DISCRETIZATIONS OF ISOMETRIES

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ABSTRACT. This paper deals with the dynamics of discretizations of isometries of $\mathbf{R}^n$, and more precisely the density of the successive images of $\mathbf{Z}^n$ by the discretizations of a generic sequence of isometries. We show that this density tends to 0 as the time goes to infinity. Thus, in general, all the information of a numerical image will be lost by applying many times a naive algorithm of rotation.

RÉSUMÉ. On étudie ici la dynamique des discrétisations d'isométries de $\mathbf{R}^n$, et plus précisément la densité des images successives du réseau $\mathbf{Z}^n$ par les discrétisations d'une suite générique d'isométries. On montre que cette densité tend vers 0 quand le temps tend vers l'infini. Ainsi, en général, on va perdre toute l'information contenue dans une image numérique en appliquant plusieurs fois un algorithme naif de rotation.

1. INTRODUCTION

In this paper, we consider the dynamical behaviour of the discretizations of (linear) isometries of a real and finite dimensional vector space $\mathbf{R}^n$. The goal is to understand how it is possible to rotate a numerical image (made of pixels) with the smallest loss of quality as possible.For example, in Figure 1, we have applied 40 successive random rotations to a 500×684 pixels picture, using a consumer software. These discretized rotations induce a very strong blur in the resulting image, thus a big loss of information.

Here, we consider the simplest algorithm that can be used to perform a discrete rotation: discretizing the rotation. More precisely, if $x \in \mathbf{Z}^2$ is a integer point (representing a pixel), then the image of x by the discretization of a rotation R will be the integer point which is the closest of $R(x)$. More precisely, in the general case of isometries we will use the following definition of a discretization.

Definition 1. We define the projection $p : \mathbf{R} \to \mathbf{Z}$ such that for every $x \in \mathbf{R}$, $p(x)$ is the unique integer $k \in \mathbf{Z}$ such that $k - 1/2 < x \le k + 1/2$. This projection induces the map

$$\pi : \quad \begin{aligned} \mathbf{R}^n &\longmapsto \mathbf{Z}^n \\ (x_i)_{1\le i\le n} &\longmapsto \big(p(x_i)\big)_{1\le i\le n} \end{aligned}$$

which is an Euclidean projection on the lattice $\mathbf{Z}^n$. For $P \in O_n(\mathbf{R})$, we denote by $\widehat{P}$ the *discretization* of P, defined by

$$\widehat{P} : \quad \begin{aligned} \mathbf{Z}^n &\longrightarrow \mathbf{Z}^n \\ x &\longmapsto \pi(Px). \end{aligned}$$

We will measure the loss of information induced by the action of discretizing by the *density* of the image set. More precisely, given a sequence $(P_k)_{k\ge 1}$ of linear isometries of $\mathbf{R}^n$, we will look for the density of the set $\Gamma_k = (\widehat{P_k} \circ \cdots \circ \widehat{P_1})(\mathbf{Z}^n)$.

FIGURE 1. Original image (left) of size 220×282 and 10 successive random rotations of this image (right), obtained with the software *Gimp* (linear interpolation algorithm).

Definition 2. For $A_1, \cdots, A_k \in O_n(\mathbf{R})$, the *rate of injectivity in time k* of this sequence is the quantity

$$\tau^k(P_1, \cdots, P_k) = \limsup_{R \to +\infty} \frac{\mathrm{Card}\left((\widehat{P_k} \circ \cdots \circ \widehat{P_1})(\mathbf{Z}^n) \cap [B_R]\right)}{\mathrm{Card}[B_R]} \in]0, 1],$$

where B_R denotes the infinite ball of radius R and $[B_R]$ the set of integral points (i.e. with integer coordinates) inside B_R. For an infinite sequence $(P_k)_{k \geq 1}$ of isometries, as the previous quantity is decreasing, we can define the *asymptotic rate of injectivity*

$$\tau^\infty\big((P_k)_{k \geq 1}\big) = \lim_{k \to +\infty} \tau^k(P_1, \cdots, P_k) \in [0, 1].$$

An example of the sets Γ_k for a random draw of isometries P_m is presented on Figure 2. In particular, we observe that the density of these sets seems to decrease when k goes to infinity: the images get whiter and whiter.
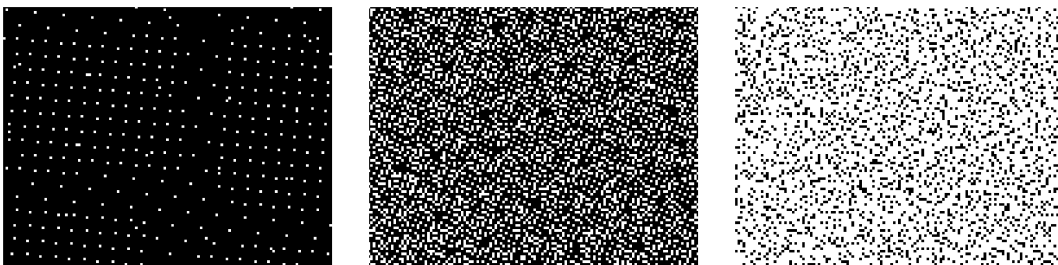

FIGURE 2. Successive images of $\mathbf{Z}^2$ by discretizations of random rotations, a point is black if it belongs to $(\widehat{R_{\theta_k}} \circ \cdots \circ \widehat{R_{\theta_1}})(\mathbf{Z}^2)$, where the θ_i are chosen uniformly randomly in $[0, 2\pi]$. From left to right and top to bottom, $k = 2, 5, 50$.

This phenomenon is confirmed when we plot the density of the intersection between these image sets Γ_k and a big ball of $\mathbf{R}^n$ (see Figure 3): this density seems to tend to 0 as the time k goes to infinity.

We would like to explain theoretically this phenomenon. Of course, if we take $P_m = \mathrm{Id}$, then we will have $\Gamma_k = \mathbf{Z}^n$ and the rates of injectivity will be equal to 0. To avoid this kind of "exceptional cases", we will study the asymptotic rate of injectivity of a *generic* sequence of matrices of $O_n(\mathbf{R})$, in the following sense.
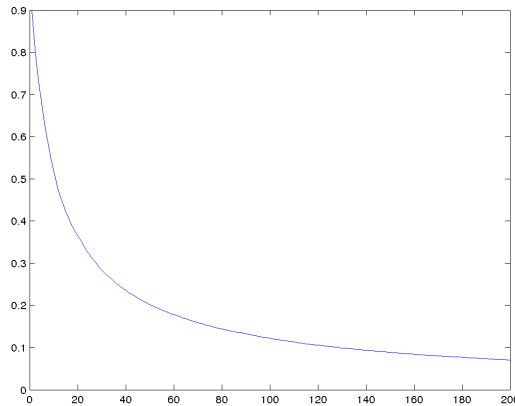
FIGURE 3. Expectation of the rate of injectivity of a random sequences of rotations: the graphic represents the mean of the rate of injectivity $\tau^k(R_{\theta_k}, \cdots, R_{\theta_1})$ depending on k, $1 \leq k \leq 200$, for 50 random draws of sequences of angles $(\theta_i)_i$, with each θ_i chosen independently and uniformly in $[0, 2\pi]$. Note that the behaviour is not exponential.

Definition 3. We fix once for all a norm $\|\cdot\|$ on $M_n(\mathbf{R})$. For any sequence $(P_k)_{k \geq 1}$ of matrices of $O_n(\mathbf{R})$, we set

$$\|(P_k)_k\|_\infty = \sup_{k \geq 1} \|P_k\|.$$

In other words, we consider the space $\ell^\infty(O_n(\mathbf{R}))$ of uniformly bounded sequences of linear isometries endowed with this natural metric.

This metric is complete, thus there is a good notion of genericity on the set of linear isometries: a set $\mathcal{U} \subset (O_n(\mathbf{R}))^{\mathbf{N}}$ is *generic* if it is a countable intersection of open and dense subsets of $\ell^\infty(O_n(\mathbf{R}))$. The main theorem of this paper studies the asymptotic rate of injectivity in this context.

Theorem 1. *Let $(P_k)_{k \geq 1}$ be a generic sequence of matrices of $O_n(\mathbf{R})$. Then $\tau^\infty((P_k)_k) = 0$.*

The proof of this theorem will even show that for every $\varepsilon > 0$, there exists an open and dense subset of $\ell^\infty(O_n(\mathbf{R}))$ on which τ^∞ is smaller than ε. This theorem expresses that for "most of" the sequences of isometries, the loss of information is total. Thus, for a generic sequence of rotations, we will not be able to avoid the blur observed in Figure 1.

Note that we do not know what is the rate of injectivity of a sequence of isometries made of independent identically distributed random draws (for example with respect to the Haar measure on $O_n(\mathbf{R})$).

