THE p-ADIC LOCAL LANGLANDS CORRESPONDENCE FOR $\mathrm{GL}_2(\mathbb{Q}_p)$

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Abstract. The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is given by an exact functor from unitary Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to representations of the absolute Galois group $G_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. We prove, using characteristic 0 methods, that this correspondence induces a bijection between absolutely irreducible non-ordinary representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and absolutely irreducible 2-dimensional representations of $G_{\mathbb{Q}_p}$. This had already been proved, by characteristic p methods, but only for $p \geq 5$.

1. Introduction

1.1. **The p-adic local Langlands correspondence.** Let p be a prime number and let $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let L be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathscr{O}$, residue field k and uniformizer ϖ.

Let $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ be the category of admissible unitary L-Banach representations of G. Any $\Pi \in \mathrm{Ban}_G^{\mathrm{adm}}(L)$ has an open, bounded and G-invariant lattice Θ and $\Theta \otimes_{\mathscr{O}} k$ is an admissible smooth k-representation of G. We say that Π in $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ is *residually of finite length* if for any (equivalently, one) such lattice Θ, the G-representation $\Theta \otimes_{\mathscr{O}} k$ is of finite length. In this case the semi-simplification of $\Theta \otimes_{\mathscr{O}} k$ is independent of the choice of Θ, and we denote it by $\overline{\Pi}^{\mathrm{ss}}$. We say that an absolutely irreducible[1] $\Pi \in \mathrm{Ban}_G^{\mathrm{adm}}(L)$ is *ordinary* if it is a subquotient of a unitary parabolic induction of a unitary character.

Let $\mathrm{Rep}_L(G)$ be the full subcategory of $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ consisting of representations Π having a central character and which are residually of finite length. Let $\mathrm{Rep}_L(\mathscr{G}_{\mathbb{Q}_p})$ be the category of finite dimensional continuous L-representations of $\mathscr{G}_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. In [20, ch. IV] is constructed an exact, covariant functor (that some people call the Montreal functor) $\Pi \mapsto \mathbf{V}(\Pi)$ from $\mathrm{Rep}_L(G)$ to $\mathrm{Rep}_L(\mathscr{G}_{\mathbb{Q}_p})$. We prove that this functor has all the properties needed to be called the p-adic local Langlands correspondence for G.

Theorem 1.1. *The functor $\Pi \mapsto \mathbf{V}(\Pi)$ induces a bijection between the isomorphism classes of :*

- *absolutely irreducible non-ordinary $\Pi \in \mathrm{Ban}_G^{\mathrm{adm}}(L)$,*
- *2-dimensional absolutely irreducible continuous L-representations of $\mathscr{G}_{\mathbb{Q}_p}$.*

Implicit in the statement of the theorem is the fact that absolutely irreducible $\Pi \in \mathrm{Ban}_G^{\mathrm{adm}}(L)$ are residually of finite length so that one can apply the functor $\mathbf{V}$ to them.

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[1]This means that $\Pi \otimes_L L'$ is topologically irreducible for all finite extensions L' of L.

One corollary of the theorem, and of the explicit construction of the representation $\Pi(V)$ of G corresponding to a representation V of $\mathscr{G}_{\mathbb{Q}_p}$ (see below), is the compatibility between the p-adic local Langlands correspondence and local class field theory: we let $\varepsilon : \mathscr{G}_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ be the cyclotomic character and we view unitary characters of $\mathbb{Q}_p^\times$ as characters of $\mathscr{G}_{\mathbb{Q}_p}$ via class field theory[2] (for example, ε corresponds to $x \mapsto x|x|$). Note that, by Schur's lemma [27], any absolutely irreducible object of $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ admits a central character.

Corollary 1.2. *If Π is an absolutely irreducible non-ordinary object of $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ with central character δ, then $\mathbf{V}(\Pi)$ has determinant $\delta\varepsilon$.*

The next result shows that the p-adic local Langlands correspondence is a refinement of the classical one (that such a statement could be true was Breuil's starting point for his investigations on the existence of a p-adic local Langlands correspondence [10]).

Let π be an admissible, absolutely irreducible, infinite dimensional, smooth L-representation of G, and let W be an algebraic representation of G (so there exist $a \in \mathbb{Z}$ and $k \geq 1$ such that $W = \mathrm{Sym}^{k-1} L^2 \otimes \det^a$). Let Δ be the Weil representation corresponding to π via the classical local Langlands correspondence; we view Δ as a $(\varphi, \mathscr{G}_{\mathbb{Q}_p})$-module[3] [40, 13]. Let $\mathscr{F}(\Delta, W)$ be the space of isomorphism classes of weakly admissible, absolutely irreducible, filtered $(\varphi, N, \mathscr{G}_{\mathbb{Q}_p})$-modules [39, Chap. 4] whose underlying $(\varphi, \mathscr{G}_{\mathbb{Q}_p})$-module is isomorphic to Δ and the jumps of the filtration are $-a$ and $-a-k$: if $\mathscr{L} \in \mathscr{F}(\Delta, W)$, the corresponding [22] representation $V_{\mathscr{L}}$ of $\mathscr{G}_{\mathbb{Q}_p}$ is absolutely irreducible and its Hodge-Tate weights are a and $a+k$. If $\mathscr{F}(\Delta, W)$ is not empty, it is either a point if π is principal series or $\mathbf{P}^1(L)$ if π is supercuspidal or a twist of the Steinberg representation[4].

Theorem 1.3. (i) *If Π is an admissible, absolutely irreducible, non-ordinary, unitary completion of $\pi \otimes W$, then $\mathbf{V}(\Pi)$ is potentially semi-stable with Hodge-Tate weights a and $a+k$ and the underlying $(\varphi, \mathscr{G}_{\mathbb{Q}_p})$-module of $D_{\mathrm{pst}}(\mathbf{V}(\Pi))$ is isomorphic to Δ.*

(ii) *The functor $\Pi \mapsto D_{\mathrm{pst}}(\mathbf{V}(\Pi))$ induces a bijection between the admissible, absolutely irreducible, non-ordinary, unitary completions of $\pi \otimes W$ and $\mathscr{F}(\Delta, W)$.*

The theorem follows from the combination of theorem 1.1, [20, th. 0.20] (or [30]), [20, th. VI.6.42] and Emerton's local-global compatibility[5] ([34], th. 3.2.22).

If $p \geq 5$, the results are not new; they were proven in [54], building upon [20, 43], via characteristic p methods, but these methods seemed to be very difficult to extend to the case $p = 2$ (and also $p = 3$ in a special case). That we are able to prove the

[2]Normalized so that uniformizers correspond to geometric Frobenii.

[3]In general, this can require to extend scalars to a finite unramified extension of L, but we assume that this is already possible over L.

[4] In this last case, the filtration corresponding to $\infty \in \mathbf{P}^1(L)$ makes the monodromy operator N on Δ vanish and V_∞ is crystalline (up to twist by a character) whereas, if $\mathscr{L} \neq \infty$, $V_{\mathscr{L}}$ is semi-stable non-crystalline (up to twist by a character).

[5] It is a little bit frustrating to have to use global considerations to prove it. By purely local considerations, one could prove it when π is a principal series or a twist of the Steinberg representation. When π is supercuspidal, one could show that there is a set S_π of $(\varphi, \mathscr{G}_{\mathbb{Q}_p})$-modules Δ such that the functor $\Pi \mapsto D_{\mathrm{pst}}(\mathbf{V}(\Pi))$ induces a bijection between the admissible, absolutely irreducible, unitary completions of $\pi \otimes W$ and the union of the $\mathscr{F}(\Delta, W)$, for $\Delta \in S_\pi$, but we would not know much about S_π except for the fact that $S_\pi \cap S_{\pi'} = \emptyset$ if $\pi \ncong \pi'$.

theorem in full generality relies on a shift to characteristic 0 methods and an array of results which were not available at the time [54] was written:

• The computation [52] of the blocks of the mod p representations of G, in the case $p = 2$; this computation also uses characteristic 0 methods.

• Schur's lemma for unitary Banach representations of p-adic Lie groups ([27] which uses results of Ardakov and Wadsley [1].)

• The computation [21, 44] of the locally analytic vectors of unitary principal series representations of G.

• The computation [30] of the infinitesimal action of G on locally analytic vectors of objects of $\mathrm{Rep}_L(G)$.

There are 3 issues to tackle if one wants to establish theorem 1.1: one has to prove that absolutely irreducible objects of $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ are residually of finite length and bound this length, and one has to prove surjectivity and injectivity.

1.2. **Residual finiteness.** Before stating the result, let us introduce some notations. Let B be the (upper) Borel subgroup of G and let $\omega : \mathbb{Q}_p^\times \to k^\times$ be the character $x \mapsto x|x| \pmod p$. If $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$ are (not necessarily distinct) smooth characters, we let

$$\pi\{\chi_1, \chi_2\} = (\mathrm{Ind}_B^G \chi_1 \otimes \chi_2\omega^{-1})_{\mathrm{sm}}^{\mathrm{ss}} \oplus (\mathrm{Ind}_B^G \chi_2 \otimes \chi_1\omega^{-1})_{\mathrm{sm}}^{\mathrm{ss}}.$$

Then $\pi\{\chi_1, \chi_2\}$ is typically of length 2, but it may be of length 3, and even 4 when $p = 2$ or $p = 3$ (lemma 2.14 gives an explicit description of $\pi\{\chi_1, \chi_2\}$). Recall that a smooth irreducible k-representation is called *supersingular* if it is not isomorphic to a subquotient of some representation $\pi\{\chi_1, \chi_2\}$.

Theorem 1.4. *Let Π be an absolutely irreducible object of $\mathrm{Ban}_G^{\mathrm{adm}}(L)$. Then Π is residually of finite length and, after possibly replacing L by a quadratic unramified extension, $\overline{\Pi}^{\mathrm{ss}}$ is either absolutely irreducible supersingular or a subrepresentation of some $\pi\{\chi_1, \chi_2\}$.*

For $p \geq 5$, this theorem is proved in [54]. The starting points of the proofs in [54] and in this paper are the same: one starts from an absolutely irreducible mod p representation π of G and considers the projective envelope P of its dual (in a suitable category, cf. § 2.1). Then we are led to try to understand the ring E of endomorphisms of P, as this gives a description of the Banach representations of G which have π as a Jordan-Hölder component of their reduction mod p. After this the strategies of proof differ completely and only some formal parts of [54] are used in this paper (mainly §4 on Banach representations).

To illustrate the differences, let us consider the simplest case where π is supersingular, so that $\mathbf{V}(\pi)$ is an irreducible representation of $\mathscr{G}_{\mathbb{Q}_p}$. The key point in both approaches to theorem 1.4 is to prove that the ring $E[1/p]$ is commutative. This is done as follows.

In [54], the functor $\Pi \mapsto \mathbf{V}(\Pi)$ is used to show that E surjects onto the universal deformation ring of $\mathbf{V}(\pi)$, which is commutative. It is then shown that this map is an isomorphism by showing that it induces an isomorphism on the graded rings of E and the universal deformation ring of $\mathbf{V}(\pi)$ with respect to the maximal ideals. To control the dimension of the graded pieces of $\mathrm{gr}^\bullet E$, one needs to be able to compute the dimension of Ext-groups of mod p representations of G. These computations become hard to handle for $p = 3$ and very hard for $p = 2$. Moreover, the argument uses that the universal deformation ring of $\mathbf{V}(\pi)$ is formally smooth, which fails if $p=2$ and in one case if $p=3$.

In this paper we use the functor $\Pi \mapsto \mathrm{m}(\Pi)$, defined in [54, § 4], from $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ to the category of finitely generated $E[1/p]$-modules. We show that if Π is the universal unitary completion of locally algebraic unramified principal series representation of G, then the image of $E[1/p] \to \mathrm{End}_L(\mathrm{m}(\Pi))$ is commutative and then show[6] that the map

$$E[1/p] \to \prod_i \mathrm{End}_L(\mathrm{m}(\Pi_i)) \tag{1}$$

is injective, where the product is taken over all such representations. The argument uses the work of Berger–Breuil [5], that the Π_i are admissible and absolutely irreducible, and we can control their reductions modulo p [4, 23]. The injectivity of (1) is morally equivalent to the density of crystalline representations in the universal deformation ring of $\mathbf{V}(\pi)$, and is more or less saying that "polynomials are dense in continuous functions", an observation that was used by Emerton [34] in a global context. However, in our local situation, P is not finitely generated over $\mathscr{O}[[\mathrm{GL}_2(\mathbb{Z}_p)]]$, and things are more complicated than what the above sketch would suggest; we refer the reader to §2.1 for a more detailed overview of the proof of the theorem.

Remark 1.5. The approach developed in [54], when it works, gives more information than theorem 1.4: one gets a complete description of finite length objects of $\mathrm{Ban}_G^{\mathrm{adm}}(L)$, not only of its absolutely irreducible objects and also a complete description of the category of smooth locally admissible representations of G on $\mathscr{O}$-torsion modules. However, the fact that E is commutative is a very useful piece of information, and, when π is either supersingular or generic principal series, in a forthcoming paper [55] we will extend the results of [54] to the cases when $p=2$ and $p=3$.

