ON THE COHOMOLOGY OF p-ADIC ANALYTIC SPACES, II: THE C_{st}-CONJECTURE

PIERRE COLMEZ AND WIESŁAWA NIZIOŁ

Abstract. Long ago, Fontaine formulated conjectures (now theorems) relating étale and de Rham cohomologies of algebraic varieties over p-adic fields. In an earlier work we have shown that pro-étale and de Rham cohomologies of analytic varieties in the two extreme cases: proper and Stein, are also related. In the proper case, the comparison theorems are similar to those for algebraic varieties, but for Stein varieties they are quite different.

In this paper, we state analogs of Fontaine's conjectures for general smooth dagger varieties, that interpolate between these two extreme cases, and we prove them for "small" varieties (which include quasi-compact varieties and their naive interiors, and analytifications of algebraic varieties). The proof uses a "geometrization" of all involved cohomologies in terms of quasi-Banach-Colmez spaces (qBC's for short), quasi- because we relax the finiteness conditions. The heart of the proof is a delicate induction argument starting from the case of affinoids and exploiting properties of qBC's in the inductive step. These properties should be of independent interest and we have devoted a large part of the paper to their study.

Contents

Date: July 22, 2022.

The authors' research was supported in part by the grant NR-19-CE40-0015-02 COLOSS and the NSF grant No. DMS-1440140.

1. Introduction

Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of characteristic p. Let $\overline{K}$ be an algebraic closure of K, let C be its p-adic completion, and let $\mathscr{O}_{\overline{K}}$ denote the integral closure of $\mathscr{O}_K$ in $\overline{K}$. Let $W(k)$ be the ring of Witt vectors of k with fraction field F (i.e, $W(k)=\mathscr{O}_F$) and let φ be the absolute Frobenius on $W(\overline{k})$. Set $\mathscr{G}_K=\operatorname{Gal}(\overline{K}/K)$.

1.1. **Comparison theorems for algebraic varieties.** In order to put our results into perspective, let us first recall what is known for algebraic varieties. The story started with Tate conjecturing [41] the existence of a Hodge-like decomposition for the étale cohomology of smooth and proper algebraic (or even rigid analytic) varieties over K and proving the existence of such a decomposition for abelian varieties with good reduction. One upshot of Tate's results is that the p-adic periods of algebraic varieties do not live in C. Fontained constructed [22, 23, 24] rings $\mathbf{B}_{\rm cris}$, $\mathbf{B}_{\rm st}$, $\mathbf{B}_{\rm dR}$ that should contain these periods and refined Tate's conjecture by conjecturing (first [22, 23] in the case of X with good reduction and then [25] in the case of X with semistable reduction) the existence of functorial period isomorphisms relating étale and de Rham cohomologies of smooth algebraic varieties.

Conjecture 1.1. (Fontaine) *Let X be a proper and smooth algebraic variety over K admitting a semistable model over $\mathscr{O}_K$. Let $i\geq 0$.*

(i) (Conjecture $C_{\rm dR}$) *We have a functorial $\mathscr{G}_K$-equivariant isomorphism preserving filtrations*

$$H^i_{\text{ét}}(X_C,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\rm dR}\simeq H^i_{\rm dR}(X)\otimes_K\mathbf{B}_{\rm dR}$$

(ii) (Conjecture $C_{\rm st}$) *We have a functorial $\mathscr{G}_K$-equivariant isomorphism commuting with φ and N*

$$H^i_{\text{ét}}(X_C,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\rm st}\simeq H^i_{\rm HK}(X)\otimes_F\mathbf{B}_{\rm st},$$

compatible with the de Rham period morphism, and the natural injections $\mathbf{B}_{\rm st}\subset\mathbf{B}_{\rm dR}$ and $H^i_{\rm HK}\subset H^i_{\rm dR}$.

As we explain below (Remark 1.3) these conjectures are now theorems (even without the asumptions on existence of nice models, properness, or even smoothness).

Remark 1.2. (i) All the above cohomology groups are finite dimensional over the appropriate field ($\mathbf{Q}_p$ for étale cohomology, K for de Rham, and F for Hyodo-Kato).

(ii) The existence of the period isomorphism implies in particular that

$$\dim_{\mathbf{Q}_p}H^i_{\text{ét}}(X_C,\mathbf{Q}_p)=\dim_K H^i_{\rm dR}(X).$$

(iii) The Hyodo-Kato cohomology group in (ii) is an F-vector space with a semilinear Frobenius φ and a monodromy operator N satisfying $N\varphi=p\varphi N$. Moreover, there is a Hyodo-Kato isomorphism

$$\iota_{\rm HK}:H^i_{\rm HK}(X)\otimes_F K\xrightarrow{\sim}H^i_{\rm dR}(X)$$

(iv) In $C_{\rm dR}$, the filtration on the left hand side is the one coming from $\mathbf{B}_{\rm dR}$; the one on the right hand side is the tensor product of the Hodge filtration on $H^i_{\rm dR}(X)$ and the filtration on $\mathbf{B}_{\rm dR}$. In $C_{\rm st}$, the φ and N on the left hand side are those coming form $\mathbf{B}_{\rm st}$; on the right hand side they are the tensor products of those coming from $H^i_{\rm HK}(X)$ and $\mathbf{B}_{\rm st}$.

(v) Galois properties of the rings $\mathbf{B}_{\rm st}$ and $\mathbf{B}_{\rm dR}$ make it possible to recover de Rham cohomology from étale cohomology by taking fixed points by $\mathscr{G}_K$: we have functorial "étale-to-de Rham" isomorphisms

$$\begin{aligned} H^i_{\rm dR}(X) &\simeq (H^i_{\rm \acute{e}t}(X_C,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\rm dR})^{\mathscr{G}_K}, \quad \text{as filtered } K\text{-vector spaces},\\ H^i_{\rm HK}(X) &\simeq (H^i_{\rm \acute{e}t}(X_C,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\rm st})^{\mathscr{G}_K}, \quad \text{as } F\text{-vector spaces with a } \varphi \text{ and a } N.\end{aligned}$$

Moreover, the Hydo-Kato isomorphism is induced by the inclusion $\mathbf{B}_{\rm st}\subset\mathbf{B}_{\rm dR}$. Note that, instead of tensor products, we could have considered $\mathscr{G}_K$-equivariant homomorphisms: we have functorial isomorphisms

$$\begin{aligned} H^i_{\rm dR}(X)^* &\simeq \mathrm{Hom}_{\mathscr{G}_K}(H^i_{\rm \acute{e}t}(X_C,\mathbf{Q}_p),\mathbf{B}_{\rm dR}), \quad \text{as filtered } K\text{-vector spaces},\\ H^i_{\rm HK}(X)^* &\simeq \mathrm{Hom}_{\mathscr{G}_K}(H^i_{\rm \acute{e}t}(X_C,\mathbf{Q}_p),\mathbf{B}_{\rm st}), \quad \text{as } F\text{-vector spaces with a } \varphi \text{ and a } N.\end{aligned}$$

(vi) It is possible to extract $\mathbf{Q}_p$ from $\mathbf{B}_{\rm st}$, using the extra structures. This induces a description of étale cohomology by de Rham cohomology (with the extra structures coming from Hydo-Kato cohomology), i.e., it gives a "de Rham-to-étale" bicartesian diagram

$$\begin{tikzcd} H^i_{\rm \acute{e}t}(X_C,\mathbf{Q}_p) \arrow[r] \arrow[d] & (H^i_{\rm HK}(X)\otimes_F\mathbf{B}_{\rm st})^{N=0,\varphi=1} \arrow[d] \\ F^0(H^i_{\rm dR}(X)\otimes_K\mathbf{B}_{\rm dR}) \arrow[r] & H^i_{\rm dR}(X)\otimes_K\mathbf{B}_{\rm dR} \end{tikzcd}$$

One can refine this diagram by taking a large enough twist, which makes it possible to remove denominators[1] in t: if $r\geq i$, we have a bicartesian diagram

$$\begin{tikzcd} H^i_{\rm \acute{e}t}(X_C,\mathbf{Q}_p(r)) \arrow[r] \arrow[d] & (H^i_{\rm HK}(X)\otimes_F\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^r} \arrow[d] \\ F^r(H^i_{\rm dR}(X)\otimes_K\mathbf{B}^+_{\rm dR}) \arrow[r] & H^i_{\rm dR}(X)\otimes_K\mathbf{B}^+_{\rm dR} \end{tikzcd}$$

(vii) The pair $(H^i_{\rm HK}(X),H^i_{\rm dR}(X))$ is a filtered (φ,N)-*module* in the sense of Fontaine; the fact that the above diagram is bicartesian and $H^i_{\rm \acute{e}t}(X_C,\mathbf{Q}_p)$ is finite dimensional implies, in particular, that this filtered (φ,N)-module is *weakly admissible*, a condition that can be described purely in terms of the interplay between φ, N and the filtration.

Remark 1.3. (i) As we have mentioned above, Fontaine's conjecture is now a theorem. There have been essentially four lines of attack: the almost étale approach [18], the syntomic approach [42], the motivic approach [33], and the derived geometry approach [2]. The resulting period isomorphisms are compared in [34]. The most comprehensive results are those of Beilinson [2]: there is no assumption of properness, existence of good models or smoothness.

(ii) If we don't assume X to have a semistable model over $\mathscr{O}_K$ then $H^i_{\rm HK}(X)$ has to be replaced with $H^i_{\rm HK}(X_{\overline{K}})$, which is not an F-vector space anymore but a $F^{\rm nr}$-vector space, where $F^{\rm nr}$ is the maximal unramified extension of F. Moreover, $H^i_{\rm HK}(X_{\overline{K}})$ is equipped with a semi-linear action of $\mathscr{G}_K$ commuting with φ and N and the action of $\mathscr{G}_K$ is smooth (i.e., any element x is fixed by $\mathscr{G}_L$ for some finite extension L of K that depends on x). In this case, the isomorphisms involving $H^i_{\rm HK}(X)$ in (v) of Remark 1.2 involve smooth vectors for the action of $\mathscr{G}_K$ and not only fixed vectors: i.e, we have a functorial isomorphism

$$H^i_{\rm HK}(X_{\overline{K}})^*\simeq \mathrm{Hom}^{\rm sm}_{\mathscr{G}_K}(H^i_{\rm \acute{e}t}(X_C,\mathbf{Q}_p),\mathbf{B}_{\rm st}) \quad \text{of } (\varphi,N,\mathscr{G}_K)\text{-modules over } F^{\rm nr}.$$

[1] Recall that $\mathbf{B}_{\rm dR}=\mathbf{B}^+_{\rm dR}[\frac{1}{t}]$, $\mathbf{B}_{\rm st}=\mathbf{B}^+_{\rm st}[\frac{1}{t}]$.

1.2. **Analytic varieties.** Interest in analytic varieties is more recent despite the fact that Tate [41] already formulated his conjecture for rigid analytic spaces. Scholze [38] established a version of Tate's original conjecture for smooth and proper analytic spaces over K or C and proved the $C_{\rm dR}$-conjecture for smooth and proper analytic spaces over K. We proved the $C_{\rm st}$-conjecture for proper analytic spaces with a semistable model [13] (see [30] for a simplified construction) and "de Rham-to-pro-étale" versions of the $C_{\rm st}$-conjecture for Stein varieties [11] over K (also with a semistable model).

In this paper, we will consider smooth dagger varieties over K or C (without any assumption on the existence of nice models):

• Any proper or partially proper rigid analytic variety has a natural dagger structure, and this includes, in particular, analytifications of algebraic varieties, Stein varieties (e.g. étale coverings of Drinfeld's spaces in any dimension), or, more generally, holomorphically convex varieties (proper fibrations over a Stein base).

• Dagger varieties that are not necessarily partially proper include overconvergent affinoids or, more generally, quasi-compact rigid analytic varieties with an overconvergent structure.

As we have seen in the case of algebraic varieties, Fontaine's conjectures $C_{\rm dR}$ and $C_{\rm st}$ can be split in two: a "de Rham-to-étale" direction, and a "étale-to-de Rham" direction. We are going to state analogous conjectures for analytic varieties but with étale cohomology replaced with pro-étale cohomology (i.e., we are dealing with "rational" p-adic Hodge theory and not "integral" p-adic Hodge theory; in particular, the pro-étale cohomology of the analytication of an algebraic variety is much bigger than the étale cohomology of the algebraic variety – the latter is finite dimensional).

1.2.1. *The "de Rham-to-étale" $C_{\rm dR}$ and $C_{\rm st}$ conjectures for dagger varieties.*

Conjecture 1.4. (de Rham-to-pro-étale $C_{\rm dR}+C_{\rm st}$) *Let X be a smooth dagger variety over C. Let $i\leq r$. We have a functorial bicartesian diagram:*

$$\begin{array}{ccc} H^i_{\rm pro\acute{e}t}(X,\mathbf{Q}_p(r)) & \longrightarrow & (H^i_{\rm HK}(X)\widehat{\otimes}_{F^{\rm nr}}\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^r} \\ \downarrow & & \downarrow \\ H^i(F^r{\rm R}\Gamma_{\rm dR}(X/\mathbf{B}^+_{\rm dR})) & \longrightarrow & H^i_{\rm dR}(X/\mathbf{B}^+_{\rm dR}) \end{array}$$

Remark 1.5. (i) The $\mathbf{B}^+_{\rm dR}$-cohomology group $H^i_{\rm dR}(X/\mathbf{B}^+_{\rm dR})$ is a torsion-free $\mathbf{B}^+_{\rm dR}$-module from which one recovers the usual de Rham cohomology by moding out by t. If X is defined over K then $H^i_{\rm dR}(X_C/\mathbf{B}^+_{\rm dR})\simeq H^i_{\rm dR}(X)\widehat{\otimes}_K\mathbf{B}^+_{\rm dR}$.

(ii) If X is defined over K, all spaces are endowed with a natural topology and an action of $\mathscr{G}_K$ and all maps are supposed to be $\mathscr{G}_K$-equivariant and continuous.

(iii) As we have shown in [15], if X is defined over C, then all spaces have a natural topology and are C-points of VS's (pro-étale sheaves on the category ${\rm Perf}_C$ of perfectoid spaces over C) and all maps are supposed to be evaluations of morphisms of VS's and continuous.

(iv) The Hyodo-Kato cohomology group $H^i_{\rm HK}(X)$ (see [15, Sec. 2]) is a $F^{\rm nr}$-module with a Frobenius φ, a monodromy operator N, a (pro-)smooth action of $\mathscr{G}_K$, and a Hyodo-Kato isomorphism $\iota_{\rm HK}: H^i_{\rm HK}(X)\widehat{\otimes}_{F^{\rm nr}}\mathbf{B}^+_{\rm dR}\stackrel{\sim}{\to} H^i_{\rm dR}(X/\mathbf{B}^+_{\rm dR})$. The definitions of $H^i_{\rm HK}(X)$ and $\iota_{\rm HK}$ are adapted from the ones of Beilinson in the algebraic setting and use the alterations of Hartl and Temkin to produce good local models.

(v) In the case of proper analytic varieties, all cohomology groups in the diagram are finite dimensional (as was the case in the algebraic setting) and the kernels of the horizontal arrows are 0. This is not the case for a general analytic variety and all spaces have to be seen in the derived category of locally convex topological vector spaces over $\mathbf{Q}_p$; in particular, the tensor products are (derived) completed tensor products. Even if $H^i_{\rm dR}(X)$, for X over K, is finite dimensional, $H^i({\rm Fil}^r(\mathbf{B}^+_{\rm dR}\widehat{\otimes}_K{\rm R}\Gamma_{\rm dR}(X)))$ surjects onto $\Omega^i(X_C)^{d=0}$ and hence can be huge (and then so is $H^i_{\rm pro\acute{e}t}(X_C,\mathbf{Q}_p(r))$).

1.2.2. *The "étale-to-de Rham" $C_{\rm dR}$ and $C_{\rm st}$ conjectures for dagger varieties.* In the other direction, we have the following conjectures:

Conjecture 1.6. (pro-étale-to-de Rham) *Let X be a smooth dagger variety defined over K. We have functorial isomorphims:*

$$
\begin{aligned}
&(C_{\rm dR}) \qquad H^i_{\rm dR}(X)^* \simeq \mathrm{Hom}_{\mathscr{G}_K}(H^i_{\rm pro\acute{e}t}(X_C,\mathbf{Q}_p),\mathbf{B}_{\rm dR}), \quad \text{as filtered } K\text{-vector spaces},\\
&(C_{\rm st}) \qquad H^i_{\rm HK}(X)^* \simeq \mathrm{Hom}^{\rm sm}_{\mathscr{G}_K}(H^i_{\rm pro\acute{e}t}(X_C,\mathbf{Q}_p),\mathbf{B}_{\rm st}), \quad \text{as } (\varphi,N,\mathscr{G}_K)\text{-modules over } F^{\rm nr}.
\end{aligned}
$$

Remark 1.7. As we mentioned above, even if $H^i_{\rm dR}(X)$ is finite dimensional, $H^i_{\rm pro\acute{e}t}(X_C,\mathbf{Q}_p)$ is, in general, huge. Hence it is a little bit of a miracle that one could recover $H^i_{\rm dR}(X)$ from it.

The previous conjecture uses Galois action to recover de Rham and Hyodo-Kato cohomologies from pro-étale cohomology. If X is defined over C, there is no Galois action anymore but one can use the VS structure alluded to above to recover part of the structure. This leads to the following conjecture:

Conjecture 1.8. *Let X be a smooth dagger variety defined over C. We have functorial isomorphisms*[2]*:*

$$
\begin{aligned}
&(\mathbb{C}_{\rm dR}) \qquad \mathrm{Hom}_{\mathbf{B}^+_{\rm dR}}(H^i_{\rm dR}(X/\mathbf{B}^+_{\rm dR}),\mathbf{B}_{\rm dR}) \simeq \mathrm{Hom}_{\rm VS}(\mathbb{H}^i_{\rm pro\acute{e}t}(X,\mathbf{Q}_p),\mathbb{B}_{\rm dR}), \quad \text{as } \mathbf{B}_{\rm dR}\text{-modules},\\
&(\mathbb{C}_{\rm st}) \qquad \mathrm{Hom}_{F^{\rm nr}}(H^i_{\rm HK}(X),\mathbf{B}_{\rm st}) \simeq \mathrm{Hom}_{\rm VS}(\mathbb{H}^i_{\rm pro\acute{e}t}(X,\mathbf{Q}_p),\mathbb{B}_{\rm st}), \quad \text{as } \mathbf{B}_{\rm st}\text{-modules.}
\end{aligned}
$$

Remark 1.9. (i) It is not possible to recover filtration on the $\mathbf{B}^+_{\rm dR}$-cohomology just by looking at the cohomology level because the $t^k\mathbb{B}^+_{\rm dR}$'s are all isomorphic as VS's whereas the $t^k\mathbf{B}^+_{\rm dR}$'s are all distinct as $\mathscr{G}_K$-modules.

(ii) In the same way, φ and N disappear since $M = M^{N=0,\varphi=1}\otimes_{\mathbf{B}^{\varphi=1}_{\rm cr}}\mathbf{B}_{\rm st}$ if $M = M_0\otimes_{F^{\rm nr}}\mathbf{B}_{\rm st}$, where M_0 is a finite rank (φ,N)-module over $F^{\rm nr}$.

(iii) One way to understand points (i) and (ii) of the remark is the following. Conjecture 1.4 represents $H^i_{\rm pro\acute{e}t}$ as the H^0 of a quasi-coherent sheaf on the Fargues-Fontaine curve that is obtained as the "modification" at ∞ of the sheaf associated to a φ-module. The H^0 determines the sheaf (because there is no H^1), but not the "modification" that gave rise to it. Maybe a more sophisticated formulation on the level of derived categories would allow to do that (see Remark 7.9) ?

Our main result towards these conjectures is Thorem 1.10 below, in which a *small variety* (for results pertaining to "big" varieties, see Remark 1.14) is a smooth dagger variety that is quasi-compact or can be covered by a finite number of Stein varieties whose intersections have finite dimensional de Rham cohomology (the latter case includes analytifications of algebraic varieties and naive interiors of quasi-compact varieties).

Theorem 1.10. *If X is small, then:*

- *Conjecture 1.4 is true.*
- *If X is defined over K, then Conjecture 1.6 is true, and if X is defined over C, Conjecture 1.8 is true.*

The VS's in the diagram in Conjecture 1.4 have some finiteness properties: they are extensions of finite Dimensional Vector Spaces (also known as Banach-Colmez spaces, and referred to as BC's in the rest of the text) by torsion $\mathbb{B}^+_{\rm dR}$-Modules. In particular, such objects $\mathbb{W}$ (referred to as qBC's, the "q" standing for "quasi") have a height $\mathrm{ht}(\mathbb{W})\in\mathbf{N}$ (if $\mathbb{W}$ is the qBC attached to a finite dimensional $\mathbf{Q}_p$-vector space W, then $\mathrm{ht}(\mathbb{W}) = \dim_{\mathbf{Q}_p} W$). We have the following result echoing (ii) of Remark 1.2 which is the key to the proof of all the results in Theorem 1.10; it is a little bit surprising that the pro-étale cohomology encodes this finiteness result considering how big it is if X is not proper.

[2] In all of the paper $\mathrm{Hom}_{\rm VS}$ means morphisms of VS's which are continuous on C-points.

Theorem 1.11. *If X is a quasi-compact smooth dagger variety, we have*

$$\mathrm{ht}(\mathbb{H}^i_{\text{proét}}(X_C,\mathbf{Q}_p)) = \dim_C(H^i_{\mathrm{dR}}(X_C)).$$

The fact that the diagram in Conjecture 1.4 is bicartesian implies in particular that the associated sequence is exact. The exactitude on the right can be rephrased in a way echoing (vii) of Remark 1.2:

Theorem 1.12. *If X is a quasi-compact smooth dagger variety, $(H^i_{\mathrm{HK}}(X), H^i(X/\mathbf{B}^+_{\mathrm{dR}}))$ is an acyclic[3] filtered (φ,N)-module.*

Remark 1.13. (i) The proof of Theorem 1.10 makes heavy use of the theory of BC's and, in particular, the geometric point of view advocated in Le Bras's thesis [32]. About half of the paper is devoted to proving results about BC's that are needed in the proof of our main result. Some of these results may be of independent interest.

(ii) For proper varieties, we can use the more naive theory of BC's as in [13], where we treated the case of varieties with semistable models over the integers.

(iii) For an overconvergent affinoid or a small Stein variety, one gets a direct proof from the basic comparison theorem proved in [15]. The main difficulty is to go from this case to the case of a variety covered by a finite number n of overconvergent affinoids or small Stein varieties. This relies on a delicate induction on n which deepest part is establishing Theorem 1.11 and Theorem 1.12, see Proposition 1.17 and the ensuing comments.

Remark 1.14. (i) If X is smooth and Stein over C or K, it can be written as a strict increasing union of overconvergent affinoids. We get that the horizontal rows in the diagram in Conjecture 1.4 are surjective and their kernels are $(\Omega^{i-1}(X)/\mathrm{Ker}\, d)(r-i)$. In particular, we have an exact sequence

$$0\to \Omega^{r-1}(X)/\mathrm{Ker}\, d \to H^r_{\text{proét}}(X,\mathbf{Q}_p(r)) \to (H^r_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\to 0$$

generalizing the exact sequence of [11, Th. 1.8] which was proven under the assumption that X has a semistable model. It is not difficult to deduce Conjectures 1.4, 1.6 and 1.8 in this case.

(ii) A general partially proper rigid analytic variety can be written as an increasing union of quasi-compact dagger varieties, but there are annoying problems with $\mathrm{R}^1\lim$'s that show up when you want to deduce Conjectures 1.4, 1.6 and 1.8 in general from the quasi-compact case of Theorem 1.10 (these problems do not appear in the Stein case). The main issue seems to be whether or not the Hodge filtration on de Rham cohomology is formed of closed subspaces. We have partial results (for example for a product of a Stein and a proper variety or for a proper fibration over a curve), but nothing definitive; we report on these results in [16].

1.3. **Proofs.** The main ingredients in the proofs are the results from [15] and the parallel theories of BC's [8, 9, 32] and Fontaine's almost C-representations [27, 28].

Let X be a quasi-compact smooth dagger variety over C. From [15] we use the existence of the basic comparison isomorphism with syntomic cohomology:

$$H^i_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \xrightarrow{\sim} H^i_{\text{proét}}(X,\mathbf{Q}_p(r)),\quad i\leq r, \tag{1.15}$$

and the exact sequence

$$\cdots\to \mathrm{DR}^{i-1}_r(X)\to H^i_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\longrightarrow \mathrm{HK}^i_r(X)\xrightarrow{\iota_{\mathrm{HK}}}\mathrm{DR}^i_r(X)\to\cdots, \tag{1.16}$$

where we have set:

$$\mathrm{HK}^i_r(X) := (H^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r},\quad \mathrm{DR}^i_r(X):=H^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r).$$

Very important for what follows is that this exact sequence and the isomorphism (1.15) can be promoted to the category of VS's (see [15, Th. 1.3, Th. 1.7]).

[3]This means that the associated vector bundle on the Fargues-Fontaine curve has cohomology only in degree 0. It is purely a condition on the interplay between φ, N and the filtration. Weakly admissible filtered (φ,N)-modules are acyclic.

1.3.1. *On the proof of Conjecture 1.4.* Proving Conjecture 1.4 amounts to splitting the long exact sequence (1.16) into short exact sequences, which can be done directly for dagger affinoids (Theorem 5.14), going back to the definition of syntomic cohomology. The problem is to go from affinoids to varieties covered by a finite number of affinoids. This we do by induction on the number of affinoids needed, using the theory of BC's. We already used BC's in [13] to treat the proper case (with a semistable model), but there we were helped by Scholze's theorem [38, Th. 1.1] that the $H^i_{\text{proét}}(X,\mathbf{Q}_p(r))$'s are, in that case, finite dimensional $\mathbf{Q}_p$-vector spaces; this made it possible to use the basic theory of BC's as developed in [8, 9]. In the present paper, we need to use the more powerful point of view advocated in Le Bras' thesis [32]. The key result that comes out of the theory of BC's is the following proposition (Proposition 6.11):

Proposition 1.17. *The following properties are equivalent:*

(a) *The diagram in Conjecture 1.4 is bicartesian.*

(b) $(H^i_{\rm HK}(X), F^0H^i_{\rm dR}(X/\mathbf{B}_{\rm dR}))$ *is acyclic, for $i=r-1$ and $i=r$.*

(c) *The kernel and cokernel of* $\mathbb{H}^r_{\text{proét}}(X,\mathbf{Q}_p(r))\to (H^r_{\rm HK}(X)\otimes_{F^{\rm nr}}\mathbb{B}^+_{\rm st})^{N=0,\varphi=p^r}$ *have height* 0.

(d) $\operatorname{ht}(\mathbb{H}^r_{\text{proét}}(X,\mathbf{Q}_p(r)))=\dim_C H^r_{\rm dR}(X)$.

The equivalence of (d), which is a condition only about r, and (b), which involves r and $r-1$, makes it possible to do an induction on r. Among the ingredients that go into the proof of this proposition are:

• an interpretation (Proposition 3.27) of $\operatorname{ht}(\mathbb{W})$ for a qBC $\mathbb{W}$ as the rank of $\operatorname{Hom}_{\rm VS}(\mathbb{W},\mathbb{B}_{\rm dR})$ (i.e. a *categorification* of the height ht),

• a dichotomy (Proposition 4.18) that tells us what happens if the sequence associated to the diagram is not exact on the right.

It is in this dichotomy that we use in an essential way that the degrees of involved cohomology groups are $\leq r$: this implies that the slopes of Frobenius on Hyodo-Kato cohomology are $\leq r$.

1.3.2. *On the proof of Conjectures 1.6 and 1.8.* Conjectures 1.6 and 1.8 follow from Conjecture 1.4 and results about almost C representations or BC's of the following type:

$$\operatorname{Hom}_{\mathscr{G}_K}(\mathbf{B}^+_{\rm dR}/t^k,\mathbf{B}_{\rm dR})=0\quad\text{and}\quad\operatorname{Hom}_{\rm VS}(\mathbb{B}^+_{\rm dR}/t^k,\mathbb{B}_{\rm dR})=0$$

(Proposition 2.14 for the first statement and Corollary 3.17 for the second; this type of results allow us to get rid of the de Rham terms in the sequence of Hom's that is deduced from the exact sequence associated to the bicartesian diagram of Conjecture 1.4.)

$$\begin{aligned}\operatorname{Hom}_{\mathscr{G}_K}((M\otimes_{F^{\rm nr}}\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^r},\mathbf{B}_{\rm dR})&=M^*_K\\ \operatorname{Hom}_{\rm VS}((M\otimes_{F^{\rm nr}}\mathbb{B}^+_{\rm st})^{N=0,\varphi=p^r},\mathbb{B}_{\rm dR})&=M^*\otimes_{F^{\rm nr}}\mathbf{B}_{\rm dR}\end{aligned}$$

if M is a (φ,N)-module of slopes $\leq r$ (Theorem 4.8 and corollaries for the first statement and Proposition 4.20 and 4.22 for the second).

Acknowledgments. W.N. would like to thank MSRI, Berkeley, and the Isaac Newton Institute, Cambridge, for hospitality during Spring 2019 and Spring 2020 semesters, respectively, when parts of this paper were written. We would like to thank Guido Bosco, Jean-Benoît Bost, Gabriel Dospinescu, Laurent Fargues, Marco Maculan, Jerôme Poineau, Peter Scholze for helpful conversations concerning the content of this paper.

1.3.3. *Notation and conventions.* Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field K of characteristic 0 and perfect residue field k of characteristic p. Let $\overline{K}$ be an algebraic closure of K and let $\mathscr{O}_{\overline{K}}$ denote the integral closure of $\mathscr{O}_K$ in $\overline{K}$. Let $C=\widehat{\overline{K}}$ be the p-adic completion of $\overline{K}$. Let $W(k)$ be the ring of Witt vectors of k with fraction field F (i.e., $W(k)=\mathscr{O}_F$); let $e=e_K$ be the ramification index of K over F. Set $\mathscr{G}_K=\operatorname{Gal}(\overline{K}/K)$ and let φ be the absolute Frobenius on $W(\overline{k})$. We will denote by $\mathbf{A}_{\rm cr}, \mathbf{B}_{\rm cr}, \mathbf{B}_{\rm st}, \mathbf{B}_{\rm dR}$ the crystalline, semistable, and de Rham period rings of Fontaine.

We will denote by $\mathscr{O}_K$, $\mathscr{O}_K^\times$, and $\mathscr{O}_K^0$, depending on the context, the scheme $\operatorname{Spec}(\mathscr{O}_K)$ or the formal scheme $\operatorname{Spf}(\mathscr{O}_K)$ with the trivial, the canonical (i.e., associated to the closed point), and the induced by $\mathbf{N} \to \mathscr{O}_K, 1 \mapsto 0$, log-structure, respectively. Unless otherwise stated all formal schemes are p-adic, locally of finite type, and equidimensional. For a (p-adic formal) scheme X over $\mathscr{O}_K$, let X_0 denote the special fiber of X; let X_n denote its reduction modulo p^n.

All rigid analytic spaces considered will be over K or C. We assume that they are separated, taut, and countable at infinity.

Our cohomology groups will be equipped with a canonical topology. To talk about it in a systematic way, we will work rationally in the category of locally convex K-vector spaces and integrally in the category of pro-discrete $\mathscr{O}_K$-modules. For details the reader may consult [11, Sec. 2.1, 2.3]. To summarize quickly:

(1) C_K is the category of convex K-vector spaces; it is a quasi-abelian category. We will denote the left-bounded derived ∞-category of C_K by $\mathscr{D}(C_K)$. A morphism of complexes that is a quasi-isomorphism in $\mathscr{D}(C_K)$, i.e., its cone is strictly exact, will be called a *strict quasi-isomorphism*. The associated cohomology objects are denoted by $\widetilde{H}^n(E) \in LH(C_K)$; they are called *classical* if the natural map $\widetilde{H}^n(E) \to H^n(E)$ is an isomorphism.

(2) Objects in the category PD_K of pro-discrete $\mathscr{O}_K$-modules are topological $\mathscr{O}_K$-modules that are countable inverse limits, as topological $\mathscr{O}_K$-modules, of discrete $\mathscr{O}_K$-modules M^i, $i \in \mathbf{N}$. It is a quasi-abelian category. Inside PD_K we distinguish the category PC_K of pseudocompact $\mathscr{O}_K$-modules, i.e., pro-discrete modules $M \simeq \lim_i M_i$ such that each M_i is of finite length (we note that if K is a finite extension of $\mathbf{Q}_p$ this is equivalent to M being profinite). It is an abelian category.