The proof of Theorem 1 will be the occasion to study the structure of the image sets $\Gamma_k = (\widehat{P_k} \circ \cdots \circ \widehat{P_1})(\mathbf{Z}^n)$. It appears that there is a kind of "regularity at infinity" in Γ_k. More precisely, this set is an *almost periodic pattern*: roughly speaking, for R large enough, the set $\Gamma_k \cap B_R$ determines the whole set Γ_k up to an error of density smaller than ε (see Definition 6). We prove that the image of an almost periodic pattern by the discretization of a linear map is still an almost periodic pattern (Theorem 7); thus, the sets Γ_k are almost periodic patterns.

The idea of the proof of Theorem 1 is to take advantage of the fact that for a generic sequence of isometries, we have a kind of independence of the coefficients of the matrices. Thus, for a generic isometry $P \in O_n(\mathbf{R})$, the set $P(\mathbf{Z}^n)$ is uniformly distributed modulo $\mathbf{Z}^n$. We then remark that the local pattern of the image set $\widehat{P}(\mathbf{Z}^n)$ around $\widehat{P}(x)$ is only

determined by P and the the remainder of Px modulo $\mathbf{Z}^n$: the global behaviour of $\widehat{P}(\mathbf{Z}^n)$ is coded by the quotient $\mathbf{R}^n/\mathbf{Z}^n$. This somehow reduces the study to a local problem.

As a first application of this remark, we state that the rate of injectivity in time 1 can be seen as the area of an intersection of cubes (Proposition 14). This observation is one of the two keys of the proof of Theorem 1, the second one being the study of the action of the discretizations $\widehat{P}$ on the frequencies of differences $\rho_{\Gamma_k}(v) = D\big((\Gamma_k - v) \cap \Gamma_k\big)$. Indeed, if there exists a set $\Gamma' \subset \Gamma$ of positive density, together with a vector v such that for every $x \in \Gamma'$, we have $\widehat{P}(x) = \widehat{P}(x+v)$, then we will have $D(\widehat{P}(\Gamma)) \le D(\widehat{P}) - D(\Gamma')$. This study of the frequencies of differences will include a Minkowski-type theorem for almost-periodic patterns (Theorem 13).

The particular problem of the discretization of linear maps has been quite little studied. To our knowledge, what has been made in this direction has been initiated by image processing. One wants to avoid phenomenons like loss of information (due to the fact that discretizations of linear maps are not injective) or aliasing (the apparition of undesirable periodic patterns in the image, due for example to a resonance between a periodic pattern in the image and the discretized map). To our knowledge, the existing studies are mostly interested in the linear maps with *rational coefficients* (see for example [JDC95], [Neh96] or [JDC01]), including the specific case of *rotations* (see for example [And96], [Nou06], [Thi10], [BV00]). These works mainly focus on the local behaviour of the images of $\mathbf{Z}^2$ by discretizations of linear maps: given a radius R, what pattern can follow the intersection of this set with any ball of radius R? What is the number of such patterns, what are their frequencies? Are they complex (in a sense to define) or not? Are these maps bijections? In particular, the thesis [Nou06] of B. Nouvel gives a characterization of the angles for which the discrete rotation is a bijection (such angles are countable and accumulate only on 0). Our result complements that of B. Nouvel: one the one hand it expresses that a generic sequence of discretizations is far from being a bijection, and on the other hand this remains true in any dimension.

Note that Theorem 1 will be generalized to the case of matrices of determinant 1 in [Gui15a], with more sophisticated techniques (see also [Gui15b]).

2. Almost periodic sets

In this section, we introduce the basic notions that we will use during the study of discretizations of isometries of $\mathbf{R}^n$.

We fix once for all an integer $n \ge 1$. We will denote by $[\![a, b]\!]$ the integer segment $[a, b] \cap \mathbf{Z}$. In this part, every ball will be taken with respect to the infinite norm; in particular, for $x = (x_1, \cdots, x_n)$, we will have

$$B(x, R) = B_\infty(x, R) = \big\{ y = (y_1, \cdots, y_n) \in \mathbf{R}^n \mid \forall i \in [\![1, n]\!], |x_i - y_i| < R \big\}.$$

We will also denote $B_R = B(0, R)$. Finally, we will denote by $\lfloor x \rfloor$ the biggest integer that is smaller than x and $\lceil x \rceil$ the smallest integer that is bigger than x. For a set $B \subset \mathbf{R}^n$, we will denote $[B] = B \cap \mathbf{Z}^n$.

2.1. **Almost periodic patterns: definitions and first properties.** In this subsection, we define the notion of almost periodic pattern and prove that these sets possess a uniform density.

Definition 4. Let Γ be a subset of $\mathbf{R}^n$.

- We say that Γ is *relatively dense* if there exists $R_\Gamma > 0$ such that each ball with radius at least R_Γ contains at least one point of Γ.

- We say that Γ is a *uniformly discrete* if there exists $r_\Gamma > 0$ such that each ball with radius at most r_Γ contains at most one point of Γ.

The set Γ is called a *Delone* set if it is both relatively dense and uniformly discrete.

Definition 5. For a discrete set $\Gamma \subset \mathbf{R}^n$ and $R \geq 1$, we define the uniform R-density:

$$D_R^+(\Gamma) = \sup_{x \in \mathbf{R}^n} \frac{\mathrm{Card}\left(B(x,R) \cap \Gamma\right)}{\mathrm{Card}\left(B(x,R) \cap \mathbf{Z}^n\right)},$$

and the uniform upper density:

$$D^+(\Gamma) = \limsup_{R \to +\infty} D_R^+(\Gamma).$$

Remark that if $\Gamma \subset \mathbf{R}^n$ is Delone for the parameters r_Γ and R_Γ, then its upper density satisfies:

$$\frac{1}{(2R_\Gamma + 1)^n} \leq D^+(\Gamma) \leq \frac{1}{(2r_\Gamma + 1)^n}.$$

We can now define the notion of almost periodic pattern that we will use throughout this paper. Roughly speaking, an almost periodic pattern Γ is a set for which there exists a relatively dense set of translations of Γ, where a vector v is a translation of Γ if $\Gamma - v$ is equal to Γ up to a set of upper density smaller than ε. More precisely, we state the following definition.

Definition 6. A Delone set $\Gamma \subset \mathbf{Z}^n$ is an *almost periodic pattern* if for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ and a relatively dense set $\mathcal{N}_\varepsilon$, called the *set of ε-translations* of Γ, such that

$$\forall R \geq R_\varepsilon, \ \forall v \in \mathcal{N}_\varepsilon, \ D_R^+\big((\Gamma + v)\Delta\Gamma\big) < \varepsilon. \tag{1}$$

Of course, every lattice, or every finite union of translates of a given lattice, is an almost periodic pattern. We will see in next subsection a large class of examples of almost periodic patterns: images of $\mathbf{Z}^n$ by discretizations of linear maps.

We end this introduction to almost periodic patterns by stating that the notion of almost periodic pattern is invariant under discretizations of linear isometries: the image of an almost periodic pattern by the discretization of a linear isometry is still an almost periodic pattern.

Theorem 7. *Let $\Gamma \subset \mathbf{Z}^n$ be an almost periodic pattern and $P \in O_n(\mathbf{R})$. Then the set $\widehat{P}(\Gamma)$ is an almost periodic pattern.*

This implies that, given a sequence $(P_k)_{k \geq 1}$ of isometries of $\mathbf{R}^n$, the successive images $(\widehat{P_k} \circ \cdots \circ \widehat{P_1})(\mathbf{Z}^n)$ are almost periodic patterns. See Figure 2 for an example of the successive images of $\mathbf{Z}^2$ by a random sequence of bounded matrices of $O_2(\mathbf{R})$. The proof of Theorem 7 will be done in Appendix B. Examples of sets $\widehat{P}(\mathbf{Z}^2)$ for various rotations P can be found in Figure 4, where the almost periodicity is patent.

2.2. **Differences in almost periodic patterns.** We will need to understand how behave the differences in an almost periodic pattern Γ, i.e. the vectors $x - y$ with $x, y \in \Gamma$. In fact, we will study the frequency of appearance of these differences.

