complicated when the matrices are no longer diagonal…

To overcome this difficulty, we use the same strategy of proof than in the case where $\overline{\tau} \ge 1/2$, provided that we wait for long enough time: for a generic sequence $(A_k)_{k\ge 1}$, if $\overline{\tau}^k(A_1, \cdots, A_k) > 1/\ell$, then $\overline{\tau}^{k+\ell-1}(A_1, \cdots, A_{k+\ell-1})$ is strictly smaller than $\overline{\tau}^k(A_1, \cdots, A_k)$. More precisely, we consider the maximal number of disjoint translates of $W^k + \widetilde{\Lambda}_k$ in $\mathbf{R}^{nk}$: we easily see that if the density of $W^k + \widetilde{\Lambda}_k$ is bigger than $1/\ell$, then there can not be more than ℓ disjoint translates of $W^k + \widetilde{\Lambda}_k$ in $\mathbf{R}^{nk}$(Lemma 23). At this point, Lemma 24 states that if the sequence of matrices is generic, then either the density of $W^{k+1} + \widetilde{\Lambda}_{k+1}$ is smaller than that of $W^k + \widetilde{\Lambda}_k$ (Figure 5), or there can not be more than $\ell - 1$ disjoint translates of $W^{k+1} + \widetilde{\Lambda}_{k+1}$ in $\mathbf{R}^{n(k+1)}$(see Figure 6). Applying this reasoning (at most) $\ell - 1$ times, we obtain that the density of $W^{k+\ell-1} + \widetilde{\Lambda}_{k+\ell-1}$ is smaller than that of $W^k + \widetilde{\Lambda}_k$. For example if $D_c\big(W^k + \widetilde{\Lambda}_k\big) > 1/3$, then $D_c\big(W^{k+2} + \widetilde{\Lambda}_{k+2}\big) < D\big(W^k + \widetilde{\Lambda}_k\big)$ (see Figure 7). To apply this strategy in practice, we have to obtain quantitative estimates about the loss of density we get between times k and $k + \ell - 1$.

Remark that with this strategy, we do not need to make "clever" perturbations of the matrices: provided that the coefficients of the matrices are rationally independent, the perturbation of each matrix is made independently from that of the others. However, this reasoning does not tell when exactly the rate of injectivity decreases (likely, in most of cases, the speed of decreasing of the rate of injectivity is much faster than the one obtained by this method), and does not say either where exactly the loss of injectivity occurs in the image sets.

We will indeed prove a more precise statement of Theorem 9.

Theorem 9. *For a generic sequence of matrices $(A_k)_{k\geq 1}$ of $\ell^\infty(SL_n(\mathbf{R}))$, for every $\ell \in \mathbf{N}$, there exists $\lambda_\ell \in]0,1[$ such that for every $k \in \mathbf{N}$,*

$$\tau^{\ell k}(A_1, \cdots, A_{\ell k}) \leq \lambda_\ell^k + \frac{1}{\ell}. \tag{5}$$

Also, for every $\nu < 1$, we have

$$\tau^k(A_1, \cdots, A_k) = o\big(\ln(k)^{-\nu}\big). \tag{6}$$

In particular, the asymptotic rate of injectivity $\tau^\infty\big((A_k)_{k\geq 1}\big)$ is equal to zero.

The following lemma is a generalization of Lemma 20. It expresses that if the density of $W^k + \widetilde{\Lambda}_k$ is bigger than $1/\ell$, then there can not be more than ℓ disjoint translates of $W^k + \widetilde{\Lambda}_k$, and gives an estimation on the size of these intersections.

Lemma 23. *Let $W^k =]-1/2, 1/2]^k$ and $\Lambda \subset \mathbf{R}^k$ be a lattice with covolume 1 such that $D_c(W^k + \Lambda) \geq 1/\ell$. Then, for every collection $v_1, \cdots, v_\ell \in \mathbf{R}^k$, there exists $i \neq i' \in [\![1, \ell]\!]$ such that*

$$D_c\big((W^k + \Lambda + v_i) \cap (W^k + \Lambda + v_{i'})\big) \geq 2\frac{\ell D_c(W^k + \Lambda) - 1}{\ell(\ell - 1)}.$$

Proof of Lemma 23. For every $v \in \mathbf{R}^k$, the density $D_c(W^k + \Lambda + v)$ is equal to the volume of the projection of W^k on the quotient space $\mathbf{R}^k/\Lambda$. As this volume is greater than $1/\ell$, and as the covolume of Λ is 1, the projections of the $W^k + v_i$ overlap, and the volume of the points belonging to at least two different sets is bigger than $\ell D_c(W^k + \Lambda) - 1$. As there are $\ell(\ell-1)/2$ possibilities of intersection, there exists $i \neq i'$ such that the volume of the intersection between the projections of $W^k + v_i$ and $W^k + v_{i'}$ is bigger than $2(\ell D_c(W^k + \Lambda) - 1)/(\ell(\ell - 1))$. Returning to the whole space $\mathbf{R}^k$, we get the conclusion of the lemma. □

Recall that we denote $\widetilde{\Lambda}_k$ the lattice spanned by the matrix

$$\widetilde{M}_{A_1, \cdots, A_k} = \begin{pmatrix} A_1 & -\mathrm{Id} & & & \\ & A_2 & -\mathrm{Id} & & \\ & & \ddots & \ddots & \\ & & & A_{k-1} & -\mathrm{Id} \\ & & & & A_k \end{pmatrix} \in M_{nk}(\mathbf{R}),$$

and W^k the cube $]-1/2, 1/2]^{nk}$. The proof of Theorem 9 will reduce to the following technical lemma.

Lemma 24. *For every $\delta > 0$ and every $M > 0$, there exists $\varepsilon > 0$ and an open set of matrices $\mathcal{O} \subset SL_n(\mathbf{R})$, which is δ-dense in the set of matrices of norm $\leq M$, such that if $\ell \geq 2$ and $D_0 > 0$ are such that for every collection of vectors $v_1, \cdots, v_\ell \in \mathbf{R}^n$, there exists $j, j' \in [\![1, \ell]\!]$ such that*

$$D_c\Bigg(\Big(W^k + \widetilde{\Lambda}_k + (0^{(k-1)n}, v_j)\Big) \cap \Big(W^k + \widetilde{\Lambda}_k + (0^{(k-1)n}, v_{j'})\Big)\Bigg) \geq D_0,$$

then for every $B \in \mathcal{O}$, if we denote by $\widetilde{\Lambda}_{k+1}$ the lattice spanned by the matrix $\widetilde{M}_{A_1, \cdots, A_k, B}$,

(1) either $D_c(W^{k+1} + \widetilde{\Lambda}_{k+1}) \leq D_c(W^k + \widetilde{\Lambda}_k) - \varepsilon D_0/(4\ell)$;

(2) or for every collection of vectors $w_1, \cdots, w_{\ell-1} \in \mathbf{R}^n$*, there exists* $i \neq i' \in [\![1, \ell-1]\!]$ *such that*

$$D_c\bigg(\Big(W^{k+1} + \widetilde{\Lambda}_{k+1} + (0^{kn}, w_i)\Big) \cap \Big(W^{k+1} + \widetilde{\Lambda}_{k+1} + (0^{kn}, w_{i'})\Big)\bigg) \geq \varepsilon D_0/\ell^2.$$

Remark 25. If $\ell = 2$, then we have automatically the conclusion (1) of the lemma; that easily implies Proposition 19.

In a certain sense, the conclusion (1) corresponds to an hyperbolic case, and the conclusion (2) expresses that there is a diffusion between times k and $k+1$.

Proof of Lemma 24. Let $\mathcal{O}_\varepsilon$ be the set of the matrices $B \in SL_n(\mathbf{R})$ satisfying: for any collection of vectors $w_1, \cdots, w_{\ell-1} \in \mathbf{R}^n$, there exists a set $U \subset \mathbf{R}^n/B\mathbf{Z}^n$ of measure $> \varepsilon$ such that every point of U belongs to at least ℓ different cubes of the collection $(Bv + w_i + W^1)_{v \in \mathbf{Z}^n, 1 \leq i \leq \ell-1}$. In other words[11], every $x \in \mathbf{R}^n$ whose projection $\overline{x}$ on $\mathbf{R}^n/B\mathbf{Z}^n$ belongs to U satisfies

$$\sum_{i=1}^{\ell-1} \sum_{v \in \mathbf{Z}^n} \mathbf{1}_{x \in Bv + w_i + W^1} \geq \ell. \tag{7}$$

We easily see that the sets $\mathcal{O}_\varepsilon$ are open and that the union of these sets over $\varepsilon > 0$ is dense (it contains the set of matrices B whose entries are all irrational). Thus, if we are given $\delta > 0$ and $M > 0$, there exists $\varepsilon > 0$ such that $\mathcal{O} = \mathcal{O}_\varepsilon$ is δ-dense in the set of matrices of $SL_n(\mathbf{R})$ whose norm is smaller than M.

We then choose $B \in \mathcal{O}$ and a collection of vectors $w_1, \cdots, w_{\ell-1} \in \mathbf{R}^n$. Let $x \in \mathbf{R}^n$ be such that $\overline{x} \in U$. By hypothesis on the matrix B, x satisfies Equation (7), so there exists $\ell+1$ integer vectors $v_1, \cdots, v_\ell$ and ℓ indices $i_1, \cdots, i_\ell$ such that the couples (v_j, i_j) are pairwise distinct and that

$$\forall j \in [\![1, \ell]\!], \quad x \in Bv_j + w_{i_j} + W^1. \tag{8}$$

The following formula makes the link between what happens in the n last and in the n penultimates coordinates of $\mathbf{R}^{n(k+1)}$:

$$W^{k+1} + \widetilde{\Lambda}_{k+1} + \big(0^{(k-1)n}, 0^n, w_{i_j}\big) = W^{k+1} + \widetilde{\Lambda}_{k+1} + \big(0^{(k-1)n}, -v_j, w_{i_j} + Bv_j\big), \tag{9}$$

(we add a vector belonging to $\widetilde{\Lambda}_{k+1}$).

We now apply the hypothesis of the lemma to the vectors $-v_1, \cdots, -v_{\ell+1}$: there exists $j \neq j' \in [\![1, \ell]\!]$ such that

$$D_c\bigg(\Big(W^k + \widetilde{\Lambda}_k + (0^{(k-1)n}, -v_j)\Big) \cap \Big(W^k + \widetilde{\Lambda}_k + (0^{(k-1)n}, -v_{j'})\Big)\bigg) \geq D_0. \tag{10}$$

Let y be a point belonging to this intersection. Applying Equations (8) and (10), we get that

$$(y, x) \in W^{k+1} + \big(\widetilde{\Lambda}_k, 0^n\big) + \big(0^{(k-1)n}, -v_j, w_{i_j} + Bv_j\big) \tag{11}$$

and the same for j'.

Two different cases can occur.

11. Matrices that does not possess this property for every $\varepsilon > 0$ are such that the union of cubes form a k-fold tiling. This was the subject of Furtwängler conjecture, see [Fur36], proved false by G. Hajós. R. Robinson gave a characterization of such k-fold tilings in some cases, see [Rob79] or [SS94, p. 29].