Combining theorem 1.4 and the fact [27, cor. 3.14] that an irreducible object of $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ decomposes as the direct sum of finitely many absolutely irreducible objects after a finite extension of L, we obtain the following result:

Corollary 1.6. *An object of* $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ *has finite length if and only if it is residually of finite length.*

The following result answers question (Q3) of [20, p. 297] and is an easy consequence of theorem 1.4, the exactness of the functor[7] $\Pi \mapsto \mathbf{V}(\Pi)$ and [20, th. 0.10].

Corollary 1.7. *If* $\Pi \in \mathrm{Ban}_G^{\mathrm{adm}}(L)$ *is absolutely irreducible, then* $\dim_L \mathbf{V}(\Pi) \leq 2$.

1.3. **Surjectivity.** The surjectivity was proven in [20] (for $p \geq 3$, and almost for $p=2$, see below) by constructing, for any 2-dimensional representation V of $\mathscr{G}_{\mathbb{Q}_p}$, a representation $\Pi(V)$ of G such that $\mathbf{V}(\Pi(V)) = V$ (or $\check{V}$, depending on the normalisation). The construction goes through Fontaine's equivalence of categories [41]

[6] We actually end up proving a weaker statement, which is too technical for this introduction, see the proofs of corollary 2.20 and theorem 2.21. To show the injectivity of (1) one would additionally have to show that the rings $E[1/p]/\mathfrak{a}_V$ in corollary 2.20 are reduced. We do not prove this here, but will return to this question in [55].

[7] More precisely, of integral and torsion versions of this functor.

between representations of $\mathscr{G}_{\mathbb{Q}_p}$ and (φ,Γ)-modules, as does the construction of the functor $\Pi\mapsto\mathbf{V}(\Pi)$. If D is the (φ,Γ)-module attached to V by this equivalence of categories and if δ is a character of $\mathbb{Q}_p^\times$, one can construct a G-equivariant sheaf $U\mapsto D\boxtimes_\delta U$ on $\mathbf{P}^1=\mathbf{P}^1(\mathbb{Q}_p)$. If $\delta=\delta_D$, where $\delta_D=\varepsilon^{-1}\det V$, then the global sections of this sheaf fit into an exact sequence of G-representations

$$0\to\Pi(V)^*\otimes\delta\to D\boxtimes_\delta\mathbf{P}^1\to\Pi(V)\to 0. \tag{2}$$

The proof of the existence of this decomposition is by analytic continuation, using explicit computations to deal with trianguline representations, in which case $\Pi(V)$ is the universal completion of a locally analytic principal series [5, 9, 18, 32, 48], and the Zariski density [17, 43, 7, 15] of trianguline (or even crystalline) representations in the deformation space of $\overline{V}^{\rm ss}$. That such a strategy could work was suggested by Kisin who used a variant [43] to prove surjectivity for $p\geq 5$ in a more indirect way. This Zariski density was missing when $p=2$ and $\overline{V}^{\rm ss}$ is scalar: the methods of [17, 43] prove that the Zariski closure of the trianguline (or crystalline) representations is a union of irreducible components of the space of deformations of the residual representation; so what was really missing was an identification of the irreducible components, which is not completely straightforward. This is not an issue anymore as we proved [24] that there are exactly 2 irreducible components and that the crystalline representations are dense in each of them.

1.4. **Injectivity.** The following result is a strengthening of the injectivity of the p-adic local Langlands correspondence.

Theorem 1.8. *Let $\Pi_1,\Pi_2\in\mathrm{Ban}_G^{\rm adm}(L)$ be absolutely irreducible, non-ordinary.*
(i) *If $\mathbf{V}(\Pi_1)\cong\mathbf{V}(\Pi_2)$, then $\Pi_1\cong\Pi_2$.*
(ii) *We have $\mathrm{Hom}^{\rm cont}_{L[P]}(\Pi_1,\Pi_2)=\mathrm{Hom}^{\rm cont}_{L[G]}(\Pi_1,\Pi_2)$, where P is the mirabolic subgroup of G.*

For absolutely irreducible non-ordinary objects of $\mathrm{Ban}_G^{\rm adm}(L)$, the knowledge of $\mathbf{V}(\Pi)$ is equivalent to that of the action of P (this is not true for ordinary objects). So, theorem 1.8 is equivalent to the fact that we can recover an absolutely irreducible non-ordinary object Π from its restriction to P. If we replace P with the Borel subgroup B, then the result follows from [49] (see also [23, remark III.48] for a different proof). The key difficulty is therefore controlling the central character, and, thanks to results from [20, 23], the proof reduces to showing that δ_D is the only character δ such that $D\boxtimes_\delta\mathbf{P}^1$ admits a decomposition as in (2).

So assume $D\boxtimes_\delta\mathbf{P}^1$ admits such a decomposition and set $\eta=\delta_D^{-1}\delta$. We need to prove that $\eta=1$ and this is done in two steps: we first prove that $\eta=1$ if it is locally constant, and then we prove that η is locally constant.

The proof of step one splits into two cases:

• If D is trianguline then we use techniques of [21, 28] to study locally analytic principal series appearing in the locally analytic vectors in Π_1 and Π_2, and make use of their universal unitary completions.

• If D is not trianguline then the restriction of global sections $D\boxtimes_\delta\mathbf{P}^1$ to any non-empty compact open subset of $\mathbf{P}^1$ is injective on $\Pi(V)^*\otimes\delta$, viewed as a subspace of $D\boxtimes_\delta\mathbf{P}^1$ via (2). If $\alpha\in\mathscr{O}^*$, let $\mathscr{C}^\alpha\subset D\boxtimes\mathbb{Z}_p^\times$ be the image of the eigenspace of $\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)$ for the eigenvalue α under the restriction to $\mathbb{Z}_p^\times$. This image is the same for δ and δ_D and can be described purely in terms of D as $(1-\alpha\varphi)\cdot D^{\psi=\alpha}$. Using the action of $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ on $D\boxtimes_\delta\mathbf{P}^1$ and $D\boxtimes_{\delta_D}\mathbf{P}^1$ we show that the "multiplication by η"

operator $m_\eta : D\boxtimes\mathbb{Z}_p^\times \to D\boxtimes\mathbb{Z}_p^\times$ (see n° 3.1.2 for a precise definition) sends $\mathscr{C}^\alpha$ into $\mathscr{C}^{\alpha\eta(p)}$, and using the above-mentionned injectivity, that $\eta = 1$.

To prove that η is locally constant, one can, in most cases, use the formulas [30] for the infinitesimal action of G. In the remaining cases one uses the fact that the characters η sending $\mathscr{C}^\alpha$ into $\mathscr{C}^{\alpha\eta(p)}$ for all α form a Zariski closed subgroup of the space of all characters, and such a subgroup automatically contains a non-trivial locally constant character if it is not reduced to $\{1\}$.

The reader will find a more detailed overview of the proof in §3.2.

Finally, we give a criterion for an absolutely irreducible object of $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ to be non-ordinary; this refines theorem 1.4, by describing in which case the inclusion $\overline{\Pi}^{\mathrm{ss}} \subset \pi\{\chi_1,\chi_2\}$ given by this theorem is an equality. It is a consequence of theorems 1.4 and 1.8, and of the compatibility [4, 23] of p-adic and mod p Langlands correspondences.

Theorem 1.9. *Let $\Pi \in \mathrm{Ban}_G^{\mathrm{adm}}(L)$ be absolutely irreducible. The following assertions are equivalent*

(i) $\mathbf{V}(\Pi)$ *is 2-dimensional.*

(ii) Π *is non-ordinary.*

(iii) *After possibly replacing L by a quadratic unramified extension, $\overline{\Pi}^{\mathrm{ss}}$ is either absolutely irreducible supersingular or isomorphic to some $\pi\{\chi_1,\chi_2\}$.*

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2. Residual finiteness

2.1. **Overview of the proof.** If G is any p-adic analytic group, let $\mathrm{Mod}_G^{\mathrm{sm}}(\mathscr{O})$ be the category of smooth representations of G on $\mathscr{O}$-torsion modules. Pontryagin duality induces an anti-equivalence of categories between $\mathrm{Mod}_G^{\mathrm{sm}}(\mathscr{O})$ and a certain category $\mathrm{Mod}_G^{\mathrm{pro}}(\mathscr{O})$ of linearly compact $\mathscr{O}$-modules with a continuous G-action, see [35]. In particular, if G is compact then $\mathrm{Mod}_G^{\mathrm{pro}}(\mathscr{O})$ is the category of compact $\mathscr{O}[\![G]\!]$-modules, where $\mathscr{O}[\![G]\!]$ is the completed group algebra. Let $\mathrm{Mod}_G^{?}(\mathscr{O})$ be a full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(\mathscr{O})$ closed under subquotients and arbitrary direct sums in $\mathrm{Mod}_G^{\mathrm{sm}}(\mathscr{O})$ and such that representations in $\mathrm{Mod}_G^{?}(\mathscr{O})$ are equal to the union of their subrepresentations of finite length. Let $\mathfrak{C}(\mathscr{O})$ be the full subcategory of $\mathrm{Mod}_G^{\mathrm{pro}}(\mathscr{O})$ antiequivalent to $\mathrm{Mod}_G^{?}(\mathscr{O})$ via the Pontryagin duality.

Let $\pi \in \mathrm{Mod}_G^{?}(\mathscr{O})$ be admissible and absolutely irreducible, let $P \twoheadrightarrow \pi^\vee$ be a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathscr{O})$, and let $E := \mathrm{End}_{\mathfrak{C}(\mathscr{O})}(P)$.

Let $\Pi \in \mathrm{Ban}_G^{\mathrm{adm}}(L)$ and let Θ be an open, bounded and G-invariant lattice in Π. Let $\Theta^d = \mathrm{Hom}_{\mathscr{O}}(\Theta,\mathscr{O})$ be the Schikhof dual of Θ. Endowed with the topology of pointwise convergence, Θ^d is an object of $\mathrm{Mod}_G^{\mathrm{pro}}(\mathscr{O})$, see [54, Lem.4.4]. If Θ^d is in $\mathfrak{C}(\mathscr{O})$ then Ξ^d is in $\mathfrak{C}(\mathscr{O})$ for every open bounded G-invariant lattice Ξ in Π, since Θ and Ξ are commensurable and $\mathfrak{C}(\mathscr{O})$ is closed under subquotients, see [54, Lem.4.6]. We let $\mathrm{Ban}_{\mathfrak{C}(\mathscr{O})}^{\mathrm{adm}}$ be the full subcategory of $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ consisting of those Π with Θ^d in $\mathfrak{C}(\mathscr{O})$. For $\Pi \in \mathrm{Ban}_{\mathfrak{C}(\mathscr{O})}^{\mathrm{adm}}$ we let

$$\mathrm{m}(\Pi) := \mathrm{Hom}_{\mathfrak{C}(\mathscr{O})}(P,\Theta^d)\otimes_{\mathscr{O}} L.$$

Then $\mathrm{m}(\Pi)$ is a right $E[1/p]$-module which does not depend on the choice of Θ, since any two open, bounded lattices in Π are commensurable. The functor $\Pi \mapsto \mathrm{m}(\Pi)$ from $\mathrm{Ban}^{\mathrm{adm}}_{\mathfrak{C}(\mathscr{O})}$ to the category of right $E[1/p]$-modules is exact by [54, Lem.4.9]. The proposition below is proved in [54, § 4], as we explain in §2.2.

Proposition 2.1. *For* Π *in* $\mathrm{Ban}^{\mathrm{adm}}_{\mathfrak{C}(\mathscr{O})}$ *the following assertions hold:*

(i) $\mathrm{m}(\Pi)$ *is a finitely generated* $E[1/p]$*-module;*
(ii) $\dim_L \mathrm{m}(\Pi)$ *is equal to the multiplicity with which* π *occurs as a subquotient of* $\Theta \otimes_{\mathscr{O}} k$*;*
(iii) *if* Π *is topologically irreducible, then*
  a) $\mathrm{m}(\Pi)$ *is an irreducible* $E[1/p]$*-module;*
  b) *the natural map* $\mathrm{End}^{\mathrm{cont}}_G(\Pi) \to \mathrm{End}_{E[1/p]}(\mathrm{m}(\Pi))^{op}$ *is an isomorphism.*

Let us suppose that we are given a family $\{\Pi_i\}_{i\in I}$ in $\mathrm{Ban}^{\mathrm{adm}}_G(L_i)$, where for each $i \in I$, L_i is a finite extension of L with residue field k_i. Let us further suppose that each Π_i lies in $\mathrm{Ban}^{\mathrm{adm}}_{\mathfrak{C}(\mathscr{O})}$, when considered as L-Banach representation. Suppose that $d \geq 1$ is an integer such that we can find open, bounded and G-invariant lattices Θ_i in Π_i such that $\pi \otimes_k k_i$ occurs with multiplicity $\leq d$ as a subquotient of $\Theta_i \otimes_{\mathscr{O}_{L_i}} k_i$ for all $i \in I$. Thus π occurs with multiplicity $\leq [L_i : L]d$ as a subquotient of $\Theta_i/(\varpi)$ and proposition 2.1 yields $\dim_{L_i} \mathrm{m}(\Pi_i) \leq d$. For simplicity let us further assume that $d = 1$, then we can conclude that the action of $E[1/p]$ induces a homomorphism $E[1/p] \to \mathrm{End}_{L_i}(\mathrm{m}(\Pi_i)) \cong L_i$. If $\mathfrak{a}_i$ is the kernel of this map then $E[1/p]/\mathfrak{a}_i$ is commutative, and hence if we let $\mathfrak{a} = \cap_{i\in I}\mathfrak{a}_i$, then we deduce that $E[1/p]/\mathfrak{a}$ is commutative. Let us further assume that $\mathfrak{a} = 0$. Then we can conclude that the ring E is commutative. Let Π in $\mathrm{Ban}^{\mathrm{adm}}_{\mathfrak{C}(\mathscr{O})}$ be absolutely irreducible, and let $\mathscr{E}$ be the image of $E[1/p]$ in $\mathrm{End}_L(\mathrm{m}(\Pi))$. Since $E[1/p]$ is commutative, so is $\mathscr{E}$, and using proposition 2.1 (iii) b) we deduce that $\mathscr{E}$ is a subring of $\mathrm{End}^{\mathrm{cont}}_G(\Pi)$.