Definition 8. For $v \in \mathbf{Z}^n$, we set

$$\rho_\Gamma(v) = \frac{D\{x \in \Gamma \mid x + v \in \Gamma\}}{D(\Gamma)} = \frac{D\big(\Gamma \cap (\Gamma - v)\big)}{D(\Gamma)} \in [0,1]$$

the *frequency* of the difference v in the almost periodic pattern Γ.
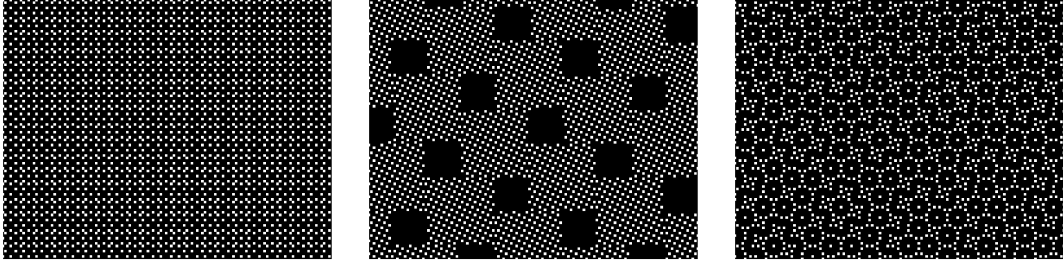
FIGURE 4. Images of $\mathbf{Z}^2$ by discretizations of rotations, a point is black if it belongs to the image of $\mathbf{Z}^2$ by the discretization of the rotation. From left to right and top to bottom, angles $\pi/4$, $\pi/5$ and $\pi/6$.

Studying frequencies of differences allows to focus on the global behaviour of an almost periodic set. The function ρ_Γ is itself almost periodic in the sense given by H. Bohr (see [Boh24]).

Definition 9. Let $f : \mathbf{Z}^n \to \mathbf{R}$. Denoting by T_v the translation of vector v, we say that f is *Bohr almost periodic* (also called *uniformly almost periodic*) if for every $\varepsilon > 0$, the set

$$\mathcal{N}_\varepsilon = \left\{ v \in \mathbf{Z}^n \mid \|f - f \circ T_v\|_\infty < \varepsilon \right\},$$

is relatively dense.

If $f : \mathbf{Z}^n \to \mathbf{R}$ is a Bohr almost periodic function, then it possesses a *mean* $\mathcal{M}(f)$ (see for example the historical paper of H. Bohr [Boh24, Satz VIII]), which satisfies: for every $\varepsilon > 0$, there exists $R_0 > 0$ such that for every $R \geq R_0$ and every $x \in \mathbf{R}^n$, we have

$$\left| \mathcal{M}(f) - \frac{1}{\mathrm{Card}[B(x,R)]} \sum_{v \in [B(x,R)]} f(v) \right| < \varepsilon.$$

The fact that ρ_Γ is Bohr almost periodic is straightforward.

Lemma 10. *If Γ is an almost periodic pattern, then the function ρ_Γ is Bohr almost periodic.*

In fact, we can compute precisely the mean of $\rho(\Gamma)$.

Proposition 11. *If Γ is an almost periodic pattern, then we have*

$$\mathcal{M}(\rho_\Gamma) = D(\Gamma).$$

The proof of this proposition will be done in Appendix A.

We now state a Minkowski-type theorem for the map ρ_Γ. To begin with, we recall the classical Minkowski theorem (see for example the book [GL87]).

Theorem 12 (Minkowski). *Let Λ be a lattice of $\mathbf{R}^n$, $k \in \mathbf{N}$ and $S \subset \mathbf{R}^n$ be a centrally symmetric convex body. If $\mathrm{Leb}(S/2) > k \, \mathrm{covol}(\Lambda)$, then S contains at least $2k$ distinct points of $\Lambda \setminus \{0\}$.*

Theorem 13. *Let $\Gamma \subset \mathbf{Z}^n$ be an almost periodic pattern of density $D(\Gamma)$. Let S be a centrally symmetric body, with $\mathrm{Leb}(S) > 4^n k$. If for every $v \in S \cap \mathbf{Z}^n$, we have $\rho_\Gamma(v) < \rho_0$, then*

$$\rho_0 \geq \frac{1}{k} \left(1 - \frac{1}{D(\Gamma)(2k+1)} \right).$$

In particular, if $k \geq \frac{1}{D(\Gamma)}$, then there exists $x \in C \cap \mathbf{Z}^n$ such that $\rho_\Gamma(x) \geq \frac{D(\Gamma)}{2}$.

Proof of Theorem 13. Minkowski theorem (Theorem 12) asserts that $S/2$ contains at least $2k + 1$ distinct points of $\mathbf{Z}^n$, denoted by u^i. By the hypothesis on the value of ρ_Γ on S, and because the set of differences of $S/2$ is included in S, we know that the density of $(\Gamma + u^i) \cap (\Gamma + u^j)$ is smaller than $\rho_0 D(\Gamma)$. Thus,

$$D\Big(\bigcup_i (\Gamma + u^i) \Big) \geq \sum_i D(\Gamma) - \sum_{i<j} D\big((\Gamma + u^i) \cap (\Gamma + u^j) \big)$$

$$\geq (2k+1)D(\Gamma) - \frac{2k(2k+1)}{2} \rho_0 D(\Gamma).$$

The theorem the follows from the fact that the left member of this inequality is smaller than 1. $\square$

3. Rate of injectivity of isometries

We now focus in more detail on the rate of injectivity of a sequence of isometries (see Definition 2).

3.1. **A geometric viewpoint on the rate of injectivity.** In this subsection, we present a geometric construction to compute the rate of injectivity of a generic matrix, and some applications of it.

Let $P \in O_n(\mathbf{R})$ and $\Lambda = P(\mathbf{Z}^n)$. The density of $\pi(\Lambda)$ is the proportion of $x \in \mathbf{Z}^n$ belonging to $\pi(\Lambda)$; in other words the proportion of $x \in \mathbf{Z}^n$ such that there exists $\lambda \in \Lambda$ whose distance to x (for $\| \cdot \|_\infty$) is smaller than $1/2$. remark that this property only depends on the value of x modulo Λ. If we consider the union

$$U = \bigcup_{\lambda \in \Lambda} B(\lambda, 1/2)$$

of balls of radius $1/2$ centred on the points of Λ (see Figure 5), then $x \in \pi(\Lambda)$ if and only if $x \in U \cap \mathbf{Z}^n$. So, if we set ν the measure of repartition of the $x \in \mathbf{Z}^n$ modulo Λ, that is

$$\nu = \lim_{R \to +\infty} \frac{1}{\mathrm{Card}(B_R \cap \mathbf{Z}^n)} \sum_{x \in B_R \cap \mathbf{Z}^n} \delta_{\mathrm{pr}_{\mathbf{R}^n/\Lambda}(x)},$$

then we obtain the following formula.

Proposition 14. *For every $P \in O_n(\mathbf{R})$ (we identify U with its projection of $\mathbf{R}^n/\Lambda$),*

$$\tau(P) = D\big(\pi(\Lambda) \big) = \nu\big(\mathrm{pr}_{\mathbf{R}^n/\Lambda}(U) \big).$$

An even more simple formula holds when the matrix P is totally irrational.

Definition 15. We say that a matrix $P \in O_n(\mathbf{R})$ is *totally irrational* if the image $P(\mathbf{Z}^n)$ is equidistributed[1] modulo $\mathbf{Z}^n$; in particular, this is true when the coefficients of P form a $\mathbf{Q}$-free family.

If the matrix P is totally irrational, then the measure ν is the uniform measure. Thus, if $\mathcal{D}$ is a fundamental domain of $\mathbf{R}^n/\Lambda$, then $\tau(P)$ is the area of $\mathcal{D} \cap U$.

With the same kind of arguments, we easily obtain a formula for $\rho_{\widehat{P}(\mathbf{Z}^n)}(v)$ (the frequency of the difference v in $\widehat{P}(\mathbf{Z}^n)$, see Definition 8).

Proposition 16. *If $P \in GL_n(\mathbf{R})$ is totally irrational, then for every $v \in \mathbf{Z}^n$,*

$$\rho_{\widehat{P}(\mathbf{Z}^n)}(v) = \mathrm{Leb}\big(B(v, 1/2) \cap U \big).$$

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[1]It is equivalent to require that it is dense instead of equidistributed.
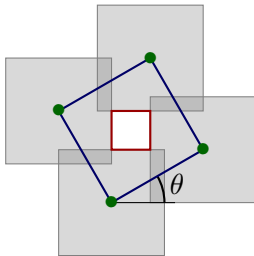
FIGURE 5. Computation of the mean rate of injectivity of a rotation of $\mathbf{R}^2$: it is equal to 1 minus the area of the interior of the red square.

Sketch of proof of Proposition 16. We want to know which proportion of points $x \in \Gamma = \widehat{P}(\mathbf{Z}^n)$ are such that $x + v$ also belongs to Γ. But modulo $\Lambda = P(\mathbf{Z}^n)$, x belongs to Γ if and only if $x \in B(0, 1/2)$. Similarly, $x + v$ belongs to Γ if and only if $x \in B(-v, 1/2)$. Thus, by equirepartition, $\rho_{\widehat{P}(\mathbf{Z}^n)}(v)$ is equal to the area of $B(v, 1/2) \cap U$. $\hspace{1em}\square$

From Proposition 14, we deduce the continuity of $\overline{\tau}$. More precisely, $\overline{\tau}$ is continuous and piecewise polynomial of degree smaller than n; moreover τ coincides with a continuous function on a generic subset of $O_n(\mathbf{R})$.