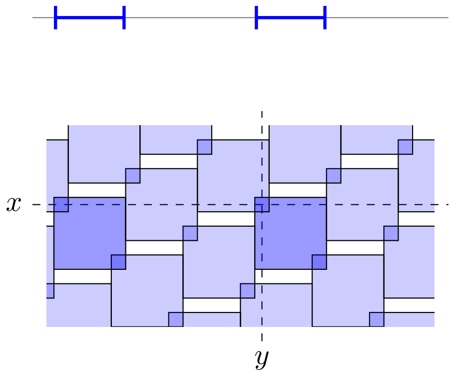

FIGURE 5. First case of Lemma 24, in the case $\ell = 3$: the set $W^{k+1} + \widetilde{\Lambda}_{k+1}$ auto-intersects.

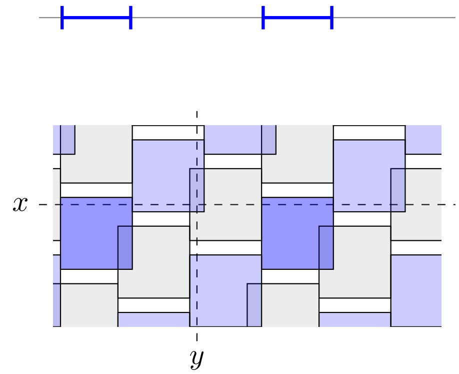

FIGURE 6. Second case of Lemma 24, in the case $\ell = 3$: two distinct vertical translates of $W^{k+1} + \widetilde{\Lambda}_{k+1}$ intersect (the first translate contains the dark blue thickening of $W^k + \widetilde{\Lambda}_k$, the second is represented in grey).

(i) Either $i_j = i_{j'}$ (that is, the translation vectors w_{i_j} and $w_{i_{j'}}$ are equal). As a consequence, applying Equation (11), we have

$$(y,x) + \big(0^{(k-1)n}, v_j, -Bv_j - w_{i_j}\big) \in \Big(W^{k+1} + \big(\widetilde{\Lambda}_k, 0^n\big)\Big) \cap \Big(W^{k+1} + \big(\widetilde{\Lambda}_k, 0^n\big) + v'\Big),$$

with

$$v' = \big(0^{(k-1)n}, -(v_{j'} - v_j), B(v_{j'} - v_j)\big) \in \widetilde{\Lambda}_{k+1} \setminus \widetilde{\Lambda}_k.$$

This implies that the set $W^{k+1} + \widetilde{\Lambda}_{k+1}$ auto-intersects (see Figure 5).

(ii) Or $i_j \neq i_{j'}$ (that is, $w_{i_j} \neq w_{i_{j'}}$). Combining Equations (11) and (9) (note that $\big(\widetilde{\Lambda}_k, 0^n\big) \subset \widetilde{\Lambda}_{k+1}$), we get

$$(y,x) \in \Big(W^{k+1} + \widetilde{\Lambda}_{k+1} + \big(0^{kn}, w_{i_j}\big)\Big) \cap \Big(W^{k+1} + \widetilde{\Lambda}_{k+1} + \big(0^{kn}, w_{i_{j'}}\big)\Big).$$

This implies that two distinct vertical translates of $W^{k+1} + \widetilde{\Lambda}_{k+1}$ intersect (see Figure 6).

We now look at the global behaviour of all the x such that $\overline{x} \in U$. Again, we have two cases.

(1) Either for more than the half of such $\overline{x}$ (for Lebesgue measure), we are in the case (i). To each of such $\overline{x}$ corresponds a translation vector w_i. We choose w_i such that the set of corresponding $\overline{x}$ has the biggest measure; this measure is bigger than $\varepsilon/\big(2(\ell-1)\big) \geq \varepsilon/(2\ell)$. Reasoning as in the proof of Proposition 19, and in particular using the notations of Corollary 22, we get that the density D_1 of the auto-intersection of $W^{k+1} + \widetilde{\Lambda}_{k+1} + (0, w_i)$ is bigger than $D_0\varepsilon/(2\ell)$. This leads to (using Corollary 22)

$$D_c(W^{k+1} + \widetilde{\Lambda}_{k+1}) < D_c(W^k + \widetilde{\Lambda}_k) - \frac{D_0 \varepsilon}{4\ell}.$$

In this case, we get the conclusion (1) of the lemma.

(2) Or for more than the half of such $\overline{x}$, we are in the case (ii). Choosing the couple $(w_i, w_{i'})$ such that the measure of the set of corresponding $\overline{x}$ is the greatest, we

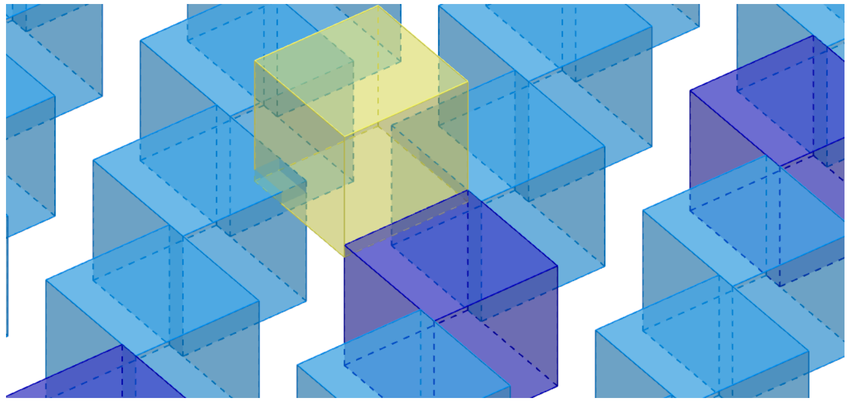

FIGURE 7. Intersection of cubes in the case where the rate is bigger than 1/3. The thickening of the cubes of $W^k+\widetilde\Lambda_k$ is represented in dark blue and the thickening of the rest of the cubes of $W^{k+1}+\widetilde\Lambda_{k+1}$ is represented in light blue; we have also represented another cube of $W^{k+2}+\widetilde\Lambda_{k+2}$ in yellow. We see that if the projection on the z-axis of the centre of the yellow cube is smaller than 1, then there is automatically an intersection between this cube and one of the blue cubes.

get

$$D_c\bigg(\Big(W^{k+1}+\widetilde\Lambda_{k+1}+(0^{kn},w_i)\Big)\cap\Big(W^{k+1}+\widetilde\Lambda_{k+1}+(0^{kn},w_{i'})\Big)\bigg)\geq\frac{D_0\varepsilon}{(\ell-1)(\ell-2)}.$$

In this case, we get the conclusion (2) of the lemma.

□

We can now prove Theorem 9.

Proof of Theorem 9. As in the proof of Proposition 19, we proceed by induction on k. Suppose that $\widetilde\Lambda_k$ is such that $D_c(W^k+\widetilde\Lambda_k)>1/\ell$. Then, Lemma 23 ensures that it is not possible to have ℓ disjoint translates of $W^k+\widetilde\Lambda_k$. Applying Lemma 24, we obtain that either $D_c(W^{k+1}+\widetilde\Lambda_{k+1})<D_c(W^k+\widetilde\Lambda_k)$, or it is not possible to have $\ell-1$ disjoint translates of $W^{k+1}+\widetilde\Lambda_{k+1}$. And so on, applying Lemma 24 at most $\ell-1$ times, there exists $k'\in[\![k+1,k+\ell-1]\!]$ such that $W^{k'}+\widetilde\Lambda_{k'}$ has additional auto-intersections. Quantitatively, combining Lemmas 23 and 24, we get

$$D_c\big(W^{k+\ell-1}+\widetilde\Lambda_{k+\ell-1}\big)\leq D\big(W^k+\widetilde\Lambda_k\big)-\frac{\varepsilon}{4\ell}\left(\frac{\varepsilon}{\ell^2}\right)^{\ell-1}2\frac{\ell D_c(W^k+\widetilde\Lambda_k)-1}{\ell(\ell-1)},$$

thus

$$D_c\big(W^{k+\ell-1}+\widetilde\Lambda_{k+\ell-1}\big)-1/\ell\leq\left(1-\frac{1}{2}\left(\frac{\varepsilon}{\ell^2}\right)^{\ell}\right)\Big(D_c\big(W^k+\widetilde\Lambda_k\big)-1/\ell\Big),$$

in other words, if we denote $\overline\tau^k=\overline\tau^k(B_1,\cdots,B_k)$ and $\lambda_\ell=1-\left(\frac{\varepsilon}{\ell^2}\right)^\ell$,

$$\overline\tau^{k+\ell-1}-1/\ell\leq\lambda_\ell\Big(\overline\tau^k-1/\ell\Big). \tag{12}$$

This implies that for every $\ell>0$, the sequence of rates $\overline\tau^k$ is smaller than a sequence converging exponentially fast to $1/\ell$: we get Equation (5). In particular, the asymptotic rate of injectivity is generically equal to zero.

We now prove the estimation of Equation (6). Suppose that $\overline\tau^k\in[1/(\ell-2),1/(\ell-1)]$, we compute how long it takes for the rate to be smaller than $1/(\ell-1)$. We apply Equation (12) j times to ℓ and get

$$\overline{\tau}^{k+j\ell} - 1/\ell \leq \lambda_\ell^j(\overline{\tau}^k - 1/\ell),$$

with $\lambda = 1 - \left(\frac{\varepsilon}{\ell^2}\right)^\ell$. In the worst case, we have $\overline{\tau}^k = 1/(\ell-2)$, thus if j' satisfies

$$\frac{1}{\ell-1} - \frac{1}{\ell} = \lambda_\ell^{j'}\left(\frac{1}{\ell-2} - \frac{1}{\ell}\right), \tag{13}$$

then $j = \lceil j' \rceil$ is such that $\overline{\tau}^{k+j\ell} \leq 1/(\ell-1)$. Equation (13) is equivalent to

$$j' = \frac{-1}{\log \lambda_\ell}\left(\log 2 - \log\left(1 - \frac{1}{\ell-1}\right)\right).$$

And for ℓ very big, we have the equivalent (recall that $\lambda_\ell = 1 - \frac{1}{2}\left(\frac{\varepsilon}{\ell^2}\right)^\ell$)

$$j' \sim \left(\frac{\ell^2}{\varepsilon}\right)^\ell 2\log 2.$$

Thus, when ℓ is large enough, the time it takes for the rate to decrease from $1/(\ell-2)$ to $1/(\ell-1)$ is smaller than $\ell^{2(\ell+1)} = e^{2(\ell+1)\log\ell}$.