(3) There is a tensor product functor from the category of pro-discrete $\mathscr{O}_K$-modules to convex K-vector spaces:
$$(-)\widehat{\otimes} K : PD_K \to C_K, \quad M \mapsto M \widehat{\otimes}_{\mathscr{O}_K} K.$$
Since C_K admits filtered inductive limits, the functor $(-)\widehat{\otimes} K$ extends to a functor $(-)\widehat{\otimes} K : \operatorname{Ind}(PD_K) \to C_K$. The functor $(-)\widehat{\otimes} K$ is right exact but not, in general, left exact. We will consider its (compatible) left derived functors
$$(-)\widehat{\otimes}^L K : \mathscr{D}^-(PD_K) \to \operatorname{Pro}(\mathscr{D}^-(C_K)), \quad (-)\widehat{\otimes}^L K : \mathscr{D}^-(\operatorname{Ind}(PD_K)) \to \operatorname{Pro}(\mathscr{D}^-(C_K)).$$
If E is a complex of torsion free and p-adically complete (i.e., $E \simeq \lim_n E/p^n$) modules from PD_K then the natural map
$$E\widehat{\otimes}^L K \to E\widehat{\otimes} K$$
is a strict quasi-isomorphism [11, Prop. 2.6].

Unless otherwise stated, we work in the derived (stable) ∞-category $\mathscr{D}(A)$ of left-bounded complexes of a quasi-abelian category A (the latter will be clear from the context). Many of our constructions will involve (pre)sheaves of objects from $\mathscr{D}(A)$. We will use a shorthand for certain homotopy limits: if $f : C \to C'$ is a map in the derived ∞-category of a quasi-abelian category, we set
$$[\, C \xrightarrow{f} C' \,] := \operatorname{holim}(C \to C' \leftarrow 0).$$
For an operator F acting on C, we will use the brackets $[C]^F$ to denote the derived eigenspaces and the brackets $(C)^F$ or simply C^F to denote the non-derived ones.

Finally, we will use freely the notation and results from [14].

2. Review of almost C-representations

We will briefly review Fontaine's theory of almost C-representations [27] (see also [28]) and some of its consequences. The theory has a satisfactory shape only when $[K:\mathbf{Q}_p]<\infty$, but some parts work for K arbitrary. Fortunately, the almost C-representations that we are going to deal with have special features and we are only going to use the results in § 2.3 for which we provide alternative proofs (working for general K).

2.1. **Notation.** A *banach* is a Banach space over $\mathbf{Q}_p$ (up to an equivalence of norms) and a *banach representation* of $\mathscr{G}_K$ is a banach with a continuous and linear action of $\mathscr{G}_K$. Denote by $\mathscr{B}(\mathscr{G}_K)$ the category of banach representations of $\mathscr{G}_K$. It has a natural exact category structure: a *short exact sequence* in $\mathscr{B}(\mathscr{G}_K)$ is a sequence

$$0\to B_1\xrightarrow{f} B_2\xrightarrow{g} B_3\to 0,$$

where g is a strict epimorphism and f is a kernel of g.

A $\mathbf{Q}_p$*-representation of* $\mathscr{G}_K$ is a finite dimensional $\mathbf{Q}_p$-vector space with a continuous and linear action of $\mathscr{G}_K$. Similarly, a C*-representation of* $\mathscr{G}_K$ is a finite dimensional C-vector space with a continuous and semilinear action of $\mathscr{G}_K$. We will denote by $\mathrm{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, resp. $\mathrm{Rep}_C(\mathscr{G}_K)$, the category of $\mathbf{Q}_p$-representations, resp. C-representations. More generally, one can define the category $\mathrm{Rep}_{\mathbf{B}^+_{\rm dR}}(\mathscr{G}_K)$ of finite length $\mathbf{B}^+_{\rm dR}$-representations.

Fontaine proved the following surprising theorem:

Theorem 2.1. (Fontaine, [27, Th. A, Th. A']) *If* $[K:\mathbf{Q}_p]<\infty$*, the forgetful functors*

$$\mathrm{Rep}_C(\mathscr{G}_K)\to\mathscr{B}(\mathscr{G}_K),\quad \mathrm{Rep}_{\mathbf{B}^+_{\rm dR}}(\mathscr{G}_K)\to\mathscr{B}(\mathscr{G}_K)$$

are fully faithful.

In other words, if W_1,W_2 are two C-representations of $\mathscr{G}_K$, all $\mathbf{Q}_p$-linear continuous $\mathscr{G}_K$-equivariant maps, of W_1 to W_2 are necessarily C-linear. Similarly, if W_1,W_2 are two $\mathbf{B}^+_{\rm dR}$-representations of $\mathscr{G}_K$, all $\mathbf{Q}_p$-linear continuous $\mathscr{G}_K$-equivariant maps, of W_1 to W_2 are necessarily $\mathbf{B}^+_{\rm dR}$-linear.

Remark 2.2. The proof uses Sen's theory [39] and gives a stronger result: one has the same statements for E-linear maps between E-representations for which the Hodge-Tate-Sen weights are algebraic over E (a condition that is automatic for $E=\mathbf{Q}_p$, if $[K:\mathbf{Q}_p]<\infty$, whence the theorem). In particular, we have the following fundamental result, valid for arbitrary K:

$$\mathrm{Hom}_{\mathscr{G}_K}(C,C)\simeq K$$

2.2. Almost C-representations.

2.2.1. *The general theory.* Two banach representations W_1 and W_2 are *almost isomorphic* if there exist two finite dimensional $\mathbf{Q}_p$-vector spaces $V_i\subset W_i$, $i=1,2$, stable under $\mathscr{G}_K$ such that $W_1/V_1\simeq W_2/V_2$. An *almost C-representation* is a banach representation which is almost isomorphic to C^d for some $d\in\mathbf{N}$. Denote by $\mathscr{C}(\mathscr{G}_K)$ the category of almost C-representations.

Remark 2.3. The above definition makes sense for arbitrary K, but the theory has a satisfactory shape only for $[K:\mathbf{Q}_p]<\infty$; maybe the point of view developped in [28] would lead to satisfactory theory for arbitrary K?

From now on, assume that $[K:\mathbf{Q}_p]<\infty$. Then $\mathscr{C}(\mathscr{G}_K)$ is an abelian subcategory of the exact category $\mathscr{B}(\mathscr{G}_K)$. If $W/V_2\cong C^d/V_1$, one sets $\dim W=d$ and $\mathrm{ht}(W)=\dim_{\mathbf{Q}_p}V_2-\dim_{\mathbf{Q}_p}V_1$. This is independent of choices and yields additive functions [27, Th. B] – a nontrivial fact whose proof uses the theory of BC's[4] [8], [9]. We have $d(C)=1, \mathrm{ht}(C)=0$ and $d(V)=0, \mathrm{ht}(V)=\dim_{\mathbf{Q}_p}(V)$ if V is a $\mathbf{Q}_p$-representation. The category $\mathscr{C}(\mathscr{G}_K)$ contains all $\mathbf{B}_{\mathrm{dR}}^+$-representations and if W is a $\mathbf{B}_{\mathrm{dR}}^+$-representation of length d then it is almost isomorphic to C^d [27, Th. C]; we have $d(W)=d$ and $\mathrm{ht}(W)=0$. In particular, the $\mathbf{B}_{\mathrm{dR}}^+$-representations C and $C(1)$ are almost isomorphic.

Remark 2.4. The category $\mathscr{C}(\mathscr{G}_K)$ modulo almost isomorphisms is semi-simple with a single isomorphism class of simple objects, the class of C.

Fontaine reduced the computation of Ext-groups in the category $\mathscr{C}(\mathscr{G}_K)$ to the computation of Ext-groups in the category $\mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}^+}(\mathscr{G}_K)$ and the computation of Ext-groups in the category $\mathrm{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$ via the following fact (which relies on [27, Prop. 5.5] and [27, Prop. 5.6]):

Proposition 2.5. (Fontaine, [27, Prop. 6.4, Prop. 6.5])

(i) *Let $X, Y \in \mathrm{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$. Then we have a canonical isomorphism*

$$\mathrm{Ext}^i_{\mathscr{G}_K}(X,Y) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathscr{C}(\mathscr{G}_K)}(X,Y), \quad i \geq 0.$$

(ii) *Let $X, Y \in \mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}^+}(\mathscr{G}_K)$. Then we have a canonical isomorphism*

$$\mathrm{Ext}^i_{\mathbf{B}_{\mathrm{dR}}^+(\mathscr{G}_K)}(X,Y) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathscr{C}(\mathscr{G}_K)}(X,Y), \quad i \geq 0.$$

He proved the following result:

Theorem 2.6. (Fontaine, [27, Th. 6.1, Prop. 6.8, Prop. 6.9]) *Let $X, Y \in \mathscr{C}(\mathscr{G}_K)$.*

(i) *The $\mathbf{Q}_p$-vector spaces $\mathrm{Ext}^i_{\mathscr{C}(\mathscr{G}_K)}(X,Y)$ have finite rank and are trivial for $i \geq 3$.*

(ii) $\sum_{i=0}^2 (-1)^i \dim_{\mathbf{Q}_p} \mathrm{Ext}^i_{\mathscr{C}(\mathscr{G}_K)}(X,Y) = -[K:\mathbf{Q}_p]\,\mathrm{ht}(X)\,\mathrm{ht}(Y)$.

(iii) *There exists a natural trace map $\mathrm{Ext}^2_{\mathscr{C}(\mathscr{G}_K)}(X, X(1)) \to \mathbf{Q}_p$ and, for $0 \leq i \leq 2$, the map*

$$\mathrm{Ext}^i_{\mathscr{C}(\mathscr{G}_K)}(X,Y) \times \mathrm{Ext}^{2-i}_{\mathscr{C}(\mathscr{G}_K)}(Y, X(1)) \to \mathrm{Ext}^2_{\mathscr{C}(\mathscr{G}_K)}(X, X(1)) \to \mathbf{Q}_p$$

defines a perfect duality.

2.2.2. *$\mathbf{B}_{\mathrm{dR}}^+$-representations.* $\mathbf{B}_{\mathrm{dR}}^+$-representations are objects of $\mathscr{C}(\mathscr{G}_K)$ and we have a recipe for computing Ext groups between these objects. We still assume $[K:\mathbf{Q}_p] < \infty$, but the results below are valid in greater generality (see Remark 2.2).

Let χ be the cyclotomic character. Let $K_\infty \subset K(\mu_{p^\infty})$ denote the cyclotomic $\mathbf{Z}_p$-extension of K. Let γ be a topological generator of $\mathrm{Gal}(K_\infty/K)$. We choose a sequence $\{\zeta_{p^n}\}_{n\geq 1}$ of primitive p^n'th roots of unity $\zeta_{p^n}, n \geq 1$, such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. Let $t \in \mathbf{B}_{\mathrm{dR}}^+$ be the uniformizer associated to $\{\zeta_{p^n}\}_{n\geq 1}$. We will also use its twisted form $t' := t/\pi_t$ defined in [27, Sec. 2.1]; it is a uniformizer of $\mathbf{B}_{\mathrm{dR}}^+$ as well, fixed by $\mathrm{Gal}(\overline{K}/K_\infty)$ whereas t is only fixed by $\mathrm{Gal}(\overline{K}/K(\mu_{p^\infty}))$.

Proposition 2.7. *Let W be a $\mathbf{B}_{\mathrm{dR}}^+$-representation. The groups $\mathrm{Ext}^i_{\mathscr{C}(\mathscr{G}_K)}(C,W)$ are computed by the complex*

$$W_{(0)} \xrightarrow{x \mapsto (t'x, (\gamma-1)x)} W_{(1)} \oplus W_{(0)} \xrightarrow{(x,y) \mapsto (t'y - (\chi^{-1}(\gamma)\gamma - 1)x)} W_{(1)},$$

where $W_{(0)}$ is the space of generalized invariants[5] *and $W_{(1)} = t'((t')^{-1}W)_{(0)}$.*

Proof. By Proposition 2.5, the natural map

$$\mathrm{Ext}^i_{\mathbf{B}_{\mathrm{dR}}^+(\mathscr{G}_K)}(C,W) \to \mathrm{Ext}^i_{\mathscr{C}(\mathscr{G}_K)}(C,W) \tag{2.8}$$

from the Ext-groups in the category of $\mathbf{B}_{\mathrm{dR}}^+$-representations is an isomorphism. Our proposition is now [27, Th. 2.14]. $\square$

[4]More specifically, the proof uses the result which says that, if S is an effective BC of dimension 1 and height h and if $f: S \to C$ is a morphism of BC's whose image is not finite-dimensional, then f is surjective and its kernel has dimension 0 and height h.

[5]The elements killed by $(g_1-1)\cdots(g_r-1)$ for all $g_1, \ldots, g_r \in \mathscr{G}_K$, for r big enough.

We will state two simple consequences of the above proposition.

Corollary 2.9. *If $N \geq 0$ and $1 \leq i < j$, then*
$$\mathrm{Hom}_{\mathscr{C}(\mathscr{G}_K)}(\mathbf{B}_{\mathrm{dR}}^+/t^i, t^{-N}\mathbf{B}_{\mathrm{dR}}^+/t^j) = 0 \quad \text{and} \quad \mathrm{Ext}^1_{\mathscr{C}(\mathscr{G}_K)}(\mathbf{B}_{\mathrm{dR}}^+/t^i, t^{-N}\mathbf{B}_{\mathrm{dR}}^+/t^j) = 0.$$

Proof. By devissage (and twisting by χ^{-j} for $0 \leq j \leq i-1$), we can reduce to the case $i = 1$. If $W = t^{-N}\mathbf{B}_{\mathrm{dR}}^+/t^j$, we have $W_{(0)} = K$ and $W_{(1)} = Kt'$, and our result follows from Proposition 2.7 (multiplication by t' induces an isomorphism of $W_{(0)}$ with $W_{(1)}$ and the other maps are identically zero, and it follows that the complex in the proposition is acyclic). $\square$

Lemma 2.10. *Let $j \in \mathbf{Z}, k \geq 1$. Then*
$$\mathrm{Ext}^0_{\mathscr{C}(\mathscr{G}_K)}(\mathbf{B}_{\mathrm{dR}}^+/t^k, C(j)) \simeq \begin{cases} 0 & \text{if } j \neq 0 \\ K & \text{if } j = 0, \end{cases}$$
$$\mathrm{Ext}^1_{\mathscr{C}(\mathscr{G}_K)}(\mathbf{B}_{\mathrm{dR}}^+/t^k, C(j)) \simeq \begin{cases} 0 & \text{if } j \neq 0, k \\ K & \text{if } j = 0, k, \end{cases}$$
$$\mathrm{Ext}^2_{\mathscr{C}(\mathscr{G}_K)}(\mathbf{B}_{\mathrm{dR}}^+/t^k, C(j)) \simeq \begin{cases} 0 & \text{if } j \neq k \\ K & \text{if } j = k. \end{cases}$$

Proof. We will use Proposition 2.7 with $W = (\mathbf{B}_{\mathrm{dR}}^+/t^k)(1-j)$ as well as the duality between $\mathrm{Ext}^i_{\mathscr{C}(\mathscr{G}_K)}(C, W)$ and $\mathrm{Ext}^{2-i}_{\mathscr{C}(\mathscr{G}_K)}(W, C(1))$ from Theorem 2.6. We have $W_{(0)} = Kt^{j-1}(1-j)$ if $1 \leq j \leq k$, $W_{(0)} = 0$ if $j \leq 0$ or $j \geq k+1$, and $W_{(1)} \simeq Kt't^{j-1}(1-j)$ if $0 \leq j \leq j-1$, $W_{(1)} = 0$ if $j \leq -1$ or $j \geq k$. In the complex from Proposition 2.7 computing the Ext-groups $\mathrm{Ext}^i_{\mathscr{C}(\mathscr{G}_K)}(C, W)$, the only nonzero maps are the multiplications by $t' : W_{(0)} \to W_{(1)}$ which are isomorphisms unless exactly one of the two groups is trivial (i.e., $j = 0$ or $j = k$). Our result follows. $\square$

Remark 2.11. Every non-trivial extension of $\mathbf{B}_{\mathrm{dR}}^+/t^k$ by $C(k)$ is isomorphic to $\mathbf{B}_{\mathrm{dR}}^+/t^{k+1}$.

Example 2.12. *Extensions of Tate twists.* We have $\mathrm{Ext}^1_{\mathscr{C}(\mathscr{G}_K)}(C, C) \simeq K$ (the K-vector space is generated by the class of $C \otimes (\mathbf{Q}_p \oplus \mathbf{Q}_p \log t)$). Since $\mathrm{Ext}^0_{\mathscr{C}(\mathscr{G}_K)}(C, C) \simeq K$ by Remark 2.2, this implies that $\mathrm{Ext}^2_{\mathscr{C}(\mathscr{G}_K)}(C, C) = 0$ by Theorem 2.6. We also have, by duality, $\mathrm{Ext}^1_{\mathscr{C}(\mathscr{G}_K)}(C, C(1)) \simeq K$ (generated by the class of $\mathbf{B}_{\mathrm{dR}}^+/t^2$) and $\mathrm{Ext}^2_{\mathscr{C}(\mathscr{G}_K)}(C, C(1)) \simeq K$, and all the other $\mathrm{Ext}^i_{\mathscr{C}(\mathscr{G}_K)}(C, C(j))$ are trivial.

2.3. **Morphisms of $\mathbf{B}_{\mathrm{dR}}^+$-representations.** In this section, K is arbitrary. We are going to derive consequences of the following two results which are valid for such K. These are the results that we will use in the rest of the paper.

Proposition 2.13. (i) $\mathrm{Hom}_{\mathscr{G}_K}(C, C) \simeq K$.
(ii) $\overline{K}$ *is dense in* $\mathbf{B}_{\mathrm{dR}}^+$.

Proof. Fontaine's proof [26] of (i) works for arbitrary K (the alternative proof in [29], which uses class field theory, works only for $[K : \mathbf{Q}_p] < \infty$). For (ii), see [10]. $\square$

Proposition 2.14. *We have*
(i) $\mathrm{Hom}_{\mathscr{G}_K}(C, C) \simeq K$, $\mathrm{Hom}_{\mathscr{G}_K}(C, C(j)) = 0$ *if* $j \neq 0$.
(ii) $\mathrm{Hom}_{\mathscr{G}_K}(\mathbf{B}_{\mathrm{dR}}^+, t^{-N}\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+) \simeq K$ *for* $N \geq 0$, $k \geq 1$*; compatibly in* k.
(iii) $\mathrm{Hom}_{\mathscr{G}_K}(\mathbf{B}_{\mathrm{dR}}^+, \mathbf{B}_{\mathrm{dR}}^+) \simeq K$, $\mathrm{Hom}_{\mathscr{G}_K}(\mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{dR}}) \simeq K$.
(iv) $\mathrm{Hom}_{\mathscr{G}_K}(t^j\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+, \mathbf{B}_{\mathrm{dR}}) = 0$, $\mathrm{Hom}_{\mathscr{G}_K}(\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+, \mathbf{B}_{\mathrm{dR}}^+/t^\ell\mathbf{B}_{\mathrm{dR}}^+) = 0$ *if* $\ell > k$.

Proof. The first claim of (i) is Fontaine's theorem. The second claim of (i) follows from the fact that $C(j)$ does not have elements on which the action of $\mathscr{G}_K$ factors through a finite quotient and hence $\lambda \in \mathrm{Hom}_{\mathscr{G}_K}(C, C(j))$ is identically zero on $\overline{K}$, and thus on C by continuity and density of $\overline{K}$ in C.

Let us prove (ii). Any map $\lambda : \mathbf{B}_{\rm dR}^+ \to t^{-N}\mathbf{B}_{\rm dR}^+/t^k\mathbf{B}_{\rm dR}^+$ send $\overline{K}$ to $\overline{K}$ since elements of $\overline{K}$ are the smooth vectors for the action of $\mathscr{G}_K$ in $\mathbf{B}_{\rm dR}^+$ and $t^{-N}\mathbf{B}_{\rm dR}^+/t^k\mathbf{B}_{\rm dR}^+$. By continuity, and density of $\overline{K}$ in $\mathbf{B}_{\rm dR}^+$ one sees that $\lambda(\mathbf{B}_{\rm dR}^+) \subset \mathbf{B}_{\rm dR}^+/t^k\mathbf{B}_{\rm dR}^+$.

Now, if $k = 1$, $\lambda(t\overline{K})$ is sent to 0 in $\mathbf{B}_{\rm dR}^+/t = C$ since $C(-1)$ does not have elements on which the action of $\mathscr{G}_K$ factors through a finite quotient. But $t\overline{K}$ is dense in $t\mathbf{B}_{\rm dR}^+$, hence λ factors through $\mathbf{B}_{\rm dR}^+/t = C$. It follows from (i) that λ is the multiplication by $\kappa \in K$ (composed with $\mathbf{B}_{\rm dR}^+ \to C$). Now, if $k \geq 1$, we can compose with $\mathbf{B}_{\rm dR}^+/t^k \to C$ to deduce that there exist $\kappa \in K$ such that $\lambda - \kappa$ has values in $t\mathbf{B}_{\rm dR}^+/t^k\mathbf{B}_{\rm dR}^+$. But the image of $\overline{K}$ by $\lambda - \kappa$ is identically 0 for the same reasons as above; hence $\lambda = \kappa$ and we are done.

The first statement of (iii) follows from (ii) since

$$\mathrm{Hom}_{\mathscr{G}_K}(\mathbf{B}_{\rm dR}^+, \mathbf{B}_{\rm dR}^+) = \lim_k \mathrm{Hom}_{\mathscr{G}_K}(\mathbf{B}_{\rm dR}^+, \mathbf{B}_{\rm dR}^+/t^k\mathbf{B}_{\rm dR}^+)$$

For the second statement, let $\lambda \in \mathrm{Hom}_{\mathscr{G}_K}(\mathbf{B}_{\rm dR}, \mathbf{B}_{\rm dR})$. By the same arguments as above, $\lambda(\overline{K}) \subset \overline{K}$; hence, by continuity, $\lambda(\mathbf{B}_{\rm dR}^+) \subset \mathbf{B}_{\rm dR}^+$ and the restriction of λ to $\mathbf{B}_{\rm dR}^+$ is the multiplication by an element κ of K. But λ_N defined by $\lambda_N(x) = t^N\lambda(t^{-N}x)$ also belongs to $\mathrm{Hom}_{\mathscr{G}_K}(\mathbf{B}_{\rm dR}, \mathbf{B}_{\rm dR})$ and is the multiplication by κ on $t^N\mathbf{B}_{\rm dR}^+$. It follows that λ_N is the multiplication by κ on $\mathbf{B}_{\rm dR}^+$, and λ is the multiplication by κ on $t^{-N}\mathbf{B}_{\rm dR}^+$. Hence the second claim of (iii).

Finally, for (iv) we may assume $j = 0$ by twisting. Then the same arguments as for (ii) show that $\lambda(\mathbf{B}_{\rm dR}^+/t^k) \subset \mathbf{B}_{\rm dR}^+$ for the first claim, and that $\theta \circ \lambda$ is $\kappa\theta$ for some $\kappa \in K$ for both claims. One deduces that $\lambda = \kappa$ on $\overline{K}$. But we can find a sequence $(x_n)_{n\in\mathbf{N}}$ of elements of $\overline{K}$ converging to t^k in $\mathbf{B}_{\rm dR}^+$ hence to 0 in $\mathbf{B}_{\rm dR}^+/t^k$, and continuity of λ implies that $0 = \lambda(0) = \kappa t^k$. Hence $\kappa = 0$ (since $\ell > k$ for the second claim), and $\lambda = 0$, which finishes the proof. $\square$

3. The categories $\mathscr{BC}$ and $q\mathscr{BC}$

In this chapter we recall the definition of BC's[6] and introduce qBC's (categories $\mathscr{BC}$ and $q\mathscr{BC}$). We will study properties of both categories, in particular, the canonical filtration and its relation to the Harder-Narasimhan filtration. Moreover, we will partially categorify height of BC's and introduce a notion of acyclic (φ, N)-modules that will play an important role later on in the paper.

3.1. **The category $\mathscr{BC}$.**

We will discuss now basic properties of the category $\mathscr{BC}$ of BC's and the canonical filtration of its objects.

3.1.1. *Definitions and basic properties.* Recall [8] that a BC[7] $\mathbb{W}$ is, morally, a finite dimensional C-vector space up to a finite dimensional $\mathbf{Q}_p$-vector space. It has a Dimension $\mathrm{Dim}\,\mathbb{W} = (a, b)$, where $a = \dim \mathbb{W} \in \mathbf{N}$, the *dimension of* $\mathbb{W}$, is the dimension of the C-vector space and $b = \mathrm{ht}\,\mathbb{W} \in \mathbf{Z}$, the *height of* $\mathbb{W}$, is the dimension of the $\mathbf{Q}_p$-vector space.

More precisely, a *Vector Space* (VS for short) $\mathbb{W}$ is a pro-étale sheaf of $\mathbf{Q}_p$-vector spaces on Perf_C: $\Lambda \mapsto \mathbb{W}(\Lambda)$. Trivial examples of VS's are:

- finite dimensional $\mathbf{Q}_p$-vector spaces V: $\Lambda \mapsto V$ for all Λ,
- $\mathbb{V}^d$, for $d \in \mathbf{N}$, with $\mathbb{V}^d(\Lambda) = \Lambda^d$, for all Λ.

More interesting examples are provided by Fontaine's rings:

- $\mathbb{B}_{\rm cris}^+$, $\mathbb{B}_{\rm st}^+$, $\mathbb{B}_{\rm dR}^+$, $\mathbb{B}_{\rm cris}$, $\mathbb{B}_{\rm st}$, $\mathbb{B}_{\rm dR}$ are naturally VS's (and even Rings).
- If $m \geq 1$, then $\mathbb{B}_m := \mathbb{B}_{\rm dR}^+/t^m\mathbb{B}_{\rm dR}^+$ is a VS (and also a Ring).
- Let $h \geq 1$ and $d \in \mathbf{Z}$. Then $\mathbb{U}_{h,d} = (\mathbb{B}_{\rm cris}^+)^{\varphi^h = p^d}$ if $d \geq 0$, and $\mathbb{U}_{h,d} = \mathbb{B}_d/\mathbf{Q}_{p^h}$ if $d < 0$, are VS's.

A Vector Space $\mathbb{W}$ is said to be finite Dimensional (a BC for short) if it "is equal to $\mathbb{V}^d$, for some $d \in \mathbf{N}$, up to finite dimensional $\mathbf{Q}_p$-vector spaces": there exists finite dimensional $\mathbf{Q}_p$-vector

[6]Often called Banach-Colmez spaces.

[7]Called in [8] "finite dimensional Banach Space".

spaces V_1, V_2 and exact sequences[8]

$$0 \to V_1 \to \mathbb{Y} \to \mathbb{V}^d \to 0, \quad 0 \to V_2 \to \mathbb{Y} \to \mathbb{W} \to 0,$$

so that $\mathbb{W}$ is obtained from $\mathbb{V}^d$ by "adding V_1 and moding out by V_2". Then $\dim \mathbb{W} = d$ and $\operatorname{ht} \mathbb{W} = \dim_{\mathbf{Q}_p} V_1 - \dim_{\mathbf{Q}_p} V_2$.

Remark 3.1. (i) We are, in general, only interested in $W = \mathbb{W}(C)$ but, without the extra structure, it would be impossible to speak of its Dimension (for example, C and $C \oplus \mathbf{Q}_p$ are isomorphic as topological $\mathbf{Q}_p$-vector spaces).

(ii) The functor $\mathbb{W} \mapsto \mathbb{W}(C)$ of C-points is faithful for BC's. Also, if $\mathbb{W}$ is a BC, then $\mathbb{W}(\Lambda)$ is a $\mathbf{Q}_p$-Banach, for all Λ, and if $\mathbb{W}_1 \to \mathbb{W}_2$ is a morphism of BC's, then $\mathbb{W}_1(\Lambda) \to \mathbb{W}_2(\Lambda)$ is continuous and strict, for all Λ.

We quote [13, Prop. 5.16]:

Proposition 3.2. (i) *The Dimension of a BC is independent of the choices made in its definition.*

(ii) *If $f : \mathbb{W}_1 \to \mathbb{W}_2$ is a morphism of BC's, then* $\operatorname{Ker} f$, $\operatorname{Coker} f$ *and* $\operatorname{Im} f$ *are BC's, and we have*

$$\operatorname{Dim} \mathbb{W}_1 = \operatorname{Dim} \operatorname{Ker} f + \operatorname{Dim} \operatorname{Im} f \quad \text{and} \quad \operatorname{Dim} \mathbb{W}_2 = \operatorname{Dim} \operatorname{Coker} f + \operatorname{Dim} \operatorname{Im} f.$$

(iii) *If* $\dim \mathbb{W} = 0$*, then* $\operatorname{ht} \mathbb{W} \geq 0$.

(iv) *If $\mathbb{W}$ has an increasing filtration such that the successive quotients are $\mathbb{V}^1$, and if $\mathbb{W}'$ is a sub-BC of $\mathbb{W}$, then* $\operatorname{ht} \mathbb{W}' \geq 0$.

We will denote by $\mathscr{BC}$ the category of BC's. It is an abelian category.

Example 3.3. The Spaces $\mathbb{B}_m$ and $\mathbb{U}_{h,d}$ defined above are BC's. Their Dimensions are

$$\operatorname{Dim} \mathbb{B}_m = (m, 0), \quad \operatorname{Dim} \mathbb{U}_{h,d} = \begin{cases} (d, h) & \text{if } d \geq 0, \\ (-d, -h) & \text{if } d < 0. \end{cases}$$

3.1.2. *Canonical filtration.* In his thesis [35], [36], Plût introduced a filtration on objects of $\mathscr{BC}$ and stated a number of results about this filtration. We will review them briefly here.

Remark 3.4. Most of these results can be recovered from the relation of $\mathscr{BC}$ to vector bundles on the Fargues-Fontaine curve and the Harder-Narasimhan filtration studied in Le Bras' thesis (see Section 3.2.4).

Definition 3.5. (*Curvature*) Le $\mathbb{W} \in \mathscr{BC}$. We say that $\mathbb{W}$ has:

- curvature > 0, if $\operatorname{Hom}(\mathbb{W}, \mathbb{V}^1) = 0$,
- curvature $= 0$, or is *affine*, if it is a successive extension of $\mathbb{V}^1$,
- curvature < 0, if it injects into $\mathbb{B}_{\mathrm{dR}}^d$ (or, equivalently, into $(\mathbb{B}_{\mathrm{dR}}^+)^d$),
- curvature ≤ 0, if it injects into a $\mathbb{B}_{\mathrm{dR}}^+$-module,
- curvature ≥ 0, if $\operatorname{Hom}(\mathbb{W}, \mathbb{B}_{\mathrm{dR}}^+) = 0$.

Remark 3.6. (i) If $\mathbb{W}$ has curvature > 0 (resp. ≥ 0) and $\mathbb{W}'$ has curvature ≤ 0 (resp. < 0), then $\operatorname{Hom}_{\mathscr{BC}}(\mathbb{W}, \mathbb{W}') = 0$.

(ii) A sub-VS of an VS with curvature ≤ 0 (resp. < 0) has curvature ≤ 0 (resp. < 0).

(iii) A quotient of an VS with curvature ≥ 0 (resp. > 0) has curvature ≥ 0 (resp. > 0).

Proposition 3.7. (The canonical filtration) *Every $\mathbb{W} \in \mathscr{BC}$ have a unique filtration, called the canonical filtration,*

$$\mathbb{W}_{>0} \subset \mathbb{W}_{\geq 0} \subset \mathbb{W}$$

[8]In fact, by [32, Prop. 7.8], a sequence $0 \to \mathbb{W}_1 \to \mathbb{W}_2 \to \mathbb{W}_3 \to 0$ of BC's is exact if and only if $0 \to \mathbb{W}_1(\Lambda) \to \mathbb{W}_2(\Lambda) \to \mathbb{W}_3(\Lambda) \to 0$ is exact for all sympathetic algebras Λ. This implies that the latter sequence is actually strictly exact.

such that:

- $\mathbb{W}_{>0}$ *has curvature* >0,
- $\mathbb{W}_{\geq 0}/\mathbb{W}_{>0}$ *has curvature* 0,
- $\mathbb{W}/\mathbb{W}_{\geq 0}$ *has curvature* <0.

Proof. One defines $\mathbb{W}_{>0}$ as the intersection of the kernels of all morphisms $\alpha : \mathbb{W} \to \mathbb{B}_m$, for $m \geq 1$, and $\mathbb{W}_{\geq 0}$ as the intersection of the kernels of all morphisms $\alpha : \mathbb{W} \to \mathbb{B}_{\rm dR}$. □

Remark 3.8. (i) $\mathbb{W}_{\leq 0:} = \mathbb{W}/\mathbb{W}_{>0}$ is the largest quotient of curvature ≤ 0 of $\mathbb{W}$.

(ii) $\mathbb{W}^{=0} := \mathbb{W}_{\geq 0}/\mathbb{W}_{>0}$ is the largest affine sub-VS of $\mathbb{W}^{\leq 0}$.

3.2. **The category $\mathscr{BC}$ and coherent sheaves on the Fargues-Fontaine curve.** The canonical filtration of BC's is closely related to the Harder-Narasimhan filtration defined, using the presentation of $\mathscr{BC}$ via coherent sheaves on the Fargues-Fontaine curve, by Le Bras [32]. We will now explain this relation.