It also allows to compute simply the mean rate of injectivity of some examples of matrices: for $\theta \in [0, \pi/2]$, the mean rate of injectivity of a rotation of $\mathbf{R}^2$ of angle θ is (see Figure 5).

$$\overline{\tau}(R_\theta) = 1 - (\cos(\theta) + \sin(\theta) - 1)^2.$$

3.2. **Diffusion process.** In this paragraph, we study the action of a discretization of a matrix on the set of differences of an almost periodic pattern Γ; more precisely, we study the link between the functions ρ_Γ and $\rho_{\widehat{P}(\Gamma)}$.

For $u \in \mathbf{R}^n$, we define the function φ_u, which is a "weighted projection" of u on $\mathbf{Z}^n$.

Definition 17. Given $u \in \mathbf{R}^n$, the function $\varphi_u = \mathbf{Z}^n \to [0, 1]$ is defined by

$$\varphi_u(v) = \begin{cases} 0 & \text{if } d_\infty(u, v) \geq 1 \\ \prod_{i=1}^n (1 - |u_i + v_i|) & \text{if } d_\infty(u, v) < 1. \end{cases}$$

In particular, the function φ_u satisfies $\sum_{v \in \mathbf{Z}^n} \varphi_u(v) = 1$, and is supported by the vertices of the integral unit cube[2] that contains[3] u. Figure 6 gives a geometric interpretation of this function φ_u.

The following property asserts that the discretization $\widehat{P}$ acts "smoothly" on the frequency of differences. In particular, when $D(\Gamma) = D(\widehat{P}\Gamma)$, the function $\rho_{\widehat{P}\Gamma}$ is obtained from the function ρ_Γ by applying a linear operator $\mathcal{A}$, acting on each Dirac function δ_v such that $\mathcal{A}\delta_u(v) = \varphi_{P(u)}(v)$. Roughly speaking, to compute $\mathcal{A}\delta_v$, we take δ_{Pv} and apply a diffusion process. In the other case where $D(\widehat{P}\Gamma) < D(\Gamma)$, we only have inequalities involving the operator $\mathcal{A}$ to compute the function $\rho_{\widehat{P}\Gamma}$.

Proposition 18. *Let $\Gamma \subset \mathbf{Z}^n$ be an almost periodic pattern and $P \in O_n(\mathbf{R})$ be a generic matrix.*

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[2]An integral cube has vertices with integer coordinates and its faces parallel to the canonical hyperplanes of $\mathbf{R}^n$.

[3]More precisely, the support of φ_u is the smallest integral unit cube of dimension $n' \leq n$ which contains u.
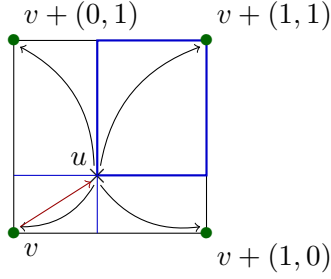
FIGURE 6. The function φ_u in dimension 2: its value on one vertex of the square is equal to the area of the opposite rectangle; in particular, $\varphi_u(v)$ is the area of the rectangle with the vertices u and $v+(1,1)$ (in bold).
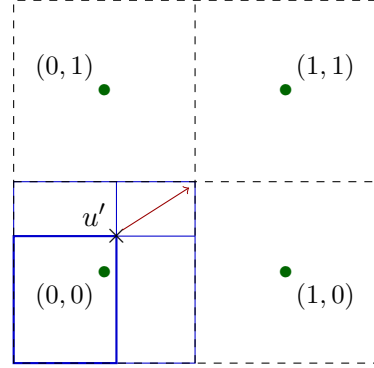
FIGURE 7. The red vector is equal to that of Figure 6 for $u = Pv$. If Px belongs to the bottom left rectangle, then $\pi(Px + Pv) = y \in \mathbf{Z}^2$; if Px belongs to the top left rectangle, then $\pi(Px + Pv) = y + (0,1)$ etc.

(i) If $D(\widehat{P}(\Gamma)) = D(\Gamma)$, then for every $u \in \mathbf{Z}^n$,

$$\rho_{\widehat{P}(\Gamma)}(u) = \sum_{v \in \mathbf{Z}^n} \varphi_{P(v)}(u)\rho_\Gamma(v).$$

(ii) In the general case, for every $u \in \mathbf{Z}^n$, we have

$$\frac{D(\Gamma)}{D(\widehat{P}(\Gamma))} \sup_{v \in \mathbf{Z}^n} \varphi_{P(v)}(u)\rho_\Gamma(v) \le \rho_{\widehat{P}(\Gamma)}(u) \le \frac{D(\Gamma)}{D(\widehat{P}(\Gamma))} \sum_{v \in \mathbf{Z}^n} \varphi_{P(v)}(u)\rho_\Gamma(v).$$

Proof of Proposition 18. We begin by proving the first point of the proposition. Suppose that $P \in O_n(\mathbf{R})$ is generic and that $D(\widehat{P}(\Gamma)) = D(\Gamma)$. Let $x \in \Gamma \cap (\Gamma - v)$. We consider the projection y' of $y = Px$, and the projection u' of $u = Pv$, on the fundamental domain $]-1/2, 1/2]^n$ of $\mathbf{R}^n/\mathbf{Z}^n$. We have

$$P(x + v) = \pi(Px) + \pi(Pv) + y' + u'.$$

Suppose that y' belongs to the parallelepiped whose vertices are $(-1/2, \cdots, -1/2)$ and u' (in bold in Figure 7), then $y'+u' \in [-1/2, 1/2[^n$. Thus, $\pi(P(x+v)) = \pi(Px)+\pi(Pv)$. The same kind of results holds for the other parallelepipeds whose vertices are u' and one vertex of $[-1/2, 1/2[^n$.

We set $\Gamma = \widehat{P}(\mathbf{Z}^n)$. The genericity of P ensures that for every $v \in \mathbf{Z}^n$, the set $\Gamma \cap (\Gamma - v)$, which has density $D(\Gamma)\rho_\Gamma(v)$ (by definition of ρ_Γ), is equidistributed modulo $\mathbf{Z}^n$ (by Lemma 23). Thus, the points x' are equidistributed modulo $\mathbf{Z}^n$. In particular, the difference v will spread into the differences which are the support of the function φ_{Pv}, and each of them will occur with a frequency given by $\varphi_{Pv}(x)\rho_\Gamma(v)$. The hypothesis about the fact that the density of the sets does not decrease imply that the contributions of each difference of Γ to the differences of $\widehat{P}(\Gamma)$ add.

In the general case, the contributions to each difference of Γ may overlap. However, applying the argument of the previous case, we can easily prove the second part of the proposition. □

Remark 19. We also remark that:

(i) the density strictly decreases (that is, $D(\widehat{P}(\Gamma)) < D(\Gamma)$) if and only if there exists $v_0 \in \mathbf{Z}^n$ such that $\rho_\Gamma(v_0) > 0$ and $\|Pv_0\|_\infty < 1$;

(ii) if there exists $v_0 \in \mathbf{Z}^n$ such that
$$\sum_{v \in \mathbf{Z}^n} \varphi_{P(v)}(v_0)\rho_\Gamma(v_0) > 1,$$

then the density strictly decreases by at least $\sum_{v \in \mathbf{Z}^n} \varphi_{P(v)}(v_0)\rho_\Gamma(v_0) - 1$.

3.3. **Rate of injectivity of a generic sequence of isometries.** We now come to the proof of the main theorem of this paper (Theorem 1). We will begin by applying the Minkowski theorem for almost periodic patterns (Theorem 13), which gives *one* nonzero difference whose frequency is positive. The rest of the proof of Theorem 1 consists in using again an argument of equidistribution. More precisely, we apply successively the following lemma, which asserts that given an almost periodic pattern Γ of density D_0, a sequence of isometries and $\delta > 0$, then, perturbing each isometry of at most δ if necessary, we can make the density of the k_0-th image of Γ smaller than $\lambda_0 D_0$, with k_0 and λ_0 depending only on D_0 and δ. The proof of this lemma involves the study of the action of the discretizations on differences made in Proposition 18

Lemma 20. *Let $(P_k)_{k \geq 1}$ be a sequence of matrices of $O_n(\mathbf{R})$ and $\Gamma \subset \mathbf{Z}^n$ an almost periodic pattern. Given $\delta > 0$ and $D > 0$ such that $D(\Gamma) \geq D$, there exists $k_0 = k_0(D)$ (decreasing in D), $\lambda_0 = \lambda_0(D, \delta) < 1$ (decreasing in D and in δ), and a sequence $(Q_k)_{k \geq 1}$ of totally irrational matrices of $O_n(\mathbf{R})$, such that $\|P_k - Q_k\| \leq \delta$ for every $k \geq 1$ and*
$$D\big((\widehat{Q_{k_0}} \circ \cdots \circ \widehat{Q_1})(\Gamma)\big) < \lambda_0 D(\Gamma).$$

We begin by proving that this lemma implies Theorem 1.