On the other hand, if we set $f(k) = (\log k)^{-\nu}$, the time it takes for f to go from $1/(\ell-2)$ to $1/(\ell-1)$ is equal to

$$e^{(\ell-1)^{1/\nu}} - e^{(\ell-2)^{1/\nu}} = e^{(\ell-1)^{1/\nu}}\left(1 - e^{(\ell-2)^{1/\nu}-(\ell-1)^{1/\nu}}\right) \sim e^{(\ell-1)^{1/\nu}}$$

when l goes to infinity, thus smaller than $e^{(\ell-1)^{1/\nu}+1}$ when ℓ is large enough. But we have $2(\ell+1)\log\ell = o\big((\ell-1)^{1/\nu}+1\big)$. So, when ℓ is large enough, it takes arbitrarily much more time for $\overline{\tau}^k$ to decrease from $1/(\ell-2)$ to $1/(\ell-1)$ than for f to decrease from $1/(\ell-2)$ to $1/(\ell-1)$. As a consequence, $\overline{\tau}(k) = o(f(k))$. □

3. A local-global formula for C^r-generic expanding maps and C^r-generic diffeomorphisms

The goal of this section is to study the rate of injectivity of generic C^r-expanding maps and C^r-generic diffeomorphisms of the torus $\mathbf{T}^n$. Here, the term expanding map is taken from the point of view of discretizations: we say that a linear map A is expanding if there does not exist two distinct integer points $x, y \in \mathbf{Z}^n$ such that $\widehat{A}(x) = \widehat{A}(y)$. This condition is satisfied in particular if for every $x \neq 0$, we have $\|Ax\|_\infty \leq \|x\|_\infty$; thus when $n = 1$ this definition coincides with the classical definition of expanding map.

The main result of this section is that the rate of injectivity of both generic C^1-diffeomorphisms and generic C^r-expanding maps of the torus $\mathbf{T}^n$ is obtained from a local-global formula (Theorems 26 and 31). Let us begin by explaining the case of diffeomorphisms.

Theorem 26. *Let $r \geq 1$, and $f \in \mathrm{Diff}^r(\mathbf{T}^n)$ (or $f \in \mathrm{Diff}^r(\mathbf{T}^n, \mathrm{Leb})$) be a generic diffeomorphism. Then $\tau^k(f)$ is well defined (that is, the limit superior in (1) is a limit) and satisfies:*

$$\tau^k(f) = \int_{\mathbf{T}^n} \tau^k\left(Df_x, \cdots, Df_{f^{k-1}(x)}\right) \mathrm{d}\,\mathrm{Leb}(x).$$

Moreover, the function τ^k is continuous in f.

The idea of the proof of this theorem is very simple: locally, the diffeomorphism is almost equal to a linear map. This introduces an intermediate *mesoscopic* scale on the torus:

– at the macroscopic scale, the discretization of f acts as f;

– at the intermediate mesoscopic scale, the discretization of f acts as a linear map;
– at the microscopic scale, we are able to see that the discretization is a finite map and we see that the phase space is discrete.

This remark is formalized by Taylor's formula: for every $\varepsilon > 0$ and every $x \in \mathbf{T}^n$, there exists $\rho > 0$ such that f and its Taylor expansion at order 1 are ε-close on $B(x,\rho)$. We then suppose that the derivative Df_x is "good": the rate of injectivity of any of its C^1-small perturbations can be seen on a ball B_R of $\mathbf{R}^n$ (with R uniform in x). Then, the proof of the local-global formula is made in two steps.

– Prove that "a lot" of maps of $SL_n(\mathbf{R})$ are "good". This is formalized by Lemma 32, which gives estimations of the size of the perturbations of the linear map allowed, and of the size of the ball B_R. Its proof is quite technical and uses crucially the formalism of model sets, and an improvement of Weyl's criterion.
– Prove that for a generic diffeomorphism, the derivative satisfies the conditions of Lemma 32 at almost every point. This follows easily from Thom's transversality theorem.

As the case of expanding maps is more complicated but similar, we will prove the local-global formula only for expanding maps; the adaptation of it for diffeomorphisms is straightforward.

Remark that the hypothesis of genericity is necessary to get Theorem 26. For example, it can be seen that if we set

$$f_0 = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{2} & 1 \end{pmatrix},$$

then $\tau(f_0) = 1/2$ whereas $\tau(f_0 + (1/4, 3/4)) = 3/4$. Thus, if g is a diffeomorphism of the torus which is equal to $f_0 + v$ on an open subset of $\mathbf{T}^2$, with v a suitable translation vector, then the conclusions of Theorem 26 does not hold (see Example 11.4 of [Gui15c] for more explanations).

The definition of the linear analogue of the rate of injectivity of an expanding map in time k is more complicated than for diffeomorphisms: in this case, the set of preimages has a structure of d-ary tree. We define the rate of injectivity of a tree — with edges decorated by linear expanding maps — as the probability of percolation of a random graph associated to this decorated tree (see Definition 29). In particular, if all the expanding maps were equal, then the connected component of the root of this random graph is a Galton-Watson tree. We begin by the definition of the set of expanding maps.

Definition 27. For $r \geq 1$ and $d \geq 2$, we denote by $\mathcal{D}^r(\mathbf{T}^n)$ the set of C^r "$\mathbf{Z}^n$-expanding maps" of $\mathbf{T}^n$ for the infinite norm. More precisely, $\mathcal{D}^r(\mathbf{T}^n)$ is the set of maps $f : \mathbf{T}^n \to \mathbf{T}^n$, which are local diffeomorphisms, such that the derivative $f^{(\lfloor r \rfloor)}$ is well defined and belongs to $C^{r-\lfloor r \rfloor}(\mathbf{T}^n)$ and such that for every $x \in \mathbf{T}^n$ and every $v \in \mathbf{Z}^n \setminus \{0\}$, we have $\|Df_x v\|_\infty \geq 1$.

In particular, for $f \in \mathcal{D}^r(\mathbf{T}^n)$, the number of preimages of any point of $\mathbf{T}^n$ is equal to a constant, that we denote by d.

Remark that in dimension $n = 1$, the set $\mathcal{D}^r(\mathbf{S}^1)$ coincides with the classical set of expanding maps: $f \in \mathcal{D}^r(\mathbf{S}^1)$ if and only if it belongs to $C^r(\mathbf{S}^1)$ and $f'(x) \geq 1$ for every $x \in \mathbf{S}^1$.

We now define the linear setting corresponding to a map $f \in \mathcal{D}(\mathbf{T}^n)$.

Definition 28. We set (see also Figure 8)

$$I_k = \bigsqcup_{m=1}^{k} [\![1,d]\!]^m$$

the set of m-tuples of integers of $[\![1,d]\!]$, for $1 \leq m \leq k$.

For $\boldsymbol{i} = (i_1, \cdots, i_m) \in [\![1,d]\!]^m$, we set $\text{length}(\boldsymbol{i}) = m$ and $\text{father}(\boldsymbol{i}) = (i_1, \cdots, i_{m-1}) \in [\![1,d]\!]^{m-1}$ (with the convention $\text{father}(i_1) = \emptyset$).

The set I_k is the linear counterpart of the set $\bigsqcup_{m=1}^{k} f^{-m}(y)$. Its cardinal is equal to $d(1-d^k)/(1-d)$.

Definition 29. Let $k \in \mathbf{N}$. The *complete tree of order* k is the rooted d-ary tree T_k whose vertices are the elements of I_k together with the root, denoted by $\emptyset$, and whose edges are of the form $(\text{father}(\boldsymbol{i}), \boldsymbol{i})_{\boldsymbol{i} \in I_k}$ (see Figure 8).

Let $(p_{\boldsymbol{i}})_{\boldsymbol{i} \in I_k}$ be a family of numbers belonging to $[0,1]$. These probabilities will be seen as decorations of the edges of the tree T_k. We will call *random graph associated to* $(p_{\boldsymbol{i}})_{\boldsymbol{i} \in I_k}$ the random subgraph $G_{(p_{\boldsymbol{i}})_{\boldsymbol{i}}}$ of T_k, such that the laws of appearance the edges $(\text{father}(\boldsymbol{i}), \boldsymbol{i})$ of $G_{(p_{\boldsymbol{i}})_{\boldsymbol{i}}}$ are independent Bernoulli laws of parameter $p_{\boldsymbol{i}}$. In other words, $G_{(p_{\boldsymbol{i}})_{\boldsymbol{i}}}$ is obtained from T_k by erasing independently each vertex of T_k with probability $1 - p_{\boldsymbol{i}}$.

We define the *mean density* $\overline{D}((p_{\boldsymbol{i}})_{\boldsymbol{i}})$ of $(p_{\boldsymbol{i}})_{\boldsymbol{i} \in I_k}$ as the probability that in $G_{(p_{\boldsymbol{i}})_{\boldsymbol{i}}}$, there is at least one path linking the root to a leaf.

Remark that if the probabilities $p_{\boldsymbol{i}}$ are constant equal to p, the random graph $G_{(p_{\boldsymbol{i}})_{\boldsymbol{i}}}$ is a Galton-Watson tree, where the probability for a vertex to have i children is equal to $\binom{d}{i} p^i (1-p)^{d-i}$.

Definition 30. By the notation $\overline{D}((\det Df_x^{-1})_{x \in f^{-m}(y),\, 1 \leq m \leq k})$, we will mean that the mean density is taken with respect to the random graph $G_{f,y}$ associated to the decorated tree whose vertices are the $f^{-m}(y)$ for $0 \leq m \leq k$, and whose edges are of the form $(f(x), x)$ for $x \in f^{-m}(y)$ with $1 \leq m \leq k$, each one being decorated by the number $\det Df_x^{-1}$ (see Figure 9).

Recall that the rates of injectivity are defined by (see also Definition 5)

$$\tau^k(f_N) = \frac{\text{Card}\big((f_N)^k(E_N)\big)}{\text{Card}(E_N)} \qquad \text{and} \qquad \tau^k(f) = \limsup_{N \to +\infty} \tau^k(f_N).$$

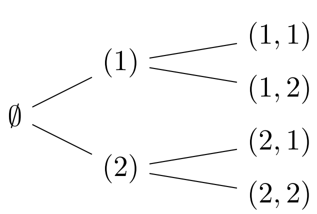

FIGURE 8. The tree T_2 for $d = 2$.

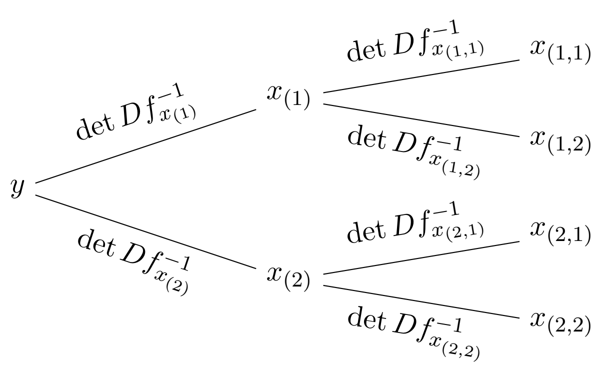

FIGURE 9. The probability tree associated to the preimages of y, for $k = 2$ and $d = 2$. We have $f(x_{(1,1)}) = f(x_{(1,2)}) = x_{(1)}$, etc.