3.2.1. *The Fargues-Fontaine curve.* The (algebraic) Fargues-Fontaine curve $X = X_{\rm FF}$ is the projective scheme attached to the graded $\mathbf{Q}_p$-algebra $\oplus_{d\geq 0}{\rm U}_d$, where ${\rm U}_d = (\mathbf{B}^+_{\rm cris})^{\varphi=p^d}$. The closed points of X are in bijection with the $\mathbf{Q}_p$-lines of ${\rm U}_1$; if $x\in X$, we fix a basis t_x of the corresponding line. The field C corresponds to a specific point ∞ of X, and $t = t_\infty$ is Fontaine's p-adic $2\pi i$. We denote by C_x the residue field at x; it is an algebraically close field, complete for v_p, and $C_x^\flat = C^\flat$, but C_x is not necessarily isomorphic to C. The residue field at ∞ is C itself.

The completed local ring $\widehat{\mathscr{O}}_{X,x}$ at x is the ring $\mathbb{B}^+_{\rm dR}(C_x)$, and t_x is a uniformizer (if $x=\infty$, then $\mathbb{B}^+_{\rm dR}(C_x) = \mathbf{B}^+_{\rm dR}$).

3.2.2. *Harder-Narasimhan categories.* A *Harder-Narasimhan category* is an exact category with two real valued functions rk and deg (rank and degree) – which are additive in short exact sequences – satisfying extra conditions (see [20, 5.5.1], [1]). This allows to define a slope function $\mu = \frac{\deg}{\rm rk}$ taking values in $\mathbf{R}\coprod\{\pm\infty\}$ (endowed with the obvious ordering).

An object $\mathscr{E}$ is *of slope* λ if $\mu(\mathscr{E}) = \lambda$. It is *semistable* if $\mu(\mathscr{E}') \leq \mu(\mathscr{E})$ for all strict subobjects $\mathscr{E}'\subset\mathscr{E}$. It is *stable* if $\mu(\mathscr{E}') < \mu(\mathscr{E})$ for all strict subobjects $\mathscr{E}'\subset\mathscr{E}$. Any object $\mathscr{E}$ has a canonical decreasing filtration (the *Harder-Narasimhan filtration*) by strict subobjects $\mathscr{E}^{\geq\lambda}$ (with $\mathscr{E}^{\geq\lambda}\subset\mathscr{E}^{\geq\mu}$ if $\lambda\geq\mu$), such that, if $\mathscr{E}^{>\lambda} = \cup_{\mu>\lambda}\mathscr{E}^{\geq\mu}$, then $\mathscr{E}^{\geq\lambda}/\mathscr{E}^{>\lambda}$ is semistable of slope λ.

We say that $\mathscr{E}$ has slopes $\geq\lambda$ (resp. $>\lambda$) if $\mathscr{E} = \mathscr{E}^{\geq\lambda}$ (resp. $\mathscr{E} = \mathscr{E}^{>\lambda}$), and has slopes $\leq\lambda$ (resp. $<\lambda$) if $\mathscr{E}^{>\lambda} = 0$ (resp. $\mathscr{E}^{\geq\lambda} = 0$).

This has a number of consequences:

- If $\mathscr{E}_1$ is of slopes $\geq\lambda$ and $\mathscr{E}_2$ is of slopes $<\lambda$, then ${\rm Hom}(\mathscr{E}_1,\mathscr{E}_2) = 0$.
- A quotient of an object of slopes $\geq\lambda$ has slopes $\geq\lambda$.
- A subobject of an object of slopes $\leq\lambda$ has slopes $\leq\lambda$.
- Il $\lambda_1\leq\lambda_2$, if $\mathscr{E}_1$ is of slopes $\geq\lambda_1$ and $\mathscr{E}_2$ of slopes $\leq\lambda_2$, then an extension of $\mathscr{E}_2$ by $\mathscr{E}_1$ has slopes in $[\lambda_1,\lambda_2]$.

3.2.3. *Vector bundles.* To a vector bundle $\mathscr{E}$ on X, one can attach its rank ${\rm rk}(\mathscr{E})\in\mathbf{N}$, its degree $\deg(\mathscr{E})\in\mathbf{Z}$, and its slope $\mu(\mathscr{E}) = \frac{\deg(\mathscr{E})}{{\rm rk}(\mathscr{E})}$. These definitions can be extended to torsion coherent sheaves by additivity in short exact sequences. In particular, torsion sheaves have rank 0, degree >0 and slope $+\infty$. Endowed with rk and deg, the category ${\rm Coh}_X$ of coherent sheaves on X is a Harder-Narasimhan category.

The following result [20] is fundamental:

Theorem 3.9. (Fargues-Fontaine)

(i) *If* $\lambda = \frac{d}{h}$ (in lowest terms)*, there exists, up to isomorphism, a unique stable vector bundle* $\mathscr{O}(\lambda)$ *of slope* λ*; its rank is* h *and its degree* d.

(ii) *Every vector bundle* $\mathscr{E}$ *on* X *is a direct sum*

$$\mathscr{E} = \mathscr{O}(\lambda_1) \oplus \cdots \oplus \mathscr{O}(\lambda_r).$$

In particular the Harder-Narasimhan filtration splits (non canonically).

(iii) *Every coherent sheaf* $\mathscr{E}$ *on* X *is a direct sum*

$$\mathscr{E} = \mathscr{O}(\lambda_1) \oplus \cdots \oplus \mathscr{O}(\lambda_r) \oplus \big(\oplus_{x\in X} \mathscr{F}_x \big),$$

where $\mathscr{F}_x$ *is a torsion coherent sheaf, supported at* x *and zero for almost all* x.

The $\lambda_1, \dots, \lambda_r$ above are *the slopes of* $\mathscr{E}$ (to which one has to add $+\infty$ if one of the $\mathscr{F}_x$ is non zero).

We have, by [20, Prop. 8.2.3],

$$\begin{aligned} &H^0(X, \mathscr{O}(\lambda)) = (\mathbf{B}^+_{\mathrm{cris}})^{\varphi^h = p^d}, \quad H^1(X, \mathscr{O}(\lambda)) = 0, \quad \text{if } \lambda \geq 0, \\ &H^0(X, \mathscr{O}(\lambda)) = 0, \quad H^1(X, \mathscr{O}(\lambda)) = \mathbf{B}^+_{\mathrm{dR}}/(t^d \mathbf{B}^+_{\mathrm{dR}} \oplus \mathbf{Q}_{p^h}), \quad \text{if } \lambda < 0. \end{aligned} \tag{3.10}$$

Note that $H^i(X, \mathscr{O}(\lambda))$, for $i = 0, 1$, is the space of C-points of a BC $\mathbb{H}^i(X, \mathscr{O}(\lambda))$, and we have

$$\begin{aligned} &\mathrm{Dim}(\mathbb{H}^0(X, \mathscr{O}(\lambda))) = (d, h), \quad \mathrm{Dim}(\mathbb{H}^1(X, \mathscr{O}(\lambda))) = 0, \quad \text{if } \lambda \geq 0, \\ &\mathrm{Dim}(\mathbb{H}^0(X, \mathscr{O}(\lambda))) = 0, \quad \mathrm{Dim}(\mathbb{H}^1(X, \mathscr{O}(\lambda))) = (d, -h), \quad \text{if } \lambda < 0. \end{aligned}$$

Note also that $\mathbb{H}^i(X, \mathscr{O}(\lambda))$, for $i = 0, 1$, clearly depends only on $C^\flat$ if $\lambda \geq 0$; this is less clear when $\lambda < 0$ (since t depends on C) but it is still true.

3.2.4. *The category* Coh^-_X. In his thesis [32], Le Bras shows that $\mathscr{BC}$ is the smallest sub-abelian category of the category of VS's, stable by extensions, and containing $\mathbf{Q}_p$ and $\mathbb{V}^1$; this gives an efficient alternative definition of $\mathscr{BC}$ (in particular, it shows that a VS extension of two BC's is a BC).

We note Coh^-_X the sub-category of $D^b(\mathrm{Coh}_X)$ of complexes $\mathscr{E}_\bullet$ such that $H^i(\mathscr{E}_\bullet) = 0$ if $i \neq -1, 0$, $H^{-1}(\mathscr{E}_\bullet)$ has slopes < 0 and $H^0(\mathscr{E}_\bullet)$ has slopes ≥ 0. Any object of Coh^-_X can be represented by a complex $\mathscr{E}_{-1} \xrightarrow{0} \mathscr{E}_0$ of coherent sheaves such that $H^0(X, \mathscr{E}_{-1}) = 0$ (i.e., $\mathscr{E}_{-1}$ has slopes < 0) and $H^1(X, \mathscr{E}_0) = 0$ (i.e., $\mathscr{E}_0$ has slopes ≥ 0). Le Bras defines an exact functor

$$\mathrm{BC} : \mathrm{Coh}^-_X \to \mathscr{BC}$$

(denoted by $R^0\tau_*$ in [32, 6.2]): for a complex of coherent sheaves $\mathscr{F}$ on X, $\mathrm{BC}(\mathscr{F})$ is the sheaf associated to the presheaf $S \mapsto H^0(X_S, \mathscr{F}_S)$. By definition of this functor[9], one gets an exact sequence in $\mathscr{BC}$:

$$0 \to \mathbb{H}^1(X, \mathscr{E}_{-1}) \to \mathrm{BC}(\mathscr{E}_{-1} \xrightarrow{0} \mathscr{E}_0) \to \mathbb{H}^0(X, \mathscr{E}_0) \to 0.$$

If $\mathscr{E}_{-1} \xrightarrow{0} \mathscr{E}_0$ et $\mathscr{F}_{-1} \xrightarrow{0} \mathscr{F}_0$ are objects of Coh^-_X, then

$$\mathrm{Hom}_{\mathrm{BC}}(\mathrm{BC}(\mathscr{E}_{-1} \xrightarrow{0} \mathscr{E}_0), \mathrm{BC}(\mathscr{F}_{-1} \xrightarrow{0} \mathscr{F}_0)) = \begin{pmatrix} \mathrm{Hom}(\mathscr{E}_{-1}, \mathscr{F}_{-1}) & \mathrm{Ext}^1(\mathscr{E}_0, \mathscr{F}_{-1}) \\ 0 & \mathrm{Hom}(\mathscr{E}_0, \mathscr{F}_0) \end{pmatrix}$$

The following result is the main result of [32].

Theorem 3.11. (Le Bras, [32, Th. 1.2]) *The functor* BC *realizes an equivalence of categories*

$$\mathrm{Coh}^-_X \simeq \mathscr{BC}.$$

[9]More generally, $\mathrm{BC}(\mathscr{E}_\bullet)$ is an extension of $\mathbb{H}^0(X, H^0(\mathscr{E}^\bullet))$ by $\mathbb{H}^1(X, H^{-1}(\mathscr{E}^\bullet))$.

3.2.5. *Harder-Narasimhan filtration on BC's.* One endows Coh_X^- with functions rank rk^-, degree $\deg^-$ and slope $\mu^- = \frac{\deg^-}{\mathrm{rk}^-}$, by setting:

$$\mathrm{rk}^-(\mathscr{E}_{-1} \xrightarrow{0} \mathscr{E}_0) = \deg(\mathscr{E}_0) - \deg(\mathscr{E}_{-1}), \quad \deg^-(\mathscr{E}_{-1} \xrightarrow{0} \mathscr{E}_0) = \mathrm{rk}(\mathscr{E}_{-1}) - \mathrm{rk}(\mathscr{E}_0),$$

which turn it into a Harder-Narasimhan category.

By transport of structure, this endows also $\mathscr{BC}$ with functions rank rk^-, degree $\deg^-$ and slope μ^-; we have

$$\mathrm{rk}^- = \dim \quad \text{and} \quad \deg^- = -\mathrm{ht}.$$

If $\mathscr{F}_x$ is torsion, then $\mu^-(\mathrm{BC}(0 \to \mathscr{F}_x)) = 0$.

If $\lambda = \frac{d}{h}$ (in lowest terms), denote by $\mathbb{U}_\lambda$ the BC defined by $\mathbb{U}_\lambda := \mathbb{U}_{h,d}$. Note that, if $\lambda = \frac{d}{h}$ is in lowest terms and $e \geq 1$, then $\mathbb{U}_{eh,ed} = \mathbb{U}_\lambda^e$. Then we have

$$\mathbb{U}_\lambda = \begin{cases} \mathbb{H}^0(X, \mathscr{O}(\lambda)), & \text{if } \lambda \geq 0, \\ \mathbb{H}^1(X, \mathscr{O}(\lambda)), & \text{if } \lambda < 0. \end{cases}$$

Alternatively,

$$\mathbb{U}_\lambda = \begin{cases} \mathrm{BC}(0 \to \mathscr{O}(\lambda)), & \text{if } \lambda \geq 0, \\ \mathrm{BC}(\mathscr{O}(\lambda) \to 0), & \text{if } \lambda < 0. \end{cases}$$

Then

$$\mathrm{rk}^-(\mathbb{U}_\lambda) = \mathrm{sign}(\lambda)\, d, \quad \deg^-(\mathbb{U}_\lambda) = -\mathrm{sign}(\lambda)\, h, \quad \mu^-(\mathbb{U}_\lambda) = \frac{-1}{\lambda}.$$

Remark 3.12. (i) Since $\mathbf{Q}_p = \mathbb{U}_0$, we have $\mu^-(\mathbf{Q}_p) = -\infty$.

(ii) BC's are naturally diamonds (and were amongst the first non trivial examples of diamonds) and, as such, have connected components. If $\mathbb{W}$ is a BC, then $\mathbb{W}^{>-\infty}$ is the connected component of 0 and the quotient $\mathbb{W}^{-\infty}$ is the largest étale quotient (group of connected components, a finite dimensional $\mathbf{Q}_p$-vector space).

(iii) The Harder-Narasimhan filtration splits (non canonically), and every BC can be decomposed as

$$\mathbb{W} = \mathbb{U}_{-1/\lambda_1} \oplus \cdots \oplus \mathbb{U}_{-1/\lambda_r} \oplus \big(\oplus_x \mathbb{H}^0(X, \mathscr{F}_x)\big), \tag{3.13}$$

where the λ_i are non zero elements of $\mathbf{Q} \cup \{-\infty\}$, $\mathbb{U}_{-1/\lambda_i}$ is of slope λ_i, and $\mathscr{F}_x$ is a torsion coherent sheaf, supported at x, zero for almost all x, and $\mathbb{H}^0(X, \mathscr{F}_x)$ is of slope 0. The λ_i are *the slopes of* $\mathbb{W}$ (to which one has to add 0 if one of the $\mathscr{F}_x$ is non zero).

(iv) In the exact sequence

$$0 \to \mathbb{H}^1(X, \mathscr{E}_{-1}) \to \mathrm{BC}(\mathscr{E}_{-1} \xrightarrow{0} \mathscr{E}_0) \to \mathbb{H}^0(X, \mathscr{E}_0) \to 0,$$

the term on the left represents the subspace of slopes > 0 of $\mathrm{BC}(\mathscr{E}_{-1} \xrightarrow{0} \mathscr{E}_0)$.

Remark 3.14. (i) The existence of the exact sequence $0 \to \mathbb{W}^{>-\infty} \to \mathbb{W} \to \mathbb{W}^{-\infty} \to 0$ makes it possible to show that a decreasing sequence of BC's is stationary: indeed, if $(\mathbb{W}_n)_{n \in \mathbf{N}}$ is such a sequence, then $\dim(\mathbb{W}_n)$ is decreasing and bounded below by 0, hence is constant for $n \geq N$. It follows that $\mathbb{W}_N/\mathbb{W}_n$ is of dimension 0 and hence is a quotient of $\mathbb{W}_N^{-\infty}$. Since $\mathrm{ht}(\mathbb{W}_N/\mathbb{W}_n)$ is increasing and bounded by $\mathrm{ht}(\mathbb{W}_N^{-\infty}) < \infty$, one concludes that $\mathbb{W}_N/\mathbb{W}_n$ is constant for n big enough, and that so is $\mathbb{W}_n$.

(ii) One can also use a presentation to reduce to the case $\mathbb{W} = \mathbb{V}^d$, and then argue by induction on d using the fact that a sub-BC of $\mathbb{V}^1$ is either $\mathbb{V}^1$ of a finite dimensional $\mathbf{Q}_p$-vector space. This proof applies verbatim to almost C-representations.

Lemma 3.15. ([9, Prop. 2.4]) *A sub-BC* $\mathbb{W}$ *of* $\mathbb{V}^N$*, containing no* $\mathbb{V}^1$*, satisfies* $\dim(\mathbb{W}) < \mathrm{ht}(\mathbb{W})$ *or, equivalently, has slopes* < -1.

Proof. We will use the equivalence $\mathrm{Coh}_X^- \simeq \mathscr{BC}$. By decomposing $\mathbb{W}$ as a direct sum of stable BC's, it suffices to prove the statement for $\mathbb{U}_\lambda$, with $\lambda = \frac{d}{h} > 0$ (here we used the fact that $\mathbb{W}$ does not contain $\mathbb{V}^1$). This amounts to showing that, if $f : \mathbb{U}_\lambda \to \mathbb{V}^N$ is an injective map then $h > d$. Passing to the category Coh_X^-, we see that we need to show that if a map $f^\flat : \mathscr{O}(\lambda) \to \iota_{\infty,*}C^N$ is injective on global sections then $h > d$. But this map factors as

$$\begin{tikzcd} & \mathscr{O}(\lambda) \arrow[rr, "f^\flat"] \arrow[d] & & \iota_{\infty,*}C^N \\ \widehat{\mathscr{O}}_{X,\infty} \otimes \mathscr{O}(\lambda) \arrow[r] & (\iota_{\infty,*}C) \otimes \mathscr{O}(\lambda) \arrow[rr, "\sim"] & & \iota_{\infty,*}C^h \arrow[u] \end{tikzcd}$$

Tracing this diagram from the left upper corner first vertically then horizontally we obtain a map $\overline{f} : \mathscr{O}(\lambda) \to \iota_{\infty,*}C^h$, which is injective on global sections. Now, passing back (from the category Coh_X^-) to the category $\mathscr{BC}$ we get an injective map $\widetilde{f} : \mathbb{U}_\lambda \to \mathbb{V}^h$. Since $\mathbb{U}_\lambda$ is of Dimension (d, h) and $\mathbb{V}^h$ of Dimension $(h, 0)$, the existence of an injection implies $h < d$, as wanted. $\square$

3.2.6. *Morphisms.* We can describe Hom and Ext^1 in the category $\mathscr{BC}$ using the curve. For example:

(1) If D_λ is the division algebra with center $\mathbf{Q}_p$ and invariant λ, we have
$$\mathrm{End}_{\mathscr{B}C}(\mathbb{U}_\lambda) = \mathrm{End}(\mathscr{O}(\lambda)) = D_\lambda.$$
(2) Recall that, if $\lambda_1 = \frac{d_1}{h_1}$, $\lambda_2 = \frac{d_2}{h_2}$, and $\lambda_1 + \lambda_2 = \frac{d}{h}$ in the minimal form, we have $\mathscr{O}(\lambda_1) \otimes \mathscr{O}(\lambda_2) = \mathscr{O}(\lambda_1 + \lambda_2)^{n(\lambda_1,\lambda_2)}$, where $n(\lambda_,\lambda_2) = \frac{h_1}{h_2}h$. Also:
$$\mathrm{Hom}(\mathscr{O}(\lambda_1), \mathscr{O}(\lambda_2)) = \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\lambda_2 - \lambda_1)), \quad \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\lambda)) = H^0(X, \mathscr{O}(\lambda)).$$
(3) There is a bijection $x \mapsto T_x$ between the closed points of X and the $\mathbf{Q}_p$-lines of $\mathbb{U}_1(C)$. If $x \in X$, then $\mathrm{End}(\mathbb{U}_1/T_x) = C_x$, where C_x is the residue field of the local ring of x in X (recall that C_x is an algebraically closed field, complete for the p-adic valuation, an untilt of $C^\flat$; but C_x is not always isomorphic to C).
(4) Similarly, $\mathbb{U}_d/T_x^{\otimes d} = \mathrm{BC}(i_{x,*}\mathbb{B}_d(C_x))$, and thus $\mathrm{End}(\mathbb{U}_d/T_x^{\otimes d}) = \mathbb{B}_d(C_x)$.
(5) If $\lambda = \frac{d}{h} \geq 0$ then $\mathrm{Hom}(\mathbb{U}_\lambda, \mathbb{V}^1)$ is the C-module of rank h generated by $\theta \circ \varphi^i$, for $0 \leq i \leq h - 1$.

3.2.7. *Slope* 0 *and curvature* 0. The map $\mathscr{F} \mapsto H^0(X, \mathscr{F})$ induces an equivalence of categories from the category of torsion coherent sheaves, supported at x, to the category of finite length $\mathbb{B}_{\mathrm{dR}}^+(C_x)$-modules. A finite length $\mathbb{B}_{\mathrm{dR}}^+(C_x)$-module is a direct sum of $\mathbb{B}_m(C_x) = \mathbb{B}_{\mathrm{dR}}^+(C_x)/t_x^m$, and the sheaf $i_{x,*}\mathbb{B}_m$ corresponding to $\mathbb{B}_m(C_x)$ (where i_x denotes the inclusion of x in X) lives in an exact sequence
$$0 \to \mathscr{O} \xrightarrow{t_x^m} \mathscr{O}(m) \to i_{x,*}\mathbb{B}_m \to 0$$
Passing to the sequence of H^0's (which is exact as $H^1(X, \mathscr{O}) = 0$), this gives an isomorphism
$$\mathbb{H}^0(X, i_{x,*}\mathbb{B}_m) = \mathbb{U}_m/\mathbf{Q}_p t_x^m.$$
One deduces from the equivalence $\mathrm{Coh}_X^- \simeq \mathscr{BC}$ that
$$\mathrm{End}_{\mathscr{BC}}(\mathbb{U}_m/\mathbf{Q}_p t_x^m) \simeq \mathrm{End}_{\mathrm{Coh}_X}(i_{x,*}\mathbb{B}_m(C_x)) \simeq \mathrm{End}_{\mathbb{B}_{\mathrm{dR}}^+(C_x)}(\mathbb{B}_m(C_x)) \simeq \mathbb{B}_m(C_x).$$
In particular, for $m = 1$, one obtains
$$\mathrm{End}_{\mathscr{BC}}(\mathbb{U}_1/\mathbf{Q}_p t_x) \simeq C_x.$$
Note also that, if $x \neq \infty$, then
$$\mathrm{Hom}_{\mathscr{BC}}(\mathbb{U}_1/\mathbf{Q}_p t_x, \mathbb{V}^1) \simeq \mathrm{Hom}_{\mathrm{Coh}_X}(i_{x,*}\mathbb{B}_m, i_{\infty,*}\mathbb{B}_m) = 0,$$
since the two sheaves are supported at distinct points.

In the special case $x=\infty$, which will be crucial for our results, $t_x=t$ and $\mathbb{U}_m/\mathbf{Q}_p t^m=\mathbb{B}_m$, and one can describe directly the object $\mathbb{M}$ of $\mathscr{BC}$ attached to a $\mathbf{B}_{\rm dR}^+$-module of finite length M: we have

$$\mathbb{M}=M\otimes_{\mathbf{B}_{\rm dR}^+}\mathbb{B}_{\rm dR}^+.$$

This can be summarized by the following result.

Proposition 3.16. *The functor $M\mapsto\mathbb{M}=M\otimes_{\mathbf{B}_{\rm dR}^+}\mathbb{B}_{\rm dR}^+$ defines an equivalence of categories between the category of $\mathbf{B}_{\rm dR}^+$-modules of finite length and the subcategory of $\mathscr{BC}$ of objects of curvature* 0.

Corollary 3.17. (i) *The kernel and cokernel of a morphism of objects of curvature* 0 *are of curvature* 0.

(ii) *If $\mathbb{W}$ is a torsion $\mathbb{B}_{\rm dR}^+$-module, then* $\operatorname{Hom}_{\rm VS}(\mathbb{W},\mathbb{B}_{\rm dR}^+)=0$ *and* $\operatorname{Hom}_{\rm VS}(\mathbb{W},\mathbb{B}_{\rm dR})=0$.

Proof. Point (i) is a direct consequence of Proposition 3.16. To prove point (ii) we may assume $\mathbb{W}$ to be finitely generated (since, in any case, it is an inductive limit of finitely generated $\mathbb{B}_{\rm dR}^+$-modules). Then, we can use Proposition 3.16 to write $\mathbb{W}$ as $W\otimes_{\mathbf{B}_{\rm dR}^+}\mathbb{B}_{\rm dR}^+$ for some torsion $\mathbf{B}_{\rm dR}^+$-module W. Now we have, using Proposition 3.16 and the fact that $\mathbb{B}_{\rm dR}^+=\lim_k\mathbb{B}_{\rm dR}^+/t^k$,

$$\operatorname{Hom}_{\rm VS}(\mathbb{W},\mathbb{B}_{\rm dR}^+)=\lim_k\operatorname{Hom}_{\rm VS}(\mathbb{W},\mathbb{B}_{\rm dR}^+/t^k)=\lim_k\operatorname{Hom}_{\mathbf{B}_{\rm dR}^+}(W,\mathbf{B}_{\rm dR}^+/t^k)=\operatorname{Hom}_{\mathbf{B}_{\rm dR}^+}(W,\mathbf{B}_{\rm dR}^+)=0$$

This proves the result for $\mathbb{B}_{\rm dR}^+$. To prove it for $\mathbb{B}_{\rm dR}$, note that W is naturally a $\mathbf{Q}_p$-banach and $\mathbf{B}_{\rm dR}$ is an inductive limit of the $t^{-N}\mathbf{B}_{\rm dR}^+$ which are $\mathbf{Q}_p$-Fréchet's. Hence there exists N such that W maps to $t^{-N}\mathbf{B}_{\rm dR}^+$ and then $\mathbb{W}$ maps to $t^{-N}\mathbb{B}_{\rm dR}^+$. This makes it possible to use the case of $\mathbb{B}_{\rm dR}^+$ to finish the proof. ☐

3.2.8. *Canonical and Harder-Narasimhan filtrations.* The relation between the decomposition (3.13) and the filtration of Proposition 3.7 is given by:

$$\begin{aligned}
&\mathbb{W}_{>0}\simeq\big(\oplus_{\lambda_i>0}\mathbb{U}_{-1/\lambda_i}\big)\oplus\big(\oplus_{x\neq\infty}\mathbb{H}^0(X,\mathscr{F}_x)\big),\\
&\mathbb{W}_{\leq 0}\simeq\big(\oplus_{\lambda_i<0}\mathbb{U}_{-1/\lambda_i}\big)\oplus\mathbb{H}^0(X,\mathscr{F}_\infty)=\mathbb{H}^0\big(X,\mathscr{F}_\infty\oplus\big(\oplus_{\lambda_i<0}\mathscr{O}(-1/\lambda_i)\big)\big),\\
&\mathbb{W}_{<0}\simeq\oplus_{\lambda_i<0}\mathbb{U}_{-1/\lambda_i},\quad \mathbb{W}_{=0}\simeq\mathbb{H}^0(X,\mathscr{F}_\infty).
\end{aligned}$$

From the properties of Harder-Narasimhan filtrations and the above decompositions, we can deduce the following results.

Corollary 3.18. (i) *$\mathbb{W}$ is of curvature <0 (resp. ≤ 0) if and only if $\mathbb{W}\simeq H^0(X,\mathscr{E})$, where $\mathscr{E}$ is a vector bundle of slopes ≥ 0 (resp. the sum of a vector bundle of slopes ≥ 0 and a torsion sheaf supported at ∞).*

(ii) *An extension of two BC's of curvature <0 (resp. ≤ 0) is of curvature <0 (resp. ≤ 0).*

Corollary 3.19. The sign of the curvature determines the sign of the height*:*

(a) *curvature* 0 *implies height* 0*;*

(b) *curvature <0 implies height >0;*

(c) *curvature >0 implies height ≤ 0.*

Corollary 3.20. The curvature decreases by going to a subobject and increases by taking a quotient:

(i) *A sub-BC of a BC of curvature ≤ 0 (resp. <0) has curvature ≤ 0 (resp. <0).*

(ii) *A sub-BC of height* 0 *of a BC of curvature ≤ 0 has curvature* 0.

(iii) *A quotient of a BC of curvature ≥ 0 (resp. >0) has curvature ≥ 0 (resp. >0).*

(iv) *A quotient of height* 0 *of a BC of curvature ≥ 0 has curvature* 0.

Remark 3.21. An important consequence of (ii) of Corollary 3.20 is that a sub-BC $\mathbb{U}$ of a torsion $\mathbb{B}_{\mathrm{dR}}^+$-module $\mathbb{W}$ satisfies $\mathrm{ht}(\mathbb{U}) \geq 0$ and $\mathbb{U}$ is itself a torsion $\mathbb{B}_{\mathrm{dR}}^+$-module if and only if $\mathrm{ht}(\mathbb{U}) = 0$.

This can be proven, without the Harder-Narasimhan decomposition, by induction on the length of $\mathbb{W}$, using the fact that a sub-BC of $\mathbb{V}^1$ is either $\mathbb{V}^1$ or a finite dimensional $\mathbf{Q}_p$-vector space and the fact that an extension of $\mathbb{B}_{\mathrm{dR}}^+$-modules is itself a $\mathbb{B}_{\mathrm{dR}}^+$-module. This proof extends verbatim to almost C-representations thanks to Proposition 2.5.

3.2.9. *BC's of curvature* ≤ 0. Plût says that a BC of curvature ≤ 0 is *constructible*, but we will not use this terminology.

Lemma 3.22. *The following conditions are equivalent:*

(i) $\mathbb{W}$ *is of curvature* ≤ 0.

(ii) *There is an exact sequence*

$$0 \to V \to \mathbb{W} \to \mathbb{M} \to 0, \tag{3.23}$$

where $\mathbb{M}$ *is of curvature* 0 *and* V *is finite dimensional over* $\mathbf{Q}_p$.

Proof. Implication (i)⇒(ii) follows from the fact that, if $\mathbb{W}$ is of curvature ≤ 0, then

$$\mathbb{W} = \mathbb{H}^0(X, \mathscr{F}_\infty) \oplus \big(\oplus_{d_i/h_i \geq 0} \mathbb{U}_{d_i/h_i} \big)$$

and we have an exact sequence $0 \to \mathbf{Q}_{p^{h_i}} \to \mathbb{U}_{d_i/h_i} \to \mathbb{B}_{d_i} \to 0$. Then $V = \oplus_{d_i/h_i} \mathbf{Q}_{p^{h_i}}$ gives what we want.

The converse implication (ii)⇒(i) follows from the fact that $\mathbb{M}$ is of slope 0 and V of slope $-\infty$ by assumption, so any extension has slopes in $[-\infty, 0]$. □

3.3. **Categorification of height.** We will introduce now a partial categorification of height of BC's.

3.3.1. *Useful lemma.* We will need the following two lemmas.

Lemma 3.24. *Let* $\lambda : \prod_{n\geq 0} W_n \to \mathbf{B}_{\mathrm{dR}}$ *be continuous, where the* W_n*'s are* $\mathbf{Q}_p$*-Banach spaces.*

(i) *There exists* N *such that* $\lambda(\prod_{n\geq 0} W_n) \subset t^{-N}\mathbf{B}_{\mathrm{dR}}^+$.

(ii) *If* $j \in \mathbf{N}$, *there exists* $N(j)$ *such that* $\lambda(\prod_{n\geq N(j)} W_n) \subset t^j\mathbf{B}_{\mathrm{dR}}^+$.

Proof. (i) is a consequence of the fact that $\prod_{n\geq 0} W_n$ is Fréchet and $\mathbf{B}_{\mathrm{dR}}$ is the inductive limit of the Fréchet's $t^{-N}\mathbf{B}_{\mathrm{dR}}^+$.

(ii) is a consequence of the fact that $t^{-N}\mathbf{B}_{\mathrm{dR}}^+/t^j\mathbf{B}_{\mathrm{dR}}^+$ is a Banach; hence $\lambda_j : \prod_{n\geq 0} W_n \to t^{-N}\mathbf{B}_{\mathrm{dR}}^+/t^j\mathbf{B}_{\mathrm{dR}}^+$ factors through $\prod_{n<N(j)} W_n$. □

Lemma 3.25. *Let* $W = \varprojlim_n W_n$, *where the* W_n*'s are* $\mathbf{Q}_p$*-Banach spaces, the transition maps are strict and the system is Mittag-Leffler.*

(i) *If* Y *is a* $\mathbf{Q}_p$*-Banach space,* $\mathrm{Hom}(W, Y) = \varinjlim_n \mathrm{Hom}(W_n, Y)$.

(ii) $\mathrm{Hom}(W, t^j\mathbf{B}_{\mathrm{dR}}^+) = \varprojlim_k \varinjlim_n \mathrm{Hom}(W_n, t^j\mathbf{B}_{\mathrm{dR}}^+/t^{j+k}\mathbf{B}_{\mathrm{dR}}^+))$.

(iii) $\mathrm{Hom}(W, \mathbf{B}_{\mathrm{dR}}) = \mathrm{Hom}(W, \mathbf{B}_{\mathrm{dR}}^+) \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathbf{B}_{\mathrm{dR}}$.

