MAX-PLANCK-INSTITUT FÜR MATHEMATIK

(19) 49. 228. 4021

Uwe Jannsen
Gottfried-Claren-Straße 26
5300 Bonn 3
Telefon (02 28) 4021
Durchwahl 402-

Deutsche Bank
(BLZ 380 700 59) Kto. 0533570

Max-Planck-Institut für Mathematik · Gottfried-Claren-Straße 26, 5300 Bonn 3

Prof. J. M. Fontaine
Université de Grenoble I
Institut Fourier
B. P. 74
38402 Saint-Martin d'Heres / FRANCE

Bonn, 23. 10. 87

Dear Professor Fontaine,

do you remember my questions at Wuppertal, concerning a "crystalline conjecture" in the case of bad reduction? I think that I now have a proposal, see the copied notes pp. 42 ff.. These ideas are very recent, and I haven't had time to make the conjecture more precise. In any case, I would like to know your opinion about this.

Let me change notations compared with my notes. Let $K / \mathbb{Q}_p$ be a local field with residue field $\varkappa$ and let $K^0 = \mathrm{Quot}(W(\varkappa))$, $W(\varkappa)$ the ring of Witt vectors for $\varkappa$. Let X be smooth and proper over K (no assumption about the reduction). I am not quite sure whether one can expect an isomorphism (over a finite extension K' of K)

(1) $$H^i(\bar{X}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris,N} \cong D \otimes_{K'^0} B_{cris,N}$$

of objects in $MF^N_{K'}$ with Galois action, as in the case $N=0$. If this is possible (with a certain field $B_{cris,N}$

in $MF_{K'}^{N}$ with Galois action, generalising B_{cris} for $MF_{K'}$), could you imagine a description for $B_{cris,N}$? Perhaps one should have $B_{cris} = (B_{cris,N})^{N}$ (kernel of N on $B_{cris,N}$).

I have tried to consider an example and come to the following question. Look at extensions of $G_{\mathbb{Q}_p}$-representation

(2) $$0 \to \mathbb{Q}_p(1) \to V \longrightarrow \mathbb{Q}_p \to 0$$

(for example, these arise from elliptic curves with multiplicative reduction). These are described by classes in

(3) $$H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p(1)),$$

which is a 2-dimensional vector space. It follows from Wintenberger's work that the corresponding space of extensions of Tate objects in $MF^{f}_{\mathbb{Q}_p}$ has dimension 1:

(4) $$\mathrm{Ext}^1_{MF^{f}_{\mathbb{Q}_p}}(\overset{S^0}{\overset{\shortparallel}{\mathbb{Q}_p}}, \overset{S^{-1}}{\overset{\shortparallel}{\mathbb{Q}_p(1)}}) \cong \mathbb{Q}_p/(\phi - p)F^1\mathbb{Q}_p = \mathbb{Q}_p .$$

Can you identify these extensions in (3) (in other words, the extensions (2) which are crystalline)? This should (after proving some compatibilities) amount to the description of the morphism

$$\mathbb{Q}_p/(\phi - p)F^1\mathbb{Q}_p \hookrightarrow H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p(1))$$

arising as the connecting morphism of the $G_{\mathbb{Q}_p}$-cohomology sequence associated to

(5) $$0 \to \mathbb{Q}_p(1) \to F^0(S^{-1} \otimes B_{cris}) \xrightarrow{\phi - 1} S^{-1} \otimes B_{cris} \to 0 .$$

A related question: for an abelian variety A/K with <u>bad</u> reduction the Tate module $T_p A$ should <u>not</u> be crystalline – do you know a proof for this?

Let me finish with some remarks on my conjecture.

1.) The formula $\phi N \phi^{-1} = p \cdot N$ implies that N vanishes mod p. Thus, $H^1(\bar{X}, \mathbb{Z}/p)$ should be crystalline, also in the case of bad reduction, and the N does not appear in the classification of mod p representations involved in Serre's conjecture.

2.) My conjecture would imply that the G_K-representation $H^1(\bar{X}, \mathbb{Q}_p)$ gives rise to a K^0-representation of the Weil-Deligne group W_K', and it should be the "same" as the $\mathbb{Q}_\ell$-representation of W_K' attached to $H^1(\bar{X}, \mathbb{Q}_\ell)$ for $\ell \neq p$ (see Deligne's article in SLN 349).

3.) S. Sen (Invent. math. 62) has constructed a generalised Hodge-Tate decomposition for <u>all</u> $\mathbb{Q}_p$-representations of G_K, and determined the Hodge-Tate representations among these as the ones having "integral weights". It would be nice to have a construction associating to <u>every</u> $\mathbb{Q}_p$-representation V a certain filtered module D with N (not necessarily weakly admissible) such that the N-crystalline representations V are those for which D is "mixed" in the sense that $Gr_r^M D$ is pure (in terms of the eigenvalues of $\phi^{[K^0 : \mathbb{Q}_p]}$, not in terms of the slopes).

I am visiting the Max-Planck-Institut at Bonn from October 1^{st}, 1987 till March 31^{st}, 1988, and on November 13 I'll give a talk at the K-theory seminar in Paris (I'll also stay for the Bourbaki seminar that weekend). Since I also would like to talk about these things, I would be very happy to get an answer from you before. Otherwise, I hope to perhaps see you in Paris.

With best regards

Uwe Jannsen

MAX-PLANCK-INSTITUT FÜR MATHEMATIK

Uwe Jannsen
Gottfried-Claren-Straße 26
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Telefon (02 28) 4021
Durchwahl 402-264

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Max-Planck-Institut für Mathematik · Gottfried-Claren-Straße 26, 5300 Bonn 3

Prof. J.-M. Fontaine
Université de Grenoble I
Institut Fourier (laboratoire
ass. aux C.N.R.S) B.P. 74
38402 SAINT-MARTIN d'HERES

Bonn, 23. 10. 87

Dear Professor Fontaine,

one hour after sending away my letter to you I realised how one can probably answer one of my questions: Since $V_p(A)$, for an abelian variety A over the local field $K/\mathbb{Q}_p$, is Hodge-Tate of weights $0, 1$, your remarks in your astérisque 65 article seem to suggest that $V_p(A)$ should be crystalline if and only if it comes from a p-divisible group G over $\mathcal{O}_K$, hence (by SGA 7 $\boxed{\text{IX}}$ 5.13), if and only if A has good reduction.

In loc. cit. you prove the mentioned statement for $e=1$ and an algebraically closed residue field, so the question now rather is: can one remove the assumption $e=1$ and is the last assumption essential?

Sincerely Yours

Uwe Jannsen