Theorem 31. *Let $r \geq 1$, f a generic element of $\mathcal{D}^r(\mathbf{T}^n)$ and $k \in \mathbf{N}$. Then, $\tau^k(f)$ is a limit (that is, the sequence $(\tau^k(f_N))_N$ converges), and we have*

$$\tau^k(f) = \int_{\mathbf{T}^n} \overline{D}\big((\det Df_x^{-1})_{\substack{1\leq m\leq k\\ x\in f^{-m}(y)}}\big)\, \mathrm{d}\,\mathrm{Leb}(y). \tag{14}$$

Moreover, the map $f \mapsto \tau^k(f)$ is continuous in f.

The proof of Theorem 31 is mainly based on the following lemma, which treats the linear corresponding case. Its statement is divided into two parts, the second one being a quantitative version of the first.

Lemma 32. *Let $k \in \mathbf{N}$, and a family $(A_{\boldsymbol{i}})_{\boldsymbol{i}\in I_k}$ of invertible matrices, such that for any $\boldsymbol{i} \in I_k$ and any $v \in \mathbf{Z}^n \setminus \{0\}$, we have $\|A_{\boldsymbol{i}} v\|_\infty \geq 1$.*

If the image of the map

$$\mathbf{Z}^n \ni x \mapsto \bigoplus_{\boldsymbol{i}\in I_k} A_{\boldsymbol{i}}^{-1} A_{\mathrm{father}(\boldsymbol{i})}^{-1} \cdots A_{\mathrm{father}^{\mathrm{length}(\boldsymbol{i})}(\boldsymbol{i})}^{-1} x$$

projects on a dense subset of the torus $\mathbf{R}^{n\,\mathrm{Card}\, I_k}/\mathbf{Z}^{n\,\mathrm{Card}\, I_k}$, then we have

$$D\left(\bigcup_{\boldsymbol{i}\in[\![1,d]\!]^k} \big(\widehat{A}_{\mathrm{father}^{k-1}(\boldsymbol{i})} \circ \cdots \circ \widehat{A}_{\boldsymbol{i}}\big)(\mathbf{Z}^n)\right) = \overline{D}\big((\det A_{\boldsymbol{i}}^{-1})_{\boldsymbol{i}}\big).$$

More precisely, for every $\ell', c \in \mathbf{N}$, there exists a locally finite union of positive codimension submanifolds V_q of $(GL_n(\mathbf{R}))^{\mathrm{Card}\, I_k}$ such that for every $\eta' > 0$, there exists a radius $R_0 > 0$ such that if $(A_{\boldsymbol{i}})_{\boldsymbol{i}\in I_k}$ satisfies $d((A_{\boldsymbol{i}})_{\boldsymbol{i}}, V_q) > \eta'$ for every q, then for every $R \geq R_0$, and every family $(v_{\boldsymbol{i}})_{\boldsymbol{i}\in I_k}$ of vectors of $\mathbf{R}^n$, we have[12]

$$\left| D_R^+\left(\bigcup_{\boldsymbol{i}\in[\![1,d]\!]^k} \big(\pi(A_{\mathrm{father}^{k-1}(\boldsymbol{i})} + v_{\mathrm{father}^{k-1}(\boldsymbol{i})}) \circ \cdots \circ \pi(A_{\boldsymbol{i}} + v_{\boldsymbol{i}})\big)(\mathbf{Z}^n)\right) - \overline{D}\big((\det A_{\boldsymbol{i}}^{-1})_{\boldsymbol{i}}\big)\right| < \frac{1}{\ell'} \tag{15}$$

(the density of the image set is "almost invariant" under perturbations by translations), and for every $m \leq k$ and every $\boldsymbol{i} \in [\![1,d]\!]^k$, we have[13]

$$D_R^+\left\{x \in \big(A_{\mathrm{father}^m(\boldsymbol{i})} + v_{\mathrm{father}^m(\boldsymbol{i})}\big)(\mathbf{Z}^n) \;\middle|\; d\big(x, (\mathbf{Z}^n)'\big) < \frac{1}{c\ell'(2n+1)}\right\} < \frac{1}{c\ell'} \tag{16}$$

(there is only a small proportion of the points of the image sets which are obtained by discretizing points close to $(\mathbf{Z}^n)'$).

The local-global formula (14) will later follow from this lemma, an appropriate application of Taylor's theorem and Thom's transversality theorem (Lemma 35).

The next lemma uses the strategy of proof of Weyl's criterion to get a uniform convergence in Birkhoff's theorem for rotations of the torus $\mathbf{T}^n$ whose rotation vectors are outside of a neighbourhood of a finite union of hyperplanes.

Lemma 33 (Weyl)**.** *Let dist be a distance generating the weak-* topology on $\mathcal{P}$ the space of Borel probability measures on $\mathbf{T}^n$. Then, for every $\varepsilon > 0$, there exists a locally finite family of affine hyperplanes $H_i \subset \mathbf{R}^n$, such that for every $\eta > 0$, there exists $M_0 \in \mathbf{N}$, such that for every $\boldsymbol{\lambda} \in \mathbf{R}^n$ satisfying $d(\boldsymbol{\lambda}, H_q) > \eta$ for every q, and for every $M \geq M_0$, we have*

$$\mathrm{dist}\left(\frac{1}{M}\sum_{m=0}^{M-1} \bar{\delta}_{m\boldsymbol{\lambda}},\ \mathrm{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}\right) < \varepsilon,$$

12. The map $\pi(A+v)$ is the discretization of the affine map $A+v$.
13. Where $(\mathbf{Z}^n)'$ stands for the set of points of $\mathbf{R}^n$ at least one coordinate of which belongs to $\mathbf{Z}+1/2$.

where $\bar\delta_x$ *is the Dirac measure of the projection of* x *on* $\mathbf{R}^n/\mathbf{Z}^n$*.*

Proof of Lemma 33. As dist generates the weak-* topology on $\mathcal{P}$, it can be replaced by any other distance also generating the weak-* topology on $\mathcal{P}$. So we consider the distance dist_W defined by:

$$\mathrm{dist}_W(\mu,\nu) = \sum_{\boldsymbol{k}\in\mathbf{N}^n} \frac{1}{2^{k_1+\cdots+k_n}} \left| \int_{\mathbf{R}^n/\mathbf{Z}^n} e^{i2\pi\boldsymbol{k}\cdot x} \,\mathrm{d}(\mu-\nu)(x)\right|;$$

there exists $K>0$ and $\varepsilon'>0$ such that if a measure $\mu\in\mathcal{P}$ satisfies

$$\forall \boldsymbol{k}\in\mathbf{N}^n : 0<k_1+\cdots+k_n\le K, \qquad \left|\int_{\mathbf{R}^n/\mathbf{Z}^n} e^{i2\pi\boldsymbol{k}\cdot x}\,\mathrm{d}\,\mu(x)\right| < \varepsilon', \tag{17}$$

then $\mathrm{dist}(\mu,\mathrm{Leb})<\varepsilon$.

For every $\boldsymbol{k}\in\mathbf{N}^n\setminus\{0\}$ and $j\in\mathbf{Z}$, we set

$$H^j_{\boldsymbol{k}} = \{\boldsymbol{\lambda}\in\mathbf{R}^n \mid \boldsymbol{k}\cdot\boldsymbol{\lambda} = j\}.$$

Remark that the family $\{H^j_{\boldsymbol{k}}\}$, with $j\in\mathbf{Z}$ and $\boldsymbol{k}$ such that $0<k_1+\cdots+k_n\le K$, is locally finite. We denote by $\{H_q\}_q$ this family, and choose $\boldsymbol{\lambda}\in\mathbf{R}^n$ such that $d(\boldsymbol{\lambda},H_q)>\eta$ for every q. We also take

$$M_0 \ge \frac{2}{\varepsilon'\,|1-e^{i2\pi\eta}|}. \tag{18}$$

Thus, for every $\boldsymbol{k}\in\mathbf{N}^n$ such that $k_1+\cdots+k_n\le K$, and every $M\ge M_0$, the measure

$$\mu = \frac{1}{M}\sum_{m=0}^{M-1}\bar\delta_{m\boldsymbol{\lambda}}.$$

satisfies

$$\left|\int_{\mathbf{R}^n/\mathbf{Z}^n} e^{i2\pi\boldsymbol{k}x}\,\mathrm{d}\,\mu(x)\right| = \frac{1}{M}\left|\frac{1-e^{i2\pi M\boldsymbol{k}\cdot\boldsymbol{\lambda}}}{1-e^{i2\pi\boldsymbol{k}\cdot\boldsymbol{\lambda}}}\right| \le \frac{2}{M_0}\frac{1}{|1-e^{i2\pi\boldsymbol{k}\cdot\boldsymbol{\lambda}}|}.$$

By (18) and the fact that $d(\boldsymbol{k}\cdot\boldsymbol{\lambda},\mathbf{Z})\ge\eta$, we deduce that

$$\left|\int_{\mathbf{R}^n/\mathbf{Z}^n} e^{i2\pi\boldsymbol{k}x}\,\mathrm{d}\,\mu(x)\right| \le \varepsilon'.$$

Thus, the measure μ satisfies the criterion (17), which proves the lemma. □

Proof of Lemma 32. To begin with, let us treat the case $d=1$. Let $A_1,\cdots,A_k$ be k invertible matrices. We want to compute the rate of injectivity of $\widehat{A}_k\circ\cdots\circ\widehat{A}_1$. Recall that we set

$$\widetilde{M}_{A_1,\cdots,A_k} = \begin{pmatrix} A_1 & -\mathrm{Id} & & & \\ & A_2 & -\mathrm{Id} & & \\ & & \ddots & \ddots & \\ & & & A_{k-1} & -\mathrm{Id} \\ & & & & A_k \end{pmatrix} \in M_{nk}(\mathbf{R}),$$

$\widetilde{\Lambda}_k = \widetilde{M}_{\lambda_1,\cdots,\lambda_k}\mathbf{Z}^{nk}$ and $W^k =]-1/2,1/2]^{nk}$. Resuming the proof of Proposition 12, we see that $x\in\big(\widehat{A}_k\circ\cdots\circ\widehat{A}_1\big)(\mathbf{Z}^n)$ if and only if $(0^{n(k-1)},x)\in W^k+\widetilde{\Lambda}_k$. This implies the following statement.

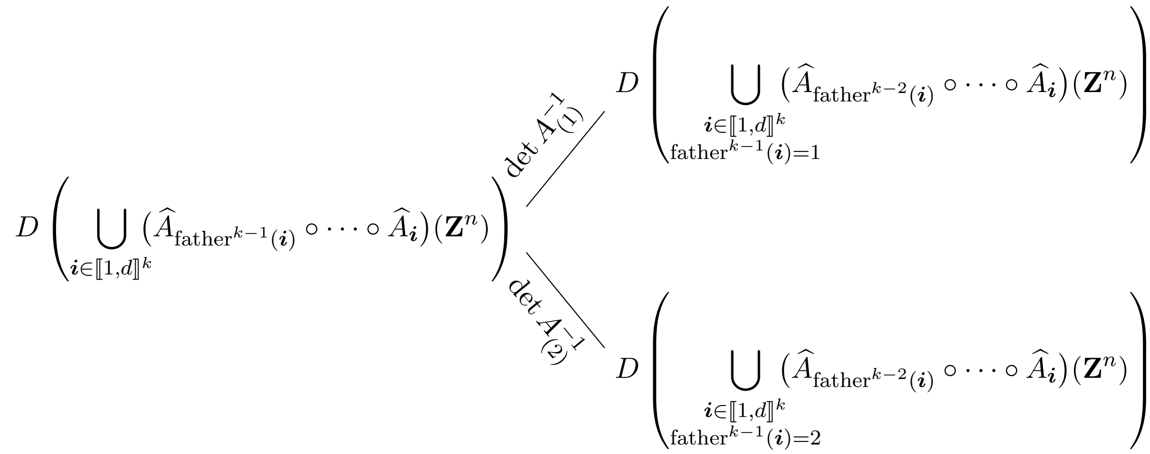

FIGURE 10. Calculus of the density of the image set at the level k according to the density of its sons.