Proof. Since the system is Mittag-Leffler, one can replace W_n by the image of $W_{n+k} \to W_n$ for k big enough (this image is still a Banach space by the strictness assumption) without changing W or $\varinjlim_n \mathrm{Hom}(W_n, Y)$. Hence we may assume $W_{n+1} \to W_n$ to be surjective, and we can choose a supplementary Banach subspace W_n'' of $W_n' = \mathrm{Ker}\,(W_{n+1} \to W_n)$ inside W_{n+1}: then $W_n'' \to W_n$ is an isomorphism of Banach spaces, hence $W \simeq \prod_n W_n'$. This implies $\mathrm{Hom}(W, Y) = \oplus_n \mathrm{Hom}(W_n', Y)$ from which (i) follows (since $W_n = \prod_{i\leq n} W_i'$).

Having written W as $\prod_n W_n'$, we can apply Lemma 3.24 to deduce (ii) and (iii) from (i), since $t^j\mathbf{B}_{\mathrm{dR}}^+/t^{j+k}\mathbf{B}_{\mathrm{dR}}^+$ is a $\mathbf{Q}_p$-Banach space. □

3.3.2. *The functor* $\mathbb{W}\mapsto h(\mathbb{W})$. If $\mathbb{W}$ is a topological VS, set

$$h(\mathbb{W}) := \mathrm{Hom}(\mathbb{W},\mathbb{B}_{\mathrm{dR}})$$

(Let us remind that this includes the requirement that $\mathbb{W}(C)\to\mathbf{B}_{\mathrm{dR}}$ is continuous.) This is a $\mathbf{B}_{\mathrm{dR}}$-module; hence $h(-)$ is a functor from the category $\mathscr{BC}$ to the category of vector spaces over $\mathbf{B}_{\mathrm{dR}}$. In most cases of interest, because of Lemmas 3.24 and 3.25 and Corollary 3.17, we have in fact

$$h(\mathbb{W}) = \mathrm{Hom}^{\natural}(\mathbb{W},\mathbb{B}_{\mathrm{dR}}) := \mathrm{colim}_{k\geq 0}\,\mathrm{Hom}_{\mathrm{VS}}(\mathbb{W},t^{-k}\mathbb{B}_{\mathrm{dR}}^+) = \mathrm{Hom}(\mathbb{W},\mathbb{B}_{\mathrm{dR}}^+)\otimes_{\mathbf{B}_{\mathrm{dR}}^+}\mathbf{B}_{\mathrm{dR}}$$

We also set

$$\mathrm{Ext}^{1,\natural}(\mathbb{W},\mathbb{B}_{\mathrm{dR}}) := \mathrm{colim}_{k\geq 0}\,\mathrm{Ext}^1_{\mathrm{VS}}(\mathbb{W},t^{-k}\mathbb{B}_{\mathrm{dR}}^+).$$

This is a $\mathbf{B}_{\mathrm{dR}}$-module as well.

Lemma 3.26. (i) *If* $\mathbb{W}$ *is of curvature* ≤ 0*, then* $\mathrm{Ext}^{1,\natural}(\mathbb{W},\mathbb{B}_{\mathrm{dR}})=0$.

(ii) *If* $0\to\mathbb{W}_1\to\mathbb{W}\to\mathbb{W}_2\to 0$ *is an exact sequence of BC's of curvature* ≤ 0*, the sequence* $0\to h(\mathbb{W}_2)\to h(\mathbb{W})\to h(\mathbb{W}_1)\to 0$ *is exact.*

Proof. Claim (ii) follows immediately from (i).

Now, the exact sequence (3.23) shows that it suffices to prove (i) for an affine and for $\mathbf{Q}_p$. That $\mathrm{Ext}^1_{\mathrm{VS}}(\mathbf{Q}_p,\mathbb{B}_{\mathrm{dR}}^+)=0$ follows easily, by devissage, from the fact that the maps $\mathbb{B}_{\mathrm{dR}}^+/t^{m+1}\to\mathbb{B}_{\mathrm{dR}}^+/t^m$ are surjective and $\mathrm{Ext}^1_{\mathrm{VS}}(\mathbf{Q}_p,\mathbb{V}^1)=0$ (see [32, Th. 4.1]). This implies that

$$\mathrm{Ext}^{1,\natural}(\mathbf{Q}_p,\mathbb{B}_{\mathrm{dR}}) = \mathrm{colim}_{k\geq 0}\,\mathrm{Ext}^1_{\mathrm{VS}}(\mathbf{Q}_p,t^{-k}\mathbb{B}_{\mathrm{dR}}^+)=0,$$

as wanted.

To show that $\mathrm{Ext}^{1,\natural}(\mathbb{W},\mathbb{B}_{\mathrm{dR}})=0$ for $\mathbb{W}$ affine, it suffices, again by devissage, to show that $\mathrm{Ext}^{1,\natural}(\mathbb{V}^1,\mathbb{B}_{\mathrm{dR}})=0$. To show the latter fact we use the exact sequence

$$0\to t^{-k}\mathbb{B}_{\mathrm{dR}}^+\to t^{-k-1}\mathbb{B}_{\mathrm{dR}}^+\to t^{-k-1}\mathbb{V}^1\to 0$$

and the fact that $\mathrm{Hom}_{\mathrm{VS}}(\mathbb{V}^1,\mathbb{B}_{\mathrm{dR}}^+)=0$. This gives us the exact sequence

$$0\to\mathrm{Hom}_{\mathrm{VS}}(\mathbb{V}^1,t^{-k-1}\mathbb{V}^1)\to\mathrm{Ext}^1_{\mathrm{VS}}(\mathbb{V}^1,t^{-k}\mathbb{B}_{\mathrm{dR}}^+)\to\mathrm{Ext}^1_{\mathrm{VS}}(\mathbb{V}^1,t^{-k-1}\mathbb{B}_{\mathrm{dR}}^+).$$

Since the first two terms are isomorphic to $\mathbf{B}_{\mathrm{dR}}^+/t$ as $\mathbf{B}_{\mathrm{dR}}^+$-modules, $\mathrm{Ext}^1_{\mathrm{VS}}(\mathbb{V}^1,t^{-k}\mathbb{B}_{\mathrm{dR}}^+)\to\mathrm{Ext}^1_{\mathrm{VS}}(\mathbb{V}^1,t^{-k-1}\mathbb{B}_{\mathrm{dR}}^+)$ is zero. This finishes the proof. ☐

Proposition 3.27. (i) *If* $\mathbb{W}$ *is of curvature* ≤ 0 *then* $\mathrm{rk}(h(\mathbb{W}))=\mathrm{ht}(\mathbb{W})$.

(ii) *In general,* $\mathrm{rk}(h(\mathbb{W}))=\mathrm{ht}(\mathbb{W})+\mathrm{rk}(\mathrm{Ext}^{1,\natural}(\mathbb{W},\mathbb{B}_{\mathrm{dR}}))$.

Proof. For (i), by assumption, we have the exact sequence $0\to V\to\mathbb{W}\to\mathbb{M}\to 0$, where $\mathbb{M}$ is affine. This yields the exact sequence

$$0\to h(\mathbb{M})\to h(\mathbb{W})\to h(V)\to\mathrm{Ext}^{1,\natural}(\mathbb{M},\mathbb{B}_{\mathrm{dR}}).$$

Since $h(\mathbb{M})=0$ and we have $\mathrm{Ext}^{1,\natural}(\mathbb{M},\mathbb{B}_{\mathrm{dR}})=0$, by Lemma 3.26, this sequence implies that $\mathrm{rk}(h(\mathbb{W}))=\dim_{\mathbf{Q}_p}V$. We are done because $\mathrm{ht}(\mathbb{W})=\dim_{\mathbf{Q}_p}V$.

(ii) follows via the exact sequence of $\mathrm{Ext}^{\natural}$ of a presentation: if $0\to V\to\mathbb{W}'\to\mathbb{W}\to 0$ represents $\mathbb{W}$, where $\mathbb{W}'$ is an extension of $\mathbb{V}^d$ by a $\mathbf{Q}_p$-vector space V' of finite dimension, arguing as for (1) we get the following diagram with exact row:

$$\begin{tikzcd}
0 \arrow[r] & h(\mathbb{W}) \arrow[r] & h(\mathbb{W}') \arrow[r] \arrow[d, "\wr"] & h(V) \arrow[r] & \mathrm{Ext}^{1,\natural}(\mathbb{W},\mathbb{B}_{\mathrm{dR}}) \arrow[r] & \mathrm{Ext}^{1,\natural}(\mathbb{W}',\mathbb{B}_{\mathrm{dR}}) \arrow[r] & 0 \\
& & h(V') & & & &
\end{tikzcd}$$

Since $\mathbb{W}'$ is of curvature ≤ 0, by Lemma 3.26, we have $\mathrm{Ext}^{1,\natural}(\mathbb{W}',\mathbb{B}_{\mathrm{dR}})=0$. It follows that

$$\mathrm{rk}(h(\mathbb{W}))=\mathrm{rk}(h(V'))-\mathrm{rk}(h(V))+\mathrm{rk}(\mathrm{Ext}^{1,\natural}(\mathbb{W},\mathbb{B}_{\mathrm{dR}})),$$

which gives us what we wanted because $\mathrm{ht}(\mathbb{W})=\mathrm{rk}(h(V'))-\mathrm{rk}(h(V))$. ☐

3.4. **The category $q\mathscr{BC}$.** We need to enlarge the category $\mathscr{BC}$ to allow extensions by arbitrary $\mathbb{B}_m$-Modules, for $m \geq 1$.

3.4.1. *Definitions.* A qBC (q stands for quasi) is a VS $\mathbb{W}$ such that there exists $m \geq 1$ and a sub-$\mathbb{B}_m$-Module $\mathbb{W}_0$ of $\mathbb{W}$ such that $\mathbb{W}/\mathbb{W}_0$ is a BC. We will denote by $q\mathscr{BC}$ the full subcategory of VS's consisting of qBC's.

Remark 3.28. Proposition 3.16 extends to arbitrary $\mathbb{B}_m$-modules: if $\mathbb{W}$ is a $\mathbb{B}_m$-Module, then $\mathbb{W}(C)$ is a $\mathbf{B}_m$-module, and can be written as $\oplus_{i\in I}(\mathbf{B}_m/t^{j_i})e_i$ (with $j_i \leq m$). The natural map of $\mathbb{B}_m$-Modules $\mathbb{B}_m \otimes_{\mathbf{B}_m} \mathbb{W}(C) \to \mathbb{W}$ gives an isomorphism when evaluated on C-points, hence is an isomorphism (the kernel and cokernel are 0 since their C-points are 0). It follows that $\mathbb{W} \simeq \oplus_{i\in I}(\mathbb{B}_m/t^{j_i})e_i$, and one can deduce the result for arbitrary $\mathbb{B}_m$-modules from its counterpart for finite type ones.

For $\mathbb{W} \in q\mathscr{BC}$, we define:

$$\mathrm{ht}(\mathbb{W}) := \mathrm{ht}(\mathbb{W}/\mathbb{W}_0).$$

This does not depend on the choice of $\mathbb{W}_0$: if $\mathbb{W}_0'$ is another choice, then $\mathbb{W}_0'' = \mathbb{W}_0 \cap \mathbb{W}_0'$ is also a possible choice, and we have exact sequences in $\mathscr{BC}$:

$$0 \to \mathbb{W}_0/\mathbb{W}_0'' \to \mathbb{W}/\mathbb{W}_0'' \to \mathbb{W}/\mathbb{W}_0 \to 0, \quad 0 \to \mathbb{W}_0'/\mathbb{W}_0'' \to \mathbb{W}/\mathbb{W}_0'' \to \mathbb{W}/\mathbb{W}_0' \to 0;$$

now, since $\mathbb{W}_0/\mathbb{W}_0''$ and $\mathbb{W}_0'/\mathbb{W}_0''$ are finite lenght $\mathbb{B}_m$-Modules, their height is 0, hence

$$\mathrm{ht}(\mathbb{W}/\mathbb{W}_0) = \mathrm{ht}(\mathbb{W}/\mathbb{W}_0'') = \mathrm{ht}(\mathbb{W}/\mathbb{W}_0'),$$

as wanted.

Definition 3.29. We say that $\mathbb{W}$ has curvature ≤ 0 (resp. 0, resp. ≥ 0) if $\mathbb{W}/\mathbb{W}_0$ has.

This does not depend on the choice of $\mathbb{W}_0$ for the same reasons as above, because $\mathbb{W}_0'/\mathbb{W}_0''$ and $\mathbb{W}_0/\mathbb{W}_0''$ have curvature 0, hence $\mathbb{W}/\mathbb{W}_0''$ has curvature ≤ 0 (resp. 0, resp. ≥ 0) if and only if $\mathbb{W}/\mathbb{W}_0'$ has, and if and only if $\mathbb{W}/\mathbb{W}_0$ has.

Remark 3.30. We have analogs of Corollary 3.19 and Corollary 3.20 in the category $q\mathscr{BC}$. Indeed, in the case of the first corollary this is clear. In the case of the other one, this follows easily from the independence of height and curvature from the presentation of qBC's and the analogous results for BC's.

Lemma 3.31. *Let $\mathbb{W}$ be a VS. Assume that $\mathbb{W}$ has a presentation*

$$0 \to \mathbb{W}' \to \mathbb{W} \to \mathbb{V}_1 \to 0, \quad 0 \to \mathbb{V}_2 \to \mathbb{Y} \to \mathbb{W}' \to 0,$$

where $\mathbb{V}_1$ and $\mathbb{V}_2$ are BC's, and $\mathbb{Y}$ is a $\mathbb{B}_m$-Module. Then $\mathbb{W}$ is in $q\mathscr{BC}$, and $\mathrm{ht}(\mathbb{W}) = \mathrm{ht}(\mathbb{V}_1) - \mathrm{ht}(\mathbb{V}_2)$.

Proof. We have to produce a sub-$\mathbb{B}_m$-Module $\mathbb{W}_0$ of $\mathbb{W}$ such that $\mathbb{W}/\mathbb{W}_0$ is a BC. If we can do the same for $\mathbb{W}'$, then $\mathbb{W}_0 = \mathbb{W}_0'$ will work.

Hence, we can assume $\mathbb{V}_1 = 0$, in which case we have an exact sequence

$$0 \to \mathbb{V}_2 \to \mathbb{Y} \to \mathbb{W} \to 0.$$

Since $\mathbb{Y}$ is a $\mathbb{B}_m$-Module, $\mathbb{V}_2$ has slopes ≤ 0 (hence is isomorphic to $\mathbb{H}^0(X, \mathscr{F})$, for a coherent sheaf $\mathscr{F}$), and the arrow $\mathbb{V}_2 \to \mathbb{Y}$ factors through $\mathbb{H}^0(X, \widehat{\mathscr{O}}_{X,\infty} \otimes \mathscr{F})$, which is a $\mathbb{B}_{\mathrm{dR}}^+$-Module of finite type. Hence the image of this arrow lands in a sub-$\mathbb{B}_m$-Module of finite type. This $\mathbb{B}_m$-Module can be included in a direct factor, still of finite type, and one takes for $\mathbb{W}_0$ a complementary subspace of this direct factor.

The equality of heights is clear since everything contributing lives inside a BC. $\square$

3.4.2. *qBC's and morphisms.*

Lemma 3.32. *Let $\pi : \mathbb{W} \to \mathbb{W}'$ be a morphism of VS's, with $\mathbb{W}$ a $\mathbb{B}_m$-Module and $\mathbb{W}'$ a BC. Then,* $\operatorname{Ker}\pi$ *contains a sub-$\mathbb{B}_m$-Module $\mathbb{W}_0$ such that $\mathbb{W}/\mathbb{W}_0$ is a BC.*

Proof. After quotienting $\mathbb{W}$ by a maximal sub-$\mathbb{B}_m$-Module of $\operatorname{Ker}\pi$, one can assume that $\operatorname{Ker}\pi$ contains no non-zero sub-$\mathbb{B}_m$-Module (the inverse image of a $\mathbb{B}_m$-Module of $\operatorname{Ker}\pi/\mathbb{W}_0$ is a-$\mathbb{B}_m$-Module of $\operatorname{Ker}\pi$). We want to infer that then $\mathbb{W}$ is of finite length or, equivalently, that its t-torsion sub-Module is of finite rank over $\mathbb{B}_1$.

If not, this sub-Module contains an increasing sequence of sub-Modules $\mathbb{X}_n$, with $\mathbb{X}_n \simeq \mathbb{V}^n$. Denote by $\mathbb{Y}_n$ the intersection of $\operatorname{Ker}\pi$ and $\mathbb{X}_n$, and by $\mathbb{I}_n$ the image of $\mathbb{X}_n$ in $\mathbb{W}'$. Since $\mathbb{X}_{n+1}/\mathbb{X}_n \simeq \mathbb{V}^1$, we have an exact sequence

$$0 \to \mathbb{Y}_{n+1}/\mathbb{Y}_n \to \mathbb{V}^1 \to \mathbb{I}_{n+1}/\mathbb{I}_n \to 0.$$

It follows that one has either $\mathbb{Y}_{n+1}/\mathbb{Y}_n \simeq \mathbb{V}^1$ and $\mathbb{I}_{n+1} = \mathbb{I}_n$, or $\mathbb{Y}_{n+1}/\mathbb{Y}_n$ is of finite dimension over $\mathbf{Q}_p$ and $\dim \mathbb{I}_{n+1} = \dim \mathbb{I}_n + 1$. Since $\dim \mathbb{W}' < \infty$, the second case can only happen for a finite number of n's, hence $\mathbb{Y}_{n+1}/\mathbb{Y}_n \simeq \mathbb{V}^1$, if n is big enough. In particular,

$$\operatorname{ht}(\mathbb{Y}_{n+1}) = \operatorname{ht}(\mathbb{Y}_{n_0}) \quad \text{and} \quad \dim(\mathbb{Y}_{n+1}) = \dim(\mathbb{Y}_{n_0}) + n + 1 - n_0, \qquad \text{if } n \geq n_0.$$

Now, $\mathbb{Y}_n$ is a sub-module of $\operatorname{Ker}\pi$, hence contains no $\mathbb{V}^1$ by assumption. Since $\mathbb{Y}_n$ is a sub-BC of $\mathbb{V}^n$, Lemma 3.15 gives a contradiction for $n \geq \operatorname{ht}(\mathbb{Y}_{n_0}) - \dim(\mathbb{Y}_{n_0}) + n_0$, which concludes the proof. □

Lemma 3.33. *Let $\pi : \mathbb{W} \to \mathbb{W}'$ be a morphism of VS's, with $\mathbb{W}$, $\mathbb{W}'$ $q\mathscr{BC}$'s. If $\mathbb{W}'_0$ is a sub-$\mathbb{B}_m$-Module of $\mathbb{W}'$ such that $\mathbb{W}'/\mathbb{W}'_0$ is a BC, there exists a sub-$\mathbb{B}_m$-Module $\mathbb{W}_0$ of $\mathbb{W}$ such that $\mathbb{W}/\mathbb{W}_0$ is a BC and $\pi(\mathbb{W}_0) \subset \mathbb{W}'_0$.*

Proof. By assumption, there exists a sub-$\mathbb{B}_m$-Module $\mathbb{W}_1$ of $\mathbb{W}$ such that $\mathbb{W}/\mathbb{W}_1$ is a BC. Applying Lemma 3.32 to $\overline{\pi} : \mathbb{W}_1 \to \mathbb{W}'/\mathbb{W}'_0$ produces $\mathbb{W}_0$ such that $\mathbb{W}_1/\mathbb{W}_0$ is a BC and $\pi(\mathbb{W}_0) \subset \mathbb{W}'_0$. But then $\mathbb{W}/\mathbb{W}_0$ is also BC as an extension of the two BC's $\mathbb{W}/\mathbb{W}_1$ and $\mathbb{W}_1/\mathbb{W}_0$. □

Proposition 3.34. *Il $\pi : \mathbb{W} \to \mathbb{W}'$ is a morphism in $q\mathscr{BC}$, then* $\operatorname{Ker}\pi$, $\operatorname{Im}\,\pi$, *and* $\operatorname{Coker}\pi$ *are in $q\mathscr{BC}$ and* $\operatorname{ht}(\mathbb{W}) = \operatorname{ht}(\operatorname{Ker}\pi) + \operatorname{ht}(\operatorname{Im}\,\pi)$.

Proof. Lemma 3.33 gives us sub-$\mathbb{B}_m$-Modules $\mathbb{W}_0$ and $\mathbb{W}'_0$ of $\mathbb{W}$ and $\mathbb{W}'$ such that $\overline{\mathbb{W}} = \mathbb{W}/\mathbb{W}_0$ and $\overline{\mathbb{W}}' = \mathbb{W}'/\mathbb{W}'_0$ are BC's and $\pi(\mathbb{W}_0) \subset \mathbb{W}'_0$. Then the restriction of π to $\mathbb{W}_0$ is $\mathbb{B}_m$-linear, hence its kernel $\mathbb{K}^0$ and its cokernel $\mathbb{C}^0$ are $\mathbb{B}_m$-Modules. The snake Lemma give us exact sequences $0 \to \mathbb{K}^0 \to \operatorname{Ker}\pi \to \mathbb{V}_1 \to 0$ and $0 \to \mathbb{V}_2 \to \mathbb{C}^0 \to \operatorname{Coker}\pi \to \mathbb{V}_3 \to 0$, with $\mathbb{V}_1$, $\mathbb{V}_2$, $\mathbb{V}_3$ BC's. This implies that $\operatorname{Ker}\pi$ is in $q\mathscr{BC}$ and, using Lemma 3.31, that so is $\operatorname{Coker}\pi$.

Now $\operatorname{Im}\,\pi$ is the cokernel of $\operatorname{Ker}\pi \to \mathbb{W}$, hence is also a qBC. Finally the formula for the heights is easily deduced from the analogous formula for BC's. □

Remark 3.35. (i) The above proposition implies that the category $q\mathscr{BC}$ is abelian.

(ii) It follows that a sequence $\mathbb{W}_1 \xrightarrow{f} \mathbb{W}_2 \xrightarrow{g} \mathbb{W}_3$ is exact if and only if the associated sequence $\mathbb{W}_1(C) \xrightarrow{f} \mathbb{W}_2(C) \xrightarrow{g} \mathbb{W}_3(C)$ is exact ($\mathbb{H} = (\operatorname{Ker} g + \operatorname{Coker} f)/(\operatorname{Ker} g \cap \operatorname{Coker} f)$ is a qBC, hence $\mathbb{H} = 0$ if and only if $\mathbb{H}(C) = 0$).

3.4.3. *Exactness of the functor $h(-)$.*

Lemma 3.36. *Let $0 \to \mathbb{W}_1 \to \mathbb{W}_2 \to \mathbb{W}_3 \to 0$ be an exact sequence of qBC's. Then there exist sub-$\mathbb{B}_m$-Modules $\mathbb{W}'_i$ of $\mathbb{W}_i$, such $\mathbb{W}_i/\mathbb{W}'_i$ are BC's, pour $i = 1, 2, 3$, and such that we have an exact sequence $0 \to \mathbb{W}'_1 \to \mathbb{W}'_2 \to \mathbb{W}'_3 \to 0$.*

Proof. Start with a $\mathbb{B}_m$-Module $\mathbb{W}_3'' \subset \mathbb{W}_3$ such that $\mathbb{W}_3/\mathbb{W}_3''$ is a BC. Lemma 3.33 provides $\mathbb{W}_2' \subset \mathbb{W}_2$ with image $\mathbb{W}_3'$ incuded in $\mathbb{W}_3''$. Then $\mathbb{W}_3'$ is a $\mathbb{B}_m$-Module since $\mathbb{W}_2' \to \mathbb{W}_3''$ is a morphism of $\mathbb{B}_m$-Modules. And $\mathbb{W}_3/\mathbb{W}_3'$ is a quotient of $\mathbb{W}_2/\mathbb{W}_2'$, hence is BC. Finally, if $\mathbb{W}_1'$ is the kernel of $\mathbb{W}_2' \to \mathbb{W}_3'$, the wanted sequence is exact, $\mathbb{W}_1'$ is a $\mathbb{B}_m$-Module as kernel of a morphism of $\mathbb{B}_m$-Modules, and $\mathbb{W}_1/\mathbb{W}_1'$ is a subobject of $\mathbb{W}_2/\mathbb{W}_2'$, hence is a BC. □

Corollary 3.37. *If $0 \to \mathbb{W}_1 \to \mathbb{W}_2 \to \mathbb{W}_3 \to 0$ is an exact sequence of qBC's, of curvatures ≤ 0, then the sequence $0 \to h(\mathbb{W}_3) \to h(\mathbb{W}_2) \to h(\mathbb{W}_1) \to 0$ is exact.*

Proof. Choose $\mathbb{W}_i'$, for $i = 1, 2, 3$, fulfilling the conclusions of Lemma 3.36; set $\mathbb{W}_i'' = \mathbb{W}_i/\mathbb{W}_i'$. Then $0 \to \mathbb{W}_1'' \to \mathbb{W}_2'' \to \mathbb{W}_3'' \to 0$ is an exact sequence of BC's of curvature ≤ 0, hence the sequence $0 \to h(\mathbb{W}_3'') \to h(\mathbb{W}_2'') \to h(\mathbb{W}_1'') \to 0$ is exact by Lemma 3.26. One concludes remarking that $h(\mathbb{W}_i') = 0$ (since $\mathbb{W}_i'$ is a $\mathbb{B}_m$-Module), hence $h(\mathbb{W}_i'') \to h(\mathbb{W}_i)$ is an isomorphism. □

4. Filtered (φ, N)-modules

In this chapter we study filtered $(\varphi, N, \mathscr{G}_K)$-modules over K or C and their relations to the categories of almost C-representations and BC's. In particular, we introduce the notion of acyclic $(\varphi, N, \mathscr{G}_K)$-modules as a generalization of weakly-admissible $(\varphi, N, \mathscr{G}_K)$-modules. While the $(\varphi, N, \mathscr{G}_K)$-modules $(H^i_{\rm HK}(X_C), H^i_{\rm dR}(X))$ coming from algebraic geometry tend to be weakly admissible those coming from overconvergent geometry tend to be only acyclic (as shown later in this paper).

The results of this chapter will be crucial for the proofs of our results towards the $C_{\rm st}$-conjecture. In particular, Theorem 4.8 and its corollaries (resp. Propositions4.22) will be used to study the pro-étale-to-de Rham part of the $C_{\rm st}$-conjecture for varieties over K (resp. over C). The dichotomy of Proposition 4.18 will play a big role in the proof of the de Rham-to-pro-étale part of the $C_{\rm st}$-conjecture.

4.1. Filtered $(\varphi, N, \mathscr{G}_K)$-modules over K.

4.1.1. *Filtered $(\varphi, N, \mathscr{G}_K)$-modules.*

• *$(\varphi, N, \mathscr{G}_K)$-modules.* A (φ, N)-module over F or $F^{\rm nr}$ is a finite dimensional F-module or $F^{\rm nr}$-module M endowed with a Frobenius $\varphi : M \to M$, semilinear with respect to the absolute Frobenius on F or $F^{\rm nr}$, and a linear map $N : M \to M$ satisfying $N\varphi = p\,\varphi N$.

More generally, a $(\varphi, N, \mathscr{G}_K)$-module over $F^{\rm nr}$ is a (φ, N)-module over $F^{\rm nr}$ endowed with a smooth[10] semilinear action of $\mathscr{G}_K$ which commutes with φ and N.

If M is a $(\varphi, N, \mathscr{G}_K)$-module over F or $F^{\rm nr}$, we define its dual M^* as $\mathrm{Hom}_F(M, F)$ endowed with actions of φ, N and $\mathscr{G}_K$ given by

$$\langle\varphi(\mu), v\rangle = \varphi(\langle\mu, \varphi^{-1}(v)\rangle), \quad \langle N(\mu), v\rangle = -\langle\mu, N(v)\rangle, \quad \langle\sigma(\mu), v\rangle = \sigma(\langle\mu, \sigma^{-1}(v)\rangle), \text{ if } \sigma \in \mathscr{G}_K.$$

• *Filtered modules.* A filtered module $(M, \mathrm{Fil}^\bullet)$ over K is a K-module M together with a descending filtration $\mathrm{Fil}^\bullet$ on $M_K = K \otimes_F M$ by sub-K-modules $\mathrm{Fil}^i M_K$, with $\mathrm{Fil}^i M_K = M_K$ if $i \ll 0$ and $\mathrm{Fil}^i M_K = 0$ if $i \gg 0$.

If $(M, \mathrm{Fil}^\bullet)$ is a filtered module over K, we define the dual filtered module $(M^*, \mathrm{Fil}^\bullet_\perp)$ by endowing the K-dual $M^* = \mathrm{Hom}_K(M, K)$ of M, with the filtration

$$\mathrm{Fil}^i_\perp M^* = (\mathrm{Fil}^{1-i} M)^\perp.$$

If the filtration is obvious from the context, we don't indicate it in the notation; for example, the de Rham cohomology of a variety X over K is a filtered module over K if we endow it with the Hodge filtration, and will just be denoted by $H^\bullet_{\rm dR}(X)$, the Hodge filtration being taken for granted.

• *Filtered $(\varphi, N, \mathscr{G}_K)$-modules.* A filtered (φ, N)-module $(M, \mathrm{Fil}^\bullet)$ over K is a (φ, N)-module M over F with a structure of filtered module over K on $M_K = M \otimes_F K$.

[10] This means that the stabilizers of elements of M are open in $\mathscr{G}_K$.

A filtered $(\varphi, N, \mathscr{G}_K)$-module $(M, \mathrm{Fil}^\bullet)$ over K is a $(\varphi, N, \mathscr{G}_K)$-module M over F^{nr} with a structure of filtered module over K on $M_K = (M \otimes_{F^{\mathrm{nr}}} \overline{K})^{\mathscr{G}_K}$.

If $(M, \mathrm{Fil}^\bullet)$ is a filtered $(\varphi, N, \mathscr{G}_K)$-module over K, we define its dual $(M^*, \mathrm{Fil}^\bullet_\perp)$ as the dual $(\varphi, N, \mathscr{G}_K)$-module M^* with the module $M^*_K = (M_K)^*$ endowed with the filtration $\mathrm{Fil}^\bullet_\perp$.

As before, if the filtration is obvious from the context, we will use sometimes just M to denote a filtered (φ, N)-module $(M, \mathrm{Fil}^\bullet)$ over K or sometimes (M, M_K) as in the case of de Rham cohomology: if X is a smooth quasi-compact dagger variety over K then $(H^i_{\mathrm{HK}}(X_C), H^i_{\mathrm{dR}}(X))$ is a filtered $(\varphi, N, \mathscr{G}_K)$-module over K thanks to the Hyodo-Kato isomorphism.

4.1.2. *Acyclicity and admissibility.* If M is a filtered $(\varphi, N, \mathscr{G}_K)$-module over K, *the rank* $\mathrm{rk}(M)$ *of* M is the dimension of M over F^{nr}. If M has rank 1, one defines the *degree* $\deg(M)$ *of* M by the formula

$$\deg(M) := t_N(M) - t_H(M),$$

where $t_N(M)$ et $t_H(M)$ are defined by choosing a basis e of M over F^{nr}:

- there exists $\lambda \in (F^{\mathrm{nr}})^*$ such that $\varphi(e) = \lambda e$, and we set $t_N(M) = v_p(\lambda)$;
- there exists $i \in \mathbf{Z}$, unique, such that $e \in M^i_K - M^{i+1}_K$, and we set $t_H(M) = i$.

If M has rank $r \geq 2$, then $\det M = \wedge^r M$ is of rank 1, and one defines *the degree of* M by:

$$\deg(M) := \deg(\det M) = t_N(M) - t_H(M),$$
$$t_N(M) := t_N(\det(M)), \quad t_H(M) := t_H(\det M) = \sum_{i \in \mathbf{Z}} i \dim_K M^i_K / M^{i+1}_K.$$

Endowed with the rank and degree functions, the category of filtered $(\varphi, N, \mathscr{G}_K)$-modules over K is a Harder-Narasimhan $\otimes$-category.

Definition 4.1. A filtered $(\varphi, N, \mathscr{G}_K)$-module over K is said to be *weakly admissible* if it is semi-stable of slope 0 (a reformulation [19] of the original notion [21]). It is said to be *acyclic* if its Harder-Narasimhan slopes are ≥ 0.

Remark 4.2. A weakly admissible filtered $(\varphi, N, \mathscr{G}_K)$-module is acyclic; conversely an acyclic filtered $(\varphi, N, \mathscr{G}_K)$-module is weakly admissible if and only if it is of degree 0.