Proof of Theorem 1. Suppose that Lemma 20 is true. Let $\tau_0 \in]0, 1[$ and $\delta > 0$. We want to prove that we can perturb the sequence $(P_k)_k$ into a sequence $(Q_k)_k$ of isometries, which is δ-close to $(P_k)_k$ and is such that its asymptotic rate is smaller than τ_0 (and that this remains true on a whole neighbourhood of these matrices).

Thus, we can suppose that $\tau^\infty((P_k)_k) > \tau_0$. We apply Lemma 20 to obtain the parameters $k_0 = k_0(\tau_0/2)$ (because $k_0(D)$ is decreasing in D) and $\lambda_0 = \lambda_0(\tau_0/2, \delta)$ (because $\lambda_0(D, \delta)$ is decreasing in D). Applying the lemma ℓ times, this gives a sequence $(Q_k)_k$ of isometries, which is δ-close to $(P_k)_k$, such that, as long as $\tau^{\ell k_0}(Q_0, \cdots, Q_{\ell k_0}) > \tau_0/2$, we have $\tau^{\ell k_0}(Q_1, \cdots, Q_{\ell k_0}) < \lambda_0^\ell D(\mathbf{Z}^n)$. But for ℓ large enough, $\lambda_0^\ell < \tau_0$, which proves the theorem. $\qquad\square$

Proof of Lemma 20. The idea of the proof is the following. Firstly, we apply the Minkowski-type theorem for almost periodic patterns (Theorem 13) to find a uniform constant $C > 0$ and a point $u_0 \in \mathbf{Z}^n \setminus \{0\}$ whose norm is "not too big", such that $\rho_\Gamma(u_0) > CD(\Gamma)$. Then, we apply Proposition 18 to prove that the difference u_0 in Γ eventually goes to 0; that is, that there exists $k_0 \in \mathbf{N}^*$ and an almost periodic pattern $\widetilde{\Gamma}$ of positive density (that can be computed) such that there exists a sequence $(Q_k)_k$ of isometries, with $\|Q_i - P_i\| \leq \delta$, such that for every $x \in \widetilde{\Gamma}$,
$$(\widehat{Q_{k_0}} \circ \cdots \circ \widehat{Q_1})(x) = (\widehat{Q_{k_0}} \circ \cdots \circ \widehat{Q_1})(x + u_0).$$
This makes the density of the k_0-th image of Γ decrease:
$$D\big((\widehat{Q_{k_0}} \circ \cdots \circ \widehat{Q_1})(\Gamma)\big) \leq D(\Gamma) - D(\widetilde{\Gamma});$$
a precise estimation of the density of $\widetilde{\Gamma}$ will then prove the lemma.

We begin by applying the Minkowski-like theorem for almost periodic patterns (Theorem 13) to a *Euclidean* ball B'_R such that (recall that $[B]$ denotes the set of integer

points inside B)

$$\text{Leb}(B_R') = V_n R^n = 4^n \left\lfloor \frac{1}{D(\Gamma)} \right\rfloor, \tag{2}$$

where V_n denotes the measure of the unit ball on $\mathbf{R}^n$. Then, Theorem 13 says that there exists $u_0 \in B_R' \cap \mathbf{Z}^n \setminus \{0\}$ such that

$$\rho_\Gamma(u_0) \geq \frac{D(\Gamma)}{2}. \tag{3}$$

We now perturb each matrix P_k into a totally irrational matrix Q_k such that for every point $x \in [B_R'] \setminus \{0\}$, the point $Q_k(x)$ is far away from the lattice $\mathbf{Z}^n$. More precisely, as the set of matrices $Q \in O_n(\mathbf{R})$ such that $Q([B_R']) \cap \mathbf{Z}^n \neq \{0\}$ is finite, there exists a constant $d_0(R, \delta)$ such that for every $P \in O_n(\mathbf{R})$, there exists $Q \in O_n(\mathbf{R})$ such that $\|P - Q\| \leq \delta$ and for every $x \in [B_R'] \setminus \{0\}$, we have $d_\infty(Q(x), \mathbf{Z}^n) > d_0(R, \delta)$. Applying Lemma 23 (which states that if the sequence $(Q_k)_k$ is generic, then the matrices Q_k are "non resonant"), we build a sequence $(Q_k)_{k \geq 1}$ of totally irrational[4] matrices of $O_n(\mathbf{R})$ such that for every $k \in \mathbf{N}^*$, we have:

- $\|P_k - Q_k\| \leq \delta$;
- for every $x \in [B_R'] \setminus \{0\}$, we have $d_\infty(Q_k(x), \mathbf{Z}^n) > d_0(R, \delta)$;
- the set $(Q_k \circ \widehat{Q_{k-1}} \circ \cdots \circ \widehat{Q_1})(\Gamma)$ is equidistributed modulo $\mathbf{Z}^n$.

We then consider the difference u_0 (given by Equation (3)). We denote by $\lfloor P \rfloor(u)$ the point of the smallest integer cube of dimension $n' \leq n$ that contains $P(u)$ which has the smallest Euclidean norm (that is, the point of the support of $\varphi_{P(u)}$ with the smallest Euclidean norm). In particular, if $P(u) \notin \mathbf{Z}^n$, then $\|\lfloor P \rfloor(u)\|_2 < \|P(u)\|_2$ (where $\|\cdot\|_2$ is the Euclidean norm). Then, the point (ii) of Proposition 18 shows that

$$\rho_{\widehat{Q_1}(\Gamma)}(\lfloor Q_1 \rfloor(u_0)) \geq \frac{D(\Gamma)}{D(\widehat{Q_1}(\Gamma))} \varphi_{Q_1(\lfloor Q_1 \rfloor(u_0))}(u_0) \rho_\Gamma(u_0)$$

$$\geq \frac{(d_0(R, \delta))^n}{2} D(\Gamma),$$

(applying Equation (3)) and so on, for every $k \in \mathbf{N}^*$,

$$\rho_{(\widehat{Q_k} \circ \cdots \circ \widehat{Q_1})(\Gamma)}\big((\lfloor Q_k \rfloor \circ \cdots \circ \lfloor Q_1 \rfloor)(u_0)\big) \geq \left(\frac{(d_0(R, \delta))^n}{2} \right)^k D(\Gamma).$$

We then remark that the sequence

$$\big\|(\lfloor Q_k \rfloor \circ \cdots \circ \lfloor Q_1 \rfloor)(u_0)\big\|_2$$

is decreasing and can only take a finite number of values (it lies in $\sqrt{\mathbf{Z}}$). Then, there exists $k_0 \leq R^2$ such that

$$\big(\lfloor Q_{k_0} \rfloor \circ \cdots \circ \lfloor Q_1 \rfloor\big)(u_0) = 0;$$

in particular, by Equation (2), we have

$$k_0 \leq \left(\frac{4^n}{V_n} \left\lfloor \frac{1}{D(\Gamma)} \right\rfloor \right)^{2/n}.$$

Then, point (ii) of Remark 19 applied to $v_0 = 0$ implies that the density of the image set satisfies

$$D\big((\widehat{Q_k} \circ \cdots \circ \widehat{Q_1})(\Gamma)\big) \leq \left(1 - \left(\frac{(d_0(R, \delta))^n}{2} \right)^{k_0} \right) D(\Gamma).$$

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[4]See Definition 15.

The conclusions of the lemma are obtained by setting $\lambda_0 = 1 - \left(\frac{(d_0(R,\delta))^n}{2}\right)^{k_0}$. $\square$

APPENDIX A. TECHNICAL LEMMAS

Let us begin by the proof of Proposition 11.

Proof of Proposition 11. This proof lies primarily in an inversion of limits.