Lemma 34. *We have*

$$\det(A_k\cdots A_1)D\big(\widehat{A}_k\circ\cdots\circ\widehat{A}_1\big)(\mathbf{Z}^n)=\nu(\mathrm{pr}_{\mathbf{R}^{nk}/\widetilde{\Lambda}_k}(W^k)),\tag{19}$$

where ν is the uniform measure on the submodule $\mathrm{pr}_{\mathbf{R}^{nk}/\widetilde{\Lambda}_k}(0^{n(k-1)},\mathbf{Z}^n)$ of $\mathbf{R}^{nk}/\widetilde{\Lambda}_k$.

In particular, if the image of the map

$$\mathbf{Z}^n\ni x\mapsto\bigoplus_{m=1}^{k}(A_m)^{-1}\cdots(A_k)^{-1}x$$

projects on a dense subset of the torus $\mathbf{R}^{nk}/\mathbf{Z}^{nk}$, then the quantity (19) is equal to the volume of the intersection between the projection of W^k on $\mathbf{R}^{nk}/\widetilde{\Lambda}_k$ and a fundamental domain of $\widetilde{\Lambda}_k$ (see the end of the proof of Proposition 12 and in particular the form of the matrix $\widetilde{M}^{-1}_{A_1,\cdots,A_k}$). By the hypothesis made on the matrices A_m — that is, for any $v\in\mathbf{Z}^n\setminus\{0\}$, $\|A_mv\|_\infty\geq 1$ — this volume is equal to 1 (simply because the restriction to W^k of the projection $\mathbf{R}^{nk}\mapsto\mathbf{R}^{nk}/\widetilde{\Lambda}_k$ is injective). Thus, the density of the set $\big(\widehat{A}_k\circ\cdots\circ\widehat{A}_1\big)(\mathbf{Z}^n)$ is equal to $1/(\det(A_k\cdots A_1))$.

We now consider the general case where d is arbitrary. We take a family $(A_{\boldsymbol{i}})_{\boldsymbol{i}\in I_k}$ of invertible matrices, such that for any $\boldsymbol{i}\in I_k$ and any $v\in\mathbf{Z}^n\setminus\{0\}$, we have $\|A_{\boldsymbol{i}}v\|_\infty\geq 1$. A point $x\in\mathbf{Z}^n$ belongs to

$$\bigcup_{\boldsymbol{i}\in[\![1,d]\!]^k}\big(\widehat{A}_{\mathrm{father}^{k-1}(\boldsymbol{i})}\circ\cdots\circ\widehat{A}_{\boldsymbol{i}}\big)(\mathbf{Z}^n)$$

if and only if there exists $\boldsymbol{i}\in[\![1,d]\!]^k$ such that $(0^{m-1},x)\in W^k+\widetilde{\Lambda}_{\boldsymbol{i}}$. Equivalently, a point $x\in\mathbf{Z}^n$ does not belong to the set

$$\big(\widehat{A}_{\mathrm{father}^{k-1}(\boldsymbol{i})}\circ\cdots\circ\widehat{A}_{\boldsymbol{i}}\big)(\mathbf{Z}^n)$$

if and only if for every $\boldsymbol{i}\in[\![1,d]\!]^k$, we have $(0^{n(k-1)},x)\notin W^k+\widetilde{\Lambda}_{\boldsymbol{i}}$. Thus, if the image of the map

$$\mathbf{Z}^n\ni x\mapsto\bigoplus_{\boldsymbol{i}\in I_k}A_{\boldsymbol{i}}^{-1}A_{\mathrm{father}(\boldsymbol{i})}^{-1}\cdots A_{\mathrm{father}^{\mathrm{length}(\boldsymbol{i})}(\boldsymbol{i})}^{-1}x$$

projects on a dense subset of the torus $\mathbf{R}^{n\,\mathrm{Card}\,I_k}/\mathbf{Z}^{n\,\mathrm{Card}\,I_k}$, then the events $x \in S_i$, with

$$S_i = \bigcup_{\substack{\boldsymbol{i}\in[\![1,d]\!]^k\\ \mathrm{father}^{k-1}(\boldsymbol{i})=i}} \big(\widehat{A}_{\mathrm{father}^{k-1}(\boldsymbol{i})} \circ \cdots \circ \widehat{A}_{\boldsymbol{i}}\big)(\mathbf{Z}^n)$$

are independent (see Figure 10), meaning that for every $F \subset [\![1,d]\!]$, we have

$$D\left(\bigcap_{i\in F} S_i\right) = \prod_{i\in F} D(S_i). \tag{20}$$

Thus, by the inclusion-exclusion principle, we get

$$D\left(\bigcup_{i\in[\![1,d]\!]} S_i\right) = \sum_{\emptyset\neq F\subset[\![1,d]\!]} (-1)^{\mathrm{Card}(F)+1} \prod_{i\in F} D(S_i).$$

Moreover, the fact that for any $\boldsymbol{i} \in I_k$ and any $v \in \mathbf{Z}^n \setminus \{0\}$, we have $\|A_{\boldsymbol{i}} v\|_\infty \geq 1$ leads to

$$D(S_i) = \det A^{-1}_{\mathrm{father}^{k-1}(\boldsymbol{i})}\; D\left(\bigcup_{\substack{\boldsymbol{i}\in[\![1,d]\!]^k\\ \mathrm{father}^{k-1}(\boldsymbol{i})=i}} \big(\widehat{A}_{\mathrm{father}^{k-2}(\boldsymbol{i})} \circ \cdots \circ \widehat{A}_{\boldsymbol{i}}\big)(\mathbf{Z}^n)\right).$$

These facts imply that the density we look for follows the same recurrence relation as $\overline{D}\big((\det A_{\boldsymbol{i}}^{-1})_{\boldsymbol{i}}\big)$, thus

$$D\left(\bigcup_{\boldsymbol{i}\in[\![1,d]\!]^k} \big(\widehat{A}_{\mathrm{father}^{k-1}(\boldsymbol{i})} \circ \cdots \circ \widehat{A}_{\boldsymbol{i}}\big)(\mathbf{Z}^n)\right) = \overline{D}\big((\det A_{\boldsymbol{i}}^{-1})_{\boldsymbol{i}}\big).$$

The second part of the lemma is an effective improvement of the first one. To obtain the bound (15), we combine Lemma 33 with Lemma 34 to get that for every $\varepsilon > 0$, there exists a locally finite collection of submanifolds V_q of $(GL_n(\mathbf{R}))^{\mathrm{Card}\,I_k}$ with positive codimension, such that for every $\eta' > 0$, there exists $R_0 > 0$ such that if $d((A_{\boldsymbol{i}})_{\boldsymbol{i}}, V_q) > \eta'$ for every q, then Equation (20) is true up to ε.

The other bound (16) is obtained independently from the rest of the proof by a direct application of Lemma 33 and of Lemma 34 applied to $k = 1$. □

Lemma 35 (Perturbations in C^r topology)**.** *Let $1 \leq r \leq +\infty$ and f a generic element of $\mathcal{D}^r(\mathbf{T}^n)$. Then, for every $k \in \mathbf{N}$, every $\ell' \in \mathbf{N}$ and every finite collection (V_q) of submanifolds of positive codimension of $(GL_n(\mathbf{R}))^{dm}$, there exists $\eta > 0$ such that the set*

$$T_\eta = \left\{ y \in \mathbf{T}^n \;\middle|\; \forall q,\ d\Big(\big(Df_x\big)_{\substack{1\leq m\leq k\\ x\in f^{-m}(y)}},\ V_q\Big) > \eta \right\}$$

contains a finite disjoint union of cubes[14]*, whose union has measure bigger than $1 - 1/\ell'$.*

Proof of Lemma 35. By Thom's transversality theorem, for a generic map $f \in \mathcal{D}^r(\mathbf{T}^n)$, the set

$$\left\{ y \in \mathbf{T}^n \;\middle|\; \forall q,\ \big(Df_x\big)_{\substack{1\leq m\leq k\\ x\in f^{-m}(y)}} \in V_q \right\}$$

14. here, a cube is just any ball for the infinite norm.

if finite. Thus, the sets $T_\eta^\complement$ are compact sets and their (decreasing) intersection over η is a finite set. So, there exists $\eta > 0$ such that $T_\eta^\complement$ is close enough to this finite set for Hausdorff topology to have the conclusions of the lemma. □

We can now begin the proof of Theorem 31.

Proof of Theorem 31. Let $f \in \mathcal{D}^r(\mathbf{T}^n)$. The idea is to cut the torus $\mathbf{T}^n$ into small pieces on which f is very close to its Taylor expansion at order 1.

Let $m \in \mathbf{N}$, and $\mathcal{U}_\ell$ ($\ell \in \mathbf{N}^*$) be the set of maps $f \in \mathcal{D}^r(\mathbf{T}^n)$ such that the set of accumulation points of the sequence $(\tau_N^m(f))_N$ is included in the ball of radius $1/\ell$ and centre

$$\int_{\mathbf{T}^n} \overline{D}\big((Df_x)_{\substack{1\le m\le k\\ x\in f^{-m}(y)}}\big)\,\mathrm{d}\,\mathrm{Leb}(y).$$

(that is, the right side of Equation (14)). We want to show that $\mathcal{U}_\ell$ contains an open and dense subset of $\mathcal{D}^r(\mathbf{T}^n)$. In other words, we pick a map f, an integer ℓ and $\delta > 0$, and we want to find another map $g \in \mathcal{D}^r(\mathbf{T}^n)$ which is δ-close to f for the C^r distance, and which belongs to the interior of $\mathcal{U}_\ell$.