Lemma 4.3. *The following conditions are equivalent for a filtered* $(\varphi, N, \mathscr{G}_K)$*-module* $(M, \mathrm{Fil}^\bullet)$ *over* K*:*

(a) $(M, \mathrm{Fil}^\bullet)$ *is acyclic.*

(b) *There exists a filtration* $\mathrm{Fil}^\bullet_1$ *on* M_K *such that* $\mathrm{Fil}^i_1 M_K \subset \mathrm{Fil}^i M_K$ *for all* i*, and* $(M, \mathrm{Fil}^\bullet_1)$ *is weakly admissible.*

Proof. To prove (a)$\Rightarrow$(b), it is enough to show that one can find a filtration such that $\mathrm{Fil}^i_1 M_K \subset \mathrm{Fil}^i M_K$ for all i, there exist i with $\mathrm{Fil}^i_1 M_K \neq \mathrm{Fil}^i M_K$, and $(M, \mathrm{Fil}^\bullet_1)$ is acyclic. Indeed the degree of $(M, \mathrm{Fil}^\bullet_1)$ is then strictly smaller than that of $(M, \mathrm{Fil}^\bullet)$. Hence by repeating the process one ends up with a filtration such that $\mathrm{Fil}^i_n M_K \subset \mathrm{Fil}^i M_K$ for all i, $(M, \mathrm{Fil}^\bullet_n)$ is acyclic and of degree 0, hence it is of Harder-Narasimhan slope 0, i.e., it is weakly admissible.

To construct such a $\mathrm{Fil}^\bullet_1$, let M^1 be the largest subobject of Harder-Narasimhan slope 0, and let M^2 be the quotient M/M_1. Then M^2 has Harder-Narasimhan slope > 0, and if we pick up any filtration $\mathrm{Fil}^\bullet_1$ on M^2_K, such that $\mathrm{Fil}^i_1 M^2_K = \mathrm{Fil}^i M^2_K$ for all i except i_0, for which $\mathrm{Fil}^{i_0} M^2_K / \mathrm{Fil}^{i_0}_1 M^2_K$ is of dimension 1, then $(M^2, \mathrm{Fil}^\bullet_1)$ has Harder-Narasimhan slope ≥ 0 since the degree of any subobject has decreased by at most 1 and hence is ≥ 0. Then defining $\mathrm{Fil}^i_1 M_K$ as the inverse image of $\mathrm{Fil}^i_1 M^2_K$ in $\mathrm{Fil}^i M_K$ gives a filtration with the desired properties.

The converse implication is obvious: the Harder-Narasimhan slope of $(M, \mathrm{Fil}^\bullet)$ is greater or equal to that of $(M, \mathrm{Fil}^\bullet_1)$. $\square$

Lemma 4.4. *Let $(M,\mathrm{Fil}^\bullet)$ be an acyclic filtered $(\varphi,N,\mathscr{G}_K)$-module over K, with φ-slopes in $[0,r]$, and $\mathrm{Fil}^0 M_K = M_K$, $\mathrm{Fil}^{r+1}M_K = 0$. Then there exists a filtration $\mathrm{Fil}_1^\bullet$ on M_K such that $(M,\mathrm{Fil}_1^\bullet)$ is weakly admissible and $\mathrm{Fil}_1^i M_K \subset \mathrm{Fil}^i M_K$ for all i, $\mathrm{Fil}_1^0 M_K = M_K$, $\mathrm{Fil}_1^{r+1} M_K = 0$.*

Proof. In the proof of Lemma 4.3, the φ-slopes of M^2 are in $[0,r]$, and since $\deg(M^2) > 0$, this implies that $\mathrm{Fil}^1 M_K \neq 0$. Let i_0 be the largest integer with $\mathrm{Fil}^{i_0} M_K \neq 0$; define $\mathrm{Fil}_1^\bullet$ by $\mathrm{Fil}_1^{i_0} M_K$ of codimension 1 in $\mathrm{Fil}^{i_0} M_K$, and $\mathrm{Fil}_1^i M_K = \mathrm{Fil}^i M_K$ is $i \neq 0$. Then, as in the proof of Lemma 4.3, $(M,\mathrm{Fil}_1^\bullet)$ is acyclic, $\mathrm{Fil}_1^0 M_K = M_K$, $\mathrm{Fil}_1^{r+1} M_K = 0$, and $\deg(M,\mathrm{Fil}_1^\bullet) < \deg(M,\mathrm{Fil}^\bullet)$. Iterating the process gives the wanted filtration. □

4.1.3. *The complex attached to a filtered $(\varphi,N,\mathscr{G}_K)$-module.* If $(M,\mathrm{Fil}^\bullet)$ is a filtered $(\varphi,N,\mathscr{G}_K)$-module over K, we set

$$X_{\rm st}(M,\mathrm{Fil}^\bullet) := (M \otimes_{F^{\rm nr}} \mathbf{B}_{\rm st})^{N=0,\varphi=1}, \quad X_{\rm dR}(M,\mathrm{Fil}^\bullet) := (M_K \otimes_K \mathbf{B}_{\rm dR})/\mathrm{Fil}^0(M_K \otimes_K \mathbf{B}_{\rm dR}).$$

Then $X_{\rm st}(M,\mathrm{Fil}^\bullet)$ and $X_{\rm dR}(M,\mathrm{Fil}^\bullet)$ are inductive limits of the $X_{\rm st}^{(r)}(M,\mathrm{Fil}^\bullet)$ and $X_{\rm dR}^{(r)}(M,\mathrm{Fil}^\bullet)$ (defined by replacing $\mathbf{B}_{\rm st}$ and $\mathbf{B}_{\rm dR}$ by $t^{-r}\mathbf{B}_{\rm st}^+$ and $t^{-r}\mathbf{B}_{\rm dR}^+$), which are objects of $\mathscr{C}(\mathscr{G}_K)$.

The complex $X_{\rm st}(M,\mathrm{Fil}^\bullet) \to X_{\rm dR}(M,\mathrm{Fil}^\bullet)$ is called [12, § 5.3] the "fundamental complex associated to M". Its H^0 is denoted by $V_{\rm st}(M,\mathrm{Fil}^\bullet)$. Hence, we have an exact sequence

$$0 \to V_{\rm st}(M,\mathrm{Fil}^\bullet) \to X_{\rm st}(M,\mathrm{Fil}^\bullet) \to X_{\rm dR}(M,\mathrm{Fil}^\bullet)$$

Remark 4.5. (i) The cohomology of the fundamental complex is equal to that of the complex $X_{\rm st}^{(r)}(M,\mathrm{Fil}^\bullet) \to X_{\rm dR}^{(r)}(M,\mathrm{Fil}^\bullet)$, for r big enough. It follows that its cohomology groups are objects of $\mathscr{C}(\mathscr{G}_K)$.

(ii) The pair $(X_{\rm st}(M,\mathrm{Fil}^\bullet), X_{\rm dR}(M,\mathrm{Fil}^\bullet))$ is a B-pair in the sense of Berger [3, 4]; attached to it is a $\mathscr{G}_K$-equivariant vector bundle $\mathscr{E}(M,\mathrm{Fil}^\bullet)$ on the Fargues-Fontaine curve [20, § 10.1] and the fundamental complex computes the cohomology (not the $\mathscr{G}_K$-equivariant) of this vector bundle. Since the HN-slope of $\mathscr{E}(M,\mathrm{Fil}^\bullet)$ is that of $(M,\mathrm{Fil}^\bullet)$, the vector bundle $\mathscr{E}(M,\mathrm{Fil}^\bullet)$ has vanishing H^1 if and only if $(M,\mathrm{Fil}^\bullet)$ is acyclic (see Theorem 3.9 and formulas (3.10)). It follows that, if $(M,\mathrm{Fil}^\bullet)$ is acyclic, we have an exact sequence

$$0 \to V_{\rm st}(M,\mathrm{Fil}^\bullet) \to X_{\rm st}(M,\mathrm{Fil}^\bullet) \to X_{\rm dR}(M,\mathrm{Fil}^\bullet) \to 0$$

(iii) If $(M,\mathrm{Fil}^\bullet)$ is acyclic, then $V_{\rm st}(M,\mathrm{Fil}^\bullet)$ is a finite dimensional $\mathbf{Q}_p$-vector space if and only if $(M,\mathrm{Fil}^\bullet)$ is of slope 0 (i.e., is weakly admissible). This implies it *is admissible*: the natural maps give $\mathscr{G}_K$-equivariant isomorphisms

$$\begin{aligned} &V_{\rm st}(M,\mathrm{Fil}^\bullet) \otimes \mathbf{B}_{\rm st} \xrightarrow{\sim} M \otimes_F \mathbf{B}_{\rm st} && \text{of } \mathbf{B}_{\rm st}\text{-modules commuting with } \varphi \text{ and } N,\\ &V_{\rm st}(M,\mathrm{Fil}^\bullet) \otimes \mathbf{B}_{\rm dR} \xrightarrow{\sim} M_K \otimes_K \mathbf{B}_{\rm dR} && \text{of filtered } \mathbf{B}_{\rm dR}\text{-modules,}\end{aligned}$$

and $V := V_{\rm st}(M,\mathrm{Fil}^\bullet)$ is a potentially semi-stable representation of $\mathscr{G}_K$, with $D_{\rm dR}(V) = M_K$ and $D_{\rm pst}(V) = M$.

(iv) If $(M,\mathrm{Fil}^\bullet)$ is acyclic, but not admissible, the natural map

$$V_{\rm st}(M,\mathrm{Fil}^\bullet) \otimes \mathbf{B}_{\rm dR}^+ \to \mathrm{Fil}^0(M_K \otimes_K \mathbf{B}_{\rm dR})$$

is not an isomorphism (the kernel is huge), but it is surjective. Indeed, one can pick a filtration $\mathrm{Fil}_1^\bullet$ on M_K, such that $\mathrm{Fil}_1^i M_K \subset \mathrm{Fil}^i M_K$ for all i, and $(M,\mathrm{Fil}_1^\bullet)$ is admissible. Then we have an exact sequence

$$0 \to V_{\rm st}(M,\mathrm{Fil}_1^\bullet) \to V_{\rm st}(M,\mathrm{Fil}^\bullet) \to \mathrm{Fil}^0(M_K \otimes_K \mathbf{B}_{\rm dR})/\mathrm{Fil}_1^0(M_K \otimes_K \mathbf{B}_{\rm dR}) \to 0 \tag{4.6}$$

and $\mathbf{B}_{\rm dR}^+ \cdot V_{\rm st}(M,\mathrm{Fil}^\bullet)$ contains $\mathbf{B}_{\rm dR}^+ \cdot V_{\rm st}(M,\mathrm{Fil}_1^\bullet)$ which, by admissibility of $(M,\mathrm{Fil}_1^\bullet)$, is equal to $\mathrm{Fil}_1^0(M_K \otimes_K \mathbf{B}_{\rm dR})$ and the above exact sequence gives the desired surjectivity.

4.2. **Filtered $(\varphi,N,\mathscr{G}_K)$-modules and almost C-representations.**

4.2.1. *The functors $D^*_{\rm dR}$ and $D^*_{\rm st}$.* The definitions of the classical contravariant functors $D^*_{\rm dR}$, $D^*_{\rm st}$ and $D^*_{\rm pst}$ for finite dimensional $\mathbf{Q}_p$-representations of $\mathscr{G}_K$ extend to objects of $\mathscr{C}(\mathscr{G}_K)$ contrarily to those of the (more commonly used) covariant functors $D_{\rm dR}$, $D_{\rm st}$ and $D_{\rm pst}$. If $W\in\mathscr{C}(\mathscr{G}_K)$, set

$$D^*_{\rm st}(W) := {\rm Hom}_{\mathscr{G}_K}(W,\mathbf{B}_{\rm st}),\quad D^*_{\rm dR}(W) := {\rm Hom}_{\mathscr{G}_K}(W,\mathbf{B}_{\rm dR}).$$

Then $D^*_{\rm st}(W)$ is a (φ,N)-module over K (with $\langle\varphi(\mu),v\rangle=\varphi(\langle\mu,v\rangle)$ and $\langle N(\mu),v\rangle = N(\langle\mu,v\rangle)$) and $D^*_{\rm dR}(W)$ is a filtered module over K (with ${\rm Fil}^i D^*_{\rm dR}(W)=\{\mu,\ \mu(W)\subset t^i\mathbf{B}^+_{\rm dR}\}$). We also define $D^*_{\rm pst}(W)$ as:

$$D^*_{\rm pst}(W) := {\rm Hom}^{\rm sm}_{\mathscr{G}_K}(W,\mathbf{B}_{\rm st}) := \varinjlim_{[L:K]<\infty}{\rm Hom}_{\mathscr{G}_L}(W,\mathbf{B}_{\rm st})$$

This is a $(\varphi,N,\mathscr{G}_K)$-module over $F^{\rm nr}$.

Remark 4.7. The natural maps

$$D^*_{\rm st}(W)\otimes_F K\to D^*_{\rm dR}(W),\quad (D^*_{\rm pst}(W)\otimes_{F^{\rm nr}}\overline{K})^{\mathscr{G}_K}\to D^*_{\rm dR}(W)$$

induced by the injections $\mathbf{B}_{\rm st}\otimes_F K\hookrightarrow\mathbf{B}_{\rm dR}$ and $\mathbf{B}_{\rm st}\otimes_{F^{\rm nr}}\overline{K}\hookrightarrow\mathbf{B}_{\rm dR}$ are injective.

The following result is an extension to acyclic filtered (φ,N)-modules of a classical result for admissible filtered (φ,N)-modules.

Theorem 4.8. *If $(M,{\rm Fil}^\bullet)$ is an acyclic filtered (φ,N)-module over K, then*

$$D^*_{\rm st}(V_{\rm st}(M,{\rm Fil}^\bullet))\simeq M^* \qquad \text{as } (\varphi,N)\text{-modules over } F,$$
$$D^*_{\rm dR}(V_{\rm st}(M,{\rm Fil}^\bullet))\simeq (M^*_K,{\rm Fil}^\bullet_\perp) \qquad \text{as filtered } K\text{-modules.}$$

Proof. We start with noticing that the natural pairing injects M^* into ${\rm Hom}_{\mathscr{G}_K}(X_{\rm st}(M),\mathbf{B}_{\rm st})$ (as a (φ,N)-module) and M^*_K into ${\rm Hom}_{\mathscr{G}_K}(X_{\rm st}(M),\mathbf{B}_{\rm dR})$. Moreover, the natural map from $K\otimes_F {\rm Hom}_{\mathscr{G}_K}(X_{\rm st}(M),\mathbf{B}_{\rm st})$ to ${\rm Hom}_{\mathscr{G}_K}(X_{\rm st}(M),\mathbf{B}_{\rm dR})$ is injective (see Remark 4.7), and one can deduce that $D^*_{\rm st}\to M^*$ is an isomorphism from the same result for $D^*_{\rm dR}$.

Hence, we just have to prove the result for $D^*_{\rm dR}$. To do so, pick a filtration ${\rm Fil}^\bullet_1$ as in (iv) of Remark 4.5: ${\rm Fil}^i_1 M_K\subset {\rm Fil}^i M_K$ for all i and $(M,{\rm Fil}^\bullet_1)$ is admissible. Let $V:=V_{\rm st}(M,{\rm Fil}^\bullet)$, $V_1:=V_{\rm st}(M,{\rm Fil}^\bullet_1)$, and $W={\rm Fil}^0(M_K\otimes_K\mathbf{B}_{\rm dR})/{\rm Fil}^0_1(M_K\otimes_K\mathbf{B}_{\rm dR})$, so that the exact sequence (4.6) becomes $0\to V_1\to V\to W\to 0$.

Now, W is a direct sum of factors of the form $t^{k_1}\mathbf{B}^+_{\rm dR}/t^{k_2}\mathbf{B}^+_{\rm dR}$. Hence $D^*_{\rm dR}(W)=0$ thanks to Proposition 2.14, and we get injections $M^*_K\hookrightarrow D^*_{\rm dR}(V)\hookrightarrow D^*_{\rm dR}(V_1)$. But $\dim_K D^*_{\rm dR}(V_1)\leq \dim_{\mathbf{Q}_p}V_1=\dim_K M_K$ (the first inequality is true for any finite dimensional $\mathbf{Q}_p$-representation of $\mathscr{G}_K$ and the second equality is true because V_1 is de Rham). It follows that these injections are in fact isomorphisms, which proves what we want except for the equality of the filtrations on M^*_K and $D^*_{\rm dR}(V)$.

So let $\mu\in M^*_K$. One can extend μ to a $\mathbf{B}_{\rm dR}$-linear map $\mathbf{B}_{\rm dR}\otimes_K M_K\to\mathbf{B}_{\rm dR}$. Thanks to (iv) of Remark 4.5, one sees that $\mu(V)\subset t^i\mathbf{B}^+_{\rm dR}$ is equivalent to $\mu({\rm Fil}^0(M_K\otimes_K\mathbf{B}_{\rm dR}))\subset t^i\mathbf{B}^+_{\rm dR}$. But ${\rm Fil}^0(M_K\otimes_K\mathbf{B}_{\rm dR})=\sum_n {\rm Fil}^n M_K\otimes_K t^{-n}\mathbf{B}^+_{\rm dR}$, and $\mu(M_K)\subset K$, hence $\mu({\rm Fil}^0(M_K\otimes_K\mathbf{B}_{\rm dR}))\subset t^i\mathbf{B}^+_{\rm dR}$ if and only if $\mu({\rm Fil}^n M_K)=0$ for $n\geq 1-i$. By definition this translates into $\mu\in{\rm Fil}^i_\perp M^*_K$, as wanted. □

Corollary 4.9. *If $(M,{\rm Fil}^\bullet)$ is an acyclic filtered $(\varphi,N,\mathscr{G}_K)$-module over K, then*

$$D^*_{\rm pst}(V_{\rm st}(M,{\rm Fil}^\bullet))\simeq M^* \qquad \text{as } (\varphi,N,\mathscr{G}_K)\text{-modules over } F^{\rm nr},$$
$$D^*_{\rm dR}(V_{\rm st}(M,{\rm Fil}^\bullet))\simeq (M^*_K,{\rm Fil}^\bullet_\perp) \qquad \text{as filtered } K\text{-modules.}$$

Proof. As in the proof of the above theorem, we have an injection of the right hand sides into the left hand sides, and to check that these are isomorphisms, we just have to bound the dimensions of the left hand sides. This can be achieved by passing to a finite extension L of K such that the action of $\mathscr{G}_L$ on M is unramified, and applying the proposition to a (φ, N)-module $M(L)$ over F_L such that $M = F^{\mathrm{nr}} \otimes_{F_L} M(L)$. $\square$

Corollary 4.10. *Let $(M, \mathrm{Fil}^\bullet)$ be an acyclic filtered $(\varphi, N, \mathscr{G}_K)$-module over K, with φ-slopes in $[0, r]$, and $\mathrm{Fil}^0 M_K = M_K$, $\mathrm{Fil}^{r+1} M_K = 0$. Set*

$$V^r_{\mathrm{st}}(M, \mathrm{Fil}^\bullet) := \mathrm{Ker}\big((M \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \to (M_K \otimes_K \mathbf{B}^+_{\mathrm{dR}})/\mathrm{Fil}^r\big)$$

Then[11]

$$\begin{aligned} &D^*_{\mathrm{pst}}(V^r_{\mathrm{st}}(M, \mathrm{Fil}^\bullet)) \simeq M^*\{r\} && \textit{as } (\varphi, N, \mathscr{G}_K)\textit{-modules over } F^{\mathrm{nr}}, \\ &D^*_{\mathrm{dR}}(V^r_{\mathrm{st}}(M, \mathrm{Fil}^\bullet)) \simeq (M^*_K, \mathrm{Fil}^\bullet_\perp\{r\}) && \textit{as filtered } K\textit{-modules.} \end{aligned}$$

Proof. The conditions imply that $V^r_{\mathrm{st}}(M, \mathrm{Fil}^\bullet) = t^r V_{\mathrm{st}}(M, \mathrm{Fil}^\bullet)$. Hence

$$\mathrm{Hom}_{\mathscr{G}_K}(V^r_{\mathrm{st}}(M, \mathrm{Fil}^\bullet), \mathbf{B}_?) = t^r \mathrm{Hom}_{\mathscr{G}_K}(V_{\mathrm{st}}(M, \mathrm{Fil}^\bullet), \mathbf{B}_?)$$

and the result follows. $\square$

Example 4.11. If M in Corollary 4.10 satisfies $\mathrm{Fil}^r M_K = M_K$, then $(M_K \otimes_K \mathbf{B}^+_{\mathrm{dR}})/\mathrm{Fil}^r = 0$. Hence

$$V^r_{\mathrm{st}}(M, \mathrm{Fil}^\bullet) = X^r_{\mathrm{st}}(M) := (M \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}$$

The corollary then becomes:

$$\begin{aligned} &\mathrm{Hom}^{\mathrm{sm}}_{\mathscr{G}_K}(X^r_{\mathrm{st}}(M), \mathbf{B}_{\mathrm{st}}) \simeq M^*, \quad \text{as } (\varphi, N, \mathscr{G}_K)\text{-modules over } F^{\mathrm{nr}}, \\ &\mathrm{Hom}_{\mathscr{G}_K}(X^r_{\mathrm{st}}(M), t^j \mathbf{B}^+_{\mathrm{dR}}) \simeq \begin{cases} M_K & \text{if } j \le 0, \\ 0 & \text{if } j \ge 1. \end{cases} \end{aligned}$$

Lemma 4.12. *Under the hypothesis of Corollary 4.10, we have $\mathrm{Hom}_{\mathscr{G}_K}(V^r_{\mathrm{st}}(M, \mathrm{Fil}^\bullet), C(j)) = 0$, for $j \ge r+1$.*

Proof. We can write $V := V^r_{\mathrm{st}}(M, \mathrm{Fil}^\bullet)$ in the form $0 \to V_1 \to V \to W \to 0$, as in the proof of Theorem 4.8, with V_1 of dimension $\dim(M)$, de Rham with Hodge-Tate weights in $[0, r]$, and W a finite type $\mathbf{B}^+_{\mathrm{dR}}$-module, sum of $t^a \mathbf{B}^+_{\mathrm{dR}}/t^b \mathbf{B}^+_{\mathrm{dR}}$'s, with $0 \le a \le b \le r+1$.

We have an exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{C}(\mathscr{G}_K)}(W, C(j)) \to \mathrm{Hom}_{\mathscr{C}(\mathscr{G}_K)}(V, C(j)) \to \mathrm{Hom}_{\mathscr{C}(\mathscr{G}_K)}(V_1, C(j))$$

Since $j \ge r+1$, by Lemma 2.10, we have $\mathrm{Hom}_{\mathscr{C}(\mathscr{G}_K)}(t^a \mathbf{B}^+_{\mathrm{dR}}/t^b \mathbf{B}^+_{\mathrm{dR}}, C(j)) = 0$ if $b \le r+1$; hence $\mathrm{Hom}_{\mathscr{C}(\mathscr{G}_K)}(W, C(j)) = 0$. Since the Hodge–Tate weights of V_1 are $\le r$, we also have $\mathrm{Hom}_{\mathscr{C}(\mathscr{G}_K)}(V, C(j)) = 0$.

This concludes the proof. $\square$

4.3. **Filtered (φ, N)-modules over C.**

4.3.1. *Filtered (φ, N)-modules and vector bundles.* A filtered (φ, N)-module (M, M^+_{dR}) over C is a tuple $(M, \varphi, N, M^+_{\mathrm{dR}})$, where:

(1) M is a finite dimensional F^{nr}-vector space;
(2) $\varphi : M \to M$ is a Frobenius map;
(3) $N : M \to M$ is a F^{nr}-linear monodromy map such that $N\varphi = p\varphi N$;
(4) $M^+_{\mathrm{dR}} \subset M \otimes_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{dR}}$ is a sub-$\mathbf{B}^+_{\mathrm{dR}}$-lattice.

[11] The notations $M\{r\}$ and $\mathrm{Fil}^\bullet_\perp\{r\}$ mean that the action of φ is multiplied by p^r and that the filtration is shifted by r: we have $\mathrm{Fil}^i_\perp\{r\} = \mathrm{Fil}^{i-r}_\perp$.

To such an $(M, M_{\rm dR}^+)$ one can attach a vector bundle $\mathscr{E}(M, M_{\rm dR}^+)$ on $X_{\rm FF}$, characterized by

$$H^0(X_{\rm FF}\setminus\{\infty\}, \mathscr{E}(M, M_{\rm dR}^+)) = (M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm st})^{N=0,\varphi=1},\quad \mathscr{E}(M, M_{\rm dR}^+)\otimes\widehat{\mathscr{O}}_{X_{\rm FF},\infty} = M_{\rm dR}^+.$$

We set

$$V_{\rm st}(M, M_{\rm dR}^+) := H^0(X_{\rm FF}, \mathscr{E}(M, M_{\rm dR}^+)) = {\rm Ker}\big((M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm st})^{N=0,\varphi=1}\to (M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm dR})/M_{\rm dR}^+\big)$$

Definition 4.13. $(M, M_{\rm dR}^+)$ is *weakly admissible* if $\mathscr{E}(M, M_{\rm dR}^+)$ is semistable, of slope 0. It is *acyclic* if the slopes of $\mathscr{E}(M, M_{\rm dR}^+)$ are ≥ 0.

Remark 4.14. (i) If $(M, {\rm Fil}^\bullet)$ is a weakly admissible (resp. acyclic) filtered (φ, N)-module over K, the induced filtered (φ, N)-module $(M\otimes_F F^{\rm nr}, {\rm Fil}^0(M_K\otimes_K\mathbf{B}_{\rm dR}))$ over C is weakly admissible (resp. acyclic).

(ii) It follows from the classification of vector bundles on $X_{\rm FF}$ that $(M, M_{\rm dR}^+)$ is weakly admissible if and only if it is admissible (i.e., a direct sum of trivial line bundles $\mathscr{O}_{X_{\rm FF}}$). This translates into the following: $(M, M_{\rm dR}^+)$ is weakly admissible if and only if $V_{\rm st}(M, M_{\rm dR}^+)$ is finite dimensional over $\mathbf{Q}_p$ and the sequence

$$0\to V_{\rm st}(M, M_{\rm dR}^+)\to (M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm st})^{N=0,\varphi=1}\to (M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm dR})/M_{\rm dR}^+\to 0$$

is exact. Moreover, if this is the case, the triviality of $\mathscr{E}(M, M_{\rm dR}^+)$ implies that the natural maps

$$V_{\rm st}(M, M_{\rm dR}^+)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\rm st}\to M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm st},\quad V_{\rm st}(M, M_{\rm dR}^+)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\rm dR}^+\to M_{\rm dR}^+$$

are isomorphisms (for the second map this is also equivalent to the map $V_{\rm st}(M, M_{\rm dR}^+)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\rm dR}\to M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm dR}$ being a filtered isomorphism).

(iii) The following conditions are equivalent:

- $(M, M_{\rm dR}^+)$ is acyclic,
- $H^1(X, \mathscr{E}(M, M_{\rm dR}^+)) = 0$ (and hence $\mathscr{E}(M, M_{\rm dR}^+)$ is acyclic),
- $(M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm st})^{N=0,\varphi=1}\to (M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm dR})/M_{\rm dR}^+$ is surjective.
- There exists a sub-$\mathbf{B}_{\rm dR}^+$ lattice $N_{\rm dR}^+\subset M_{\rm dR}^+$ such that $(M, N_{\rm dR}^+)$ is weakly admissible.

(The first two points are equivalent by Theorem 3.9 and formulas (3.10), the second and the third are equivalent because

$$H^1(X, \mathscr{E}(M, M_{\rm dR}^+)) = {\rm Coker}\big((M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm st})^{N=0,\varphi=1}\to (M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm dR})/M_{\rm dR}^+\big)$$

the first and last points are equivalent by the same arguments as in Lemma 4.3.)

Remark 4.15. Since $(M, M_{\rm dR}^+)\mapsto\mathscr{E}(M, M_{\rm dR}^+)$ commutes with tensor products, and since the slope of $\mathscr{E}\otimes\mathscr{E}'$ is equal to the sum of slopes of $\mathscr{E}$ and $\mathscr{E}'$, the tensor product of two acyclic filtered φ-modules is again acyclic.

Remark 4.16. Assume that the φ-slopes are in $[0, r]$ and $M\otimes t^r\mathbf{B}_{\rm dR}^+\subset t^rM_{\rm dR}^+\subset M\otimes\mathbf{B}_{\rm dR}^+$. Then the following conditions are equivalent:

- $(M, M_{\rm dR}^+)$ is acyclic,
- $(M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm st})^{N=0,\varphi=p^r}\to (M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm dR})/t^rM_{\rm dR}^+$ is surjective,
- for all $k\geq 0$, $(M\otimes_{F^{\rm nr}}t^{-k}\mathbf{B}_{\rm st}^+)^{N=0,\varphi=p^r}\to (M\otimes_{F^{\rm nr}}t^{-k}\mathbf{B}_{\rm dR}^+)/t^rM_{\rm dR}^+$ is surjective.

This follows from the fact that $\frac{(M\otimes t^{-k}\mathbf{B}_{\rm st}^+)^{N=0,\varphi=p^r}}{(M\otimes t^{1-k}\mathbf{B}_{\rm st}^+)^{N=0,\varphi=p^r}}\simeq\frac{M\otimes t^{-k}\mathbf{B}_{\rm dR}^+}{M\otimes t^{1-k}\mathbf{B}_{\rm dR}^+}$, for $k\geq 1$, as can be shown, for exemple, by a Dimension of BC argument. For the same reasons, for all $k\geq 0$,

$$t^rV_{\rm st}(M, M_{\rm dR}^+) = {\rm Ker}\big((M\otimes_{F^{\rm nr}}t^{-k}\mathbf{B}_{\rm st}^+)^{N=0,\varphi=p^r}\to (M\otimes_{F^{\rm nr}}t^{-k}\mathbf{B}_{\rm dR}^+)/t^rM_{\rm dR}^+\big)$$

4.3.2. *Acyclicity and curvature.* The following results supply key arguments in the proof of our main comparison theorem (Theorem 6.14).

Lemma 4.17. *Let M be a (φ, N)-module over $F^{\rm nr}$ whose φ-slopes are in $[0, r]$. Then the $\mathbf{Q}_p$-module $(M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm st}^+)^{N=0,\varphi=p^r}$ generates the $\mathbf{B}_{\rm dR}^+$-module $M\otimes_{F^{\rm nr}}\mathbf{B}_{\rm dR}^+$.*

Proof. Since $\mathbf{B}_{\mathrm{dR}}^+$ is a local ring , with residue field C, it suffices to show that $(M\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}^+)^{N=0,\varphi=p^r}$ generates the C-module $M\otimes_{F^{\mathrm{nr}}}C$.

Now, the map $x\mapsto x-u\,Nx+\frac{u^2}{2!}\,N^2x-\frac{u^3}{3!}\,N^3x+\cdots$, for $u\in\mathbf{B}_{\mathrm{st}}^+$ maping to $\log([p^\flat]/p)$ in $\mathbf{B}_{\mathrm{dR}}^+$, induces a φ-equivariant isomorphism $M\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{cris}}^+\to(M\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}^+)^{N=0}$. Since u has image 0 in C, we are reduced to proving that $(M\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{cris}}^+)^{\varphi=p^r}$ generates the C-module $M\otimes_{F^{\mathrm{nr}}}C$.

Since $\mathbf{B}_{\mathrm{cris}}^+$ contains $W(k_C)$, the theorem of Dieudonné-Manin allows us to reduce to the case where M is elementary with slopes $\frac{a}{h}\leq r$, i.e., it is generated by $e_1,\ldots,e_h$ with

$$\varphi(e_1)=e_2,\,\varphi(e_2)=e_3,\ldots,\,\varphi(e_h)=p^ae_1.$$

The map $x\mapsto xe_1+p^{-r}\varphi(x)e_2+\cdots+p^{-(h-1)r}\varphi^{h-1}(x)e_h$ induces an isomorphism of $U_{h,rh-a}:=(\mathbf{B}_{\mathrm{cris}}^+)^{\varphi^h=p^{rh-a}}$ with $(M\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{cris}}^+)^{\varphi=p^r}$. Hence we are reduced to proving that the image of $U_{h,rh-a}$ by the map $x\mapsto(\theta(x),\theta(\varphi(x)),\ldots,\theta(\varphi^{h-1}(x)))$ does not lie inside a proper sub-C-module of C^h. Assume that it does. Then there exists $\lambda_0,\ldots,\lambda_{h-1}\in C$, not all zero, such that $\lambda_0\theta(x)+\lambda_1\theta(\varphi(x))+\cdots+\lambda_{h-1}\theta(\varphi^{h-1}(x))=0$ for all $x\in U_{h,rh-a}$. In particular, one can apply this to αx, for $\alpha\in\mathbf{Q}_{p^h}$. Setting $\mu_i(x)=\lambda_i\theta(\varphi^i(x))$, we get $\mu_0(x)\alpha+\mu_1(x)\varphi(\alpha)+\cdots+\mu_{h-1}(x)\varphi^{h-1}(\alpha)=0$, for all $\alpha\in\mathbf{Q}_{p^h}$. Linear independence of characters implies that $\mu_i(x)=0$ for all i and x, and we get our contradiction. □

Proposition 4.18. *Let M be a (φ,N)-module over F^{nr} with φ-slopes in $[0,r]$. Let M_{dR}^+ be a $\mathbf{B}_{\mathrm{dR}}^+$-lattice in $M\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{dR}}$, with $M\otimes_{F^{\mathrm{nr}}}t^r\mathbf{B}_{\mathrm{dR}}^+\subset t^rM_{\mathrm{dR}}^+\subset M\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{dR}}^+$. Set $\mathbb{M}_{\mathrm{dR}}^+=M_{\mathrm{dR}}^+\otimes_{\mathbf{B}_{\mathrm{dR}}^+}\mathbb{B}_{\mathrm{dR}}^+$. Then one and only one of the following holds:*

(a) *The map $(M\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}^+)^{N=0,\varphi=p^r}\to(M\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{dR}}^+)/t^rM_{\mathrm{dR}}^+$ is surjective.*

(b) *The image of the map $(M\otimes_{F^{\mathrm{nr}}}\mathbb{B}_{\mathrm{st}}^+)^{N=0,\varphi=p^r}\to(M\otimes_{F^{\mathrm{nr}}}\mathbb{B}_{\mathrm{dR}}^+)/t^r\mathbb{M}_{\mathrm{dR}}^+$ has height >0.*

Proof. Since $(M\otimes_{F^{\mathrm{nr}}}\mathbb{B}_{\mathrm{dR}}^+)/t^r\mathbb{M}_{\mathrm{dR}}^+$ has curvature 0, all its sub-BC's have curvature ≤ 0. Hence, if we are not in case (b), this implies that the curvature of the image is 0 (use Corollary 3.19), and hence that the image is a $\mathbb{B}_{\mathrm{dR}}^+$-module (see Proposition 3.16). Now Lemma 4.17 implies that we are in case (a), as wanted. □

Remark 4.19. In case (b), $\mathrm{Coker}\big((M\otimes_{F^{\mathrm{nr}}}\mathbb{B}_{\mathrm{st}}^+)^{N=0,\varphi=p^r}\to(M\otimes_{F^{\mathrm{nr}}}\mathbb{B}_{\mathrm{dR}}^+)/t^r\mathbb{M}_{\mathrm{dR}}^+\big)$ has height <0 and thus does not have curvature ≤ 0.