Now comes a new ingredient, not available at the time of writing [54]: by Schur's lemma [27], since Π is absolutely irreducible we have $\mathrm{End}^{\mathrm{cont}}_G(\Pi) = L$, hence $\mathscr{E} = L$. Since $\mathrm{m}(\Pi)$ is an irreducible $E[1/p]$-module by proposition 2.1 (iii) a), we conclude that $\dim_L \mathrm{m}(\Pi) = 1$, and hence by part (ii) of the proposition we conclude that π occurs with multiplicity 1 as subquotient of Θ/ϖ. If $d > 1$ one can still run the same argument concluding that π occurs with multiplicity at most d as subquotient of $\Theta/(\varpi)$ by using rings with polynomial identity.

All the previous constructions and the strategy of proof explained above work in great generality (G was any p-adic analytic group), provided certain conditions are satisfied, the hardest of which is finding a family $\{\Pi_i\}_{i\in I}$, which enjoys all these nice properties. From now on we let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, and let $\mathrm{Mod}^{?}_G(\mathscr{O})$ be the category of locally admissible representations $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathscr{O})$. This category, introduced by Emerton in [35], consists of all representations in $\mathrm{Mod}^{\mathrm{sm}}_G(\mathscr{O})$, which are equal to the union of their admissible subrepresentations. For the family $\{\Pi_i\}_{i\in I}$ we take all the Banach representations corresponding to 2-dimensional crystalline representations of $\mathscr{G}_{\mathbb{Q}_p}$. It follows from the explicit description [4] of $\overline{\Pi}_i^{\mathrm{ss}}$, that π can occur as a subquotient with multiplicity at most 2, and multiplicity one if π is either supersingular or generic principal series.

The statement $\cap_{i\in I}\mathfrak{a}_i = 0$ morally is the statement "crystalline points are dense in the universal deformation ring", so one certainly expects it to be true, since on the Galois side this statement is known [17, 43] to be true[8]. In fact, Emerton has proved an analogous global statement "classical crystalline points are dense in the big Hecke algebra" by using GL_2–methods, [34, cor. 5.4.6]. The Banach representation denoted by $\Pi(P)$ in §2.2 is a local analog of Emerton's completed cohomology. However, Emerton's argument does not seem to carry over directly, since although P is projective in $\mathrm{Mod}_K^{\mathrm{pro}}(\mathscr{O})$, it is not a finitely generated $\mathscr{O}[\![K]\!]$–module, and in our context the locally algebraic vectors in $\Pi(P)$ are not a semi-simple representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Because of this we do not prove directly that $\cap_i \mathfrak{a}_i = 0$, but a weaker statement, which suffices for the argument to work. To get around the issue that P is not finitely generated over $\mathscr{O}[\![K]\!]$ we have to perform some tricks, see propositions 2.18, 2.19.

2.2. **Proof of proposition 2.1.**

Proof. Part (i) is [54, prop. 4.17].

Part (ii) follows from the proof of [54, Lem. 4.15], which unfortunately assumes $\Theta^d \otimes_{\mathscr{O}} k$ to be of finite length. This assumption is not necessary: since Θ^d is an object of $\mathfrak{C}(\mathscr{O})$ we may write $\Theta^d \otimes_{\mathscr{O}} k \cong \varprojlim M_i$, where the projective limit is taken over all the finite length quotients. Since P is projective we obtain an isomorphism $\mathrm{Hom}_{\mathfrak{C}(\mathscr{O})}(P, \Theta^d \otimes_{\mathscr{O}} k) \cong \varprojlim \mathrm{Hom}_{\mathfrak{C}(\mathscr{O})}(P, M_i)$. Since M_i are of finite length, [54, Lem. 3.3] says that $\dim_k \mathrm{Hom}_{\mathfrak{C}(\mathscr{O})}(P, M_i)$ is equal to the multiplicity with which $\pi^\vee$ occurs in M_i as a subquotient, which is the same as multiplicity with which π occurs in $M_i^\vee$ as a subquotient. Dually we obtain $\Theta \otimes_{\mathscr{O}} k \cong (\Theta^d \otimes_{\mathscr{O}} k)^\vee \cong \varinjlim M_i^\vee$, which allows to conclude that π occurs with finite multiplicity in $\Theta \otimes_{\mathscr{O}} k$ if and only if $\dim_k \mathrm{Hom}_{\mathfrak{C}(\mathscr{O})}(P, \Theta^d \otimes_{\mathscr{O}} k)$ is finite, in which case both numbers coincide. Since P is a compact flat $\mathscr{O}$-module and a projective object in $\mathfrak{C}(\mathscr{O})$, it follows that $\mathrm{Hom}_{\mathfrak{C}(\mathscr{O})}(P, \Theta^d)$ is a compact, flat $\mathscr{O}$-module, which is congruent to $\mathrm{Hom}_{\mathfrak{C}(\mathscr{O})}(P, \Theta^d/(\varpi))$ modulo ϖ, thus $\dim_L \mathrm{Hom}_{\mathfrak{C}(\mathscr{O})}(P, \Theta^d) \otimes_{\mathscr{O}} L = \dim_k \mathrm{Hom}_{\mathfrak{C}(\mathscr{O})}(P, \Theta^d \otimes_{\mathscr{O}} k)$.

Part (iii) a) is [54, prop. 4.18 (ii)] and Part (iii) b) is [54, prop. 4.19]. □

2.3. **Rings with polynomial identity.**

Definition 2.2. Let R be a (possibly non-commutative) ring and let n be a natural number. We say that R satisfies the standard identity s_n if for every n-tuple $\Phi = (\phi_1, \dots, \phi_n)$ of elements of R we have $s_n(\Phi) := \sum_\sigma \mathrm{sgn}(\sigma)\phi_{\sigma(1)} \dots \phi_{\sigma(n)} = 0$, where the sum is taken over all the permutations of the set $\{1, \dots, n\}$.

Remark 2.3. (i) Since $s_2(\phi_1, \phi_2) = \phi_1\phi_2 - \phi_2\phi_1$, the ring R satisfies the standard identity s_2 if and only if R is commutative.

(ii) We note that R satisfies s_n if and only if the opposite ring R^{op} satisfies s_n.

(iii) Let $\{\mathfrak{a}_i\}_{i \in I}$ be a family of ideals of R such that $\bigcap_{i \in I} \mathfrak{a}_i = 0$. Then R satisfies s_n if and only if $R/\mathfrak{a}_i$ satisfies s_n for all $i \in I$.

(iv) By a classical result of Amitsur and Levitzki [46, Thm.13.3.3], for any commutative ring A, the ring $M_n(A)$ satisfies the standard identity s_{2n}.

Lemma 2.4. *Let A be a commutative ring, $n \geq 1$ and let M be an A-module which is a quotient of A^n. Then $\mathrm{End}_A(M)$ satisfies the standard identity s_{2n}.*

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[8] If we assume that $p \geq 5$ then using results of [54] one may show that the assertion on the Galois side implies the assertion on the $\mathrm{GL}_2(\mathbb{Q}_p)$-side.

Proof. By hypothesis there are $e_1, ..., e_n \in M$ generating M as an A-module. Let $\phi_1, ..., \phi_{2n} \in \mathrm{End}_A(M)$ and let $X^{(1)}, ..., X^{(2n)} \in M_n(A)$ be matrices such that $\phi_k(e_i) = \sum_{j=1}^n X_{ji}^{(k)} e_j$ for all $i \leq n$ and $k \leq 2n$. Setting $X = s_{2n}(X^{(1)}, ..., X^{(2n)})$, for all $i \leq n$ we have

$$s_{2n}(\phi_1, ..., \phi_{2n})(e_i) = \sum_{j=1}^{n} X_{ji} e_j.$$

By remark 2.3 (iv) we have $X = 0$ and the result follows. □

Lemma 2.5. *Let A be a commutative noetherian ring, let M be an A-module, such that every finitely generated submodule is of finite length, and let n be an integer. If $\dim_{\kappa(\mathfrak{m})} M[\mathfrak{m}] \leq n$ for every maximal ideal $\mathfrak{m}$ of A then $\mathrm{End}_A(M)$ satisfies the standard identity s_{2n}.*

Proof. The assumption on M implies that $M \cong \oplus_{\mathfrak{m}} M[\mathfrak{m}^\infty]$, where the sum is taken over all the maximal ideals in A and $M[\mathfrak{m}^\infty] = \varinjlim M[\mathfrak{m}^n]$, where $M[\mathfrak{m}^n] = \{m \in M : am = 0, \forall a \in \mathfrak{m}^n\}$. Since $M[\mathfrak{m}^\infty]$ is only supported on $\{\mathfrak{m}\}$, if $\mathfrak{m}_1 \neq \mathfrak{m}_2$ then $\mathrm{Hom}_A(M[\mathfrak{m}_1^\infty], M[\mathfrak{m}_2^\infty]) = 0$. Thus $\mathrm{End}_A(M) \cong \prod_{\mathfrak{m}} \mathrm{End}_A(M[\mathfrak{m}^\infty])$ and it is enough to show the assertion in the case when $M = M[\mathfrak{m}^\infty]$, which we now assume.

Since in this case $\mathrm{End}_A(M) = \mathrm{End}_{\hat{A}}(M)$, where $\hat{A}$ is the $\mathfrak{m}$-adic completion of A, we may further assume that $(A, \mathfrak{m})$ is a complete local ring. Let $E(\kappa(\mathfrak{m}))$ be an injective envelope of $\kappa(\mathfrak{m})$ in the category of A-modules. The functor $(*)^\vee := \mathrm{Hom}_A(*, E(\kappa(\mathfrak{m})))$ induces an anti-equivalence of categories between artinian and noetherian A-modules, see [38, Thm. A.35]. Hence, $\mathrm{End}_A(M) \cong \mathrm{End}_A(M^\vee)^{op}$. Since $M[\mathfrak{m}] \hookrightarrow M$ is essential, we may embed $M \hookrightarrow E(\kappa(\mathfrak{m}))^{\oplus d}$, where $d = \dim_{\kappa(\mathfrak{m})} M[\mathfrak{m}]$. Since $E(\kappa(\mathfrak{m}))^\vee \cong A$ by [38, Thm. A.31], we obtain a surjection $A^{\oplus d} \twoheadrightarrow M^\vee$. We deduce from lemma 2.4 that $\mathrm{End}_A(M^\vee)$ satisfies the standard identity s_{2n}. □

2.4. **Density.** Let K be a pro-finite group with an open pro-p group. Let $\mathscr{O}[\![K]\!]$ be the completed group algebra, and let $\mathrm{Mod}_K^{\mathrm{pro}}(\mathscr{O})$ be the category of compact linear-topological $\mathscr{O}[\![K]\!]$-modules. Let $\{V_i\}_{i \in I}$ be a family of continuous representations of K on finite dimensional L-vector spaces, and let $M \in \mathrm{Mod}_K^{\mathrm{pro}}(\mathscr{O})$.

Definition 2.6. We say that $\{V_i\}_{i \in I}$ *captures* M if the smallest quotient $M \twoheadrightarrow Q$, such that $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(Q, V_i^*) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M, V_i^*)$ for all $i \in I$ is equal to M.