To do that, we first set $\ell' = 3\ell$ and $c = d(1-d^k)/(1-d) = \mathrm{Card}(I_k)$, and use Lemma 32 to get a locally finite union of positive codimension submanifolds V_q of $(GL_n(\mathbf{R}))^{\mathrm{Card}(I_k)}$. We then apply Lemma 35 to these submanifolds, to the δ we have fixed at the beginning of the proof and to $\ell' = 4\ell$; this gives us a parameter $\eta > 0$ and a map $g \in \mathcal{D}^r(\mathbf{T}^n)$ such that $d_{C^r}(f,g) < \delta$, and such that the set

$$\left\{ y \in \mathbf{T}^n \;\middle|\; \forall q,\ d\Big(\big(Dg_x\big)_{\substack{1\le m\le k\\ x\in g^{-m}(y)}}, V_q\Big) < \eta \right\}$$

is contained in a disjoint finite union $\mathcal{C}$ of cubes, whose union has measure bigger than $1-1/(4\ell)$. Finally, we apply Lemma 32 to $\eta' = \eta/2$; this gives us a radius $R_0 > 0$ such that if $(A_{\boldsymbol{i}})_{\boldsymbol{i}\in I_k}$ is a family of matrices of $GL_n(\mathbf{R})$ satisfying $d((A_{\boldsymbol{i}})_{\boldsymbol{i}}, V_q) > \eta/2$ for every q, then for every $R \ge R_0$, and every family $(v_{\boldsymbol{i}})_{\boldsymbol{i}\in I_k}$ of vectors of $\mathbf{R}^n$, we have

$$\left| D_R^+\left(\bigcup_{\boldsymbol{i}\in[\![1,d]\!]^k} \big(\pi(A_{\mathrm{father}^{k-1}(\boldsymbol{i})} + v_{\mathrm{father}^{k-1}(\boldsymbol{i})}) \circ\cdots\circ \pi(A_{\boldsymbol{i}} + v_{\boldsymbol{i}})\big)(\mathbf{Z}^n)\right) - \overline{D}\big((\det A_{\boldsymbol{i}}^{-1})_{\boldsymbol{i}}\big)\right| < \frac{1}{3\ell}, \tag{21}$$

and for every i, j,

$$D_R^+\left\{ x \in \big(A_{j^m(\boldsymbol{i})} + v_{j^m(\boldsymbol{i})}\big)(\mathbf{Z}^n) \;\middle|\; d\big(x, (\mathbf{Z}^n)'\big) < \frac{1}{3\ell(2n+1)\,\mathrm{Card}\, I_k} \right\} < \frac{1}{3\ell\,\mathrm{Card}\, I_k}. \tag{22}$$

We now take a map $h \in \mathcal{D}^r(\mathbf{T}^n)$ such that $d_{C^1}(g,h) < \delta'$, and prove that if δ' is small enough, then h belongs to the interior of $\mathcal{U}_\ell$. First of all, we remark that if δ' is small enough, then the set

$$\left\{ y \in \mathbf{T}^n \;\middle|\; \forall q,\ d\Big(\big(Dh_x\big)_{\substack{1\le m\le k\\ x\in h^{-m}(y)}}, V_q\Big) > \eta/2 \right\}$$

contains a set $\mathcal{C}'$, which is a finite union of cubes whose union has measure bigger than $1 - 1/(3\ell)$.

Let C be a cube of $\mathcal{C}'$, $y \in C$ and $x \in f^{-m}(y)$, with $1 \le m \le k$. We write the Taylor expansion of order 1 of h at the neighbourhood of x; by compactness we obtain

$$\sup\left\{\frac{1}{\|z\|}\big\|h(x+z)-h(x)-Dh_x(z)\big\|\;\middle|\; x\in C,\, z\in B(0,\rho)\right\}\underset{\rho\to 0}{\longrightarrow} 0.$$

Thus, for every $\varepsilon>0$, there exists $\rho>0$ such that for all $x\in C$ and all $z\in B(0,\rho)$, we have

$$\big\|h(x+z)-\big(h(x)+Dh_x(z)\big)\big\|<\varepsilon\|z\|\leq\varepsilon\rho. \tag{23}$$

We now take $R\geq R_0$. We want to find an order of discretization N such that the error made by linearizing h on $B(x,R/N)$ is small compared to N, that is, for every $z\in B(0,R/N)$, we have

$$\big\|h(x+z)-\big(h(x)+Dh_x(z)\big)\big\|<\frac{1}{3\ell(2n+1)\operatorname{Card} I_k}\cdot\frac{1}{N}.$$

To do that, we apply Equation (23) to

$$\varepsilon=\frac{1}{3R\ell(2n+1)\operatorname{Card} I_k},$$

to get a radius $\rho>0$ (we can take ρ as small as we want), and we set $N=\lceil R/\rho\rceil$ (thus, we can take N as big as we want). By (23), for every $z\in B(0,R)$, we obtain the desired bound:

$$\big\|h(x+z/N)-\big(h(x)+Dh_x(z/N)\big)\big\|<\frac{1}{3\ell(2n+1)\operatorname{Card} I_k}\cdot\frac{1}{N}.$$

Combined with (22), this leads to

$$\frac{\operatorname{Card}\Big(h_N\big(B(x,R/N)\big)\,\Delta\, P_N\big(h(x)+Dh_x(B(0,R/N))\big)\Big)}{\operatorname{Card}\big(B(x,R/N)\cap E_N\big)}\leq\frac{1}{3\ell\operatorname{Card} I_k}; \tag{24}$$

in other words, on every ball of radius R/N, the image of E_N by h_N and the discretization of the linearization of h are almost the same (that is, up to a proportion $1/(3\ell\operatorname{Card} I_k)$ of points).

We now set $R_1=R_0\|f'\|_\infty^m$, and choose $R\geq R_1$, to which is associated a number $\rho>0$ and an order $N=\lceil R/\rho\rceil$, that we can choose large enough so that $2R/N\leq\|f'\|_\infty$. We also choose $y\in C$. As

$$\operatorname{Card}\big(h_N^m(E_N)\cap B(y,R/N)\big)=\operatorname{Card}\left(\bigcup_{x\in h^{-m}(y)} h_N^m\big(B(x,R/N)\cap E_N\big)\cap B(y,R/N)\right),$$

and using the estimations (21) and (24), we get

$$\left|\frac{\operatorname{Card}\big(h_N^m(E_N)\cap B(y,R/N)\big)}{\operatorname{Card}\big(B(y,R/N)\cap E_N\big)}-\overline{D}\Big((\det Df_x^{-1})_{\substack{1\leq m\leq k\\ x\in f^{-m}(y)}}\Big)\right|<\frac{2}{3\ell}.$$

As such an estimation holds on a subset of $\mathbf{T}^n$ of measure bigger than $1-1/(3\ell)$, we get the conclusion of the theorem. □

We can easily adapt the proof of Lemma 32 to the case of sequences of matrices, without the hypothesis of expansivity.

Lemma 36. *For every $k\in\mathbf{N}$ and every $\ell',c\in\mathbf{N}$, there exists a locally finite union of positive codimension submanifolds V_q of $(GL_n(\mathbf{R}))^k$ (respectively $(SL_n(\mathbf{R}))^k$) such that for every $\eta'>0$, there exists a radius $R_0>0$ such that if $(A_m)_{1\leq m\leq k}$ is a finite sequence of matrices of $(GL_n(\mathbf{R}))^k$ (respectively $(SL_n(\mathbf{R}))^k$) satisfying*

$d((A_m)_m, V_q) > \eta'$ for every q, then for every $R \geq R_0$, and every family $(v_m)_{1\leq m\leq k}$ of vectors of $\mathbf{R}^n$, we have

$$\left| D_R^+\Big(\big(\widehat{A}_k \circ \cdots \circ \widehat{A}_1\big)(\mathbf{Z}^n)\Big) - \det(A_k^{-1}\cdots A_1^{-1})\overline{\tau}^k(A_1, \cdots, A_k)\right| < \frac{1}{\ell'}$$

(the density of the image set is "almost invariant" under perturbations by translations), and for every $m \leq k$, we have[15]

$$D_R^+\left\{ x \in \big(A_m + v_m\big)(\mathbf{Z}^n) \;\middle|\; d\big(x, (\mathbf{Z}^n)'\big) < \frac{1}{c\ell'(2n+1)} \right\} < \frac{1}{c\ell'}$$

(there is only a small proportion of the points of the image sets which are obtained by discretizing points close to $(\mathbf{Z}^n)'$).

With the same proof as Theorem 31, Lemma 36 leads to the local-global formula for C^r-diffeomorphisms (Theorem 26).

4. Asymptotic rate of injectivity for a generic dissipative diffeomorphism

First of all, we tackle the issue of the asymptotic rate of injectivity of generic dissipative diffeomorphisms. Again, we will consider the torus $\mathbf{T}^n$ equipped with Lebesgue measure Leb and the canonical measures E_N, see Section A for a more general setting where the result is still true. The study of the rate of injectivity for generic dissipative diffeomorphisms is based on the following theorem of A. Avila and J. Bochi.

Theorem 37 (Avila, Bochi). *Let f be a generic C^1 maps of $\mathbf{T}^n$. Then for every $\varepsilon > 0$, there exists a compact set $K \subset \mathbf{T}^n$ and an integer $m \in \mathbf{N}$ such that*

$$\mathrm{Leb}(K) > 1 - \varepsilon \qquad \textit{and} \qquad \mathrm{Leb}(f^m(K)) < \varepsilon.$$

This statement is obtained by combining Lemma 1 and Theorem 1 of [AB06].

Remark 38. As C^1 expanding maps of $\mathbf{T}^n$ and C^1 diffeomorphisms of $\mathbf{T}^n$ are open subsets of the set of C^1 maps of $\mathbf{T}^n$, the same theorem holds for generic C^1 expanding maps and C^1 diffeomorphisms of $\mathbf{T}^n$ (this had already been proved in the case of C^1-expanding maps by A. Quas in [Qua99]).

This theorem can be used to compute the asymptotic rate of injectivity of a generic diffeomorphism.

Corollary 39. *The asymptotic rate of injectivity of a generic dissipative diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n)$ is equal to 0. In particular, the degree of recurrence $D(f_N)$ of a generic dissipative diffeomorphism tends to 0 when N goes to infinity.*

Remark 40. the same statement holds for generic C^1 expanding maps.

Proof of Corollary 39. The proof of this corollary mainly consists in stating what good properties can be supposed to possess the compact set K of Theorem 37. Thus, for f a generic diffeomorphism and $\varepsilon > 0$, there exists $m > 0$ and a compact set K such that $\mathrm{Leb}(K) > 1 - \varepsilon$ and $\mathrm{Leb}(f^m(K)) < \varepsilon$.

First of all, it can be easily seen that Theorem 37 is still true when the compact set K is replaced by an open set O: simply consider an open set $O' \supset f^m(K)$ such that $\mathrm{Leb}(O') < \varepsilon$ (by regularity of the measure) and set $O = f^{-m}(O') \supset K$. We

15. Recall that $(\mathbf{Z}^n)'$ stands for the set of points of $\mathbf{R}^n$ at least one coordinate of which belongs to $\mathbf{Z} + 1/2$.

then approach the set O by unions of dyadic cubes of $\mathbf{T}^n$: we define the cubes of order 2^M on $\mathbf{T}^n$

$$C_{M,i} = \prod_{j=1}^{n} \left[\frac{i_j}{2^M}, \frac{i_j+1}{2^M}\right],$$

and set

$$U_M = \operatorname{Int}\left(\overline{\bigcup_{C_{M,i}\subset O} C_{M,i}}\right),$$

where Int denotes the interior. Then, the union $\bigcup_{M\in\mathbf{N}} U_M$ is increasing in M and we have $\bigcup_{M\in\mathbf{N}} U_M = O$. In particular, there exists $M_0 \in \mathbf{N}$ such that $\operatorname{Leb}(U_{M_0}) > 1-\varepsilon$, and as $U_{M_0} \subset O$, we also have $\operatorname{Leb}(f^m(U_{M_0})) < \varepsilon$. We denote $U = U_{M_0}$. Finally, as U is a finite union of cubes, and as f is a diffeomorphism, there exists $\delta > 0$ such that the measure of the δ-neighbourhood of $f^m(U)$ is smaller than ε. We call V this δ-neighbourhood.