4.4. **Filtered (φ,N)-modules and BC's.** The following computations supply key arguments in the proofs of our pro-étale-to-de Rham comparison theorems in Chapter 7.

4.4.1. *Finite rank (φ,N)-modules.*

Proposition 4.20. *Let M be a (φ,N)-module with φ-slopes in $[0,r]$, and let*

$$\mathbb{X}_{\mathrm{st}}^r(M):=(M\otimes_{F^{\mathrm{nr}}}\mathbb{B}_{\mathrm{st}}^+)^{N=0,\varphi=p^r}.$$

Then:

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbb{X}_{\mathrm{st}}^r(M),\mathbb{B}_{\mathrm{dR}})\simeq M^*\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{dR}},\quad \mathrm{Hom}_{\mathrm{VS}}(\mathbb{X}_{\mathrm{st}}^r(M),\mathbb{B}_{\mathrm{st}})\simeq M^*\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}$$

Proof. $\mathbb{X}_{\mathrm{st}}^r(M)$ is of curvature ≤ 0, and the condition on the slopes imply ([13, Ex. 5.18]) that $\mathrm{ht}(\mathbb{X}_{\mathrm{st}}^r(M))=\mathrm{rk}(M)$. It follows from Proposition 3.27 that $\mathrm{rk}(h(\mathbb{X}_{\mathrm{st}}^r(M)))=\mathrm{rk}(M)$. On the other hand, the inclusion $\mathbb{B}_{\mathrm{st}}^+\hookrightarrow\mathbb{B}_{\mathrm{dR}}$ induces a natural map $M^*\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{dR}}\to h(\mathbb{X}_{\mathrm{st}}^r(M))$. Lemma 4.17 implies that this map is injective. Since the two modules have the same rank over the field $\mathbf{B}_{\mathrm{dR}}$, this natural map is an isomorphism, which provides our first isomorphism.

For the second isomorphism, the injection of the right hand side into the left hand side is obvious (it follows, for example, from the first isomorphism). To prove the converse inclusion, it is enough, granting the first isomorphism, to show that if $\lambda\in M^*\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{dR}}$ satisfies $\lambda(X_{\mathrm{st}}^r(M))\subset\mathbf{B}_{\mathrm{st}}$, then $\lambda\in M^*\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}$. For this, pick a weakly admissible filtration M_{dR}^+ on $M\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{dR}}$, and let $V:=V_{\mathrm{st}}(M,M_{\mathrm{dR}}^+)$ so that $V\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{st}}\simeq M\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}$ (see Remark 4.14). Since $t^rV\subset X_{\mathrm{st}}^r(M)$, we have $\lambda(V) \subset \mathbf{B}_{\rm st}$, and since V generates $M \otimes_{F^{\rm nr}} \mathbf{B}_{\rm st}$, we have $\lambda(M) \subset \mathbf{B}_{\rm st}$. This implies $\lambda \in M^* \otimes_{F^{\rm nr}} \mathbf{B}_{\rm st}$, as wanted. $\square$

Remark 4.21. The map $x \mapsto x - \frac{u}{1!}Nx + \frac{u^2}{2!}N^2x - \frac{u^3}{3!}N^3x + \cdots$ induces isomorphisms

$$\mathbb{X}^r_{\rm cris}(M) := (M \otimes \mathbb{B}^+_{\rm cris})^{\varphi=p^r} \xrightarrow{\sim} \mathbb{X}^r_{\rm st}(M), \quad \mathrm{Hom}_{\rm VS}(\mathbb{X}^r_{\rm st}(M), \mathbb{B}^+_{\rm cris}) \simeq \mathrm{Hom}_{\rm VS}(\mathbb{X}^r_{\rm cris}(M), \mathbb{B}^+_{\rm cris})$$

Since $\mathbb{B}_{\rm st} = \mathbb{B}^+_{\rm cris}[u, \frac{1}{t}]$, we obtain:

$$\begin{aligned} \mathrm{Hom}_{\rm VS}(\mathbb{X}^r_{\rm st}(M), \mathbb{B}_{\rm st}) =& \mathrm{Hom}_{\rm VS}(\mathbb{X}^r_{\rm st}(M), \mathbb{B}^+_{\rm cris}) \otimes_{\mathbf{B}^+_{\rm cris}} \mathbf{B}_{\rm st} \\ M^* \otimes_{F^{\rm nr}} \mathbf{B}^+_{\rm cris} \subset& \mathrm{Hom}_{\rm VS}(\mathbb{X}^r_{\rm cris}(M), \mathbb{B}^+_{\rm cris}) \subset M^* \otimes_{F^{\rm nr}} t^{-r}\mathbf{B}^+_{\rm cris} \end{aligned}$$

(For the inclusion on the right, use the fact that V in the proof of Proposition 4.20 is included in $t^{-r}X^r_{\rm cris}(M)$ and $M \subset V \otimes \mathbf{B}^+_{\rm cris}$.)

Proposition 4.22. *Let $(M, M^+_{\rm dR})$ be an acyclic filtered (φ, N)-module over C with φ-slopes in $[0, r]$, with $t^r\mathbf{B}^+_{\rm dR} \otimes M \subset t^r M^+_{\rm dR} \subset \mathbf{B}^+_{\rm dR} \otimes M$. Set $\mathbb{M}^+_{\rm dR} := \mathbb{B}^+_{\rm dR} \otimes_{\mathbf{B}^+_{\rm dR}} M^+_{\rm dR}$ and*

$$\mathbb{V}^r_{\rm st}(M, M^+_{\rm dR}) := \mathrm{Ker}((M \otimes_{F^{\rm nr}} \mathbb{B}_{\rm st})^{N=0,\varphi=p^r} \to (M \otimes_{F^{\rm nr}} \mathbb{B}^+_{\rm dR})/t^r\mathbb{M}^+_{\rm dR}).$$

Then

$$\mathrm{Hom}_{\rm VS}(\mathbb{V}^r_{\rm st}(M, M^+_{\rm dR}), \mathbb{B}_{\rm dR}) \simeq M^* \otimes_{F^{\rm nr}} \mathbf{B}_{\rm dR}, \quad \mathrm{Hom}_{\rm VS}(\mathbb{V}^r_{\rm st}(M, M^+_{\rm dR}), \mathbb{B}_{\rm st}) \simeq M^* \otimes_{F^{\rm nr}} \mathbf{B}_{\rm st}$$

Proof. Let $\mathbb{X}^r_{\rm dR}(M) := (M \otimes_{F^{\rm nr}} \mathbb{B}^+_{\rm dR})/t^r\mathbb{M}^+_{\rm dR}$; this is a $\mathbb{B}^+_{\rm dR}$-module killed by t^r. The hypothesis give us an exact sequence

$$0 \to \mathbb{V}^r_{\rm st}(M, M^+_{\rm dR}) \to \mathbb{X}^r_{\rm st}(M) \to \mathbb{X}^r_{\rm dR}(M) \to 0$$

The first isomorphism is then a consequence of Proposition 4.20 and vanishing of $\mathrm{Hom}_{\rm VS}(\mathbb{X}^r_{\rm dR}(M), \mathbb{B}_{\rm dR})$ and $\mathrm{Ext}^{1,\natural}_{\rm VS}(\mathbb{X}^r_{\rm dR}(M), \mathbb{B}_{\rm dR})$ (Corollary 3.17 and Lemma 3.26).

For the second isomorphism pick up a weakly admissible filtration $N^+_{\rm dR}$ containing $M^+_{\rm dR}$ (this is possible because $(M, M^+_{\rm dR})$ is acyclic), set $V := V_{\rm st}(M, N^+_{\rm dR})$, and argue as in Proposition 4.20. $\square$

Remark 4.23. As in Remark 4.21, we have

$$\begin{aligned} \mathrm{Hom}_{\rm VS}(\mathbb{V}^r_{\rm st}(M, M^+_{\rm dR}), \mathbb{B}_{\rm st}) =& \mathrm{Hom}_{\rm VS}(\mathbb{V}^r_{\rm st}(M, M^+_{\rm dR}), \mathbb{B}^+_{\rm cris}) \otimes_{\mathbf{B}^+_{\rm cris}} \mathbf{B}_{\rm st} \\ M^* \otimes_{F^{\rm nr}} \mathbf{B}^+_{\rm cris} \subset& \mathrm{Hom}_{\rm VS}(V^r_{\rm st}(M, M^+_{\rm dR}), \mathbb{B}^+_{\rm cris}) \subset M^* \otimes_{F^{\rm nr}} t^{-r}\mathbf{B}^+_{\rm cris} \end{aligned}$$

Remark 4.24. We could also have argued as in the proof of Theorem 4.8 to prove Propositions 4.20 and 4.22.

Proposition 4.25. *Let $(M, M^+_{\rm dR})$ be an acyclic filtered (φ, N)-module over C with φ-slopes in $[0, r]$, and $M \otimes_{F^{\rm nr}} t^r\mathbf{B}^+_{\rm dR} \subset t^r M^+_{\rm dR} \subset M \otimes_{F^{\rm nr}} \mathbf{B}^+_{\rm dR}$. Let $k \geq 2r$. Then the natural map of $\mathbf{B}^+_{\rm dR}$-modules*

$$M^* \otimes_{F^{\rm nr}} (\mathbf{B}^+_{\rm dR}/t^k) \to \mathrm{Hom}_{\rm VS}(\mathbb{V}^r_{\rm st}(M, M^+_{\rm dR}), \mathbb{B}^+_{\rm dR}/t^k)$$

induced by the inclusion $\mathbb{V}^r_{\rm st}(M, M^+_{\rm dR}) \subset M \otimes \mathbb{B}^+_{\rm dR}$, has kernel and cokernel killed by t^{2r}.

Proof. Choose a $N^+_{\rm dR}$ with $M \otimes_{F^{\rm nr}} \mathbf{B}^+_{\rm dR} \subset N^+_{\rm dR} \subset M^+_{\rm dR}$ such that $(M, N^+_{\rm dR})$ is weakly admissible. (This is possible by an adaptation of Lemma 4.4 to (φ, N)-modules over C.) Let $V_1 = \mathbb{V}^r_{\rm st}(M, N^+_{\rm dR})$, $V = \mathbb{V}^r_{\rm st}(M, M^+_{\rm dR})$, and $W = V/V_1$, so that V_1 is a finite dimensional $\mathbf{Q}_p$-vector space, and $W = t^r\mathbb{M}^+_{\rm dR}/t^r\mathbb{N}^+_{\rm dR}$ is a $\mathbb{B}^+_{\rm dR}$-module killed by t^r.

Set $h_k(-) := \mathrm{Hom}_{\rm VS}(-, \mathbb{B}^+_{\rm dR}/t^k)$. Since (we skipped the subscripts $F^{\rm nr}$)

$$M \otimes \tfrac{t^r\mathbb{B}^+_{\rm dR}}{t^k\mathbb{B}^+_{\rm dR}} \subset \tfrac{t^r\mathbb{N}^+_{\rm dR}}{M \otimes t^k\mathbb{B}^+_{\rm dR}} \subset \tfrac{t^r\mathbb{M}^+_{\rm dR}}{M \otimes t^k\mathbb{B}^+_{\rm dR}} \subset M \otimes \tfrac{\mathbb{B}^+_{\rm dR}}{t^k\mathbb{B}^+_{\rm dR}}$$

and $V_1 \subset \frac{t^r\mathbb{N}^+_{\mathrm{dR}}}{M\otimes t^k\mathbb{B}^+_{\mathrm{dR}}}$, $V \subset \frac{t^r\mathbb{M}^+_{\mathrm{dR}}}{M\otimes t^k\mathbb{B}^+_{\mathrm{dR}}}$, we have a commutative diagram

$$\begin{tikzcd} M^*\otimes(t^r\mathbf{B}^+_{\mathrm{dR}}/t^k) \arrow[r] & h_k(\frac{t^r\mathbb{M}^+_{\mathrm{dR}}}{M\otimes t^k\mathbb{B}^+_{\mathrm{dR}}}) \arrow[r] \arrow[d] & h_k(\frac{t^r\mathbb{N}^+_{\mathrm{dR}}}{M\otimes t^k\mathbb{B}^+_{\mathrm{dR}}}) \arrow[r] \arrow[d] & M^*\otimes(t^r\mathbf{B}^+_{\mathrm{dR}}/t^k)\ , \\ & h_k(V) \arrow[r] & h_k(V_1) & \end{tikzcd}$$

where the composed map $M^*\otimes(\mathbf{B}^+_{\mathrm{dR}}/t^k)\to M^*\otimes(t^r\mathbf{B}^+_{\mathrm{dR}}/t^k)$ is multiplication by t^r. Now, since V_1 is finite dimensional over $\mathbf{Q}_p$, and $\mathbb{N}^+_{\mathrm{dR}} = V_1\otimes t^{-r}\mathbb{B}^+_{\mathrm{dR}}$, we have, using Proposition 3.16,

$$h_k(\tfrac{t^r\mathbb{N}^+_{\mathrm{dR}}}{M\otimes t^k\mathbb{B}^+_{\mathrm{dR}}}) = \mathrm{Hom}_{\mathbb{B}^+_{\mathrm{dR}}}(\tfrac{t^r\mathbb{N}^+_{\mathrm{dR}}}{M\otimes t^k\mathbb{B}^+_{\mathrm{dR}}}, \mathbb{B}^+_{\mathrm{dR}}/t^k) = V_1^*\otimes(\mathbf{B}^+_{\mathrm{dR}}/t^k) = h_k(V_1).$$

It follows that:

• $\mathrm{Ker}\big(M^*\otimes(\mathbf{B}^+_{\mathrm{dR}}/t^k)\to h_k(V)\big)$ is a subobject of $\mathrm{Ker}(M^*\otimes(\mathbf{B}^+_{\mathrm{dR}}/t^k)\xrightarrow{t^r} M^*\otimes(t^r\mathbf{B}^+_{\mathrm{dR}}/t^k))$, hence is killed by t^r.

• $\mathrm{Coker}(h_k(V)\to h_k(V_1))$ is a subquotient of $\mathrm{Coker}(M^*\otimes(\mathbf{B}^+_{\mathrm{dR}}/t^k)\xrightarrow{t^r} M^*\otimes(t^r\mathbf{B}^+_{\mathrm{dR}}/t^k))$, and hence is killed by t^r. The cokernel of $M^*\otimes(\mathbf{B}^+_{\mathrm{dR}}/t^k)\to h_k(V_1)$ is also killed by t^r and, since the kernel of $h_k(V)\to h_k(V_1)$ is $h_k(W)$ which is killed by t^r, it follows that $\mathrm{Coker}(M^*\otimes(\mathbf{B}^+_{\mathrm{dR}}/t^k)\to h_k(V))$ is killed by t^{2r}.

This concludes the proof. □

5. Comparison theorems: examples and a conjecture

In this Chapter we will formulate a conjecture: the existence of the fundamental diagram for smooth dagger varieties over C. Before doing that though we will first look at examples of comparison theorems and fundamental diagrams.

5.1. **Cliffs Notes.** Here we make a small digression with a quick review of relevant results from [14] and [15].

Proposition 5.1. (Colmez-Nizioł, [14, Th. 1.1], [15, Th. 1.3])

(1) **Analytic varieties**: *To any smooth dagger or rigid analytic variety X over C there are naturally associated:*
   (a) *A pro-étale cohomology* $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))$, $r\in\mathbf{Z}$.
   (b) *A syntomic cohomology* $\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))$, $r\in\mathbf{N}$, *with a natural period morphism*

$$\alpha_r : \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)),$$

   which is a strict quasi-isomorphism after truncation $\tau_{\leq r}$. This morphism can be lifted to the derived category of Vector Spaces.
   (c) *A Hyodo-Kato cohomology*[12] $\mathrm{R}\Gamma_{\mathrm{HK}}(X)$. *This is a dg F^{nr}-algebra equipped with a Frobenius φ and a monodromy operator N. We have natural Hyodo-Kato strict quasi-isomorphisms*

$$\iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^R_{F^{\mathrm{nr}}}C \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X),\quad \iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^R_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}).$$

   (d) *A distinguished triangle*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\longrightarrow[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r$$

   that can be lifted to the derived category of Vector Spaces.
   (e) (Local-global compatibility) *In the case X has a semistable weak formal model the above constructions are compatible with their analogs defined using the model.*

[12] We take here the Hyodo-Kato cohomology defined in [15].

(2) **Compatibility**: *Let X be a smooth dagger variety over C and let $\widehat{X}$ denote its completion. Then there exist natural compatible morphisms* [14, Sec. 3.2.4]

$$\begin{aligned}\iota_{\text{proét}}:\quad & \mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(r))\to\mathrm{R}\Gamma_{\text{proét}}(\widehat{X},\mathbf{Q}_p(r)),\quad r\in\mathbf{Z},\\ \iota:\quad & \mathrm{R}\Gamma_{\text{syn}}(X,\mathbf{Q}_p(r))\to\mathrm{R}\Gamma_{\text{syn}}(\widehat{X},\mathbf{Q}_p(r)),\quad r\in\mathbf{Z},\\ & \mathrm{R}\Gamma_{\text{dR}}(X)\to\mathrm{R}\Gamma_{\text{dR}}(\widehat{X}),\quad \mathrm{R}\Gamma_{\text{dR}}(X/\mathbf{B}^+_{\text{dR}})\to\mathrm{R}\Gamma_{\text{dR}}(\widehat{X}/\mathbf{B}^+_{\text{dR}}),\\ & \mathrm{R}\Gamma_{\text{HK}}(X)\to\mathrm{R}\Gamma_{\text{HK}}(\widehat{X}).\end{aligned}$$

They are strict quasi-isomorphisms if X is partially proper.

5.2. **Proper rigid analytic varieties.** We start with smooth and proper varieties.

5.2.1. *Algebraic varieties.* Let X_K be an *algebraic* variety over K and set $X=X_{K,\overline{K}}$. Recall the comparison theorem (recall that all the cohomology groups involved have finite dimension):

Theorem 5.2. *Let $r\geq 0$. There exists a natural $\mathbf{B}_{\text{st}}$-linear Galois equivariant period isomorphism*[13]

$$\alpha_{\text{st}}:\quad H^r_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\text{st}}\simeq H^r_{\text{HK}}(X)\otimes_{F^{\text{nr}}}\mathbf{B}_{\text{st}},$$

that preserves the Frobenius and the monodromy operators, and induces a filtered isomorphism

$$\alpha_{\text{dR}}:\quad H^r_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\text{dR}}\simeq H^r_{\text{dR}}(X_K)\otimes_K\mathbf{B}_{\text{dR}}.$$

In particular, we have the natural isomorphism

$$H^r_{\text{ét}}(X,\mathbf{Q}_p)\simeq(H^r_{\text{HK}}(X)\otimes_{F^{\text{nr}}}\mathbf{B}_{\text{st}})^{\varphi=1,N=0}\cap F^0(H^r_{\text{dR}}(X_K)\otimes_K\mathbf{B}_{\text{dR}}),\quad\text{as a }\mathscr{G}_K\text{-module},$$

as well as the natural isomorphisms

$$\begin{aligned}&\mathrm{Hom}^{\text{sm}}_{\mathscr{G}_K}(H^r_{\text{ét}}(X,\mathbf{Q}_p),\mathbf{B}_{\text{st}})\simeq H^r_{\text{HK}}(X)^*,\quad\text{as a }(\varphi,N,\mathscr{G}_K)\text{-module},\\ &\mathrm{Hom}_{\mathscr{G}_K}(H^r_{\text{ét}}(X,\mathbf{Q}_p),\mathbf{B}_{\text{dR}})\simeq H^r_{\text{dR}}(X_K)^*,\quad\text{as a filtered }K\text{-module}.\end{aligned}\tag{5.3}$$

5.2.2. *Proper rigid analytic varieties over K.* Let X_K be a proper smooth rigid analytic variety over K. Let $X=X_{K,C}$. The following result generalizes [13, Cor. 1.10], where semistable reduction was assumed. We note that all the cohomology groups involved have finite dimension: for étale cohomology this is the result of Scholze [38, Th. 1.1]; for Hyodo-Kato cohomology this follows from the Hyoodo-Kato isomorphism and finitness of de Rham cohomology.

Theorem 5.4. *Let $r\geq 0$. There exists a natural $\mathbf{B}_{\text{st}}$-linear and $\mathscr{G}_K$-equivariant period isomorphism*

$$\alpha_{\text{st}}:\quad H^r_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\text{st}}\simeq H^r_{\text{HK}}(X)\otimes_{F^{\text{nr}}}\mathbf{B}_{\text{st}},$$

that preserves the Frobenius and the monodromy operators, and induces a filtered isomorphism

$$\alpha_{\text{dR}}:\quad H^r_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\text{dR}}\simeq H^r_{\text{dR},\overline{K}}(X)\otimes_{\overline{K}}\mathbf{B}_{\text{dR}}.$$

In particular, have the following natural isomorphisms

$$\begin{aligned}&\mathrm{Hom}^{\text{sm}}_{\mathscr{G}_K}(H^r_{\text{ét}}(X,\mathbf{Q}_p),\mathbf{B}_{\text{st}})\simeq H^r_{\text{HK}}(X)^*,\quad\text{as a }(\varphi,N,\mathscr{G}_K)\text{-module},\\ &\mathrm{Hom}_{\mathscr{G}_K}(H^r_{\text{ét}}(X,\mathbf{Q}_p),\mathbf{B}_{\text{dR}})\simeq H^r_{\text{dR}}(X)^*,\quad\text{as a filtered }K\text{-module}.\end{aligned}\tag{5.5}$$

Proof. Take $s>r$. To define the period maps consider the following composition

$$\alpha_{\text{st}}(s):\quad H^r_{\text{ét}}(X,\mathbf{Q}_p(s))\xrightarrow[\sim]{\alpha_s^{-1}}H^r_{\text{syn}}(X,s)\to(H^r_{\text{HK}}(X)\otimes_{F^{\text{nr}}}\mathbf{B}^+_{\text{st}})^{N=0,\varphi=p^s}\xrightarrow{p^{-s}}H^r_{\text{HK}}(X)\otimes_{F^{\text{nr}}}\mathbf{B}_{\text{st}},$$

where α_s is the period map from [15, Cor. 6.9]. Set

$$\alpha_{\text{st}}:=t^{-s}\alpha_{\text{st}}(s)\varepsilon^s,\quad \alpha_{\text{dR}}:=\iota^{-1}_{\text{BK}}(\iota_{\text{HK}}\otimes\iota)\alpha_{\text{st}},$$

[13] Note that there is no assumption on the variety. This formulation of the comparison theorem is due to Beilinson [2].

where ε is the generator of $\mathbf{Z}_p(1)$ corresponding to t.

We follow the proof of [13, Cor. 1.10], which uses BC's. We will sketch the arguments and refer the reader to [13] for details. We start with stating the isomorphism

$$(H^r_{\mathrm{dR}}(X)\otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^s \xrightarrow{\sim} (H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^s),$$

which follows from the degeneration of the Hodge-de Rham spectral sequence[14] (proved by Scholze as a corollary of the de Rham comparison theorem [38]). Using this isomorphism, we obtain the long exact sequence (obtained from the definition of overconvergent syntomic cohomology)

$$(5.6)\qquad \begin{aligned}\to (H^{r-1}_{\mathrm{dR}}(X)\otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^s \to H^r_{\mathrm{syn}}(X,\mathbf{Q}_p(s)) \to (H^r_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^s}\\ \to (H^r_{\mathrm{dR},\overline{K}}(X)\otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^s \to\end{aligned}$$

We recall that the groups $H^r_{\mathrm{HK}}(X), H^r_{\mathrm{dR}}(X)$ have finite dimension over F^{nr} and K, respectively. The above long exact sequence yields short exact sequences

$$0\to H^r_{\mathrm{syn}}(X,\mathbf{Q}_p(s)) \to (H^r_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^s} \to (H^r_{\mathrm{dR},\overline{K}}(X)\otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^s \to 0$$

To prove this, we observe that the map $f_r : (H^r_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^s} \to (H^r_{\mathrm{dR},\overline{K}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^s$ is the evaluation on C of a map of BC's. But, the syntomic cohomology group $H^r_{\mathrm{syn}}(X_C,\mathbf{Q}_p(s))$, $r\le s$, is a finite dimensional $\mathbf{Q}_p$-vector space since we have the quasi-isomorphism [15, 6.10] with étale cohomology. This implies that the cokernel of f_i, viewed as a map of BC's, is of Dimension $(0,d_i)$. On the other hand, the Space $(H^r_{\mathrm{dR},\overline{K}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^s$ is a successive extension of C-vector spaces. The theory of BC's implies now that the map $(H^r_{\mathrm{dR},\overline{K}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^s \to \operatorname{Coker} f_r$ is zero, hence $\operatorname{Coker} f_r = 0$, as wanted.

We have the Hyodo-Kato isomorphism (see [15, Cor. 4.32])

$$\iota_{\mathrm{HK}} : H^r_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}} C \simeq H^r_{\mathrm{dR}}(X).$$

Taking $\mathscr{G}_K$-smooth vectors of both sides (note that X is quasi-compact) we get the Hyodo-Kato isomorphism

$$H^r_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}} \overline{K} \simeq H^r_{\mathrm{dR}}(X_K)\otimes_K \overline{K}.$$

Hence the pair $(H^r_{\mathrm{HK}}(X), H^r_{\mathrm{dR}}(X_K))$ is a $(\varphi,N,\mathscr{G}_K)$-filtered module (in the sense of Fontaine). The above short exact sequence and a "weight" argument shows that $\mathbf{V}_{\mathrm{st}}(H^r_{\mathrm{HK}}(X), H^i_{\mathrm{dR}}(X_K)) \simeq H^r_{\text{ét}}(X,\mathbf{Q}_p)$. Here $\mathbf{V}_{\mathrm{st}}(-)$ is Fontaine's functor from filtered Frobenius modules to Galois representations. The short exact sequence and C-dimension count give also that $t_N(H^r_{\mathrm{HK}}(X)) = t_H(H^r_{\mathrm{dR}}(X_K))$, where $t_N(D)=v_p(\det\varphi)$ and $t_H(D)=\sum_{i\ge 0} i\dim_K(F^iD/F^{i+1}D)$. The theory of BC's now implies that the pair $(H^r_{\mathrm{HK}}(X), H^r_{\mathrm{dR}}(X_K))$ is weakly admissible from which our theorem follows. □

Remark 5.7. The isomorphisms (5.3) and (5.5) are strict if we put the weak topology on the Homs.

5.2.3. *Proper rigid analytic varieties over C.* Let X be a smooth rigid analytic variety over C. Its p-adic étale cohomology is finite rank by [38, Th. 1.1]. Its Hyodo-Kato cohomology is finite rank by the Hyodo-Kato isomorphism and finitness of de Rham cohomology. Its $\mathbf{B}^+_{\mathrm{dR}}$-cohomology is free, finite rank over $\mathbf{B}^+_{\mathrm{dR}}$, by the comparison isomorphism with Hyodo-Kato cohomology.

Theorem 5.8. *Let $r\ge 0$. There exists a natural $\mathbf{B}_{\mathrm{st}}$-linear period isomorphism*

$$\alpha_{\mathrm{st}}:\quad H^r_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{st}} \simeq H^r_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}},$$

that preserves the Frobenius and the monodromy operators, and induces a filtered isomorphism

$$(5.9)\qquad \alpha_{\mathrm{dR}}:\quad H^r_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}} \simeq H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\otimes_{\mathbf{B}^+_{\mathrm{dR}}}\mathbf{B}_{\mathrm{dR}}.$$

[14] Alternatively, one can use an argument analogous to the one we use in the proof of Theorem 5.8 below.

Here, the filtration on $H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ *is defined by*

$$F^i H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) := \mathrm{Im}(H^r(F^i\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})) \to H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})).$$

Remark 5.10. A de Rham comparison isomorphism as in (5.9) was constructed earlier in [7, Th. 13.1]. It did not treat filtrations.

Proof. Take $s > r$. The period maps are define as in Theorem 5.4 but by dropping the map ι_{BK}. We note that $H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^s)$ is a BC, a successive extension of C-vector spaces of finite rank. This follows from the fact that the distinguished triangle [15, 3.28] yields the distinguished triangle

$$\bigoplus_{i\leq s}\mathrm{R}\Gamma(X,\Omega^i)(s-i)[-i] \to \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^{s+1} \to \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^s$$

and $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^1 \simeq \mathrm{R}\Gamma(X,\mathscr{O}_X)$ by [15, 3.27]. Having that, the same arguments as in the case of proper varieties over K yield short exact sequence

$$0 \to H^r_{\text{ét}}(X,\mathbf{Q}_p(s)) \to (H^r_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^s} \to H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^s) \to 0 \tag{5.11}$$

Moreover, these arguments show that the canonical map

$$H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^s \to H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^s)$$

is an isomorphism. This can be proved in the following way. By (5.11), we have a surjection

$$(H^r_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^s} \twoheadrightarrow H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^s).$$

Since the above map factors through the natural map

$$H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^s \to H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^s) \tag{5.12}$$

that latter is surjective as well. But it is also injective. Indeed, we have the distinguished triangle

$$F^s\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^s$$

It yields the long exact sequence of cohomology groups

$$\to H^r(F^s\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})) \xrightarrow{f_r} H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \to H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^s) \to$$

Since $F^sH^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) = \mathrm{Im}\, f_r$, the map in (5.12) is injective. We are done.

The two isomorphisms in our theorem follow now from Remark 4.14, using the last part of Remark 4.16 (take $M = H^r_{\mathrm{HK}}(X), M^+ = F^0(H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\otimes_{\mathbf{B}^+_{\mathrm{dR}}}\mathbf{B}_{\mathrm{dR}})$). □

Remark 5.13. (i) One can restate the theorem as follows ($\mathscr{E}(-,-)$ is the associated vector bundle on the Fargues-Fontaine curve X_{FF}):

$$H^r_{\text{ét}}(X,\mathbf{Q}_p) \simeq H^0(X_{\mathrm{FF}},\mathscr{E}(H^r_{\mathrm{HK}}(X),H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))).$$

(ii) From Theorem 5.8, we get natural isomorphisms

$$\mathrm{Hom}_{\mathrm{VS}}(H^r_{\text{ét}}(X,\mathbf{Q}_p),\mathbb{B}_{\mathrm{st}}) \simeq \mathrm{Hom}_{F^{\mathrm{nr}}}(H^r_{\mathrm{HK}}(X),\mathbf{B}_{\mathrm{st}}), \quad \text{as } \mathbf{B}_{\mathrm{st}}\text{-modules},$$
$$\mathrm{Hom}_{\mathrm{VS}}(H^r_{\text{ét}}(X,\mathbf{Q}_p),\mathbb{B}_{\mathrm{dR}}) \simeq \mathrm{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}),\mathbf{B}_{\mathrm{dR}}), \quad \text{as } \mathbf{B}_{\mathrm{dR}}\text{-modules}.$$

5.3. **Dagger Stein varieties and dagger affinoids.** Having the comparison result from [15, Cor. 6.9], we can now deduce a (simplified) fundamental diagram for pro-étale cohomology from the one for overconvergent syntomic cohomology.