Lemma 2.7. *Let $N = \bigcap_\phi \mathrm{Ker}\, \phi$, where the intersection is taken over all $\phi \in \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M, V_i^*)$, for all $i \in I$. Then $\{V_i\}_{i \in I}$ captures M if and only if $N = 0$.*

Proof. It is immediate that $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M/N, V_i^*) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M, V_i^*)$ for all $i \in I$. This implies the assertion. □

Lemma 2.8. *Let M' be a closed $\mathscr{O}[\![K]\!]$-submodule of M. If $\{V_i\}_{i \in I}$ captures M, then it also captures M'.*

Proof. Let $v \in M'$ be non-zero. Since $\{V_i\}_{i \in I}$ captures M, lemma 2.7 implies that there exist $i \in I$ and $\phi \in \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M, V_i^*)$, such that $\phi(v) \neq 0$. Thus $\bigcap_\phi \mathrm{Ker}\, \phi = 0$, where the intersection is taken over all $\phi \in \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M', V_i^*)$, for all $i \in I$. Lemma 2.7 implies that $\{V_i\}_{i \in I}$ captures M'. □

Lemma 2.9. *Assume that $\{V_i\}_{i \in I}$ captures M and let $\phi \in \mathrm{End}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M)$. If ϕ kills $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M, V_i^*)$ for all $i \in I$ then $\phi = 0$.*

Proof. The assumption on ϕ implies that $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(\mathrm{Coker}\,\phi, V_i^*) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M, V_i^*)$ for all $i \in I$. Since $\{V_i\}_{i\in I}$ captures M, we deduce that $M = \mathrm{Coker}\,\phi$ and thus $\phi = 0$. $\square$

Lemma 2.10. *Let $M \in \mathrm{Mod}^{\mathrm{pro}}_K(\mathscr{O})$ be $\mathscr{O}$-torsion free, let $\Pi(M) := \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}}(M, L)$ be an L-Banach space equipped with a supremum norm. Then $\{V_i\}_{i\in I}$ captures M if and only if the image of the evaluation map $\oplus_i \mathrm{Hom}_K(V_i, \Pi(M)) \otimes_L V_i \to \Pi(M)$ is a dense subspace.*

Proof. It follows from [57, Thm.1.2] that the evaluation map $M \times \Pi(M) \to L$ induces an isomorphism

$$M \otimes_{\mathscr{O}} L \cong \mathrm{Hom}^{\mathrm{cont}}_L(\Pi(M), L). \tag{3}$$

If $\varphi \in \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M, V_i^*)$ then we define $\varphi^d \in \mathrm{Hom}_K(V_i, \Pi(M))$ by $\varphi^d(v)(m) := \varphi(m)(v)$. It follows from [57, Thm.1.2] that the map $\varphi \mapsto \varphi^d$ induces an isomorphism

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M, V_i^*) \cong \mathrm{Hom}_K(V_i, \Pi(M)). \tag{4}$$

Let $m \in M$ and let ℓ_m be the image of m in $\mathrm{Hom}^{\mathrm{cont}}_L(\Pi(M), L)$ under (3). Then for all $i \in I$ and all $\varphi \in \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M, V_i^*)$, $\varphi(m) = 0$ if and only if $\ell_m \circ \varphi^d = 0$. Using lemma 2.7 and isomorphisms (3), (4) we deduce that $\{V_i\}_{i\in I}$ does not capture M if and only if the image of the evaluation map $\oplus_i \mathrm{Hom}_K(V_i, \Pi(M)) \otimes_L V_i \to \Pi(M)$ is not a dense subspace. $\square$

Lemma 2.11. *The following assertions are equivalent:*

(i) *$\{V_i\}_{i\in I}$ captures every indecomposable projective in $\mathrm{Mod}^{\mathrm{pro}}_K(\mathscr{O})$;*
(ii) *$\{V_i\}_{i\in I}$ captures every projective in $\mathrm{Mod}^{\mathrm{pro}}_K(\mathscr{O})$;*
(iii) *$\{V_i\}_{i\in I}$ captures $\mathscr{O}[\![K]\!]$.*

Proof. (i) implies (ii). Let P be a projective object in $\mathrm{Mod}^{\mathrm{pro}}_K(\mathscr{O})$. Then $P \cong \prod_{j\in J} P_j$, where P_j is projective indecomposable for every $j \in J$, see [26, V.2.5.4]. For each $j \in J$ let $p_j : P \to P_j$ denote the projection. Since $\{V_i\}_{i\in I}$ captures P_j by assumption, it follows from lemma 2.7 that $\mathrm{Ker}\, p_j = \cap_\phi \mathrm{Ker}\, \phi \circ p_j$, where the intersection is taken over all $\phi \in \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P_j, V_i^*)$, for all $i \in I$. Since $\cap_{j\in J} \mathrm{Ker}\, p_j = 0$, we use lemma 2.7 again to deduce that $\{V_i\}_{i\in I}$ captures P.

(ii) implies (iii), as $\mathscr{O}[\![K]\!]$ is projective in $\mathrm{Mod}^{\mathrm{pro}}_K(\mathscr{O})$.

(iii) implies (i). Every indecomposable projective object in $\mathrm{Mod}^{\mathrm{pro}}_K(\mathscr{O})$ is a direct summand of $\mathscr{O}[\![K]\!]$, see for example [51, prop. 4.2]. The assertion follows from lemma 2.8. $\square$

Let G be an affine group scheme of finite type over $\mathbb{Z}_p$ such that G_L is a split connected reductive group over L. Let $\mathrm{Alg}(G)$ be the set isomorphism classes of irreducible rational representations of G_L, which we view as representations of $G(\mathbb{Z}_p)$ via the inclusion $G(\mathbb{Z}_p) \subset G(L)$.

Proposition 2.12. $\mathrm{Alg}(G)$ *captures every projective object in* $\mathrm{Mod}^{\mathrm{pro}}_{G(\mathbb{Z}_p)}(\mathscr{O})$*,*

Proof. The proof is very much motivated by [34, prop. 5.4.1], which implies the statement for $G = \mathrm{GL}_2$. Let $K = G(\mathbb{Z}_p)$ and let $\mathscr{C}(K, L)$ be the space of continuous functions from K to L. Since K is compact, the supremum norm makes $\mathscr{C}(K, L)$ into a unitary L-Banach representation of K. It is shown in [57, Lem.2.1, cor. 2.2] that the natural map $K \to \mathscr{O}[\![K]\!]$, $g \mapsto g$ induces an isometrical, K-equivariant isomorphism between $\mathscr{C}(K,L)$ and $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}}(\mathscr{O}[\![K]\!],L)$. It is shown in [52, prop. A.3] that the image of the evaluation map $\oplus \mathrm{Hom}_K(V,\mathscr{C}(K,L))\otimes V \to \mathscr{C}(K,L)$ is a dense subspace, where the sum is taken over all $V\in \mathrm{Alg}(G)$. Lemma 2.10 implies that $\mathrm{Alg}(V)$ captures $\mathscr{O}[\![K]\!]$, and the assertion follows from lemma 2.11. □

2.5. **Locally algebraic vectors in $\Pi(P)$.** From now on let $G=\mathrm{GL}_2(\mathbb{Q}_p)$, $K=\mathrm{GL}_2(\mathbb{Z}_p)$, and let π be an admissible smooth, absolutely irreducible k-representation of G. Recall that if $\chi_1,\chi_2:\mathbb{Q}_p^\times\to k^\times$ are smooth characters, then

$$\pi\{\chi_1,\chi_2\}:=(\mathrm{Ind}_B^G\chi_1\otimes\chi_2\omega^{-1})^{\mathrm{ss}}_{\mathrm{sm}}\oplus(\mathrm{Ind}_B^G\chi_2\otimes\chi_1\omega^{-1})^{\mathrm{ss}}_{\mathrm{sm}}.$$

Definition 2.13. If π is supersingular, let $d(\pi)=1$. Otherwise, there is a unique $\pi\{\chi_1,\chi_2\}$ containing π and we let $d(\pi)$ be the multiplicity of π in $\pi\{\chi_1,\chi_2\}$.

Lemma 2.14. *Let $\chi_1,\chi_2:\mathbb{Q}_p^\times\to k^\times$ be smooth characters. Then $\pi\{\chi_1,\chi_2\}$ is isomorphic to one of the following:*

(i) $(\mathrm{Ind}_B^G\chi_1\otimes\chi_2\omega^{-1})_{\mathrm{sm}}\oplus(\mathrm{Ind}_B^G\chi_2\otimes\chi_1\omega^{-1})_{\mathrm{sm}}$, *if* $\chi_1\chi_2^{-1}\neq\mathbf{1},\omega^{\pm1}$*;*
(ii) $(\mathrm{Ind}_B^G\chi\otimes\chi\omega^{-1})^{\oplus2}_{\mathrm{sm}}$, *if* $\chi_1=\chi_2=\chi$ *and* $p\geq3$*;*
(iii) $(\mathbf{1}\oplus\mathrm{Sp}\oplus\mathrm{Ind}_B^G\omega\otimes\omega^{-1})\otimes\chi\circ\det$, *if* $\chi_1\chi_2^{-1}=\omega^{\pm1}$ *and* $p\geq5$*;*
(iv) $(\mathbf{1}\oplus\mathrm{Sp}\oplus\omega\circ\det\oplus\mathrm{Sp}\otimes\omega\circ\det)\otimes\chi\circ\det$, *if* $\chi_1\chi_2^{-1}=\omega^{\pm1}$ *and* $p=3$*;*
(v) $(\mathbf{1}\oplus\mathrm{Sp})^{\oplus2}\otimes\chi\circ\det$ *if* $\chi_1=\chi_2$ *and* $p=2$.

In particular, $d(\pi)=1$ unless we are in one of the following cases, when $d(\pi)=2$:

(a) $p\geq3$ *and* $\pi\cong(\mathrm{Ind}_B^G\chi\otimes\chi\omega^{-1})_{\mathrm{sm}}$ *or*
(b) $p=2$ *and either* $\pi\cong\chi\circ\det$ *or* $\pi\cong\mathrm{Sp}\otimes\chi\circ\det$*, for some smooth character* $\chi:\mathbb{Q}_p^\times\to k^\times$.

Proof. The representation $(\mathrm{Ind}_B^G\chi_1\otimes\chi_2\omega^{-1})_{\mathrm{sm}}$ is irreducible if and only if $\chi_1\neq\chi_2\omega^{-1}$, otherwise its semi-simplification consists of a character and a twist of the Steinberg representation, see [2, Thm. 30]. The result follows. □

Let $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathscr{O})$ be the category of locally admissible representations introduced by Emerton in [35]. Proposition 2.2.18 in [35] shows that $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathscr{O})$ is closed under subquotients and arbitrary direct sums in $\mathrm{Mod}^{\mathrm{sm}}_G(\mathscr{O})$, and theorem 2.3.8 in [35] implies that every locally admissible representation is a union of its subrepresentations of finite length. So $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathscr{O})$ satisfies the conditions imposed on $\mathrm{Mod}^{?}_G(\mathscr{O})$ in §2.1. Let $\mathfrak{C}(\mathscr{O})$ be the full subcategory of $\mathrm{Mod}^{\mathrm{pro}}_G(\mathscr{O})$, which is anti-equivalent to $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathscr{O})$ via Pontryagin duality. We have $\mathrm{Ban}^{\mathrm{adm}}_{\mathfrak{C}(\mathscr{O})}=\mathrm{Ban}^{\mathrm{adm}}_G(L)$.

Let $P\twoheadrightarrow\pi^\vee$ be a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathscr{O})$ and let $E=\mathrm{End}_{\mathfrak{C}(\mathscr{O})}(P)$. Then $\pi\hookrightarrow P^\vee$ is an injective envelope of π in $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathscr{O})$. The following result is [36, cor. 3.10].

Proposition 2.15. *The restriction of $P^\vee$ to K is injective in $\mathrm{Mod}^{\mathrm{sm}}_K(\mathscr{O})$, hence P is projective in $\mathrm{Mod}^{\mathrm{pro}}_K(\mathscr{O})$.*

In particular[9], P is a torsionfree, compact linear-topological $\mathscr{O}$-module. Let

$$\Pi(P):=\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}}(P,L)$$

with the topology induced by the supremum norm. If Π is an L-Banach space and if Θ is an open, bounded lattice in Π, let $\Theta^d:=\mathrm{Hom}_{\mathscr{O}}(\Theta,\mathscr{O})$ be its Schikhof dual. Equipped with the topology of pointwise convergence, Θ^d is a torsionfree,

[9]Alternatively one may argue in the same way as in [54, cor. 5.19].

compact linear-topological $\mathscr{O}$-module and it follows from [57, Thm.1.2] that we have a natural isomorphism:

$$\mathrm{Hom}_L^{\mathrm{cont}}(\Pi, \Pi(P)) \cong \mathrm{Hom}_{\mathscr{O}}^{\mathrm{cont}}(P, \Theta^d) \otimes_{\mathscr{O}} L. \tag{5}$$

We want to use (5) in two ways, which are consequences of [57, Thm.2.3]. If Π is an admissible unitary L-Banach representation of G and Θ is an open, bounded, G-invariant lattice in Π, then Θ^d is in $\mathfrak{C}(\mathscr{O})$ and we have:

$$\mathrm{Hom}_G^{\mathrm{cont}}(\Pi, \Pi(P)) \cong \mathrm{Hom}_{\mathfrak{C}(\mathscr{O})}(P, \Theta^d) \otimes_{\mathscr{O}} L = \mathrm{m}(\Pi) \tag{6}$$

On the other hand, if V is a continuous representation of K on a finite dimensional L-vector space and if Θ is a K-invariant lattice in V, then

$$\mathrm{Hom}_K(V, \Pi(P)) \cong \mathrm{Hom}_{\mathscr{O}[\![K]\!]}^{\mathrm{cont}}(P, \Theta^d) \otimes_{\mathscr{O}} L \cong \mathrm{Hom}_{\mathscr{O}[\![K]\!]}^{\mathrm{cont}}(P, V^*). \tag{7}$$

We note that since V is finite dimensional any L-linear map is continuous.