As U is a finite union of cubes, there exists $N_0 \in \mathbf{N}$ such that if $N \geq N_0$, then the proportion of points of E_N which belong to U is bigger than $1-2\varepsilon$, and the proportion of points of E_N which belong to V is smaller than 2ε. Moreover, if N_0 is large enough, then for every $N \geq N_0$, and for every $x_N \in E_N \cap U$, we have $f_N^m(x_N) \in V$. This implies that

$$\frac{\operatorname{Card}(f_N^m(E_N))}{\operatorname{Card}(E_N)} \leq 4\varepsilon,$$

which proves the corollary. □

5. Asymptotic rate of injectivity for a generic conservative diffeomorphism

The goal of this section is to prove that the degree of recurrence of a generic conservative C^1-diffeomorphism is equal to 0.

Theorem 41. *For a generic conservative diffeomorphism $f \in \operatorname{Diff}^1(\mathbf{T}^n, \operatorname{Leb})$, we have*

$$\lim_{t\to\infty} \tau^k(f) = 0;$$

more precisely, for every $\varepsilon > 0$, the set of diffeomorphisms $f \in \operatorname{Diff}^1(\mathbf{T}^n, \operatorname{Leb})$ such that $\lim_{t\to+\infty} \tau^k(f) < \varepsilon$ is open and dense.

In particular[16]*, we have $\lim_{N\to+\infty} D(f_N) = 0$.*

It will be obtained by using the local-global formula (Theorem 26) and the result about the asymptotic rate of injectivity of a generic sequence of matrices (Theorem 9). The application of the linear results will be made through the following lemma of [ACW14], which allows to linearize locally a conservative diffeomorphism.

Lemma 42 (Avila, Crovisier, Wilkinson)**.** *Let C be the unit ball of $\mathbf{R}^n$ for $\|\cdot\|_\infty$ and $\varepsilon > 0$. Then, there exists $\delta > 0$ such that for every $g_1 \in \operatorname{Diff}^\infty(\mathbf{R}^n, \operatorname{Leb})$ such that $d_{C^1}(g_{1|C}, \operatorname{Id}_{|C}) < \delta$, there exists $g_2 \in \operatorname{Diff}^\infty(\mathbf{R}^n, \operatorname{Leb})$ such that:*

(i) $d_{C^1}(g_{2|C}, g_{1|C}) < \varepsilon$;

(ii) $g_{2|(1-\varepsilon)C} = \operatorname{Id}_{|(1-\varepsilon)C}$;

(iii) $g_{2|C^\complement} = g_{1|C^\complement}$.

16. Using Equation (2) page 4.

The proof of this lemma involves a result of J. Moser [Mos65]. The reader may refer to [ACW14, Corollary 6.9] for a complete proof [17]. By a regularization result due to A. Avila [Avi10], it is possible to weaken the hypothesis of regularity in the lemma "$g_1 \in \mathrm{Diff}^\infty(\mathbf{R}^n, \mathrm{Leb})$" into the hypothesis "$g_1 \in \mathrm{Diff}^1(\mathbf{R}^n, \mathrm{Leb})$".

Proof of theorem 41. We show that for every $\ell \in \mathbf{N}$ and every $\varepsilon > 0$, the set of conservative diffeomorphisms such that $\limsup_{t\to\infty} \tau_\infty^t < 1/\ell + \varepsilon$ contains an open dense subset of $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$. To begin with, we fix $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$ and $\delta > 0$ (which will be a size of perturbation of f). By Theorem 9, and in particular Equation (5), there exists a parameter $\lambda \in]0, 1[$ (depending only on δ, ℓ and $\|f\|_{C^1}$), such that for every sequence $(A_k)_{k\geq 1}$ of linear maps in $SL_n(\mathbf{R})$, there exists a sequence $(B_k)_{k\geq 1}$ of (generic) linear maps in $SL_n(\mathbf{R})$ such that for each k, we have $\|A_k - B_k\| \leq \delta$ and $\tau^{\ell k}(B_1, \cdots, B_{\ell k}) \leq \lambda^k + 1/\ell$ (as the sequence is generic, this property remains true on a whole neighbourhood of $(B_k)_{k\geq 1}$, see Remark 14). From that parameter λ, we deduce a time $k_0 > 0$ such that

$$\frac{1}{k_0}\sum_{k=1}^{k_0} \lambda^k = \frac{\lambda}{k_0}\frac{1-\lambda^{k_0}}{1-\lambda} < \varepsilon/100.$$

Applying a classical technique in this context (see for example [Boc02]), we use a Rokhlin tower of height k_0 with an open basis U:
- The sets $U, f(U), \cdots, f^{k_0-1}(U)$ are pairwise disjoint;
- the measure of the union of the "floors" $U \cup f(U) \cup \cdots \cup f^{k_0-1}(U)$ is bigger than $1 - \varepsilon/100$;

for the existence of such towers, see for example [Gui12, Lemme 6.8] or [Hal56, Chapter "Uniform topology"].

We then approach the basis U by a disjoint union of cubes of $\mathbf{T}^n$ (as in page 28, from now we suppose that U is a union of such cubes). If these cubes are small enough, on each cube C, it is possible to perturb f into a diffeomorphism g such that on each set $(1-\varepsilon/100)C$, g is affine and irrational, using Lemma 42, the regularization result of A. Avila [Avi10] and Franks lemma [18] (see [Fra71] or [Cro06]). We do the same thing on the $k_0 - 1$ first images of each cube and perturb g such that on each set $g^k((1-\varepsilon/100)C)$, the perturbed diffeomorphism h is linear and equal to B_k. By what we have said at the beginning of the proof, we can moreover suppose that the sequence $B_1, \cdots, B_{k_0}$ of linear maps is generic, and satisfies $\tau^k(B_1, \cdots, B_k) \leq \lambda^k + 1/\ell$ for every $k \leq k_0 - 1$. By the choice of k_0 we have made,

17. Le 12/09/2015, cette version n'est pas encore en ligne…
18. Which is valid only in the C^1 topology.

this implies that

$$\begin{aligned}
\tau^{k_0}(h) \leq & \sum_{k=0}^{k_0-1} \mathrm{Leb}(h^k(U))\tau^{k_0}\big(h_{|h^k(U)}\big) \\
& + \mathrm{Leb}\left((U\cup\cdots\cup h^{k_0-1}(U))^{\complement}\right)\tau^{k_0}\big(h_{(U\cup\cdots\cup h^{k_0-1}(U))^{\complement}}\big) \\
\leq & \sum_{k=0}^{k_0-1} \mathrm{Leb}(h^k(U))\tau^{k_0-k}\big(h_{|h^k(U)}\big) + \mathrm{Leb}\left((U\cup\cdots\cup h^{k_0-1}(U))^{\complement}\right) \\
\leq & \,\mathrm{Leb}(U)\sum_{k=0}^{k_0-1}\left(\lambda^{k_0-k}+\frac{1}{\ell}\right) + \varepsilon/100 \\
\leq & \frac{1}{k_0}\sum_{k=1}^{k_0}(\lambda^k+1)/2 + \varepsilon/100 \\
\leq & 1/\ell + \varepsilon/2.
\end{aligned}$$

Moreover, the differentials of h form generic sequences on a set of measure at least $1-\varepsilon/10$. This implies that the rate of injectivity is continuous in h when restricted to this subset of $\mathbf{T}^n$. Thus, the inequality $\tau^{k_0}(h) \leq 1/\ell + \varepsilon$ still holds on a whole neighbourhood of h. This proves the theorem. □

6. Asymptotic rate of injectivity of a generic C^r expanding map

In this section, we prove that the asymptotic rate of injectivity of a generic expanding map is equal to 0. Note that a local version of this result was already obtained by P.P. Flockermann in his thesis (Corollary 2 page 69 and Corollary 3 page 71 of [Flo02]), stating that for a generic $C^{1+\alpha}$ expanding map f of the circle, the "local asymptotic rate of injectivity" is equal to 0 almost everywhere. Some of his arguments will be used in this section. Note also that in C^1 regularity, the equality $\tau^\infty(f) = 0$ for a generic f is a consequence of Theorem 37 of A. Avila and J. Bochi (see also Corollary 39); the same theorem even proves that the asymptotic rate of injectivity of a generic C^1 endomorphism of the circle is equal to 0.

Definition 43. We define Z_m as the number of children at the m-th generation in $G_{f,y}$ (see Definition 30).

Proposition 44. *For every $r \in]1,+\infty]$, for every $f \in \mathcal{D}^r(\mathbf{S}^1)$ and every $y \in \mathbf{S}^1$, we have*

$$\mathbf{P}(Z_m > 0) \underset{m\to+\infty}{\longrightarrow} 0.$$

Equivalently,

$$\overline{D}\Big((\det Df_x^{-1})_{\substack{1\leq m\leq k\\ x\in f^{-m}(y)}}\Big) \underset{k\to+\infty}{\longrightarrow} 0.$$

Lemma 45. *The expectation of Z_m satisfies*

$$\mathbf{E}(Z_m) = (\mathcal{L}^m 1)(y),$$

where $\mathcal{L}$ is the Ruelle-Perron-Frobenius associated to f and 1 denotes the constant function equal to 1 on $\mathbf{S}^1$. In particular, there exists a constant $\Sigma_0 > 0$ such that $\mathbf{E}(Z_m) \leq \Sigma_0$ for every $m \in \mathbf{N}$.

The second part of the lemma is deduced from the first one by applying the theorem stating that for every C^r expanding map f of $\mathbf{S}^1$ ($r > 1$), the maps $\mathcal{L}^m 1$ converge uniformly towards a Hölder map, which is the density of the unique SRB measure of f (see for example [FJ03]). The first assertion of the lemma follows from the convergence of the operators f_N^* acting on $\mathcal{P}$ (the space of Borel probability measures) towards the Ruelle-Perron-Frobenius operator.

Definition 46. The *transfer operator* associated to the map f (usually called Ruelle-Perron-Frobenius operator), which acts on densities of probability measures, will be denoted by $\mathcal{L}_f$. It is defined by

$$\mathcal{L}_f\phi(y) = \sum_{x\in f^{-1}(y)} \frac{\phi(x)}{f'(x)}.$$

Lemma 45 follows directly from the following lemma.