Theorem 5.14. (Simplified fundamental diagram) *Let X be a smooth dagger Stein variety or a smooth dagger affinoid over C. Let $r \geq 0$. There is a natural map of strictly exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{r-1}(X)/\operatorname{Ker} d \arrow[r] \arrow[d, equal] & H^r_{\text{proét}}(X,\mathbf{Q}_p(r)) \arrow[r] \arrow[d, "\tilde{\beta}"] & (H^r_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}\otimes\theta"] & 0 \\
0 \arrow[r] & \Omega^{r-1}(X)/\operatorname{Ker} d \arrow[r, "d"] & \Omega^r(X)^{d=0} \arrow[r] & H^r_{\mathrm{dR}}(X) \arrow[r] & 0
\end{tikzcd}$$

Moreover, $H^r_{\text{proét}}(X,\mathbf{Q}_p(r))$ *is Fréchet or LB, respectively, the vertical maps are strict and have closed images, and* $\operatorname{Ker}\tilde{\beta}\simeq (H^r_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^{r-1}}$.

Proof. We define $\tilde{\beta}:=p^{-r}\beta\alpha_r^{-1}$, using [15, Cor. 6.9] and [15, Prop. 5.13]; the twist by p^{-r} being added to make this map compatible with symbols. The theorem follows immediately from [15, Cor. 6.9] and [15, Prop. 5.13]. $\square$

5.4. **The fundamental diagram.** We will now introduce the fundamental diagram, look at some examples, where it appears, and, finally, state a conjecture concerning it.

5.4.1. *Examples.* We start with examples.

• *Proper varieties.*

Corollary 5.15. *Let X be a smooth proper variety over C. We have the bicartesian diagram*

$$\begin{tikzcd} H^r_{\text{ét}}(X,\mathbf{Q}_p(r)) \arrow[r, "\alpha_{\mathrm{st}}(r)"] \arrow[d, "\alpha_{\mathrm{dR}}(r)"] & (H^r_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{\varphi=p^r,N=0} \arrow[d, "\iota_{\mathrm{HK}}\otimes\iota"] \\ F^rH^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r, "\mathrm{can}"] & H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \end{tikzcd} \tag{5.16}$$

Recall that this means that this is a pushout and pullback diagram, or, that the sequence

$$0\to H^r_{\text{ét}}(X,\mathbf{Q}_p(r))\xrightarrow{\alpha_{\mathrm{st}}(r)\oplus\alpha_{\mathrm{dR}}(r)}(H^r_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\oplus F^rH^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota+\mathrm{can}}H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\to 0 \tag{5.17}$$

is exact.

Proof. Follows immediately from the short exact sequence (5.11). $\square$

Remark 5.18. (i) The passage the other way, from diagram (5.16) to Theorem 5.8 is also possible: the exact sequence (5.17) yields the exact sequence (5.11) and we can finish as in the proof of Theorem 5.8.

(ii) The natural map $H^r(F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))\to F^rH^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ is an isomorphism.

• *Dagger Stein varieties and dagger affinoids.*

Corollary 5.19. *Let X be a smooth dagger Stein variety or a smooth dagger affinoid over C. Let $r\geq 0$. We have the bicartesian diagram (recall that all cohomologies are classical)*

$$\begin{tikzcd} H^r_{\text{ét}}(X,\mathbf{Q}_p(r)) \arrow[r, "\alpha_{\mathrm{st}}(r)"] \arrow[d, "\alpha_{\mathrm{dR}}(r)"] & (H^r_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \arrow[d, "\iota_{\mathrm{HK}}\otimes\iota"] \\ H^rF^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r] & H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}). \end{tikzcd} \tag{5.20}$$

Proof. Consider the following diagram of maps of distinguished triangles

$$\begin{tikzcd} \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \arrow[r, "f_1"] \arrow[d, dotted, "\beta"] & {[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}} \arrow[r, "\iota_{\mathrm{HK}}\otimes\iota"] \arrow[d, "\iota_{\mathrm{HK}}\otimes\iota"] & \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r \arrow[d, equal] \\ F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r, "f_2"] & \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r] & \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r \end{tikzcd}$$

Since $H^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r)=0$ for $i\geq r$, the maps f_1,f_2 are surjective in degrees $\geq r$. It suffices to show that $\operatorname{Ker} f_1\xrightarrow{\sim}\operatorname{Ker} f_2$ in degree r. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} f_1 \arrow[r] \arrow[d] & H^r_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \arrow[r, "f_1"] \arrow[d] & (H^r_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \arrow[d, "\iota_{\mathrm{HK}}\otimes\iota"] \\
0 \arrow[r] & \operatorname{Ker} f_2 \arrow[r] \arrow[d, "\vartheta", "\wr"'] & H^rF^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r, "f_2"] \arrow[d, "\vartheta"] & H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[d, "\vartheta"] \\
0 \arrow[r] & \operatorname{Ker} f_{2,C} \arrow[r] & H^rF^r\mathrm{R}\Gamma_{\mathrm{dR}}(X) \arrow[r, "f_{2,C}"] & H^r_{\mathrm{dR}}(X)
\end{tikzcd}$$

We claim that the map $\operatorname{Ker} f_2\to\operatorname{Ker} f_{2,C}$ is an isomorphism. Indeed, we compute

$$\begin{aligned}
&\operatorname{Ker} f_2\xleftarrow{\sim}\operatorname{Coker}(H^{r-1}_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}}\to H^{r-1}((\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r))\xrightarrow{\sim}\Omega^{r-1}(X_K)\operatorname{Ker} d\widehat{\otimes}_KC,\\
&\operatorname{Ker} f_{2,C}\xleftarrow{\sim}\Omega^{r-1}(X_K)\operatorname{Ker} d\widehat{\otimes}_KC.
\end{aligned}$$

$\square$

5.4.2. *Conjecture.* We will formulate now a conjecture describing pro-étale cohomology in terms of the de Rham complex.

Conjecture 5.21. (C_{st}-conjecture) *Let X be a smooth dagger variety over C. Let $r\geq i$. The commutative diagram*

$$\begin{tikzcd}
\widetilde{H}^i_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)) \arrow[r] \arrow[d] & (\widetilde{H}^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \arrow[d] \\
\widetilde{H}^iF^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r] & \widetilde{H}^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})
\end{tikzcd}$$

is bicartesian. That is, the following sequence

(5.22)

$$0\to\widetilde{H}^i_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))\to(\widetilde{H}^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\oplus\widetilde{H}^iF^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\to\widetilde{H}^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\to 0$$

is exact.

Remark 5.23. (i) This conjecture is known so far in the following cases:

- X is proper (see Examples 5.4.1). In this case, the two horizontal arrows are injective.
- X is Stein or affinoid (see Examples 5.4.1). In this case, the two horizontal arrows are surjective and their kernels are $\Omega^{r-1}(X)/\operatorname{Ker} d$.

(ii) Let X be a smooth dagger variety over C. If $r\geq i$, set

$$\begin{aligned}
\widetilde{H}^{r,i}&:=\widetilde{H}^i_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)) & \widetilde{X}^{r,i}&:=(\widetilde{H}^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\\
\widetilde{F}^{r,i}&:=\widetilde{H}^iF^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) & \widetilde{B}^i&:=\widetilde{H}^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})
\end{aligned}$$

and denote by $H^{r,i}$, $X^{r,i}$, etc. the images of $\widetilde{H}^{r,i}$, $\widetilde{X}^{r,i}$, etc. in $C_{\mathbf{Q}_p}$. Note that the $\widetilde{X}^{r,i}$'s and $\widetilde{B}^i$'s are classical, i.e., $\widetilde{X}^{r,i}\simeq X^{r,i}$ and $\widetilde{B}^i\simeq B^i$.

We have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] \\
\widetilde{H}^{r,i} \arrow[r] \arrow[d, "\wr"] & \widetilde{F}^{r,i} \oplus \widetilde{X}^{r,i} \arrow[rr] \arrow[d, "t", "\wr"'] & & \widetilde{B}^i \arrow[d, "t"] \\
\widetilde{H}^{r+1,i} \arrow[r] & \widetilde{F}^{r+1,i} \oplus \widetilde{X}^{r+1,i} \arrow[rr] \arrow[d] & & \widetilde{B}^i \arrow[d] \\
 & H^i_{\rm HK}(X)\widehat{\otimes}_{F^{\rm nr}}C \arrow[rr, "\sim", "\iota_{\rm HK}"'] \arrow[d] & & H^i_{\rm dR}(X) \arrow[d] \\
 & 0 & & 0
\end{tikzcd}$$

The vertical maps are multiplications by t (on pro-étale cohomology, this corresponds to the Tate twist); for the isomorphism $X^{r+1,i}/tX^{r,i} \simeq H^i_{\rm HK}(X)\widehat{\otimes}_{F^{\rm nr}}C$, see Remark 4.16; the bottom isomorphism is the Hyodo-Kato map. It follows that, for fixed i, the conjecture for r is equivalent to the conjecture for $r+1$. Hence it is enough to prove it for one $r \geq i$ (for example $r = i$).

(iii) Since the $\widetilde{B}^i$'s are actually classical, and we have a long exact sequence

$$\cdots \to \widetilde{X}^{r,i-1} \oplus \widetilde{F}^{r,i-1} \to \widetilde{B}^{i-1} \to \widetilde{H}^{r,i} \to \widetilde{X}^{r,i} \oplus \widetilde{F}^{r,i} \to \widetilde{B}^i \to \cdots$$

it is enough, thanks to an induction on i, to prove surjectivity of $X^{r,i} \oplus F^{r,i} \to B^i$: this will show that $\widetilde{X}^{r,i} \oplus \widetilde{F}^{r,i} \to \widetilde{B}^i$ is surjective (since $\widetilde{B}^i \simeq B^i$) and that the long exact sequence splits into short exact sequences, as wanted.

(iv) We will prove this conjecture for quasi-compact varieties in Theorem 6.14 below.

6. De Rham-to-pro-étale comparison theorem for small varieties

We will now prove Conjecture 5.21 for small varieties. In this chapter, until Theorem 6.14, all cohomologies, unless otherwise stated, are algebraic, i.e., we ignore topological issues (see (iii) of Remark 5.23 of why this is a reasonable thing to do).

6.1. **Conjectures.** Let X be a smooth dagger variety over C. We will first state and discuss four conjectures, a priori unrelated, on the cohomology of X:

- Conjecture 5.21 (already stated above) describes the p-adic pro-étale cohomology of X in terms of the de Rham complex.

The remaining conjectures assume X to be quasi-compact:

- Conjecture 6.1 gives a restriction on the Hodge filtration on the de Rham cohomology in terms of the slopes of Frobenius on the Hyodo-Kato cohomology.
- Conjecture 6.4 says that, even if huge, the pro-étale cohomology groups $\mathbb{H}^r_{\rm proét}(X, \mathbf{Q}_p)$ have nevertheless $\mathbf{Q}_p$-dimension equal to $\dim_C H^r_{\rm dR}(X)$.
- Conjecture 6.3 says that $\mathbb{H}^r_{\rm proét}(X, \mathbf{Q}_p)$ is of curvature ≤ 0.

Next, we proceed to proving these conjectures in the case X is quasi-compact. The proof uses:

— the period quasi-isomorphism between pro-étale cohomology and syntomic cohomology from [15, 6.10],

— the canonical distinguished triangle involving syntomic cohomology from [15, 5.12],

— delicate properties of BC's (scattered through [8, 9, 32, 35] and recalled in Section 3.1 for the convenience of the reader).

These ingredients allow us to prove that the four conjectures above are, in fact, equivalent (see Proposition 6.11 as well as Lemma 6.9 and Lemma 6.10 for precise statements). We show then Conjecture 6.4 by induction on the number n of affinoids needed to cover X; for $n = 1$, Conjecture 5.21 is exactly Corollary 5.19.

6.1.1. *Acyclicity of de Rham cohomology.* For a smooth dagger variety X over C, we set

(1) $F^iH^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}):=\mathrm{Im}(H^r(F^i\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))\to H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))$;
(2) $H^r_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}):=H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{dR}}}\mathbf{B}_{\mathrm{dR}}$ and equip it with the induced filtration.

Conjecture 6.1. *Let X be a smooth quasi-compact dagger variety over C. For all r, the map*

$$(H^r_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\to H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r$$

is surjective, i.e., the pair $(H^r_{\mathrm{HK}}(X),F^0H^r_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}))$ is acyclic.

Remark 6.2. If X is proper, then the pair $(H^r_{\mathrm{HK}}(X),F^0H^r_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}))$ is weakly admissible hence X verifies the conjecture. If X is Stein, then $H^r_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r=0$ and X clearly satisfies the conjecture.

6.1.2. *Curvature and height of pro-étale cohomology.* Let X be a smooth quasi-compact dagger variety over C.

Conjecture 6.3. *Then, for all $r\geq 0$, $\mathbb{H}^r_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p)$ has curvature ≤ 0.*

Conjecture 6.4. *For all r,*

$$\mathrm{ht}(\mathbb{H}^r_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p))=\dim_C H^r_{\mathrm{dR}}(X).$$

Remark 6.5. (i) If X is proper, Conjecture 6.4 is a theorem: we have the exact sequence of BC's (see Section 5.2.3)

$$0\to H^r_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))\to(H^r_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\to H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r)\to 0$$

We know that $H^r_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p)$ has finite dimension over $\mathbf{Q}_p$; hence its height is equal to its dimension over $\mathbf{Q}_p$. We also know that the slopes of Frobenius on $H^r_{\mathrm{HK}}(X)$ are $\leq r$, which implies that $\mathrm{ht}((H^r_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r})=\dim_{F^{\mathrm{nr}}}H^r_{\mathrm{HK}}(X)$. Now, the above short exact sequence implies that $\mathrm{ht}(\mathbb{H}^r_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p))=\dim_{F^{\mathrm{nr}}}H^r_{\mathrm{HK}}(X)$ and, since by the Hyodo-Kato isomorphism $\dim_{F^{\mathrm{nr}}}H^r_{\mathrm{HK}}(X)=\dim_C H^r_{\mathrm{dR}}(X)$, we have

$$\dim_{\mathbf{Q}_p}(H^r_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p))=\dim_C H^r_{\mathrm{dR}}(X),$$

as wanted.

(ii) Does there exist non proper dagger varieties X such that $H^r_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p)$ s finite dimensional over $\mathbf{Q}_p$ for all r? Already for $r=1$, one needs $\mathscr{O}(X)=C$.

6.2. **Equivalence of conjectures.** In this section we assume X to be a smooth dagger variety over C such that $H^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ is (free) of finite rank over $\mathbf{B}^+_{\mathrm{dR}}$ for all i (for example, X could be quasi-compact, or the interior of a quasi-compact, or the analytification of an algebraic variety, see Corollary 6.18). Conjectures 5.21, 6.1, 6.3, and 6.4 make sense in this, slightly more general, set-up.

6.2.1. *The key diagram.* Fix r and, for $i\leq r$, set

$$\begin{aligned}
&H^{r,i}:=H^i_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)), && X^{r,i}:=(H^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r},\\
&F^{r,i}:=H^i(F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})), && \mathrm{DR}^{r,i}:=H^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r),\\
&B^i:=H^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) && \mathrm{Fil}^{r,i}:=\mathrm{Im}(F^{r,i}\to B^i)
\end{aligned}$$

We also denote by

$$\mathbb{H}^{r,i},\quad \mathbb{X}^{r,i},\quad \mathbb{F}^{r,i},\quad \mathbb{DR}^{r,i},\quad \mathbb{B}^i,\quad \mathbb{Fil}^{r,i}$$

the associated VS's. Remark 3.35 makes it possible to navigate freely between these VS's and their C-points.

The isomorphism between pro-étale and syntomic cohomologies (see [15, 6.19]) yields a commutative diagram with exact rows:

$$\begin{tikzcd}
H^{r,i-1} \arrow[r] \arrow[d] & X^{r,i-1} \arrow[r] \arrow[d] & \mathrm{DR}^{r,i-1} \arrow[r] \arrow[d, equal] & H^{r,i} \arrow[r] \arrow[d] & X^{r,i} \arrow[r] \arrow[d] & \mathrm{DR}^{r,i} \arrow[d, equal] \\
F^{r,i-1} \arrow[r] \arrow[d] & B^{i-1} \arrow[r] \arrow[d] & \mathrm{DR}^{r,i-1} \arrow[r] \arrow[d, equal] & F^{r,i} \arrow[r] \arrow[d] & B^{i} \arrow[r] \arrow[d] & \mathrm{DR}^{r,i} \arrow[d, equal] \\
F^{r,i-1}/t^r \arrow[r] & B^{i-1}/t^r\mathrm{Fil}^{r,i-1} \arrow[r] & \mathrm{DR}^{r,i-1} \arrow[r] & F^{r,i}/t^r \arrow[r] & B^{i}/t^r\mathrm{Fil}^{r,i} \arrow[r] & \mathrm{DR}^{r,i}
\end{tikzcd}$$

The bottom sequence is exact because $t^rF^{r,i} \to t^r\mathrm{Fil}^{r,i}$ is an isomorphism (multiplying by t^r kills the big $\mathbf{B}_{\mathrm{dR}}^+$-torsion in $F^{r,i}$, and B^i is a free finite rank $\mathbf{B}_{\mathrm{dR}}^+$-module thanks to the Hyodo-Kato isomorphism), hence we have an isomorphism:

$$\mathrm{Ker}(F^{r,i} \to F^{r,i}/t^r) \xrightarrow{\sim} \mathrm{Ker}(B^i \to B^i/t^r\mathrm{Fil}^{r,i}). \tag{6.6}$$

All the spaces in the diagram are C-points of VS's and those in the top and the bottom rows are C-points of qBC's (this is clear for all of them except for $H^{r,i}$ (and $H^{r,i-1}$), for which one can use Lemma 3.31 and the fact that $\mathrm{DR}^{r,i}$ is a $\mathbf{B}_r$-module and $X^{r,i}$ is equal to the C-points of a BC), and the above diagram lifts to a diagram of VS's (see [15, Sec. 7]).

Lemma 6.7. *For all $i \geq 0$, we have the following isomorphisms:*

$$\mathrm{Coker}\big(X^{r,i} \oplus F^{r,i} \to B^i\big) \simeq \mathrm{Coker}\big(X^{r,i} \to B^i/\mathrm{Fil}^{r,i}\big),$$
$$\mathrm{Ker}\big(H^{r,i} \to X^{r,i} \oplus F^{r,i}\big) \simeq \mathrm{Coker}\big(X^{r,i-1} \oplus F^{r,i-1} \to B^{i-1}\big)$$

These isomorphisms can be lifted to the category of VS's and $\mathrm{Coker}\big(X^{r,i} \to B^i/\mathrm{Fil}^{r,i}\big)$ *is the C-points of a BC.*

Proof. The first isomorphism is clear. Using the snake lemma in the following commutative diagram with exact rows we prove the second isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & X^{r,i-1}/H^{r,i-1} \arrow[r] \arrow[d] & \mathrm{DR}^{r,i-1} \arrow[r] \arrow[d, equal] & \mathrm{Ker}\big(H^{r,i} \to X^{r,i}\big) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^{i-1}/\mathrm{Fil}^{r,i} \arrow[r] & \mathrm{DR}^{r,i-1} \arrow[r] & \mathrm{Ker}\big(F^{r,i} \to B^i\big) \arrow[r] & 0
\end{tikzcd}$$

The claim about BC's follows from Proposition 3.34 because the VS's corresponding to $X^{r,i}$ and $B^i/\mathrm{Fil}^{r,i}$ are BC's. $\square$

Corollary 6.8. *If* $\mathrm{Ker}\big(\mathbb{H}^{r,i} \to \mathbb{X}^{r,i} \oplus \mathbb{F}^{r,i}\big) \neq 0$, *then it has height* < 0 *hence can not have curvature* ≤ 0.

Proof. Use Lemma 6.7 to pass to

$$\mathrm{Coker}\big(\mathbb{X}^{r,i-1} \oplus \mathbb{F}^{r,i-1} \to \mathbb{B}^{i-1}\big) \xrightarrow{\sim} \mathrm{Coker}\big(\mathbb{X}^{r,i-1} \to \mathbb{B}^{i-1}/\mathbb{Fil}^{r,i-1}\big).$$

Now use Proposition 4.18 and Remark 4.19. $\square$

6.2.2. *Left exactness.* We will study now the exactness on the left of the sequence in Conjecture 5.21.

Lemma 6.9. *The following properties are equivalent:*

(a) *The map* $H^{r,r} \to X^{r,r} \oplus F^{r,r}$ *is injective.*
(a′) *The map* $H^{r,r} \to X^{r,r} \oplus (F^{r,r}/t^rF^{r,r})$ *is injective.*
(b) $(H^{r-1}_{\mathrm{HK}}(X), F^0H^{r-1}_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}))$ *is acyclic.*
(c) $\mathrm{Ker}(H^{r,r} \to X^{r,r})$ *has curvature* 0.
(c′) $\mathrm{Ker}(H^{r,r} \to X^{r,r})$ *has height* 0.
(d) $\mathbb{H}^{r,r}$ *has curvature* ≤ 0.

Proof. We note that map in (a) can be lifted to the category $q\mathscr{BC}$ and its kernel is equal to the C-points of a qBC by Proposition 3.34.

• We clearly have (a′) ⇒ (a). For the opposite implication use the isomorphism (6.6).

• The equivalence of (a) and (b) is a reformulation of the isomorphisms in Lemma 6.7 (use Remark 4.16).

• We have (c)⇒(c′) by Remark 3.30. And, since $\mathrm{Ker}\big(\mathbb{H}^{r,r}\to\mathbb{X}^{r,r}\big)$, which is a qBC, is a quotient of $\mathbb{DR}^{r,r-1}$, which has curvature 0 (since it is a $\mathbb{B}_r$-module), (c′)⇒(c) as well (use Remark 3.30).

• We have (a′)⇒(d) since $\mathbb{H}^{r,r}$ is a submodule of $\mathbb{X}^{r,r}\oplus(\mathbb{F}^{r,r}/t^r)$, the first term of which comes from a BC of curvature ≤ 0 (since this BC is a Submodule of a $\mathbb{B}_r$-Module, use Remark 3.30) and the second term is an affine qBC.

• We have (d)⇒(c) since $\mathrm{Ker}\big(\mathbb{H}^{r,r}\to\mathbb{X}^{r,r}\big)$ is a sub-VS of $\mathbb{H}^r_{\text{proét}}$ and hence has curvature ≤ 0 (since so does $\mathbb{H}^r_{\text{proét}}$ by assumption), and it is also a quotient of $\mathbb{DR}^{r,r-1}$ and hence has curvature ≥ 0 (since so does $\mathbb{DR}^{r,r-1}$). Here we have used again Remark 3.30.

• We have (c)⇒(a′) since the inclusion of the kernels implies that $\mathrm{Ker}\big(H^{r,r}\to X^{r,r}\oplus(F^{r,r}/t^r)\big)$ corresponds to a qBC of curvature ≤ 0 (by Remark 3.30), and thus it is trivial by Corollary 6.8. □

6.2.3. *Right exactness.* We pass now to the study of the exactness on the right of the sequence in Conjecture 5.21.

Lemma 6.10. *The following properties are equivalent:*

(a) *The map $X^{r,r}\oplus F^{r,r}\to B^r$ is surjective.*

(a′) *The map $X^{r,r}\oplus(F^{r,r}/t^rF^{r,r})\to B^r/t^r\mathrm{Fil}^{r,r}$ is surjective.*

(b) $(H^r_{\mathrm{HK}}(X),F^0H^r_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}))$ *is acyclic.*

(c) $\mathrm{Coker}(\mathbb{H}^{r,r}\to\mathbb{X}^{r,r})$ *has curvature* 0.

(c′) $\mathrm{Coker}(\mathbb{H}^{r,r}\to\mathbb{X}^{r,r})$ *has height* 0.

Proof. • Clearly (a) ⇒ (a′). For the opposite implication use isomorphism 6.6.

• The equivalence of (a) and (b) is a reformulation of the first isomorphism from Lemma 6.7 (use Remark 4.16).

• We clearly have (c)⇒(c′) and the opposite implication follows from the fact that the cokernel of $\mathbb{H}^{r,r}\to\mathbb{X}^{r,r}$ is a sub-qBC of $\mathbb{DR}^{r,r}$. Since the latter has curvature 0 (because it is a $\mathbb{B}_r$-Module) (c) follows from Remark 3.30.

• Let I^r be the image of $\mathbb{X}^{r,r}$ in $\mathbb{DR}^{r,r}$, and let J^r be the image of $\mathbb{X}^{r,r}$ in $\mathbb{B}^r/\mathbb{Fil}^{r,r}$. Then $\mathrm{Coker}\big(\mathbb{H}^{r,r}\to\mathbb{X}^{r,r}\big)\simeq I^r\simeq J^r$ since $\mathbb{B}^r/\mathbb{Fil}^{r,r}\to\mathbb{DR}^{r,r}$ is injective. It follows that $\mathrm{Coker}\big(\mathbb{H}^{r,r}\to\mathbb{X}^{r,r}\big)$ has curvature 0 if and only if J^r does. And thus if and only if $J^r=\mathbb{B}^r/\mathbb{Fil}^{r,r}$ by Proposition 4.18. The equivalence of (c) and (b) follows. □

6.2.4. *Bicartesian property.* We can now put Lemma 6.9 and Lemma 6.10 together:

Proposition 6.11. *The following properties are equivalent:*

(a) *The diagram in Conjecture 5.21 is bicartesian.*

(b) $(H^i_{\mathrm{HK}}(X),F^0H^i_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}))$ *is acyclic, for $i=r-1$ and $i=r$.*

(c) *The kernel and cokernel of* $\mathrm{Coker}(\mathbb{H}^{r,r}\to\mathbb{X}^{r,r})$ *have curvature* 0.

(c′) *The kernel and cokernel of* $\mathrm{Coker}(\mathbb{H}^{r,r}\to\mathbb{X}^{r,r})$ *have height* 0.

(d) $\mathrm{ht}(\mathbb{H}^{r,r})=\dim_C H^r_{\mathrm{dR}}(X)$.

Proof. • Let us start by noting that, by Remark 5.23 (ii), in (a) we can assume $i=r$.

• Note that (a), (b), (c) and (c′) are, respectively, equivalent to the conjunction of (a), (b), (c) and (c′) from Lemma 6.9 and 6.10. The equivalence of (a), (b) (c) and (c′) follows.

• (a)⇒(d): by assumption we have the exact sequence

$$0\longrightarrow H^{r,r}\longrightarrow(F^{r,r}/t^r)\oplus X^{r,r}\longrightarrow B^r/t^r\mathrm{Fil}^{r,r}\longrightarrow 0$$

This sequence can be lifted to the category $q\mathscr{BC}$. Since $\mathbb{F}^{r,r}/t^r$ and $\mathbb{B}^r/t^r\mathbb{F}\mathrm{il}^{r,r}$ have curvature 0, and hence height zero, and

$$\mathrm{ht}(\mathbb{X}^{r,r}) = \dim_{F^{\mathrm{nr}}} H^r_{\mathrm{HK}}(X) = \dim_C H^r_{\mathrm{dR}}(X),$$

(d) follows. The first equality above uses the fact that the φ-slopes are $\leq r$.

• (d)⇒(b): Since $\mathbb{DR}^{r,r-1}$ and $\mathbb{DR}^{r,r}$ are $\mathbb{B}_r$-modules, the images I^{r-1} and I^r of $\mathbb{X}^{r,r-1}$ and $\mathbb{X}^{r,r}$ have height ≥ 0. Since

$$\mathrm{ht}(\mathbb{H}^{r,r}) = (\mathrm{ht}(\mathbb{DR}^{r,r-1}) - \mathrm{ht}(I^{r-1})) + (\mathrm{ht}(\mathbb{X}^{r,r}) - \mathrm{ht}(I^r))$$

and $\mathrm{ht}(\mathbb{DR}^{r,r-1}) = 0$, the condition $\mathrm{ht}(\mathbb{H}^{r,r}) = \mathrm{ht}(\mathbb{X}^{r,r})$ (equivalent to (d) as we have seen) implies that $\mathrm{ht}(I^{r-1}) = \mathrm{ht}(I^r) = 0$. As we have seen above in proving the equivalence of (b) and (c) in Lemma 6.10, this implies (b).

This finishes the proof of our proposition. □

Corollary 6.12. *If X is quasi-compact then Conjectures 5.21, 6.1, 6.3, and 6.4 are equivalent.*

Proof. Equivalence of (a) and (d) from Proposition 6.11 shows that, already for a fixed r, Conjectures 5.21 and 6.4 are equivalent. Moreover, taken for all r, (b) from Proposition 6.11 shows that they are equivalent to Conjecture 6.3. The equivalence of (b) and (d) from Lemma 6.9 shows that Conjectures 6.1 and 6.3 are equivalent (but one has to change r). □

Remark 6.13. It is easy to see that Corollary 6.12 holds, more generally, if $H^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ is finite rank over $\mathbf{B}^+_{\mathrm{dR}}$ for all i. The proof is the same as in the quasi-compact case.

6.3. **Proof of the conjectures for small varieties.** Finally, we are ready to prove our main theorem for small varieties. We start with quasi-compact varieties.

Theorem 6.14. *Let X be a quasi-compact smooth dagger variety over C.*

(i) *For all $r \geq i$, the diagram*

$$\begin{tikzcd}
\widetilde{H}^i_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \arrow[r] \arrow[d] & (\widetilde{H}^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \arrow[d] \\
\widetilde{H}^i(F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))) \arrow[r] & \widetilde{H}^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})
\end{tikzcd}$$

is bicartesian. Equivalently, the following sequence is exact

$$0 \to \widetilde{H}^i_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \to \widetilde{H}^i(F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})))\oplus(\widetilde{H}^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \to \widetilde{H}^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \to 0$$

Moreover, for all r,

(ii) $(H^r_{\mathrm{HK}}(X), F^0H^r_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}))$ *is acyclic.*

(iii) $\mathbb{H}^r_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$ *has curvature ≤ 0.*

(iv) $\mathrm{ht}(\mathbb{H}^r_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))) = \dim_C H^r_{\mathrm{dR}}(X)$.

Proof. Let X be as in the theorem. By Remark 5.23 (iii) we can remove all the tildas in the statement of the theorem. We have seen (Corollary 6.12) that the properties (i)-(iv) for all r are equivalent (on the other hand, they are not (!) equivalent for a fixed r). It suffices thus to show (iv) for all r. This we do by induction on the number n of open affinoids necessary for covering X: the base case of $n = 1$ is Corollary 5.19.

We pass from n to $n+1$ using Mayer-Vietoris (and the fact that an intersection of two affinoids is an affinoid, from which it follows that if U_1 is covered by n open affinoids and if U_2 is an affinoid, then $U_1 \cap U_2$ is covered by n open affinoids).

Take then U_1, U_2 as above. Let $U = U_1 \cup U_2$. If $i \geq 0$, we set

$$A_{3i} = H^i_{\text{proét}}(U, \mathbf{Q}_p(r)), \quad A_{3i+1} = H^i_{\text{proét}}(U_1 \coprod U_2, \mathbf{Q}_p(r)),$$
$$A_{3i+2} = H^i_{\text{proét}}(U_1 \cap U_2, \mathbf{Q}_p(r))$$
$$B_{3i} = (H^i_{\text{HK}}(U)\widehat{\otimes}_{F^{\text{nr}}}\mathbf{B}^+_{\text{st}})^{N=0,\varphi=p^r}, \quad B_{3i+1} = (H^i_{\text{HK}}(U_1 \coprod U_2)\widehat{\otimes}_{F^{\text{nr}}}\mathbf{B}^+_{\text{st}})^{N=0,\varphi=p^r},$$
$$B_{3i+2} = (H^i_{\text{HK}}(U_1 \cap U_2)\widehat{\otimes}_{F^{\text{nr}}}\mathbf{B}^+_{\text{st}})^{N=0,\varphi=p^r}.$$

We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & A_{3r-1} \arrow[r] \arrow[d] & A_{3r} \arrow[r] \arrow[d] & A_{3r+1} \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & B_0 \arrow[r] & B_1 \arrow[r] & \cdots \arrow[r] & B_{3r-1} \arrow[r] & B_{3r} \arrow[r] & B_{3r+1} \arrow[r] & \cdots \end{tikzcd}$$

in which the rows are exact; in the top row we have qBC's and in the bottom row we have BC's.

We denote by K_i and C_i the kernel and cokernel of $A_i \to B_i$, respectively. It is clear from the definition that B_i has curvature ≤ 0; that is, it is of curvature < 0. We infer, using Proposition 3.27, that $\text{rk}(h(B_i)) = \text{ht}(B_i)$ (see Section 3.3.2 for the definition of the functor h). We know that C_i is a subobject of an affine BC and that K_i is a quotient of B_r-module. This implies in particular that $\text{ht}(A_i) \leq \text{ht}(B_i)$ with equality if and only if C_i and K_i are affine. We note that then A_i has curvature ≤ 0 because its quotient by the B_r-module K_i is a sub-BC of B_i and hence has curvature ≤ 0 (and even < 0 if is not zero).

Lemma 6.15. *If* $\text{ht}(A_k) = \text{ht}(B_k)$ *for all* $k \leq 3r+2$, $k \neq 3r$, *and if* A_{3r} *has curvature* ≤ 0, *then* $\text{ht}(A_{3r}) = \text{ht}(B_{3r})$.