Let $\mathrm{Alg}(G)$ be the set of isomorphism classes of irreducible rational representations of GL_2/L, which we view as representations of $\mathrm{GL}_2(\mathbb{Z}_p)$ via the inclusion $\mathrm{GL}_2(\mathbb{Z}_p) \subset \mathrm{GL}_2(L)$. For $V \in \mathrm{Alg}(G)$ let $A_V := \mathrm{End}_G(\text{c-Ind}_K^G V)$. It follows from [8, rem.2.1.4.2] that $A_V \cong \mathrm{End}_G(\text{c-Ind}_K^G \mathbf{1}) \cong L[t, z^{\pm 1}]$. In particular, A_V is a commutative noetherian ring. Frobenius reciprocity gives

$$\mathrm{Hom}_K(V, \Pi(P)) \cong \mathrm{Hom}_G(\text{c-Ind}_K^G V, \Pi(P)).$$

Hence, $\mathrm{Hom}_K(V, \Pi(P))$ is naturally an A_V-module. We transport the action of A_V onto $\mathrm{Hom}_{\mathscr{O}[\![K]\!]}^{\mathrm{cont}}(P, V^*)$ via (7).

Proposition 2.16. *Let $V \in \mathrm{Alg}(G)$ and let $\mathfrak{m}$ be a maximal ideal of A_V. Then*

$$\dim_{\kappa(\mathfrak{m})} \mathrm{Hom}_G(\kappa(\mathfrak{m}) \otimes_{A_V} \text{c-Ind}_K^G V, \Pi(P)) \leq d(\pi).$$

Proof. It follows from [8, prop. 3.2.1] that

$$\kappa(\mathfrak{m}) \otimes_{A_V} \text{c-Ind}_K^G V \cong (\mathrm{Ind}_B^G \delta_1 \otimes \delta_2 |\cdot|^{-1})_{\mathrm{sm}} \otimes_L V, \tag{8}$$

where $\delta_1, \delta_2 : \mathbb{Q}_p^\times \to \kappa(\mathfrak{m})^\times$ are unramified characters with $\delta_1 |\cdot| \neq \delta_2$ and the subscript sm indicates smooth induction. Let Π the universal unitary completion of $\kappa(\mathfrak{m}) \otimes_{A_V} \text{c-Ind}_K^G V$. Since the action of G on $\Pi(P)$ is unitary, the universal property of Π implies that

$$\mathrm{Hom}_G(\kappa(\mathfrak{m}) \otimes_{A_V} \text{c-Ind}_K^G V, \Pi(P)) \cong \mathrm{Hom}_G^{\mathrm{cont}}(\Pi, \Pi(P)) \overset{(6)}{\cong} \mathrm{m}(\Pi). \tag{9}$$

It is proved in [52, prop. 2.10], using results of Berger–Breuil [5] as the main input, that Π is an admissible finite length $\kappa(\mathfrak{m})$-Banach representation of G. Moreover, if Π is non-zero then $\overline{\Pi}^{ss}$ is either irreducible supersingular, or $\overline{\Pi}^{ss} \subseteq \pi\{\chi_1, \chi_2\}$ for some smooth characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k_{\kappa(\mathfrak{m})}^\times$. Lemma 2.14 implies that $\pi \otimes_k k_{\kappa(\mathfrak{m})}$ can occur in $\overline{\Pi}^{ss}$ with multiplicity at most $d(\pi)$. Hence, if Θ is an open, bounded and G-invariant lattice in Π, then π can occur as a subquotient of $\Theta/(\varpi)$ with multiplicity at most $[\kappa(\mathfrak{m}) : L] d(\pi)$. Proposition 2.1 (ii) yields $\dim_L \mathrm{m}(\Pi) \leq [\kappa(\mathfrak{m}) : L] d(\pi)$. The result follows from (9). $\square$

Corollary 2.17. *For all $V \in \mathrm{Alg}(G)$ and all maximal ideals $\mathfrak{m}$ of A_V we have*

$$\dim_{\kappa(\mathfrak{m})} \mathrm{Hom}_{\mathscr{O}[\![K]\!]}^{\mathrm{cont}}(P, V^*)[\mathfrak{m}] \leq d(\pi).$$

Proof. By (7) we have $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, V^*)[\mathfrak{m}] \cong \mathrm{Hom}_K(V, \Pi(P))[\mathfrak{m}]$. On the other hand, Frobenius reciprocity gives an isomorphism

$$\begin{aligned}\mathrm{Hom}_K(V, \Pi(P))[\mathfrak{m}] &\cong \mathrm{Hom}_G(\text{c-Ind}_K^G V, \Pi(P))[\mathfrak{m}] \\ &\cong \mathrm{Hom}_G(\kappa(\mathfrak{m}) \otimes_{A_V} \text{c-Ind}_K^G V, \Pi(P)).\end{aligned} \tag{10}$$

The result follows therefore from proposition 2.16. □

2.6. **Proof of theorem 1.4.**

Proposition 2.18. *Let $\varphi : P \twoheadrightarrow M$ be a quotient in $\mathfrak{C}(\mathscr{O})$, such that M is of finite length. Then φ factors through $\psi : P \to N$ in $\mathfrak{C}(\mathscr{O})$, such that N is a finitely generated projective $\mathscr{O}[\![K]\!]$-module.*

Proof. We claim that there exists a surjection $\theta : N \twoheadrightarrow M$ in $\mathfrak{C}(\mathscr{O})$ with N a finitely generated projective $\mathscr{O}[\![K]\!]$-module. The claim implies the assertion, since the projectivity of P implies that there exists $\psi : P \to N$, such that $\theta \circ \psi = \varphi$. The proof of the claim is a variation of the construction, which first appeared in [47], and then was generalized in [12] and [36]. Let $G^0 = \{g \in G : \det g \in \mathbb{Z}_p^\times\}$ and let $G^+ = ZG^0$, where Z is the centre of G. Since $M^\vee$ is of finite length in $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathscr{O})$, $M^\vee$ is admissible. It follows from [36, Thm.3.4] that there exists an injection $M^\vee \hookrightarrow \Omega$ in $\mathrm{Mod}^{\mathrm{adm}}_{G^0}(\mathscr{O})$, such that $M^\vee|_K \hookrightarrow \Omega|_K$ is an injective envelope of $M^\vee$ in $\mathrm{Mod}^{\mathrm{sm}}_K(\mathscr{O})$ and $\Omega \cong \Omega^c$, where Ω^c denotes the action of G^0 twisted by conjugation with an element $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. Dually we obtain a continuous, G^0-equivariant surjection $\theta^0 : \Omega^\vee \twoheadrightarrow M$, such that its restriction to K is a projective envelope of M in $\mathrm{Mod}^{\mathrm{pro}}_K(\mathscr{O})$.

We let $A := \mathscr{O}[t, t^{-1}]$ act on M by letting t act as the matrix $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$. Since M is a quotient of P, its cosocle in $\mathfrak{C}(\mathscr{O})$ is isomorphic to $\pi^\vee$, and hence is irreducible. This implies that M is indecomposable. Moreover, M is finite length by assumption. The argument of [36, cor. 3.9] shows that there exists a monic polynomial $f \in \mathscr{O}[t]$ and a natural number n, such that (ϖ, f) is a maximal ideal of A, and the action of A on M factors through $A/(f^n)$. Since f is monic $A/(f^n)$ is a free $\mathscr{O}$-module of finite rank. Hence, the restriction of $N^+ := A/(f^n) \otimes_{\mathscr{O}} \Omega^\vee$ to K is a finite direct sum of copies of $\Omega^\vee$, which implies that N^+ is a finitely generated projective $\mathscr{O}[\![K]\!]$-module. We put an action of G^+ on N^+ by using $G^+ = \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)^{\mathbb{Z}} \times G^0$. The map $t \otimes v \mapsto \theta^0(\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)v)$ induces a G^+-equivariant surjection $\theta^+ : N^+ \twoheadrightarrow M$. Let $N := \mathrm{Ind}_{G^+}^G N^+$, then by Frobenius reciprocity we obtain a surjective map $\theta : N \twoheadrightarrow M$. Since G^+ is of index 2 in G, and $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ is a representative of the non-trivial coset, we have $N|_{G^+} \cong N^+ \oplus (N^+)^c \cong N^+ \oplus N^+$, where the subscript c indicates that the action of G^+ is twisted by conjugation with $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$, and the last isomorphism follows from $\Omega \cong \Omega^c$. Hence, N satisfies the conditions of the claim. □

If V is a continuous representation of K on a finite dimensional L–vector space and if Θ is an open, bounded and K-invariant lattice in V, let $|\,.\,|$ be the norm on V^* given by $|\ell| := \sup_{v \in \Theta} |\ell(v)|$, so that $\Theta^d = \mathrm{Hom}_{\mathscr{O}}(\Theta, \mathscr{O})$ is the unit ball in V^* with respect to $|\,.\,|$. The topology on $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, V^*)$ is given by the norm $\|\phi\| := \sup_{v \in P} |\phi(v)|$, and $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, \Theta^d)$ is the unit ball in this Banach space.

Proposition 2.19. *For all V as above the submodule*

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, V^*)_{\mathrm{l.fin}} := \{\phi \in \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, V^*) : \ell_{A_V}(A_V \phi) < \infty\}$$

is dense in $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, V^*)$, *where* $\ell_{A_V}(A_V\phi)$ *is the length of* $A_V\phi$ *as an* A_V*-module.*

Proof. Let $A = A_V$. It is enough to show that for each $\phi \in \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, \Theta^d)$ and each $n \geq 1$ there exists $\psi_n \in \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, \Theta^d)$ such that the A-submodule generated by ψ_n is of finite length, and $\phi \equiv \psi_n \pmod{\varpi^n}$.

Let ϕ_n be the composition $P \xrightarrow{\phi} \Theta^d \twoheadrightarrow \Theta^d/(\varpi^n)$. Dually we obtain a map $\phi_n^\vee : (\Theta^d/(\varpi^n))^\vee \to P^\vee$. Let τ be the G-subrepresentation of $P^\vee$ generated by the image of $\phi_n^\vee$. Since $P^\vee$ is in $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathscr{O})$ any finitely generated G-subrepresentation is of finite length. Since $(\Theta^d/(\varpi^n))^\vee$ is a finite $\mathscr{O}$-module, we deduce that τ is of finite length. Thus ϕ_n factors through $P \twoheadrightarrow \tau^\vee$ in $\mathfrak{C}(\mathscr{O})$, with $\tau^\vee$ of finite length. Proposition 2.18 implies that this map factors through $\psi : P \to N$ with N finitely generated and projective $\mathscr{O}[\![K]\!]$-module. Since N is projective, using the exact sequence $0 \to \Theta^d \xrightarrow{\varpi^n} \Theta^d \to \Theta^d/(\varpi^n) \to 0$, we deduce that there exists $\theta_n \in \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(N, \Theta^d)$, which maps to $\phi_n \in \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(N, \Theta^d/(\varpi^n))$. Let $\psi_n = \theta_n \circ \psi$. Then by construction $\phi \equiv \psi_n \pmod{\varpi^n}$. Since ψ is G-equivariant, $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(N, V^*) \xrightarrow{\circ\psi} \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, V^*)$ is a map of A-modules, which contains ψ_n in the image. Since N is a finitely generated $\mathscr{O}[\![K]\!]$-module, $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(N, V^*)$ is a finite dimensional L-vector space, thus the A-submodule generated by ψ_n is of finite length. $\square$

Corollary 2.20. *For V as above, let $\mathfrak{a}_V$ be the $E[1/p]$-annihilator of* $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, V^*)$. *Then $E[1/p]/\mathfrak{a}_V$ satisfies the standard identity $s_{2d(\pi)}$ (see definition 2.13 for $d(\pi)$).*

Proof. Since the action of E preserves the unit ball in $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, V^*)$, $E[1/p]$ acts by continuous endomorphisms, which commute with the action of A_V. It follows from proposition 2.19 that $E[1/p]/\mathfrak{a}_V$ injects into $\mathrm{End}_{A_V}(\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, V^*)_{\mathrm{l.fin}})$. It follows from proposition 2.16 and lemma 2.17 that

$$\dim_{\kappa(\mathfrak{m})} \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(P, V^*)_{\mathrm{l.fin}}[\mathfrak{m}] \leq d(\pi),$$

for every maximal ideal $\mathfrak{m}$ of A_V. The assertion follows from lemma 2.5. $\square$

Theorem 2.21. *Let Π be a unitary admissible absolutely irreducible L-Banach space representation of G and let Θ be an open bounded G-invariant lattice in Π. Then π occurs with multiplicity $\leq d(\pi)$ as a subquotient of $\Theta \otimes_{\mathscr{O}} k$.*

Proof. Let $d = d(\pi)$, then it is enough to prove that $\dim_L \mathrm{m}(\Pi) \leq d$ by proposition 2.1 (ii). It follows from propositions 2.15 and 2.12 that $\mathrm{Alg}(G)$ captures P, and lemma 2.7 (ii) implies that $\bigcap_{V \in \mathrm{Alg}(G)} \mathfrak{a}_V = 0$, where $\mathfrak{a}_V$ is defined in corollary 2.20. We deduce from corollary 2.20 and remark 2.3 that $E[1/p]$ satisfies the standard identity s_{2d}. Thus, if $\mathscr{E}$ is the image of $E[1/p]$ in $\mathrm{End}_L(\mathrm{m}(\Pi))$, then $\mathscr{E}$ satisfies the standard identity s_{2d}.