Lemma 47. *Denoting* Leb_N *the uniform measure on* E_N*, for every* C^1 *expanding map of* $\mathbf{S}^1$ *and every* $m \geq 0$*, we have convergence of the measures* $(f_N^*)^m(\lambda_N)$ *towards the measure of density* $\mathcal{L}_f^m 1$ *(where* 1 *denotes the constant function equal to 1).*

The proof of this lemma is straightforward but quite long. We sketch here this proof, the reader will find a complete proof using generating functions in Section 3.4 of [Flo02] and a quantitative version of it in Section 12.2 of [Gui15c].

Sketch of proof of Lemma 47. As f is C^1, by the mean value theorem, for every segment I small enough, we have

$$\left|\mathrm{Leb}(I) - \frac{\mathrm{Leb}(f(I))}{f'(x_0)}\right| \leq \varepsilon.$$

Moreover, for every interval J,

$$\left|\mathrm{Leb}(J) - \frac{\mathrm{Card}(J\cap E_N)}{\mathrm{Card}(E_N)}\right| \leq \frac{1}{N}.$$

These two inequalities allows to prove the local convergence of the measures $f_N^*(\lambda_N)$ towards the measure with density $\mathcal{L}_f 1$. The same kind of arguments holds in arbitrary times, and allow to prove the lemma. □

Lemma 48. *We fix an integer* $K > 0$*, and denote by* A_K *the event "*$0 < Z_m \leq K$ *for an infinite number of generations* m*". Then the probability of* A_K *is equal to 0.*

Proof of Lemma 48. We remark that we have the inequality

$$\mathbf{P}(Z_{m+1} = 0 \mid Z_m \leq K) \geq \left(1 - \frac{1}{\|f'\|_\infty}\right)^{dK}.$$

In particular, if $Z_m \leq K$, then with probability bigger than $(1 - \|f'\|_\infty^{-1})^{dK}$, we will have $Z_{m'} = 0$ for every $m' > m$. Thus, the event $A_{K,M}$: "$Z_m \leq K$ for at least M generations m and $Z_{m'} > 0$ for every m'" has probability

$$\mathbf{P}(A_{K,M}) \leq \left(1 - \left(1 - \frac{1}{\|f'\|_\infty}\right)^{dK}\right)^M.$$

So, as A_K is obtained as the decreasing intersection $A_K = \bigcap_{M\in\mathbf{N}} A_{K,M}$, we have $\mathbf{P}(A_K) = 0$. □

Proof of Proposition 44. Let $\varepsilon > 0$. We denote by B the event "there exists $m \in \mathbf{N}$ such that $Z_m = 0$". It is obtained as the disjoint union

$$B = \bigsqcup_{m\in\mathbf{N}} \big((Z_{m+1} = 0) \cap (Z_m > 0)\big).$$

So there exists $M_0 \in \mathbf{N}$ such that

$$\mathbf{P}\left(\bigsqcup_{m\geq M_0}\big((Z_{m+1}=0)\cap(Z_m>0)\big)\right)=\mathbf{P}\left(\bigcup_{m\geq M_0}(Z_m>0\mid B)\right)<\varepsilon/2. \tag{25}$$

In other words, if there exists $m\in\mathbf{N}$ such that $Z_m=0$, then the smallest m such that this property holds is smaller than M_0 with probability bigger than $1-\varepsilon/2$.

We also denote by C_K the event "for all but a finite number of $m\in\mathbf{N}$, we have $Z_m>K$". By Markov inequality and Lemma 45, we have

$$\mathbf{P}(Z_m\geq 2\Sigma_0/\varepsilon)\leq\varepsilon/2.$$

Thus, we have

$$\mathbf{P}(C_{2\Sigma_0/\varepsilon})\leq\varepsilon/2. \tag{26}$$

We now take $m\geq M_0$ and $K=2\Sigma_0/\varepsilon$. We remark that

$$\begin{aligned}\mathbf{P}(Z_m>0)&=\mathbf{P}\big((Z_m>0)\cap A_K\big)+\mathbf{P}\big((Z_m>0)\cap B\big)+\mathbf{P}\big((Z_m>0)\cap C_K\big)\\&\leq\mathbf{P}(A_K)+\mathbf{P}\big((Z_m>0)\mid B\big)+\mathbf{P}(C_K).\end{aligned}$$

Using Lemma 48 and the estimations (25) and (26), this leads to

$$\mathbf{P}(Z_m>0)\leq\varepsilon,$$

which proves the proposition. □

Corollary 49. *For a generic map $f\in\mathcal{D}^r(\mathbf{S}^1)$, we have $\tau^\infty(f)=0$. In particular,* $\lim_{N\to+\infty}D(f_N)=0$.

Proof of Corollary 49. It is an easy consequence of the local-global formula (Theorem 31), Proposition 44 and the dominated convergence theorem. □

7. Numerical simulations

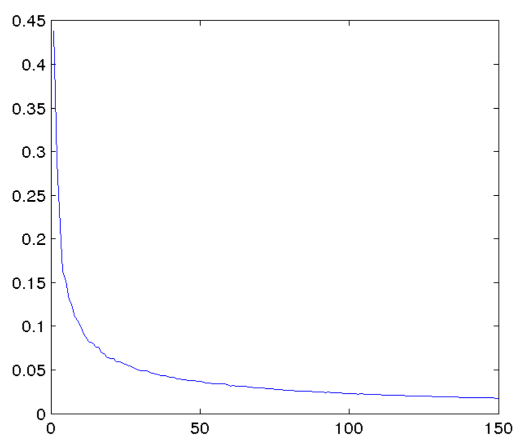

FIGURE 11. Simulation of the degree of recurrence $D(f_N)$ of the conservative diffeomorphism f, depending on N, on the grids E_N with $N=128k$, $k=1,\cdots,150$.

We have computed numerically the degree of recurrence of a diffeomorphism f, which is C^1-close to Id. It is defined by $f=Q\circ P$, with

$$P(x,y)=\big(x,y+p(x)\big)\quad\text{and}\quad Q(x,y)=\big(x+q(y),y\big),$$

$$p(x)=\frac{1}{209}\cos(2\pi\times17x)+\frac{1}{271}\sin(2\pi\times27x)-\frac{1}{703}\cos(2\pi\times35x),$$

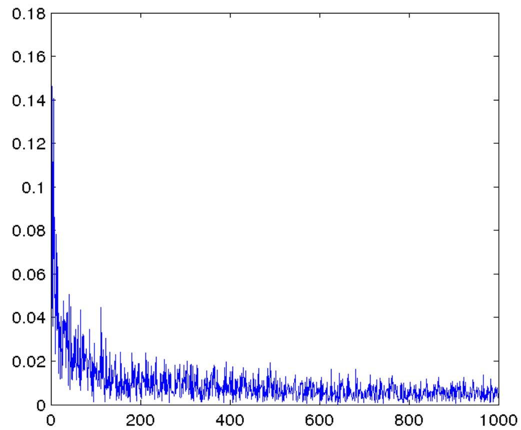

FIGURE 12. Simulation of the degree of recurrence $D(g_N)$ of the expanding map g, depending on N, on the grids E_N with $N = 128k$, $k = 1, \cdots, 1\,000$.

$$q(y) = \frac{1}{287}\cos(2\pi \times 15y) + \frac{1}{203}\sin(2\pi \times 27y) - \frac{1}{841}\sin(2\pi \times 38y).$$

On Figure 11, we have represented graphically the quantity $D(f_{128k})$ for k from 1 to 150. It appears that, as predicted by Theorem 41, this degree of recurrence goes to 0. In fact, it is even decreasing, and converges quite fast to 0: as soon as $N = 128$, the degree of recurrence is smaller than $1/2$, and if $N \gtrsim 1000$, then $D(f_N) \leq 1/10$. Note that, contrary to what is predicted by theory, this phenomenon was already observed for examples of conservative homeomorphisms of the torus which have big enough derivatives not to be considered as "typical from the C^1 case" (see [Gui15d]).

We also present the results of the numerical simulation we have conducted for the degree of recurrence of the expanding map of the circle g, defined by

$$g(x) = 2x + \varepsilon_1 \cos(2\pi x) + \varepsilon_2 \sin(6\pi x),$$

with $\varepsilon_1 = 0.127\,943\,563\,72$ and $\varepsilon_2 = 0.008\,247\,359\,61$.

On Figure 12, we have represented the quantity $D(g_{128k})$ for k from 1 to $1\,000$. It appears that, as predicted by Corollary 49, this degree of recurrence seems to tend to 0. In fact, it is even decreasing, and converges quite fast to 0: as soon as $N = 128$, the degree of recurrence is smaller than $1/5$, and if $N \gtrsim 25\,000$, then $D(g_N) \leq 1/50$.

APPENDIX A. A MORE GENERAL SETTING WHERE THE THEOREMS ARE STILL TRUE

Here, we give weaker assumptions under which the theorems of this paper are still true: the framework "torus $\mathbf{T}^n$ with grids E_N and Lebesgue measure" could be seen as a little too restrictive.

So, we take a compact smooth manifold M (possibly with boundary) and choose a partition $M_1, \cdots, M_k$ of M into closed sets [19] with smooth boundaries, such that for every i, there exists a chart $\varphi_i : M_i \to \mathbf{R}^n$. We endow $\mathbf{R}^n$ with the euclidean distance, which defines a distance on M *via* the charts ϕ_i (this distance is not

19. That is, $\bigcup_i M_i = M$, and for $i \neq j$, the intersection between the interiors of M_i and M_j are empty.

necessarily continuous). From now, we study what happens on a single chart, as what happens on the neighbourhoods of the boundaries of these charts "counts for nothing" from the Lebesgue measure viewpoint.

Finally, we suppose that the uniform measures on the grids $E_N = \bigcup_i E_{N,i}$ converge to a smooth measure λ on M when N goes to infinity.

This can be easily seen that these conditions are sufficient for Corollary 39 to be still true.

For the rest of the statements of this paper, we need that the grids behave locally as the canonical grids on the torus.

For every i, we choose a sequence $(\kappa_{N,i})_N$ of positive real numbers such that $\kappa_{N,i} \underset{N\to+\infty}{\longrightarrow} 0$. This defines a sequence $E_{N,i}$ of grids on the set M_i by $E_{N,i} = \varphi_i^{-1}(\kappa_{N,i}\mathbf{Z}^n)$. Also, the canonical projection $\pi : \mathbf{R}^n \to \mathbf{Z}^n$ (see Definition 6) allows to define the projection $\pi_{N,i}$, defined as the projection on $\kappa_{N,i}\mathbf{Z}^n$ in the coordinates given by φ_i:

$$\begin{aligned} \pi_{N,i} : \quad M_i &\longrightarrow E_{N,i} \\ x &\longmapsto \varphi_i^{-1}\Big(\kappa_{N,i}\pi\big(\kappa_{N,i}^{-1}\varphi_i(x)\big)\Big). \end{aligned}$$

We easily check that under these conditions, Theorems 26, 31 and 41 and Corollary 49 are still true, that is if we replace the torus $\mathbf{T}^n$ by M, the uniform grids by the grids E_N, the canonical projections by the projections $\pi_{N,i}$, and Lebesgue measure by the measure λ.

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