Let $\varepsilon > 0$. As Γ is an almost periodic pattern, there exists $R_0 > 0$ such that for every $R \geq R_0$ and every $x \in \mathbf{R}^n$, we have

$$\left| D(\Gamma) - \frac{\Gamma \cap [B(x,R)]}{\mathrm{Card}[B_R]} \right| \leq \varepsilon. \tag{4}$$

So, we choose $R \geq R_0$, $x \in \mathbf{Z}^n$ and compute

$$\begin{aligned}
\frac{1}{\mathrm{Card}[B_R]} \sum_{v \in [B(x,R)]} \rho_\Gamma(v) &= \frac{1}{\mathrm{Card}[B_R]} \sum_{v \in [B(x,R)]} \frac{D\big((\Gamma - v) \cap \Gamma\big)}{D(\Gamma)} \\
&= \frac{1}{\mathrm{Card}[B_R]} \sum_{v \in [B(x,R)]} \lim_{R' \to +\infty} \frac{1}{\mathrm{Card}[B_{R'}]} \sum_{y \in [B_{R'}]} \frac{\mathbf{1}_{y \in \Gamma - v} \mathbf{1}_{y \in \Gamma}}{D(\Gamma)} \\
&= \frac{1}{D(\Gamma)} \lim_{R' \to +\infty} \frac{1}{\mathrm{Card}[B_{R'}]} \sum_{y \in [B_{R'}]} \mathbf{1}_{y \in \Gamma} \frac{1}{\mathrm{Card}[B_R]} \sum_{v \in [B(x,R)]} \mathbf{1}_{y \in \Gamma - v} \\
&= \frac{1}{D(\Gamma)} \underbrace{\lim_{R' \to +\infty} \frac{1}{\mathrm{Card}[B_{R'}]} \sum_{y \in [B_{R'}]} \mathbf{1}_{y \in \Gamma}}_{\text{first term}} \underbrace{\frac{1}{\mathrm{Card}[B_R]} \sum_{v' \in [B(y+x,R)]} \mathbf{1}_{v' \in \Gamma}}_{\text{second term}}.
\end{aligned}$$

By Equation (4), the second term is ε-close to $D(\Gamma)$. Considered independently, the first term is equal to $D(\Gamma)$ (still by Equation (4)). Thus, we have

$$\left| \frac{1}{\mathrm{Card}[B(x,R)]} \sum_{v \in [B(x,R)]} \rho_\Gamma(v) - D(\Gamma) \right| \leq \varepsilon,$$

that we wanted to prove. $\square$

We now state an easy lemma which asserts that for ε small enough, the set of translations $\mathcal{N}_\varepsilon$ is "stable under additions with a small number of terms".

Lemma 21. *Let Γ be an almost periodic pattern, $\varepsilon > 0$ and $\ell \in \mathbf{N}$. Then if we set $\varepsilon' = \varepsilon/\ell$ and denote by $\mathcal{N}_{\varepsilon'}$ the set of translations of Γ and $R_{\varepsilon'} > 0$ the corresponding radius for the parameter ε', then for every $k \in [\![1, \ell]\!]$ and every $v_1, \cdots, v_\ell \in \mathcal{N}_{\varepsilon'}$, we have*

$$\forall R \geq R_{\varepsilon'}, \ D_R^+\Big(\big(\Gamma + \sum_{i=1}^{\ell} v_i\big) \Delta \Gamma\Big) < \varepsilon.$$

Proof of Lemma 21. Let Γ be an almost periodic pattern, $\varepsilon > 0$, $\ell \in \mathbf{N}$, $R_0 > 0$ and $\varepsilon' = \varepsilon/\ell$. Then there exists $R_{\varepsilon'} > 0$ such that

$$\forall R \geq R_{\varepsilon'}, \ \forall v \in \mathcal{N}_{\varepsilon'}, \ D_R^+\big((\Gamma + v) \Delta \Gamma\big) < \varepsilon'.$$

We then take $1 \le k \le \ell$, $v_1, \cdots, v_k \in \mathcal{N}_{\varepsilon'}$ and compute

$$D_R^+\Big(\big(\Gamma + \sum_{i=1}^k v_i\big)\Delta\Gamma\Big) \le \sum_{m=1}^k D_R^+\Big(\big(\Gamma + \sum_{i=1}^m v_i\big)\Delta\big(\Gamma + \sum_{i=1}^{m-1} v_i\big)\Big)$$
$$\le \sum_{m=1}^k D_R^+\Big(\big((\Gamma + v_m)\Delta\Gamma\big) + \sum_{i=1}^{m-1} v_i\Big).$$

By the invariance under translation of D_R^+, we deduce that

$$D_R^+\Big(\big(\Gamma + \sum_{i=1}^k v_i\big)\Delta\Gamma\Big) \le \sum_{m=1}^k D_R^+\big((\Gamma + v_m)\Delta\Gamma\big)$$
$$\le k\varepsilon'.$$

As $k \le \ell$, this ends the proof. $\qquad\qquad\square$

Remark 22. In particular, this lemma implies that the set $\mathcal{N}_\varepsilon$ contains arbitrarily large patches of lattices of $\mathbf{R}^n$: for every almost periodic pattern Γ, $\varepsilon > 0$ and $\ell \in \mathbf{N}$, there exists $\varepsilon' > 0$ such that for every $k_i \in [\![-\ell, \ell]\!]$ and every $v_1, \cdots, v_n \in \mathcal{N}_{\varepsilon'}$, we have

$$\forall R \ge R_{\varepsilon'}, \ D_R^+\Big(\big(\Gamma + \sum_{i=1}^n k_i v_i\big)\Delta\Gamma\Big) < \varepsilon.$$

The second lemma is more technical. It expresses that given an almost periodic pattern Γ, a generic matrix $A \in O_n(\mathbf{R})$ is non resonant with respect to Γ.

Lemma 23. *Let $\Gamma \subset \mathbf{Z}^n$ be an almost periodic pattern with positive uniform density. Then the set of $A \in O_n(\mathbf{R})$ such that $A(\Gamma)$ is equidistributed modulo $\mathbf{Z}^n$ is generic. More precisely, for every $\varepsilon > 0$, there exists an open and dense set of $A \in O_n(\mathbf{R})$ such that there exists $R_0 > 0$ such that for every $R > R_0$, the projection on $\mathbf{R}^n/\mathbf{Z}^n$ of the uniform measure on $A(\Gamma \cap B_R)$ is ε-close to Lebesgue measure on $\mathbf{R}^n/\mathbf{Z}^n$.*

Proof of Lemma 23. During this proof, we consider a distance dist on $\mathcal{P}(\mathbf{R}^n/\mathbf{Z}^n)$ which is invariant under translations, where $\mathcal{P}(\mathbf{R}^n/\mathbf{Z}^n)$ denotes the space of probability Borel measures on $\mathbf{R}^n/\mathbf{Z}^n$ endowed with weak-* topology. We also suppose that this distance satisfies the following convexity inequality: if $\mu, \nu_1, \cdots, \nu_d \in \mathcal{P}(\mathbf{R}^n/\mathbf{Z}^n)$, then

$$\mathrm{dist}\left(\mu, \frac{1}{d}\sum_{i=1}^d \nu_i\right) \le \frac{1}{d}\sum_{i=1}^d \mathrm{dist}(\mu, \nu_i).$$

For the simplicity of the notations, when μ and ν have not total mass 1, we will denote by $\mathrm{dist}(\mu, \nu)$ the distance between the normalizations of μ and ν.

We consider the set $\mathcal{U}_\varepsilon$ of matrices $A \in GL_n(\mathbf{R})$ satisfying: there exists $R_0 > 0$ such that for all $R \ge R_0$,

$$\mathrm{dist}\left(\mathrm{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{x \in B_R \cap \Gamma} \bar{\delta}_{Ax}\right) < \varepsilon,$$

where $\bar{\delta}_x$ is the Dirac measure of the projection of x on $\mathbf{R}^n/\mathbf{Z}^n$. We show that for every $\varepsilon > 0$, $\mathcal{U}_\varepsilon$ contains an open dense set. Then, the set $\bigcap_{\varepsilon > 0} \mathcal{U}_\varepsilon$ will be a G_δ dense set made of matrices $A \in GL_n(\mathbf{R})$ such that $A(\Gamma)$ is well distributed.

Let $\varepsilon > 0$, $\delta > 0$, $\ell > 0$ and $A \in GL_n(\mathbf{R})$. We apply Remark 22 to obtain a parameter $R_0 > 0$ and a family $v_1, \cdots, v_n$ of ε-translations of Γ such that the family of cubes $\big(B(\sum_{i=1}^n k_i v_i, R_0)\big)_{-\ell \le k_i \le \ell}$ is an "almost tiling" of $B_{\ell R_0}$ (in particular, each v_i is close to the vector having $2\bar{R}_0$ in the i-th coordinate and 0 in the others, see Figure 8):
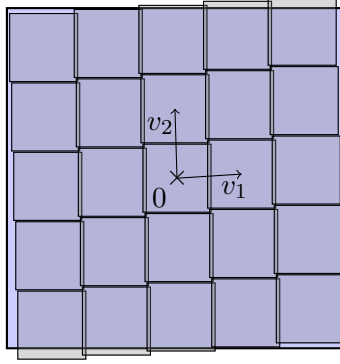
FIGURE 8. "Almost tiling" of $B_{\ell R_0}$ by cubes $B(\sum_{i=1}^n k_i v_i, R_0)$, with $-\ell \le k_i \le \ell$.