Since Π is irreducible, it follows from proposition 2.1(iii)a) that $\mathrm{m}(\Pi)$ is an irreducible $\mathscr{E}$-module, which is clearly faithful. Proposition 2.1(iii)b) shows that $\mathrm{End}_{\mathscr{E}}(\mathrm{m}(\Pi)) = \mathrm{End}^{\mathrm{cont}}_G(\Pi)^{\mathrm{op}}$. On the other hand, since Π is absolutely irreducible, Schur's lemma [27, Thm.1.1.1] yields $\mathrm{End}^{\mathrm{cont}}_G(\Pi) = L$, hence $\mathrm{End}_{\mathscr{E}}(\mathrm{m}(\Pi)) = L$. A theorem of Kaplansky, see [56, Thm.II.1.1] and [56, cor. II.1.2], implies that $\dim_L \mathrm{m}(\Pi) \leq d$, which is the desired result. $\square$

Corollary 2.22. *Let π be an absolutely irreducible smooth representation and let $P \twoheadrightarrow \pi^\vee$ be a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$, where $\mathfrak{C}(\mathcal{O})$ is the Pontryagin dual of $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$. If one of the following holds:*

(i) *π is supersingular;*
(ii) *$\pi \cong (\mathrm{Ind}_B^G \chi_1 \otimes \chi_2\omega^{-1})_{\mathrm{sm}}$ and $\chi_1\chi_2^{-1} \neq \omega^{\pm 1}$, $\mathbf{1}$;*
(iii) *$\pi \cong (\mathrm{Ind}_B^G \chi\omega \otimes \chi\omega^{-1})_{\mathrm{sm}}$ and $p \geq 5$;*
(iv) *$\pi \cong \mathrm{Sp} \otimes \chi \circ \det$ and $p \geq 3$;*
(v) *$\pi \cong \chi \circ \det$ and $p \geq 3$;*

then the ring $E := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P)$ is commutative.

Proof. In these cases $d(\pi) = 1$, and the assertion follows from the proof of theorem 2.21 and remark 2.3. □

Corollary 2.23. *Let Π be a unitary admissible absolutely irreducible L-Banach space representation of G and let Θ be an open bounded G-invariant lattice in Π. Then $\Theta \otimes_{\mathcal{O}} k$ is of finite length. Moreover, one of the following holds:*

(i) *$\Theta \otimes_{\mathcal{O}} k$ is absolutely irreducible supersingular;*
(ii) *$\Theta \otimes_{\mathcal{O}} k$ is irreducible and*

$$\Theta \otimes_{\mathcal{O}} l \cong (\mathrm{Ind}_P^G \chi \otimes \chi^\sigma \omega^{-1})_{\mathrm{sm}} \oplus (\mathrm{Ind}_P^G \chi^\sigma \otimes \chi\omega^{-1})_{\mathrm{sm}}$$

where l is a quadratic extension of k, $\chi : \mathbb{Q}_p^\times \to l^\times$ a smooth character and χ^σ is a conjugate of χ by the non-trivial element in $\mathrm{Gal}(l/k)$;
(iii) *$(\Theta \otimes_{\mathcal{O}} k)^{ss} \subseteq \pi\{\chi_1, \chi_2\}$ for some smooth characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$.*

Proof. Let π be an irreducible subquotient of $\Theta \otimes_{\mathcal{O}} k$. If π' is another irreducible subquotient of $\Theta \otimes_{\mathcal{O}} k$ then π and π' lie in the same block by [54, prop. 5.36], which means that there exist irreducible smooth k-representations $\pi = \pi_0, \ldots, \pi_n = \pi'$, such that for all $0 \leq i < n$ either $\mathrm{Ext}^1_G(\pi_i, \pi_{i+1}) \neq 0$ or $\mathrm{Ext}^1_G(\pi_{i+1}, \pi_i) \neq 0$. The blocks containing an absolutely irreducible representation have been determined in [52], and consist of either a single supersingular representation, or of all irreducible subquotients of $\pi\{\chi_1, \chi_2\}$ for some smooth characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$. These irreducible subquotients are listed explicitly in lemma 2.14. If π is absolutely irreducible, it follows from theorem 2.21 that if π is supersingular then (i) holds, if π is not supersingular then the multiplicity with which π occurs as a subquotient of $\Theta \otimes_{\mathcal{O}} k$ is less or equal to the multiplicity with which π occurs in $\pi\{\chi_1, \chi_2\}$, which implies that (iii) holds. If π is not absolutely irreducible, then arguing as in the proof of corollary 5.44 of [54] we deduce that (ii) holds. □

3. Injectivity of the functor $\Pi \mapsto \mathbf{V}(\Pi)$

In this chapter we prove theorems 1.8 and 1.9 as well as their consequences stated in the introduction. After a few preliminaries devoted to the theory of (φ, Γ)-modules and various constructions involved in the p-adic local Langlands correspondence [20], we give a detailed overview of the (rather technical) proofs. We then go on and supply the technical details of the proofs.

3.1. Preliminaries.

3.1.1. *(φ,Γ)-modules.* Let $\mathscr{O}_{\mathscr{E}}$ be the p-adic completion of $\mathscr{O}[[T]][T^{-1}]$, $\mathscr{E}=\mathscr{O}_{\mathscr{E}}[p^{-1}]$ the field of fractions of $\mathscr{O}_{\mathscr{E}}$ and let $\mathscr{R}$ be the Robba ring, consisting of those Laurent series $\sum_{n\in\mathbb{Z}} a_nT^n\in L[[T,T^{-1}]]$ which converge on some annulus $0<v_p(T)\leq r$, where $r>0$ depends on the series.

Let $\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ be the category of étale (φ,Γ)-modules over $\mathscr{E}$. These are finite dimensional $\mathscr{E}$-vector spaces D endowed with semi-linear[10] and commuting actions of φ and $\Gamma=\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)\cong\mathbb{Z}_p^\times$ such that the action of φ is étale[11]. Each $D\in\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ is naturally endowed with an operator ψ, which is left-inverse to φ and commutes with Γ.

The category $\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ is equivalent [41] to the category $\mathrm{Rep}_L(\mathscr{G}_{\mathbb{Q}_p})$ of continuous finite dimensional L-representations of $\mathscr{G}_{\mathbb{Q}_p}$. Cartier duality[12] on $\mathrm{Rep}_L(\mathscr{G}_{\mathbb{Q}_p})$ induces a Cartier duality $D\to\check{D}$ on $\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$. All these constructions have integral and torsion analogues, which will be used without further comment.

The category $\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ is equivalent, by [16, 6] and [42], to the category of (φ,Γ)-modules of slope 0 on $\mathscr{R}$. For $D\in\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ we let D_{rig} be the associated (φ,Γ)-module over $\mathscr{R}$.

3.1.2. *Analytic operations on (φ,Γ)-modules.* The monoid $P^+=\left(\begin{smallmatrix}\mathbb{Z}_p-\{0\} & \mathbb{Z}_p\\ 0 & 1\end{smallmatrix}\right)$ acts naturally on $\mathbb{Z}_p$ by $\left(\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right)x=ax+b$. Any $D\in\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ carries a P^+ action, defined by

$$\begin{pmatrix}p^ka & b\\ 0 & 1\end{pmatrix}z=(1+T)^b\varphi^k\circ\sigma_a(z)$$

for $a\in\mathbb{Z}_p^\times, b\in\mathbb{Z}_p$ and $k\in\mathbb{N}$.

D also gives rise to a P^+-equivariant sheaf $U\mapsto D\boxtimes U$ on $\mathbb{Z}_p$, whose sections on $i+p^k\mathbb{Z}_p$ are $\begin{pmatrix}p^k & i\\ 0 & 1\end{pmatrix}D\subset D=D\boxtimes\mathbb{Z}_p$ and for which the restriction map $\mathrm{Res}_{i+p^k\mathbb{Z}_p}:D\boxtimes\mathbb{Z}_p\to D\boxtimes(i+p^k\mathbb{Z}_p)$ is given by $\left(\begin{smallmatrix}1 & i\\ 0 & 1\end{smallmatrix}\right)\circ\varphi^k\circ\psi^k\circ\left(\begin{smallmatrix}1 & -i\\ 0 & 1\end{smallmatrix}\right)$.

Let U be an open compact subset of $\mathbb{Z}_p$ and let $\phi:U\to L$ be a continuous function. By [19, prop. V.2.1], the limit

$$m_\phi(z)=\lim_{N\to\infty}\sum_{i\in I_N(U)}\phi(i)\mathrm{Res}_{i+p^N\mathbb{Z}_p}(z)$$

exists for all $z\in D\boxtimes U$, and it is independent of the system of representatives $I_N(U)$ of U mod p^N. Moreover, the resulting map $m_\phi:D\boxtimes U\to D\boxtimes U$ is L-linear and continuous.

In the same vein [19, prop. V.1.3], if U,V are compact open subsets of $\mathbb{Z}_p$ and if $f:U\to V$ is a local diffeomorphism, there is a direct image operator

$$f_*:D\boxtimes U\to D\boxtimes V,\quad f_*(z)=\lim_{N\to\infty}\sum_{i\in I_N(U)}\begin{pmatrix}f'(i) & f(i)\\ 0 & 1\end{pmatrix}\mathrm{Res}_{p^n\mathbb{Z}_p}\left(\left(\begin{smallmatrix}1 & -i\\ 0 & 1\end{smallmatrix}\right)z\right).$$

The following result (see § $V.1$ and $V.2$ in [19]) summarizes the main properties of these operators (which also have integral and torsion versions, see loc.cit.).

Proposition 3.1. *Let U,V be compact open subsets of $\mathbb{Z}_p$.*

a) For all continuous maps $\phi_1,\phi_2:U\to L$ we have $m_{\phi_1}\circ m_{\phi_2}=m_{\phi_1\phi_2}$.

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[10]The rings $\mathscr{O}_{\mathscr{E}},\mathscr{E},\mathscr{R}$ are endowed with a Frobenius φ and an action of Γ defined by $\varphi(T)=(1+T)^p-1$ and $\gamma(T)=(1+T)^{\varepsilon(\gamma)}-1$.

[11]This means that the matrix of φ in some basis of D belongs to $\mathrm{GL}_d(\mathscr{O}_{\mathscr{E}})$, where $d=\dim_{\mathscr{E}}(D)$.

[12]Sending V to $\check{V}:=V^*\otimes\varepsilon$, where V^* is the L-dual of V.

b) If $f : U \to V$ is a local diffeomorphism and $\phi : V \to L$ is continuous, then

$$f_* \circ m_{\phi\circ f} = m_\phi \circ f_*.$$

c) If $f : U \to V$ and $g : V \to W$ are local diffeomorphisms, then $g_* \circ f_* = (g\circ f)_*$.
d) If $\phi : U \to L$ is a continuous map and $V \subset U$, then m_ϕ commutes with Res_V.
e) If $\phi : \mathbb{Z}_p^\times \to L$ is constant on $a + p^n\mathbb{Z}_p$ for all $a \in \mathbb{Z}_p^\times$, then

$$m_\phi = \sum_{i\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \phi(i)\mathrm{Res}_{i+p^n\mathbb{Z}_p}.$$

3.1.3. *From* $\mathrm{GL}_2(\mathbb{Q}_p)$ *to* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$*-representations and back.* Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$. We refer the reader to the introduction for the definition of the category $\mathrm{Rep}_L(G)$, and to [20, ch. IV] (or to [23, § III.2] for a summary) for the construction and study of an exact and contravariant functor $\mathbf{D} : \mathrm{Rep}_L(G) \to \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$. Composing this functor with Fontaine's [41] equivalence of categories and Cartier duality, we obtain an exact covariant functor $\Pi \mapsto \mathbf{V}(\Pi)$ from $\mathrm{Rep}_L(G)$ to $\mathrm{Rep}_L(G_{\mathbb{Q}_p})$. We will actually work with the functor $\mathbf{D}$, even though some results will be stated in terms of the more familiar functor $\mathbf{V}$.

In the opposite direction, δ being fixed, there is a functor from $\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ to the category of G-equivariant sheaves of topological L-vector spaces on $\mathbf{P}^1 = \mathbf{P}^1(\mathbb{Q}_p)$ (the space of sections on an open set U of $\mathbf{P}^1$ of the sheaf associated to D is denoted by $D \boxtimes_\delta U$). If $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$, then the restriction to $\mathbb{Z}_p$ of the sheaf $U \mapsto D \boxtimes_\delta U$ is the P^+-equivariant sheaf attached to D as in n° 3.1.2 (in particular it does not depend on δ). The space $D \boxtimes_\delta \mathbf{P}^1$ of global sections of the sheaf attached to D and δ is naturally a topological G-module.

Definition 3.2. If $\delta : \mathbb{Q}_p^\times \to \mathscr{O}^\times$ is a unitary character then we let $\mathrm{Rep}_L(\delta)$ be the full subcategory of $\mathrm{Rep}_L(G)$ consisting of representations with central character δ and we let $\mathscr{MF}(\delta)$ be the essential image of $\mathbf{D}|_{\mathrm{Rep}_L(\delta)}$.