Proof. Note that the equality $\text{ht}(A_k) = \text{ht}(B_k)$ (together with the fact that K_k is a quotient of an affine and C_k a subobject of an affine) implies that A_k and B_k have curvature ≤ 0, that K_k and C_k are affines, and that the map $h(B_k) \to h(A_k)$ is an isomorphism. All the terms of the commutative subdiagram

$$\begin{tikzcd} A_{3r-2} \arrow[r] \arrow[d] & A_{3r-1} \arrow[r] \arrow[d] & A_{3r} \arrow[r] \arrow[d] & A_{3r+1} \arrow[r] \arrow[d] & A_{3r+2} \arrow[d] \\ B_{3r-2} \arrow[r] & B_{3r-1} \arrow[r] & B_{3r} \arrow[r] & B_{3r+1} \arrow[r] & B_{3r+2} \end{tikzcd}$$

have curvature ≤ 0 and so do the kernels A'_k of $A_k \to A_{k+1}$ and B'_k of $B_k \to B_{k+1}$ (as subojects of qBC's of curvature ≤ 0). We infer, using Corollary 3.37, that the rows of the commutative diagram

$$\begin{tikzcd} h(A_{3r-2}) & h(A_{3r-1}) \arrow[l] & h(A_{3r}) \arrow[l] & h(A_{3r+1}) \arrow[l] & h(A_{3r+2}) \arrow[l] \\ h(B_{3r-2}) \arrow[u] & h(B_{3r-1}) \arrow[l] \arrow[u] & h(B_{3r}) \arrow[l] \arrow[u] & h(B_{3r+1}) \arrow[l] \arrow[u] & h(B_{3r+2}) \arrow[l] \arrow[u] \end{tikzcd}$$

are exact. As we have seen above, $h(B_k) \to h(A_k)$ is an isomorphism if $k \neq 3r$; hence also $h(B_{3r}) \to h(A_{3r})$ by the 5-Lemma. Since B_{3r} has curvature ≤ 0 and so does A_{3r} by assumption, it follows from Proposition 3.27 that $\text{ht}(A_{3r}) = \text{ht}(B_{3r})$, as wanted. □

Our theorem now follows from the following proposition. □

Proposition 6.16. *If* U_1, U_2 *et* $U_1 \cap U_2$ *satisfy Conjectures 5.21, 6.1, and 6.4* (which are equivalent)*, the same holds for* U.

Proof. By assumption, U_1, U_2 and $U_1 \cap U_2$ safisfy Conjectures 5.21, 6.1, and 6.4 for all r; we will show that the same holds for U, using induction on r. We assume thus that the result is shown up to $r-1$ (it is trivial for $r = 0$). Hence, in particular, $\text{ht}(A_k) = \text{ht}(B_k)$ for all $k \leq 3r+2$, $k \neq 3r$, and the problem is to show that $\text{ht}(A_{3r}) = \text{ht}(B_{3r})$.

Since the result is true for degree $r-1$, by Lemma 6.10 we have that $(H^{r-1}_{\mathrm{HK}}(U), F^0 H^{r-1}_{\mathrm{dR}}(U/\mathbf{B}_{\mathrm{dR}}))$ is acyclic. By Lemma 6.9, this implies that A_{3r} has curvature ≥ 0. We can thus apply Lemma 6.15, and hence $\mathrm{ht}(A_{3r}) = \mathrm{ht}(B_{3r})$. $\square$

Remark 6.17. Granting Corollary 6.12 (see Remark 6.13), the same proof applies (because the intersection of two Stein is Stein) to smooth dagger varieties that are *small*, i.e., that can be covered by a finite number of Stein varieties U_j such that $H^i_{\mathrm{dR}}(U_I/\mathbf{B}^+_{\mathrm{dR}})$ is free of finite rank over $\mathbf{B}^+_{\mathrm{dR}}$ for all degrees i and intersections U_I. We deduce the following result:

Corollary 6.18. *The statement of Theorem 6.14 applies to:*

- *analytifications of algebraic varieties,*
- *a naive interior*[15] *of a quasi-compact rigid analytic variety X.*

Proof. We just have to show that these varieties can be covered by a finite number of Stein varieties U_j such that $H^i_{\mathrm{dR}}(U_I/\mathbf{B}^+_{\mathrm{dR}})$ is free of finite rank over $\mathbf{B}^+_{\mathrm{dR}}$ for all i, I.

• In the first case, pick a covering by affine opens, and use the analytifications of these affine opens for your Stein covering.

• In the second case, cover X with a finite number of affinoids X_j and choose, for each j, a naive interior $U_j \subset X_j$. For the sake of this corollary, a *naive interior* of a smooth dagger affinoid is a Stein subvariety whose complement is open and quasi-compact. We easily check that the intersections $U_I \subset X_I$ are also naive interiors. We take $U := \cup_j U_j$. It is an admissible open of X, which we call a naive interior of X.

The U_j's are a covering of U with the desired properties: $H^i_{\mathrm{dR}}(U_I/\mathbf{B}^+_{\mathrm{dR}})$ is a free $\mathbf{B}^+_{\mathrm{dR}}$-module whose reduction modulo $\ker\theta$ is isomorphic to $H^i_{\mathrm{dR}}(U_I)$, which is of finite rank over C by [31, Th. A]. $\square$

7. Pro-étale–to–de Rham comparison theorem for small varieties

In this chapter we propose a recipe to extract, from the pro-étale cohomology of varieties defined over C, the Hyodo-Kato and de Rham cohomologies (as modules over the relevant rings) and, for varieties defined over K, to extract also Frobenius, monodromy, and the naive Hodge filtration.

7.1. **The pro-étale–to–de Rham C_{st}-conjecture for varieties over K.** In this section, we study the following conjecture extracting, for analytic spaces over K, the Hyodo-Kato and de Rham cohomologies from the pro-étale cohomology. This extends to p-adic analytic spaces the C_{st}-conjecture of Fontaine. (We go back to working with locally convex topological vector spaces.)

Conjecture 7.1. *Let X be a smooth dagger variety over K. We have natural strict isomorphisms:*

$$\mathrm{Hom}^{\mathrm{sm}}_{\mathscr{G}_K}(H^i_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p), \mathbf{B}_{\mathrm{st}}) \simeq H^i_{\mathrm{HK}}(X_C)^*, \quad \textit{as a } (\varphi, N, \mathscr{G}_K)\textit{-module,}$$
$$\mathrm{Hom}_{\mathscr{G}_K}(H^i_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p), \mathbf{B}_{\mathrm{dR}}) \simeq H^i_{\mathrm{dR}}(X)^*, \quad \textit{as a filtered } K\textit{-module.}$$

Remark 7.2. Our approach uses syntomic cohomology, which gives a description of $H^i_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))$ for $r \geq i$, rather than that of $H^i_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p)$. Hence we are going to consider the following equivalent form of Conjecture 7.1 (where the $\{r\}$ has the same meaning as in Corollary 4.10):

$$\mathrm{Hom}^{\mathrm{sm}}_{\mathscr{G}_K}(H^i_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r)), \mathbf{B}_{\mathrm{st}}) \simeq H^i_{\mathrm{HK}}(X_C)^*\{r\}, \quad \text{as a } (\varphi, N, \mathscr{G}_K)\text{-module,}$$
$$\mathrm{Hom}_{\mathscr{G}_K}(H^i_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r)), \mathbf{B}_{\mathrm{dR}}) \simeq H^i_{\mathrm{dR}}(X)^*\{r\}, \quad \text{as a filtered } K\text{-module.}$$

Our main result is the following:

Theorem 7.3. *Conjecture 7.1 holds for:*

(a) *affinoids,*
(b) *quasi-compact varieties,*

[15] See the proof for a precise definition

(c) *all other small varieties.*

The case of affinoids is included in the case of quasi-compact varieties, but the proof is considerably simpler.

7.1.1. *Dagger affinoids.* We note that it suffices to show that we have natural isomorphisms since the weak topology on the Hom-spaces is Hausdorff. Let

$$M := H^i_{\mathrm{HK}}(X_C),\quad X^i_{\mathrm{st}}(M) := (M\widehat{\otimes}^R_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^i},\quad M_K = (\overline{K}\otimes_{F^{\mathrm{nr}}} M)^{\mathscr{G}_K} \simeq H^i_{\mathrm{dR}}(X).$$

The last isomorphism follows from the Hyodo-Kato isomorphism [15, Th. 4.27]. Recall that we have the exact sequence (see Theorem 5.14)

$$0\to (\Omega^{i-1}_X/\operatorname{Ker} d)\widehat{\otimes}_K C \to H^i_{\text{proét}}(X_C,\mathbf{Q}_p(i)) \to X^i_{\mathrm{st}}(M)\to 0. \tag{7.4}$$

Applying $\mathrm{Hom}_{\mathscr{G}_K}(-,\mathbf{B}_{\mathrm{st}})$ and $\mathrm{Hom}_{\mathscr{G}_K}(-,\mathbf{B}_{\mathrm{dR}})$ to it, we get the following exact sequences

$$0 \to \mathrm{Hom}^{\mathrm{sm}}_{\mathscr{G}_K}(X^i_{\mathrm{st}}(M),\mathbf{B}_{\mathrm{st}}) \to \mathrm{Hom}^{\mathrm{sm}}_{\mathscr{G}_K}(H^i_{\text{proét}}(X_C,\mathbf{Q}_p(i)),\mathbf{B}_{\mathrm{st}}) \to \mathrm{Hom}^{\mathrm{sm}}_{\mathscr{G}_K}((\Omega^{i-1}_X/\operatorname{Ker} d)\widehat{\otimes}_K C,\mathbf{B}_{\mathrm{st}}),$$
$$0 \to \mathrm{Hom}_{\mathscr{G}_K}(X^i_{\mathrm{st}}(M),\mathbf{B}_{\mathrm{dR}}) \to \mathrm{Hom}_{\mathscr{G}_K}(H^i_{\text{proét}}(X_C,\mathbf{Q}_p(i)),\mathbf{B}_{\mathrm{dR}}) \to \mathrm{Hom}_{\mathscr{G}_K}((\Omega^{i-1}_X/\operatorname{Ker} d)\widehat{\otimes}_K C,\mathbf{B}_{\mathrm{dR}}).$$

Lemma 7.5. *We have*

$$\mathrm{Hom}^{\mathrm{sm}}_{\mathscr{G}_K}((\Omega^{i-1}_X/\operatorname{Ker} d)\widehat{\otimes}_K C,\mathbf{B}_{\mathrm{st}})=0,\quad \mathrm{Hom}_{\mathscr{G}_K}((\Omega^{i-1}_X/\operatorname{Ker} d)\widehat{\otimes}_K C,\mathbf{B}_{\mathrm{dR}})=0.$$

Proof. Since $\mathrm{Hom}^{\mathrm{sm}}_{\mathscr{G}_K}((\Omega^{i-1}_X/\operatorname{Ker} d)\widehat{\otimes}_K C,\mathbf{B}_{\mathrm{st}}) \simeq \operatorname{colim}_{L/K}\mathrm{Hom}_{\mathscr{G}_L}((\Omega^{i-1}_X/\operatorname{Ker} d)\widehat{\otimes}_K C,\mathbf{B}_{\mathrm{st}})$ and $\mathbf{B}_{\mathrm{st}}\hookrightarrow\mathbf{B}_{\mathrm{dR}}$, it suffices to show that $\mathrm{Hom}_{\mathscr{G}_L}((\Omega^{i-1}_X/\operatorname{Ker} d)\widehat{\otimes}_K C,\mathbf{B}_{\mathrm{dR}})=0$, for a finite extension L of K.

But $\mathrm{Hom}_{\mathscr{G}_L}(C,\mathbf{B}_{\mathrm{dR}})=0$ (Proposition 2.14 (iv)); hence $\mathrm{Hom}_{\mathscr{G}_L}((\Omega^{i-1}_X/\operatorname{Ker} d)\otimes_K C,\mathbf{B}_{\mathrm{dR}})=0$. Since our maps are requested to be continuous and $(\Omega^{i-1}_X/\operatorname{Ker} d)\otimes_K C$ is dense in $(\Omega^{i-1}_X/\operatorname{Ker} d)\widehat{\otimes}_K C$, we get the wanted vanishing. □

This lemma yields isomorphisms

$$\mathrm{Hom}^{\mathrm{sm}}_{\mathscr{G}_K}(X^i_{\mathrm{st}}(M),\mathbf{B}_{\mathrm{st}}) \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{sm}}_{\mathscr{G}_K}(H^i_{\text{proét}}(X_C,\mathbf{Q}_p(i)),\mathbf{B}_{\mathrm{st}}),$$
$$\mathrm{Hom}_{\mathscr{G}_K}(X^i_{\mathrm{st}}(M),\mathbf{B}_{\mathrm{dR}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{G}_K}(H^i_{\text{proét}}(X_C,\mathbf{Q}_p(i)),\mathbf{B}_{\mathrm{dR}}).$$

The filtration on M_K is concentrated in degree i (i.e., $\mathrm{Fil}^i M_K = M_K$ and $\mathrm{Fil}^{i+1}M_K=0$). Hence we can use Example 4.11 of Corollary 4.10 to finish the proof of Conjecture 7.1 in the case of dagger affinoids.

7.1.2. *Quasi-compact dagger varieties.* Let X be a quasi-compact dagger variety over K. Fix $r\geq i$. Set:

$$\widetilde{H}^{r,i} := \widetilde{H}^i_{\text{proét}}(X_C,\mathbf{Q}_p(r)),\quad \widetilde{F}^{r,i} := \widetilde{H}^i(F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})),$$
$$X^{r,i} := (H^i_{\mathrm{HK}}(X_C)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r},\quad B^i := H^i_{\mathrm{dR}}(X)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}}.$$

Let $\widetilde{A}^{r,i}$ be the kernel of the canonical map $\widetilde{F}^{r,i}\to B^i$. Then $\widetilde{A}^{r,i}$ is also canonically a subgroup of $\widetilde{H}^{r,i}$. Let

$$\overline{H}^{r,i} := \widetilde{H}^{r,i}/\widetilde{A}^{r,i},\quad \overline{F}^{r,i} := \widetilde{F}^{r,i}/\widetilde{A}^{r,i}.$$

Note that $\widetilde{F}^{r,i}/\widetilde{A}^{r,i}$ is a subgroup of B^i, hence it is classical.

Lemma 7.6. *We have, for all $i\leq r$,*

$$\mathrm{Hom}^{\mathrm{sm}}_{\mathscr{G}_K}(\widetilde{A}^{r,i},\mathbf{B}_{\mathrm{st}})=0,\quad \mathrm{Hom}_{\mathscr{G}_K}(\widetilde{A}^{r,i},\mathbf{B}_{\mathrm{dR}})=0.$$

Proof. It suffices to show that

$$\mathrm{Hom}^{\rm sm}_{\mathscr{G}_K}(A^{r,i},\mathbf{B}_{\rm st})=0,\quad \mathrm{Hom}_{\mathscr{G}_K}(A^{r,i},\mathbf{B}_{\rm dR})=0.$$

And, for that, it is enough to prove the second statement. Let

$$\mathrm{DR}^{r,i}:=H^i((\mathbf{B}^+_{\rm dR}\widehat{\otimes}\mathrm{R}\Gamma_{\rm dR})/F^r)$$

so that we have a long exact sequence $\cdots\mathrm{DR}^{r,i-1}\to F^{r,i}\to B^i\to\mathrm{DR}^{r,i}\to\cdots$ which shows that $A^{r,i}$ is a quotient of $\mathrm{DR}^{r,i-1}$. Hence it is enough to prove the same statement for $\mathrm{DR}^{r,i}$, with $i<r$.

Now $\mathrm{DR}^{r,i}$ is the i-th hypercohomology group of the complex

$$\mathrm{DR}^{r,\bullet}:=\big((\mathbf{B}^+_{\rm dR}/t^r)\widehat{\otimes}_K\mathscr{O}\to(\mathbf{B}^+_{\rm dR}/t^{r-1})\widehat{\otimes}_K\Omega^1\to\cdots\to(\mathbf{B}^+_{\rm dR}/t)\widehat{\otimes}_K\Omega^{r-1}\big).$$

Choose a covering of X by dagger affinoids, and denote by $Z^{r,i}$ the group of i-cocycles of the Čech double complex associated to this covering. Since $\mathrm{DR}^{r,i}$ is a quotient of $Z^{r,i}$, it is enough to prove that $\mathrm{Hom}_{\mathscr{G}_K}(Z^{r,i},\mathbf{B}_{\rm dR})=0$.

Denote by $Z^{j,i}_K$ the group of i-cocycles of the Čech double complex associated to the above covering and the complex

$$\mathrm{DR}^{j,\bullet}_K:=\big(\mathscr{O}\to\Omega^1\to\cdots\to\Omega^{j-1}\big).$$

We are going to prove that $\sum_{j\leq r}(t^{r-j}\mathbf{B}^+_{\rm dR}/t^r\mathbf{B}^+_{\rm dR})\otimes_K Z^{j,i}_K\to Z^{r,i}$ has dense image. This will allow us to conclude since $\mathrm{Hom}_{\mathscr{G}_K}((t^{r-j}\mathbf{B}^+_{\rm dR}/t^r\mathbf{B}^+_{\rm dR})\otimes_K Z^{j,i}_K,\mathbf{B}_{\rm dR})=0$ by Proposition 2.14 and our maps are assumed to be continuous.

To prove this density, choose a Banach basis over K of the K-Banach $\mathbf{B}^+_{\rm dR}/t^r$ of the form $(t^je_n)_{0\leq j\leq r-1,\,n\in\mathbf{N}}$ (pick a family e_n of elements of $\mathbf{B}^+_{\rm dR}/t^r$ whose images in $\mathbf{B}^+_{\rm dR}/t=C$ form a Banach basis of C over K). Then one can use $(t^je_n)_{0\leq j\leq k-1,\,n\in\mathbf{N}}$ as a Banach basis of $\mathbf{B}^+_{\rm dR}/t^k$, if $k\leq r$. This makes it possible to decompose $\mathrm{DR}^{r,\bullet}$ as a completed direct sum of the complexes $t^je_n\otimes\mathrm{DR}^{r-j,\bullet}_K$'s, and then $Z^{r,i}$ is the completion of the sum of the $t^je_n\otimes Z^{r-j,i}_K$'s. $\square$

It follows from Lemma 7.6 that we have isomorphisms

$$\mathrm{Hom}^{\rm sm}_{\mathscr{G}_K}(\overline{H}^{r,i},\mathbf{B}_{\rm st})\xrightarrow{\sim}\mathrm{Hom}^{\rm sm}_{\mathscr{G}_K}(\widetilde{H}^{r,i},\mathbf{B}_{\rm st}),\quad \mathrm{Hom}_{\mathscr{G}_K}(\overline{H}^{r,i},\mathbf{B}_{\rm dR})\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{G}_K}(\widetilde{H}^{r,i},\mathbf{B}_{\rm dR})$$

Since X is quasi-compact, Conjecture 5.21 holds and we have the exact sequence (5.22):

$$0\to\widetilde{H}^{r,i}\to X^{r,i}\oplus\widetilde{F}^{r,i}\to B^i\to 0.$$

This induces exact sequences

$$0\to\overline{H}^{r,i}\to X^{r,i}\oplus\overline{F}^{r,i}\to B^i\to 0,\quad 0\to\overline{H}^{r,i}\to X^{r,i}\to B^i/\overline{F}^{r,i}\to 0,\tag{7.7}$$

Since $\overline{F}^{r,i}$ is a subgroup of B^i, all the terms in these sequences are classical. This identifies topologically $\overline{H}^{r,i}$ with $V^r_{\rm st}(H^i_{\rm HK}(X_C),H^i_{\rm dR}(X))$. Hence we can use Corollary 4.10 to finish the proof of Conjecture 7.1 in the case of quasi-compact dagger varieties.

7.1.3. *Other small varieties.* The proof in the case of other small varieties is the same as in the quasi-compact case, using the fact that Conjecture 5.21 holds in that case (by Remark 6.17).

7.2. **The pro-étale–to–de Rham $C_{\rm st}$-conjecture for varieties over C.** The following theorem shows that one can recover de Rham cohomology (without the Hodge filtration) and Hyodo-Kato cohomology (without actions of φ and N) from pro-étale cohomology for varieties over C, despite the absence of Galois action.

Theorem 7.8. *Let X be a small dagger variety over C. Let $i\geq 0$.*

$$\mathrm{Hom}_{\rm VS}(\mathbb{H}^i_{\rm pro\acute{e}t}(X,\mathbf{Q}_p),\mathbb{B}_{\rm st})\simeq\mathrm{Hom}_{F^{\rm nr}}(H^i_{\rm HK}(X),\mathbf{B}_{\rm st}),\quad \textit{as a } \mathbf{B}_{\rm st}\textit{-module},$$
$$\mathrm{Hom}_{\rm VS}(\mathbb{H}^i_{\rm pro\acute{e}t}(X,\mathbf{Q}_p),\mathbb{B}_{\rm dR})\simeq\mathrm{Hom}_{\mathbf{B}^+_{\rm dR}}(H^i_{\rm dR}(X/\mathbf{B}^+_{\rm dR}),\mathbf{B}_{\rm dR}),\quad \textit{as a } \mathbf{B}_{\rm dR}\textit{-module}.$$

Proof. If M is a VS, we set
$$h(M) := \mathrm{Hom}_{\mathrm{VS}}(M, \mathbb{B}_{\mathrm{dR}}).$$

Fix $r \geq i$. Set:
$$\mathbb{H}^{r,i} := \mathbb{H}^i_{\mathrm{proét}}(X_C, \mathbf{Q}_p(r)), \quad \mathbb{F}^{r,i} := H^i(F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{dR}}}\mathbb{B}^+_{\mathrm{dR}})),$$
$$\mathbb{X}^{r,i} := (H^i_{\mathrm{HK}}(X_C)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbb{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}, \quad \mathbb{B}^i := H^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{dR}}}\mathbb{B}^+_{\mathrm{dR}}.$$

Let $\mathbb{A}^{r,i}$ be the kernel of the canonical map $\mathbb{F}^{r,i} \to \mathbb{B}^i$. Then $\mathbb{A}^{r,i}$ is a torsion $\mathbb{B}^+_{\mathrm{dR}}$-module; it is also canonically a subgroup of $\mathbb{H}^{r,i}$. Let
$$\overline{\mathbb{H}}^{r,i} := \mathbb{H}^{r,i}/\mathbb{A}^{r,i}, \quad \overline{\mathbb{F}}^{r,i} := \mathbb{F}^{r,i}/\mathbb{A}^{r,i}.$$

Since $\mathbb{A}^{r,i}$ is a torsion $\mathbb{B}^+_{\mathrm{dR}}$-module, $h(\mathbb{A}^{r,i}) = 0$ by Corollary 3.17; hence we have an isomorphism
$$h(\overline{\mathbb{H}}^{r,i}) \xrightarrow{\sim} h(\mathbb{H}^{r,i}).$$

We also have natural sequences
$$0 \to \mathbb{H}^{r,i} \to \mathbb{X}^{r,i} \oplus \mathbb{F}^{r,i} \to \mathbb{B}^i \to 0, \quad 0 \to \mathbb{H}^{r,i} \to \mathbb{X}^{r,i} \oplus \mathbb{F}^{r,i} \to \mathbb{B}^i \to 0$$
$$0 \to \overline{\mathbb{H}}^{r,i} \to \mathbb{X}^{r,i} \to \mathbb{B}^i/\overline{\mathbb{F}}^{r,i} \to 0$$

Since X is small, the last sequence is a sequence of BC's. It is exact (see Remark 3.35) because passing to C-points yields the sequence $0 \to \overline{H}^{r,i} \to X^{r,i} \to B^i/\overline{F}^{r,i} \to 0$ which was proven to be exact, as a consequence of the validity of Conjecture 5.21 (see (7.7)). This identifies $\overline{\mathbb{H}}^{r,i}$ with $\mathbb{V}^r_{\mathrm{st}}(H^i_{\mathrm{HK}}(X), t^{-r}\overline{F}^{r,i})$, which makes it possible to use Proposition 4.22 to prove Theorem 7.8. $\square$

Remark 7.9. (i) If X is proper, then $\mathbb{H}^i(X, \mathbf{Q}_p)$ is a finite dimensional $\mathbf{Q}_p$-vector space with no extra structure. This shows that it is hopeless to try to recover the actions of φ and N on $H^i_{\mathrm{HK}}(X)$ or the filtration on $H^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ using Theorem 7.8.

(ii) On the other hand, if X is a dagger affinoid, we know what the filtration on $H^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ is: we have $F^{i+k}H^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) = t^k H^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$. Also, as we have seen
$$(H^i_{\mathrm{HK}}(X) \otimes \mathbb{B}^+_{\mathrm{st}})^{N=0,\varphi=p^i} \simeq (H^i_{\mathrm{HK}}(X) \otimes \mathbb{B}^+_{\mathrm{cris}})^{\varphi=p^i} \tag{7.10}$$

Using Lemma 4.17, one sees that
$$\begin{aligned} H^i_{\mathrm{HK}}(X)^* \otimes \mathbf{B}^+_{\mathrm{cris}} &\subset \mathrm{Hom}_{\mathrm{VS}}((H^i_{\mathrm{HK}}(X) \otimes \mathbb{B}^+_{\mathrm{cris}})^{\varphi=p^i}, \mathbb{B}^+_{\mathrm{cris}}) \\ &\subset \{\lambda \in H^i_{\mathrm{HK}}(X)^* \otimes \mathbf{B}_{\mathrm{cris}},\ \varphi^n(\lambda) \in H^i_{\mathrm{HK}}(X)^* \otimes \mathbf{B}^+_{\mathrm{dR}},\ \forall n \geq 0\} \\ &= H^i_{\mathrm{HK}}(X)^* \otimes \mathbf{B}^+_{\mathrm{cris}} \end{aligned}$$

Since
$$\mathrm{Hom}_{\mathrm{VS}}(\mathbb{H}^i(X, \mathbf{Q}_p(i)), \mathbb{B}^+_{\mathrm{cris}}) \cong \mathrm{Hom}_{\mathrm{VS}}((H^i_{\mathrm{HK}}(X) \otimes \mathbb{B}^+_{\mathrm{cris}})^{\varphi=p^i}, \mathbb{B}^+_{\mathrm{cris}})$$
by the proof of Theorem 7.8, one sees that one can recover the action of φ on $H^i_{\mathrm{HK}}(X) \otimes \check{C}$ (by tensoring with $\check{C}$ above $\mathbf{B}^+_{\mathrm{cris}}$ and dualizing).

(iii) If X is a general smooth dagger variety over C, point (ii) suggests that one can recover a sheafified version of the actions of φ on $H^i_{\mathrm{HK}}(X)$ and of the filtration on $H^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$. Recovering the action of N seems out of reach by these methods.

References

[1] Y. André, *Slope filtrations.* Confluentes Math. **1** (2009), 1–85.

[2] A. Beilinson, *On the crystalline period map.* Preprint, arXiv:1111.3316v4. This is an extended version of Camb. J. Math. **1** (2013), 1–51.

[3] L. Berger, Construction de (φ, Γ)-modules : représentations p-adiques et B-paires, Algebra Number Theory **2** (2008), 91–120.

[4] L. Berger, Presque $\mathbf{C}_p$-représentations et (φ, Γ)-modules, J. Inst. Math. Jussieu **8** (2009), 653–668.

[5] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields.* Mathematical Surveys and Monographs, 33. American Mathematical Society, Providence, RI, 1990.
[6] B. Bhatt, D. Hansen, *The six functors for Zariski-constructible sheaves in rigid geometry.* arXiv:2101.09759 [math.AG].
[7] B. Bhatt, M. Morrow, P. Scholze, *Integral p-adic Hodge Theory.* Publ. IHES **128** (2018), 219–397.
[8] P. Colmez, *Espaces de Banach de dimension finie.* J. Inst. Math. Jussieu **1** (2002), 331–439.
[9] P. Colmez, *Espaces vectoriels de dimension finie et représentations de de Rham.* Astérisque **319** (2008), 117–186.
[10] P. Colmez, *Une construction de* $\mathbf{B}_{\rm dR}$, Rend. Sem. Mat. Univ. Padova **128** (2012), 109–130.
[11] P. Colmez, G. Dospinescu, W. Nizioł, *Cohomology of p-adic Stein spaces.* Invent. Math. **219** (2020), 873–985.
[12] P. Colmez, J.-M. Fontaine, *Construction des représentations p-adiques semi-stables.* Invent. Math. **140** (2000), 1–43.
[13] P. Colmez, W. Nizioł, *Syntomic complexes and p-adic nearby cycles.* Invent. Math. **208** (2017), 1–108.
[14] P. Colmez, W. Nizioł, *On p-adic comparison theorems for rigid analytic spaces, I.* Münster J. Math. **13** (2020) (Special Issue: In honor of Ch. Deninger), 445–507.
[15] P. Colmez, W. Nizioł, *On the cohomology of p-adic analytic spaces, I: The basic comparison theorem.* arXiv:2104.13448 [math.NT].
[16] P. Colmez, W. Nizioł, *On the $C_{\rm st}$ conjecture for analytic spaces.* Preprint, 2022.
[17] J.-F. Dat, S. Orlik, M. Rapoport, *Period domains over finite and p-adic fields.* Cambridge Tracts in Mathematics, vol. 183, Cambridge University Press, 2010.
[18] G. Faltings, Almost étale extensions, Astérisque **279** (2002), 185–270.
[19] G. Faltings, Mumford-Stabilität in der algebraischen Geometrie, in *Proceedings of the International Congress of Mathematicians (Zürich, 1994), Vol. 1*, (1995), 648–655, Birkhäuser.
[20] L. Fargues, J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique.* Astérisque **406** (2018), 51–382.
[21] J.-M. Fontaine, Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate, Astérisque **65** (1979), 3–80.
[22] J.-M. Fontaine, Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate, Ann. of Math. **115** (1982), 529–577.
[23] J.-M. Fontaine, Cohomologie de de Rham, cohomologie cristalline et représentations p-adiques, *Algebraic geometry (Tokyo/Kyoto, 1982)*, 86–108, Lecture Notes in Math. **1016**, Springer, 1983.
[24] J.-M. Fontaine, Le corps des périodes p-adiques, Astérisque **223** (1994), 59–111.
[25] J.-M. Fontaine, Représentations p-adiques semi-stables, Astérisque **223** (1994), 113–184.
[26] J.-M. Fontaine, Arithmétique des représentations galoisiennes p-adiques, Astérisque **295** (2004), 1–115.
[27] J.-M. Fontaine, *Presque C_p-représentations.* Kazuya Kato's fiftieth birthday. Doc. Math. 2003, Extra Vol., 285–385.
[28] J.-M. Fontaine, *Almost C_p Galois representations and vector bundles.* Tunisian J. Math. **2** (2020), 667–732.
[29] L. Fourquaux, *Applications $\mathbf{Q}_p$-linéaires, continues et Galois-équivariantes de C_p dans lui-même.* J. Number Theory **129** (2009), 1246–1255.
[30] S. Gilles, *Morphismes de périodes et cohomologie syntomique.* arXiv:2101.04987 [math.NT].
[31] E. Grosse-Klönne, *Finiteness of de Rham cohomology in rigid analysis.* Duke Math. J. **113** (2002), 57–91.
[32] A.-C. Le Bras, *Espaces de Banach-Colmez et faisceaux cohérents sur la courbe de Fargues-Fontaine.* Duke Math. J. **167** (2018), 3455–3532.
[33] W. Nizioł, Semistable Conjecture via K-theory, Duke Math. J. **141** (2008), 151–178.
[34] W. Nizioł, On uniqueness of p-adic period morphisms II, Compos. Math. **156** (2020), 1915–1964.
[35] J. Plût, *Espaces de Banach analytiques p-adiques et espaces de Banach-Colmez.* Thèse Orsay (2009).
[36] J. Plût, *Slope filtration on Banach-Colmez spaces.* arXiv:1610.09822 [math.NT].
[37] F. Prosmans, *Derived categories for functional analysis.* Publ. Res. Inst. Math. Sci. **36** (2000), 19–83.
[38] P. Scholze, *p-adic Hodge theory for rigid-analytic varieties.* Forum Math. Pi **1** (2013), e1, 77 pp.
[39] S. Sen, Continuous cohomology and p-adic Galois representations, Invent. math. **62** (1980/81), 89–116.
[40] The Stacks Project Authors, *Stack Project.* http://stacks.math.columbia.edu, 2020.
[41] J. Tate, p-divisible groups, *Proc. Conf. Local Fields (Driebergen, 1966)*, 158–183, Springer 1967.
[42] T. Tsuji, p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case. Invent. Math. **137** (1999), 233–411.

CNRS, IMJ-PRG, Sorbonne Université, 4 place Jussieu, 75005 Paris, France
E-mail address: pierre.colmez@imj-prg.fr

CNRS, IMJ-PRG, Sorbonne Université, 4 place Jussieu, 75005 Paris, France
E-mail address: wieslawa.niziol@imj-prg.fr