(1) this collection of cubes fills almost all $B_{\ell R_0}$:

$$\frac{\text{Card}\left(\Gamma \cap \left(\bigcup_{-\ell \le k_i \le \ell} B(\sum_{i=1}^n k_i v_i, R_0) \Delta B_{\ell R_0}\right)\right)}{\text{Card}(\Gamma \cap B_{\ell R_0})} \le \varepsilon;$$

(2) the overlaps of the cubes are not too big: for all collections (k_i) and (k_i') such that $-\ell \le k_i, k_i' \le \ell$,

$$\frac{\text{Card}\left(\Gamma \cap \left(B(\sum_{i=1}^n k_i v_i, R_0) \Delta B(\sum_{i=1}^n k_i' v_i, R_0)\right)\right)}{\text{Card}(\Gamma \cap B_{\ell R_0})} \le \varepsilon;$$

(3) the vectors $\sum_{i=1}^n k_i v_i$ are translations for Γ: for every collection (k_i) such that $-\ell \le k_i \le \ell$,

$$\frac{\text{Card}\left(\left(\Gamma \Delta(\Gamma - \sum_{i=1}^n k_i v_i)\right) \cap B_{R_0}\right)}{\text{Card}(\Gamma \cap B_{R_0})} \le \varepsilon.$$

Increasing R_0 and ℓ if necessary, there exists $A' \in GL_n(\mathbf{R})$ (respectively $SL_n(\mathbf{R})$, $O_n(\mathbf{R})$) such that $\|A - A'\| \le \delta$ and that we have

$$\text{dist}\left(\text{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{-\ell \le k_i \le \ell} \bar{\delta}_{A'(\sum_{i=1}^n k_i v_i)}\right) \le \varepsilon. \tag{5}$$

Indeed, if we denote by Λ the lattice spanned by the vectors $v_1, \cdots, v_n$, then the set of matrices A' such that $A'\Lambda$ is equidistributed modulo $\mathbf{Z}^n$ is dense in $GL_n(\mathbf{R})$ (respectively $SL_n(\mathbf{R})$ and $O_n(\mathbf{R})$).

Then, we have,

$$\text{dist}\left(\text{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{\substack{-\ell \le k_i \le \ell \\ x \in \Gamma \cap B(\sum_{i=1}^n k_i v_i, R_0)}} \bar{\delta}_{A'x}\right) \le$$

$$\text{dist}\left(\text{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{\substack{-\ell \le k_i \le \ell \\ x \in \Gamma \cap B(0, R_0)}} \bar{\delta}_{A'(\sum_{i=1}^n k_i v_i) + A'x}\right)$$

$$+ \text{dist}\left(\sum_{\substack{-\ell \le k_i \le \ell \\ x \in \Gamma \cap B(0, R_0)}} \bar{\delta}_{A'(\sum_{i=1}^n k_i v_i) + A'x}, \sum_{\substack{-\ell \le k_i \le \ell \\ x \in \Gamma \cap B(\sum_{i=1}^n k_i v_i, R_0)}} \bar{\delta}_{A'x}\right)$$

By the property of convexity of dist, the first term is smaller than

$$\frac{1}{\text{Card}\left(\Gamma \cap B(0, R_0)\right)} \sum_{x \in \Gamma \cap B(0, R_0)} \text{dist}\left(\text{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{-\ell \leq k_i \leq \ell} \bar{\delta}_{A'(\sum_{i=1}^n k_i v_i) + A'x}\right);$$

by Equation (5) and the fact that dist is invariant under translation, this term is smaller than ε. As by hypothesis, the vectors $\sum_{i=1}^n k_i v_i$ are ε-translations of Γ (hypothesis (3)), the second term is also smaller than ε. Thus, we get

$$\text{dist}\left(\text{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{\substack{-\ell \leq k_i \leq \ell \\ x \in \Gamma \cap B(\sum_{i=1}^n k_i v_i, R_0)}} \bar{\delta}_{A'x}\right) \leq 2\varepsilon$$

By the fact that the family of cubes $\left(B(\sum_{i=1}^n k_i v_i, R_0)\right)_{-\ell \leq k_i \leq \ell}$ is an almost tiling of $B_{\ell R_0}$ (hypotheses (1) and (2)), we get, for every $v \in \mathbf{R}^n$,

$$\text{dist}\left(\text{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{x \in \Gamma \cap B_{\ell R_0}} \bar{\delta}_{A'x}\right) < 4\varepsilon.$$

Remark that we can suppose that this remains true on a whole neighbourhood of A'. We use the fact that Γ is an almost periodic pattern to deduce that A' belongs to the interior of $\mathcal{U}_\varepsilon$. $\qquad\square$

Appendix B. Proof of Theorem 7

Notation 24. For $A \in GL_n(\mathbf{R})$, we denote $A = (a_{i,j})_{i,j}$. We denote by $I_{\mathbf{Q}}(A)$ the set of indices i such that $a_{i,j} \in \mathbf{Q}$ for every $j \in [\![1, n]\!]$

The proof of Theorem 7 relies on the following remark:

Remark 25. If $a \in \mathbf{Q}$, then there exists $q \in \mathbf{N}^*$ such that $\{ax \mid x \in \mathbf{Z}\} \subset \frac{1}{q}\mathbf{Z}$. On the contrary, if $a \in \mathbf{R} \setminus \mathbf{Q}$, then the set $\{ax \mid x \in \mathbf{Z}\}$ is equidistributed in $\mathbf{R}/\mathbf{Z}$.

Thus, in the rational case, the proof will lie in an argument of periodicity. On the contrary, in the irrational case, the image $A(\mathbf{Z}^n)$ is equidistributed modulo $\mathbf{Z}^n$: on every large enough domain, the density does not move a lot when we perturb the image set $A(\mathbf{Z}^n)$ by small translations. This reasoning is formalized by Lemmas 26 and 27.

More precisely, for R large enough, we would like to find vectors w such that $D_R^+\left((\pi(A\Gamma) + w)\Delta\pi(A\Gamma)\right)$ is small. We know that there exists vectors v such that $D_R^+\left((\Gamma + v)\Delta\Gamma\right)$ is small; this implies that $D_R^+\left((A\Gamma + Av)\Delta A\Gamma\right)$ is small, thus that $D_R^+\left(\pi(A\Gamma + Av)\Delta\pi(A\Gamma)\right)$ is small. The problem is that in general, we do not have $\pi(A\Gamma + Av) = \pi(A\Gamma) + \pi(Av)$. However, this is true if we have $Av \in \mathbf{Z}^n$. Lemma 26 shows that in fact, it is possible to suppose that Av "almost" belongs to $\mathbf{Z}^n$, and Lemma 27 asserts that this property is sufficient to conclude.

The first lemma is a consequence of the pigeonhole principle.

Lemma 26. *Let $\Gamma \subset \mathbf{Z}^n$ be an almost periodic pattern, $\varepsilon > 0$, $\delta > 0$ and $A \in GL_n(\mathbf{R})$. Then we can suppose that the elements of $A(\mathcal{N}_\varepsilon)$ are δ-close to $\mathbf{Z}^n$. More precisely, there exists $R_{\varepsilon,\delta} > 0$ and a relatively dense set $\widetilde{\mathcal{N}}_{\varepsilon,\delta}$ such that*

$$\forall R \geq R_{\varepsilon,\delta}, \ \forall v \in \widetilde{\mathcal{N}}_{\varepsilon,\delta}, \ D_R^+\left((\Gamma + v)\Delta\Gamma\right) < \varepsilon,$$

and that for every $v \in \widetilde{\mathcal{N}}_{\varepsilon,\delta}$, we have $d_\infty(Av, \mathbf{Z}^n) < \delta$. Moreover, we can suppose that for every $i \in I_{\mathbf{Q}}(A)$ and every $v \in \widetilde{\mathcal{N}}_{\varepsilon,\delta}$, we have $(Av)_i \in \mathbf{Z}$.

The second lemma states that in the irrational case, we have continuity of the density under perturbations by translations.