The following result follows by combining the main results of [23, chap. III].

Proposition 3.3. *If $\delta : \mathbb{Q}_p^\times \to \mathscr{O}^\times$ is a unitary character then there is a functor*

$$\mathscr{MF}(\delta^{-1}) \to \mathrm{Rep}_L(\delta), \quad D \mapsto \Pi_\delta(D)$$

such that for all $D \in \mathscr{MF}(\delta^{-1})$, we have:
(i) *If η is a unitary character, then*[13] *$D(\eta) \in \mathscr{MF}(\eta^{-2}\delta^{-1})$ and*

$$\Pi_{\eta^2\delta}(D(\eta)) \cong \Pi_\delta(D) \otimes (\eta \circ \det).$$

(ii) *$\check{D} \in \mathscr{MF}(\delta)$ and there is an exact sequence*[14]

$$0 \to \Pi_{\delta^{-1}}(\check{D})^* \to D \boxtimes_\delta \mathbf{P}^1 \to \Pi_\delta(D) \to 0.$$

(iii) *There is a canonical isomorphism $\mathbf{D}(\Pi_\delta(D)) \cong \check{D}$.*
(iv) *If $\dim(D) \geq 2$, then D is irreducible if and only if $\Pi_\delta(D)$ is irreducible.*

All these constructions have natural integral and torsion variants, which will be used without further comment: for instance, if D_0 is an $\mathscr{O}_{\mathscr{E}}$-lattice in $D \in \mathscr{MF}(\delta^{-1})$ which is stable by φ and Γ, then $\Pi_\delta(D_0)$ is an open, bounded and G-invariant lattice in $\Pi_\delta(D)$.

[13] Here $D(\eta)$ is the (φ,Γ)-module obtained by twisting the action of φ and Γ by η.
[14] Of topological G-modules, where $\Pi_{\delta^{-1}}(\check{D})^*$ is the weak dual of $\Pi_{\delta^{-1}}(\check{D})$.

The next result is the main ingredient for the proof of the surjectivity of the p-adic Langlands correspondence for G (cf. § 1.3). Note that if $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$, then $\det D$ corresponds by Fontaine's equivalence of categories to a continuous character of $\mathscr{G}_{\mathbb{Q}_p}$, which in turn can be seen as a unitary character $\det D : \mathbb{Q}_p^\times \to \mathscr{O}^\times$ by local class field theory. We define

$$\delta_D : \mathbb{Q}_p^\times \to \mathscr{O}^\times, \quad \delta_D = \varepsilon^{-1} \det D.$$

Proposition 3.4. *If $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$ is 2-dimensional, then $D \in \mathscr{MF}(\delta_D^{-1})$.*

Proof. This is a restatement of [24, prop. 10.1]. □

3.2. **Uniqueness of the central character.** In this § we explain the steps of the proof of the following theorem, which is the main result of this chapter.

Theorem 3.5. *Let $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$ be absolutely irreducible, 2-dimensional. If $D \in \mathscr{MF}(\delta^{-1})$ for some unitary character δ, then $\delta = \delta_D$.*

For the rest of this § we let $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$ be as in theorem 3.5. Let D^+ be the set of $z \in D$ such that the sequence $(\varphi^n(z))_{n\geq 0}$ is bounded in D. The module D^+ is the largest finitely generated $\mathscr{E}^+$-submodule of D, stable under φ and Γ. We say that D is *of finite height* if D^+ spans D as $\mathscr{E}$-vector space or, equivalently (since D is irreducible) if $D^+ \neq \{0\}$. The classification of representations of finite height given in [3] shows that if D is of finite height, then D is trianguline[15], so it suffices to treat the cases "D trianguline" and "$D^+ = \{0\}$".

3.2.1. *The trianguline case.* Suppose that D is trianguline and recall that $\delta_D = \varepsilon^{-1} \det D$ is seen as a unitary character of $\mathbb{Q}_p^\times$. Suppose that $D \in \mathscr{MF}(\delta^{-1})$ for some unitary character δ. Let $\Pi = \Pi_\delta(D)$ and define $\eta = \delta_D^{-1}\delta$. Also, let $\Pi^{\rm an}$ be the space of locally analytic vectors of Π, for which we refer the reader to [58] and [33].

We first prove that η is locally constant. The argument works for a much more general class of (φ, Γ)-modules, namely those corresponding to representations which are not $\mathbf{C}_p$-admissible[16] up to a twist (this condition is automatically satisfied by irreducible trianguline (φ, Γ)-modules [17, prop. 4.6]). The proof uses two ingredients

• By [23, chap. VI] and the hypothesis $D \in \mathscr{MF}(\delta^{-1})$, there is a G-equivariant sheaf of locally analytic representations $U \mapsto D_{\rm rig} \boxtimes_\delta U$ attached to $(D_{\rm rig}, \delta)$.

• Using [29, 30], one can describe the action of $\mathrm{Lie}(\mathrm{GL}_2(\mathbb{Q}_p))$ on the space $D_{\rm rig} \boxtimes_\delta \mathbf{P}^1$ of global sections of this sheaf. The fact that η is locally constant follows easily.

The second part of the proof in the trianguline case consists in analyzing the module $D_{\rm rig} \boxtimes_\delta \mathbf{P}^1$. More precisely, we prove (lemma 3.20) that if $0 \to \mathscr{R}(\delta_1) \to D_{\rm rig} \to \mathscr{R}(\delta_2) \to 0$ is a triangulation of $D_{\rm rig}$, then this exact sequence extends to an exact sequence of topological G-modules

$$0 \to \mathscr{R}(\delta_1) \boxtimes_\delta \mathbf{P}^1 \to D_{\rm rig} \boxtimes_\delta \mathbf{P}^1 \to \mathscr{R}(\delta_2) \boxtimes_\delta \mathbf{P}^1 \to 0.$$

Actually, once we know that η is locally constant, the arguments of [21, 28] (where the case $\eta = 1$ is treated) go through without any change.

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[15] Actually, more is true but will not be needed in the sequel: D is of finite height if and only if $D \in \mathscr{S}_*^{\rm cris}$, where $\mathscr{S}_*^{\rm cris}$ is defined in n° 3.3.1.

[16] That is, de Rham with Hodge-Tate weights equal to 0.

Using the description [21] of the Jordan-Hölder components of $\mathscr{R}(\delta_1)\boxtimes_\delta \mathbf{P}^1$, we deduce (lemma 3.21) the existence of a morphism with finite dimensional kernel $B^{\mathrm{an}}(\delta_1,\eta\delta_2)\to\Pi^{\mathrm{an}}$, where $B^{\mathrm{an}}(\delta_1,\eta\delta_2)$ is the locally analytic parabolic induction of the character $\eta\delta_2\otimes\varepsilon^{-1}\delta_1$. Finally, using universal completions, we prove that the morphism $B^{\mathrm{an}}(\delta_1,\eta\delta_2)\to\Pi^{\mathrm{an}}$ induces a nonzero morphism $\Pi_\delta(D')\to\Pi$ for some G-compatible pair (D',δ), where D' is a trianguline (φ,Γ)-module having a triangulation

$$0\to\mathscr{R}(\delta_1)\to D'_{\mathrm{rig}}\to\mathscr{R}(\eta\delta_2)\to 0.$$

This is the most technical part of the proof and uses results from [18, 21, 5] (and [48] suitably extended for $p=2$). Since D and D' are irreducible, the representations $\Pi_\delta(D')$ and Π are admissible and topologically irreducible, hence the morphism $\Pi_\delta(D')\to\Pi$ must be an isomorphism. Using parts (iii) and (iv) of proposition 3.3, we deduce that $D'\cong D$. In particular $\det D_{\mathrm{rig}}=\det D'_{\mathrm{rig}}$, which yields $\delta_1\delta_2=\delta_1\delta_2\eta$, hence $\eta=1$, and finishes the proof in the trianguline case.

3.2.2. *The case $D^+=\{0\}$.* Let us assume now that $D\in\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ is 2-dimensional and satisfies $D^+=\{0\}$ (then D is automatically absolutely irreducible). For each $\alpha\in\mathscr{O}^\times$ let

$$\mathscr{C}^\alpha=(1-\alpha\varphi)D^{\psi=\alpha}.$$

If $D\in\mathscr{MF}(\chi^{-1})$ for some character $\chi:\mathbb{Q}_p^\times\to\mathscr{O}^\times$, then setting $\check{\Pi}=\Pi_{\chi^{-1}}(\check{D})$ we have $\check{\Pi}^*\subset D\boxtimes_\chi\mathbf{P}^1$ (proposition 3.3), and there is [23, rem.V.14(ii)] a canonical isomorphism of $\mathscr{O}[[\Gamma]][1/p]$-modules

$$\mathrm{Res}_{\mathbb{Z}_p^\times}\big((\check{\Pi}^*)^{\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)=\alpha^{-1}}\big)\cong\mathscr{C}^\alpha. \tag{11}$$

Suppose now that $D\in\mathscr{MF}(\delta^{-1})$ and let $\eta=\delta_D^{-1}\delta$. Unravelling the isomorphism (11) for $\chi=\delta_D$ and $\chi=\delta$, we obtain the following key fact

Proposition 3.6. *For all $\alpha\in\mathscr{O}^\times$ we have $m_\eta(\mathscr{C}^\alpha)=\mathscr{C}^{\alpha\eta(p)}$.*

Proof. If $\chi\in\{\delta_D,\delta\}$, let w_χ be the restriction to $D^{\psi=0}=D\boxtimes_\chi\mathbb{Z}_p^\times$ of the action of $w=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ on $\Pi_{\chi^{-1}}(\check{D})^*\subset D\boxtimes_\chi\mathbf{P}^1$. Proposition V.12 of [23] shows that $w_\chi(\mathscr{C}^\alpha)=\mathscr{C}^{\frac{1}{\alpha\chi(p)}}$ for all $\alpha\in\mathscr{O}^\times$. On the other hand, remark II.1.3 of [20] and part b) of proposition 3.1 yield $w_\chi=w_*\circ m_\chi$ and

$$w_{\delta_D}\circ w_\delta=w_*\circ m_{\delta_D}\circ w_*\circ m_\delta=m_{\delta_D^{-1}}\circ w_*\circ w_*\circ m_\delta=m_{\delta_D^{-1}\delta}=m_\eta,$$

as $w_*\circ w_*=(w\circ w)_*=\mathrm{id}$. The result follows. $\square$

In view of proposition 3.4, theorem 3.5, in the case $D^+=0$, is equivalent to the following statement.

Proposition 3.7. *We have $\eta=1$.*

Proof. Let $\mu(\mathbb{Q}_p)$ be the set of roots of unity in $\mathbb{Q}_p$ and let

$$\hat{\mathscr{T}}^0(L)=\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^\times/\mu(\mathbb{Q}_p),1+\mathfrak{m}_L)$$

be the set of continuous characters $\chi:\mathbb{Q}_p^\times\to 1+\mathfrak{m}_L$ trivial on $\mu(\mathbb{Q}_p)$, and let

$$H=\{\chi\in\hat{\mathscr{T}}^0(L)|m_\chi(\mathscr{C}^\alpha)=\mathscr{C}^{\alpha\chi(p)}\quad\forall\alpha\in 1+\mathfrak{m}_L\}.$$

Proposition 3.31 below shows that H is a Zariski closed subgroup of $\hat{\mathscr{T}}^0(L)$ and it follows from corollary 3.35 that H is either trivial or it contains a nontrivial character of finite order (this may require replacing L by a finite extension, which we are allowed to do). We haven't used so far the hypothesis $D^+ = \{0\}$, but only the fact that D is absolutely irreducible of dimension ≥ 2. When $D^+ = \{0\}$, we prove (corollary 3.26) that H cannot contain nontrivial locally constant characters. We conclude that $H = \{1\}$, which implies that η is of finite order (since any power of η which belongs to $\hat{\mathscr{T}}^0(L)$ actually belongs to H by proposition 3.6). We conclude that $\eta = 1$ using again corollary 3.26. $\square$

Remark 3.8. Assume that the Sen operator on D is not scalar[17]. Proposition 3.16 shows that η is locally constant, and proposition 3.6 and corollary 3.26 yield directly the desired result $\eta = 1$. The key proposition 3.16 does not work if the Sen operator is scalar, which explains the more indirect approach presented above.

3.2.3. *Consequences of theorem 3.5.* Before embarking on the proof of theorem 3.5, we give a certain number of consequences of theorems 1.4 and 3.5.

If $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$, we let $\overline{D}^{\mathrm{ss}}$ be the semi-simplification of $D_0 \otimes_{\mathscr{O}} k$, where D_0 is any $\mathscr{O}_{\mathscr{E}}$-lattice in D which is stable under φ and Γ.