Lemma 27. *Let $\varepsilon > 0$ and $A \in GL_n(\mathbf{R})$. Then there exists $\delta > 0$ and $R_0 > 0$ such that for all $w \in B_\infty(0, \delta)$ (such that for every $i \in I_{\mathbf{Q}}(A)$, $w_i = 0$), and for all $R \geq R_0$, we have*

$$D_R^+\big(\pi(A\mathbf{Z}^n)\Delta\pi(A\mathbf{Z}^n + w)\big) \leq \varepsilon.$$

Remark 28. When $I_{\mathbf{Q}}(A) = \emptyset$, and in particular when A is totally irrational (see Definition 15), the map $v \mapsto \tau(A + v)$ is continuous in 0; the same proof as that of this lemma implies that this function is globally continuous.

We begin by the proofs of both lemmas, and prove Theorem 7 thereafter.

Proof of Lemma 26. Let us begin by giving the main ideas of the proof of this lemma. For R_0 large enough, the set of remainders modulo $\mathbf{Z}^n$ of vectors Av, where v is a ε-translation of Γ belonging to B_{R_0}, is close to the set of remainders modulo $\mathbf{Z}^n$ of vectors Av, where v is any ε-translation of Γ. Moreover (by the pigeonhole principle), there exists an integer k_0 such that for each ε-translation $v \in B_{R_0}$, there exists $k \leq k_0$ such that $A(kv)$ is close to $\mathbf{Z}^n$. Thus, for every ε-translation v of Γ, there exists a $(k_0 - 1)\varepsilon$-translation $v' = (k-1)v$, belonging to $B_{k_0 R_0}$, such that $A(v + v')$ is close to $\mathbf{Z}^n$. The vector $v + v'$ is then a $k_0\varepsilon$-translation of Γ (by additivity of the translations) whose distance to v is smaller than $k_0 R_0$.

We now formalize these remarks. Let Γ be an almost periodic pattern, $\varepsilon > 0$ and $A \in GL_n(\mathbf{R})$. First of all, we apply the pigeonhole principle. We partition the torus $\mathbf{R}^n/\mathbf{Z}^n$ into squares whose sides are smaller than δ; we can suppose that there are at most $\lceil 1/\delta \rceil^n$ such squares. For $v \in \mathbf{R}^n$, we consider the family of vectors $\{A(kv)\}_{0 \leq k \leq \lceil 1/\delta \rceil^n}$ modulo $\mathbf{Z}^n$. By the pigeonhole principle, at least two of these vectors, say $A(k_1 v)$ and $A(k_2 v)$, with $k_1 < k_2$, lie in the same small square of $\mathbf{R}^n/\mathbf{Z}^n$. Thus, if we set $k_v = k_2 - k_1$ and $\ell = \lceil 1/\delta \rceil^n$, we have

$$1 \leq k_v \leq \ell \quad \text{and} \quad d_\infty\big(A(k_v v), \mathbf{Z}^n\big) \leq \delta. \tag{6}$$

To obtain the conclusion in the rational case, we suppose in addition that $v \in q\mathbf{Z}^n$, where $q \in \mathbf{N}^*$ is such that for every $i \in I_{\mathbf{Q}}(A)$ and every $1 \leq j \leq n$, we have $q\, a_{i,j} \in \mathbf{Z}$ (which is possible by Remark 22).

We set $\varepsilon' = \varepsilon/\ell$. By the definition of an almost periodic pattern, there exists $R_{\varepsilon'} > 0$ and a relatively dense set $\mathcal{N}_{\varepsilon'}$ such that Equation (1) holds for the parameter ε':

$$\forall R \geq R_{\varepsilon'},\ \forall v \in \mathcal{N}_{\varepsilon'},\ D_R^+\big((\Gamma + v)\Delta\Gamma\big) < \varepsilon', \tag{1'}$$

We now set

$$P = \big\{Av \bmod \mathbf{Z}^n \mid v \in \mathcal{N}_{\varepsilon'}\big\} \quad \text{and} \quad P_R = \big\{Av \bmod \mathbf{Z}^n \mid v \in \mathcal{N}_{\varepsilon'} \cap B_R\big\}.$$

We have $\bigcup_{R>0} P_R = P$, so there exists $R_0 > R_{\varepsilon'}$ such that $d_H(P, P_{R_0}) < \delta$ (where d_H denotes Hausdorff distance). Thus, for every $v \in \mathcal{N}_{\varepsilon'}$, there exists $v' \in \mathcal{N}_{\varepsilon'} \cap B_{R_0}$ such that

$$d_\infty(Av - Av', \mathbf{Z}^n) < \delta. \tag{7}$$

We then remark that for every $v' \in \mathcal{N}_{\varepsilon'} \cap B_{R_0}$, if we set $v'' = (k_{v'} - 1)v'$, then by Equation (6), we have

$$d_\infty(Av' + Av'', \mathbf{Z}^n) = d_\infty\big(A(k_{v'}v'), \mathbf{Z}^n\big) \leq \delta.$$

Combining this with Equation (7), we get

$$d_\infty(Av + Av'', \mathbf{Z}^n) \leq 2\delta,$$

with $v'' \in B_{\ell R_0}$.

On the other hand, $k_{v'} \leq \ell$ and Equation (1') holds, so Lemma 21 (more precisely, Remark 22) implies that $v'' \in \mathcal{N}_\varepsilon$, that is

$$\forall R \geq R_{\varepsilon'}, \ D_R^+\big((\Gamma + v'')\Delta\Gamma\big) < \varepsilon.$$

In other words, for every $v \in \mathcal{N}_{\varepsilon'}$, there exists $v'' \in \mathcal{N}_\varepsilon \cap B_{\ell R_0}$ (with ℓ and R_0 independent from v) such that $d_\infty\big(A(v + v''), \mathbf{Z}^n\big) < 2\delta$. The set $\widetilde{\mathcal{N}}_{2\varepsilon,2\delta}$ we look for is then the set of such sums $v + v''$. $\qquad\square$

Proof of Lemma 27. Under the hypothesis of the lemma, for every $i \notin I_\mathbf{Q}(A)$, the sets

$$\left\{ \sum_{j=1}^n a_{i,j} x_j \mid (x_j) \in \mathbf{Z}^n \right\},$$

are equidistributed modulo $\mathbf{Z}$. Thus, for all $\varepsilon > 0$, there exists $R_0 > 0$ such that for every $R \geq R_0$,

$$D_R^+\Big\{v \in \mathbf{Z}^n \,\Big|\, \exists i \notin I_\mathbf{Q}(A) : d\big((Av)_i, \mathbf{Z} + \tfrac{1}{2}\big) \leq \varepsilon\Big\} \leq 2(n+1)\varepsilon.$$

As a consequence, for all $w \in \mathbf{R}^n$ such that $\|w\|_\infty \leq \varepsilon/(2(n+1))$ and that $w_i = 0$ for every $i \in I_\mathbf{Q}(A)$, we have

$$D_R^+\big(\pi(A\mathbf{Z}^n)\Delta\pi(A(\mathbf{Z}^n + w))\big) \leq \varepsilon.$$

Then, the lemma follows from the fact that there exists $\delta > 0$ such that $\|A(w)\|_\infty \leq \varepsilon/(2(n+1))$ as soon as $\|w\| \leq \delta$. $\qquad\square$

Proof of Theorem 7. Let $\varepsilon > 0$. Lemma 27 gives us a corresponding $\delta > 0$, that we use to apply Lemma 26 and get a set of translations $\widetilde{\mathcal{N}}_{\varepsilon,\delta}$. Then, for every $v \in \widetilde{\mathcal{N}}_{\varepsilon,\delta}$, we write $\pi(Av) = Av + \big(\pi(Av) - Av\big) = Av + w$. The conclusions of Lemma 26 imply that $\|w\|_\infty < \delta$, and that $w_i = 0$ for every $i \in I_\mathbf{Q}(A)$.

We now explain why $\hat{A}v = \pi(Av)$ is a ε-translation for the set $\widehat{A}(\Gamma)$. Indeed, for every $R \geq \max(R_{\varepsilon,\delta}, MR_0)$, where M is the maximum of the greatest modulus of the eigenvalues of A and of the greatest modulus of the eigenvalues of A^{-1}, we have

$$D_R^+\Big(\pi(A\Gamma)\Delta\big(\pi(A\Gamma) + \widehat{A}v\big)\Big) \leq D_R^+\Big(\pi(A\Gamma)\Delta\big(\pi(A\Gamma) + w\big)\Big)$$
$$+ D_R^+\Big(\big(\pi(A\Gamma) + w\big)\Delta\big(\pi(A\Gamma) + \widehat{A}v\big)\Big)$$

(where $w = \pi(Av) - Av$). By Lemma 27, the first term is smaller than ε. For its part, the second term is smaller than

$$D_R^+\big((A\Gamma + Av)\Delta A\Gamma\big) \leq M^2 D_{RM}^+\big((\Gamma + v)\Delta\Gamma\big),$$

which is smaller than ε because $v \in \mathcal{N}_\varepsilon$. $\qquad\square$

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