The functor $\Pi \mapsto \mathbf{V}(\Pi)$ has integral and torsion versions, and if Θ is an open, bounded and G-invariant lattice in $\Pi \in \mathrm{Rep}_L(G)$, then $\mathbf{V}(\Theta) = \varprojlim_n \mathbf{V}(\Theta/\varpi^n)$ and $\mathbf{V}(\Theta)/\varpi^n \cong \mathbf{V}(\Theta/\varpi^n)$ for all $n \geq 1$. The following result follows from [20, th. 0.10] (see the introduction for the definition of $\pi\{\chi_1, \chi_2\}$).

Lemma 3.9. *If π is either supersingular or $\pi\{\chi_1, \chi_2\}$ for some smooth characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$, then $\dim_k \mathbf{V}(\pi) = 2$. Moreover, if π is an irreducible subrepresentation of $\pi\{\chi_1, \chi_2\}$, then $\dim_k \mathbf{V}(\pi) \leq 1$.*

We will also need to following compatibility between the p-adic and mod p Langlands correspondences. This was first proved (in a slightly different form) in [4]. We will use the following version, taken from [23, prop. III.55, rem. III.56].

Proposition 3.10. *If $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ is 2-dimensional and if $\delta = \delta_D$, there is an isomorphism $\overline{\Pi_\delta(D)}^{\mathrm{ss}} \cong \Pi_\delta(\overline{D}^{\mathrm{ss}})$ and (possibly after replacing L with its quadratic unramified extension) this representation is either absolutely irreducible supersingular or isomorphic to $\pi\{\chi_1, \chi_2\}$ for some smooth characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$.*

Proposition 3.11. *Let $\Pi \in \mathrm{Rep}_L(\delta)$ be absolutely irreducible and let $D = \mathbf{D}(\Pi)$. Then $\dim_{\mathscr{E}} D \leq 2$ and D is absolutely irreducible. Moreover, $\dim D = 2$ if and only if Π is non-ordinary.*

Proof. The functor $\mathbf{D}$ being exact, there is a natural isomorphism $\overline{D}^{\mathrm{ss}} \cong \mathbf{D}(\overline{\Pi}^{\mathrm{ss}})$. Combined with theorem 1.4 and lemma 3.9, this yields $\dim D \leq 2$.

Next, if Π is ordinary, then $\overline{\Pi}^{\mathrm{ss}}$ is a subquotient of a smooth parabolic induction and using lemma 3.9 again we conclude that $\dim D \leq 1$. In particular, D is absolutely irreducible. If Π is not ordinary, we deduce from [23, cor. III.47] and the first paragraph that D is absolutely irreducible and 2-dimensional. The result follows. $\square$

Corollary 3.12. *If $\Pi \in \mathrm{Rep}_L(\delta)$ is absolutely irreducible non-ordinary, then $\delta = \delta_{\mathbf{D}(\Pi)}$. Thus $\det \mathbf{V}(\Pi) = \varepsilon\delta$.*

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[17] By Sen's theorem, this is equivalent to saying that inertia does not have finite image on the Galois representation associated to D.

Proof. This follows directly from proposition 3.11 and theorem 3.5. $\square$

Theorem 3.13. *Let $\Pi \in \mathrm{Rep}_L(\delta)$ be absolutely irreducible and let $D = \mathbf{D}(\Pi)$. The following assertions are equivalent*

(i) Π *is non-ordinary.*

(ii) $\dim_{\mathscr{E}} D = 2$.

(iii) *After possibly replacing L by its quadratic unramified extension, $\overline{\Pi}^{\mathrm{ss}}$ is either absolutely irreducible supersingular or isomorphic to some $\pi\{\chi_1, \chi_2\}$.*

If these assertions hold, then there is a canonical isomorphism $\Pi \cong \Pi_\delta(\check{D})$.

Proof. (i) and (ii) are equivalent by proposition 3.11. Suppose that (iii) holds. Then $\overline{D}^{\mathrm{ss}} \cong \mathbf{D}(\overline{\Pi}^{\mathrm{ss}})$ is 2-dimensional by lemma 3.9 and so $\dim D = 2$, that is (ii) holds. Finally, suppose that (i) holds. Then [23, cor. III.47] yields a canonical isomorphism $\Pi \cong \Pi_\delta(\check{D})$ and we conclude that (iii) holds using proposition 3.10. $\square$

Theorem 3.14. *Let $\Pi_1, \Pi_2 \in \mathrm{Rep}_L(G)$ be absolutely irreducible, non-ordinary.*

(i) *If $\mathbf{V}(\Pi_1) \cong \mathbf{V}(\Pi_2)$, then $\Pi_1 \cong \Pi_2$.*

(ii) *We have $\mathrm{Hom}^{\mathrm{cont}}_{L[P]}(\Pi_1, \Pi_2) = \mathrm{Hom}^{\mathrm{cont}}_{L[G]}(\Pi_1, \Pi_2)$, where P is the mirabolic subgroup of G.*

Proof. If $\mathbf{V}(\Pi_1) \cong \mathbf{V}(\Pi_2) = V$, then corollary 3.12 shows that Π_1 and Π_2 have the same central character δ and theorem 3.13 yields $\Pi_1 \cong \Pi_2 \cong \Pi_\delta(\mathbf{D}(\check{V}))$.

Let $f : \Pi_1 \to \Pi_2$ be a P-equivariant linear continuous map and let $D_j = \mathbf{D}(\Pi_j)$. Since the functor $\mathbf{D}$ uses only the restriction to P, f induces a morphism $\mathbf{D}(f) : D_2 \to D_1$ in $\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$. Since D_1 and D_2 are absolutely irreducible (proposition 3.11), $\mathbf{D}(f)$ is either 0 or an isomorphism. If $\mathbf{D}(f)$ is an isomorphism, then $\delta_1 = \delta_2$ by part (i), and we conclude that f is G-equivariant using the following diagram, in which the vertical maps are the isomorphisms given by theorem 3.13 and the map $\Pi_{\delta_1}(\check{D}_1) \to \Pi_{\delta_1}(\check{D}_2)$ is G-equivariant since induced by functoriality from the transpose of $\mathbf{D}(f)$

$$\begin{tikzcd} \Pi_1 \arrow[r, "f"] \arrow[d, "\cong"'] & \Pi_2 \arrow[d, "\cong"] \\ \Pi_{\delta_1}(\check{D}_1) \arrow[r, "\mathbf{D}(f^*)"] & \Pi_{\delta_1}(\check{D}_2). \end{tikzcd}$$

The case $\mathbf{D}(f) = 0$ is slightly trickier, since we can no longer use part (i) of the theorem to deduce that Π_1 and Π_2 have the same central character. We will prove that $f = 0$. Let Θ_j be the unit ball in Π_j and let $X_j = \mathrm{Res}_{\mathbb{Z}_p}(\Theta_j^d)$, where we use the inclusions $\Pi_j^* \subset D_j \boxtimes_{\delta_j^{-1}} \mathbf{P}^1$ (proposition 3.3). It follows from [23, cor. III.25] that the restriction of $\mathrm{Res}_{\mathbb{Q}_p} : D_j \boxtimes_{\delta_j^{-1}} \mathbf{P}^1 \to D_j \boxtimes_{\delta_j^{-1}} \mathbb{Q}_p$ to Π_j^* induces a P-equivariant isomorphism of topological vector spaces $\Pi_j^* \cong (\varprojlim_\psi X_j) \otimes L$. We have a commutative diagram

$$\begin{tikzcd} \Pi_2^* \arrow[r] \arrow[d] & \Pi_1^* \arrow[d] \\ (\varprojlim_\psi X_2) \otimes L \arrow[r] & (\varprojlim_\psi X_1) \otimes L \end{tikzcd}$$

in which the top horizontal map is the transpose f^* of f, the vertical maps are the isomorphisms explained above and the horizontal map on the bottom is induced by $\mathbf{D}(f) = 0$, and thus it is the zero map. We conclude that $f^* = 0$ and thus $f = 0$, which finishes the proof of theorem 1.8. □

The remaining sections will be devoted to the proof of theorem 3.5 in the case D trianguline (see proposition 3.24), and to the proof of the statements (namely proposition 3.31 and corollaries 3.26 and 3.35) that were used in the proof of proposition 3.7 which, as we remarked, is equivalent to theorem 3.5 in the case $D^+ = 0$.

3.3. **Trianguline representations.**

3.3.1. *Preliminaries.* If $\delta : \mathbb{Q}_p^\times \to L^\times$ is a continuous character (not necessarily unitary), let $\mathscr{R}(\delta)$ be the (φ, Γ)-module obtained by twisting the action of φ and Γ on $\mathscr{R}$ by δ. It has a canonical basis $e = 1 \otimes \delta$ for which $\varphi(e) = \delta(p)e$ and $\sigma_a(e) = \delta(a)e$ for $a \in \mathbb{Z}_p^\times$, where $\sigma_a \in \Gamma$ satisfies $\sigma_a(\zeta) = \zeta^a$ for all $\zeta \in \mu_{p^\infty}$, so that $\varepsilon(\sigma_a) = a$. Let $\mathscr{R}^+$ be the ring of analytic functions on the open unit disk, so that $\mathscr{R}^+ = \mathscr{R} \cap L[[T]]$. We define $\mathscr{R}^+(\delta)$ as the $\mathscr{R}^+$-sumodule of $\mathscr{R}(\delta)$ generated by e. We let $\kappa(\delta)$ be the derivative of δ at 1 or, equivalently (if δ is unitary), the generalized Hodge–Tate weight of the Galois character corresponding to δ by class field theory.

By[18] [21, prop. 0.2], $\mathrm{Ext}^1(\mathscr{R}(\delta_2), \mathscr{R}(\delta_1))$ has dimension 1 when $\delta_1\delta_2^{-1}$ is not of the form x^{-i} or εx^i for some $i \geq 0$, and dimension 2 in the remaining cases. Moreover, if $\mathrm{Ext}^1(\mathscr{R}(\delta_2), \mathscr{R}(\delta_1))$ is 2-dimensional, then the associated projective space is naturally isomorphic to $\mathbf{P}^1(L)$. Let $\mathscr{S}$ be the set of triples $(\delta_1, \delta_2, \mathscr{L})$, where $\delta_1, \delta_2 : \mathbb{Q}_p^\times \to L^\times$ are continuous characters and $\mathscr{L} \in \mathrm{Proj}(\mathrm{Ext}^1(\mathscr{R}(\delta_2), \mathscr{R}(\delta_1)))$ (if $\mathrm{Ext}^1(\mathscr{R}(\delta_2), \mathscr{R}(\delta_1))$ is 1-dimensional, we have $\mathscr{L} = \infty$). Each $s \in \mathscr{S}$ gives rise to an extension $0 \to \mathscr{R}(\delta_1) \to \Delta(s) \to \mathscr{R}(\delta_2) \to 0$, classified up to isomorphism by $\mathscr{L}$.

Let $\mathscr{S}_*$ be the subset of $\mathscr{S}$ consisting of those $s = (\delta_1, \delta_2, \mathscr{L})$ for which $v_p(\delta_1(p)) + v_p(\delta_2(p)) = 0$ and $v_p(\delta_1(p)) > 0$. For each $s \in \mathscr{S}_*$ let

$$u(s) = v_p(\delta_1(p)), \quad \kappa(s) = \kappa(\delta_1) - \kappa(\delta_2).$$

Let $\mathscr{S}_*^{\mathrm{cris}}$ (resp. $\mathscr{S}_*^{\mathrm{st}}$) be the set of $s \in \mathscr{S}_*$ for which $\kappa(s) \in \mathbf{N}^*$, $u(s) < \kappa(s)$ and $\mathscr{L} = \infty$ (resp. $\mathscr{L} \neq \infty$). Let $\mathscr{S}_*^{\mathrm{ng}}$ be the set of $s \in \mathscr{S}_*$ for which $\kappa(s) \notin \mathbf{N}^*$ and finally let

$$\mathscr{S}_{\mathrm{irr}} = \mathscr{S}_*^{\mathrm{cris}} \coprod \mathscr{S}_*^{\mathrm{st}} \coprod \mathscr{S}_*^{\mathrm{ng}}.$$

We say that $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ is trianguline (of rank 2) if D_{rig} is an extension of two (φ, Γ)-modules of rank 1 over $\mathscr{R}$. These are described by the following result ([21, prop. 0.3] or [17, th. 0.5]).

Proposition 3.15. *a) For any $s \in \mathscr{S}_{\mathrm{irr}}$ there is a unique $D(s) \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ such that $\Delta(s) = D(s)_{\mathrm{rig}}$, and $D(s)$ is absolutely irreducible. Moreover, if $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ is trianguline and absolutely irreducible, there exists $s \in \mathscr{S}_{\mathrm{irr}}$ such that $D \cong D(s)$.*

b) If $s = (\delta_1, \delta_2, \mathscr{L})$ and $s' = (\delta_1', \delta_2', \mathscr{L}')$ are elements of $\mathscr{S}_{\mathrm{irr}}$, then $D(s) \cong D(s')$ if and only if $s, s' \in \mathscr{S}_*^{\mathrm{cris}}$ and $\delta_1' = x^{\kappa(s)}\delta_2$, $\delta_2' = x^{-\kappa(s)}\delta_1$.

[18] Contrary to [17], all results of [21] are proved for all primes p.

3.3.2. *Infinitesimal study of the module* $D_{\mathrm{rig}} \boxtimes_\delta \mathbf{P}^1$. In this § we let D be any object of $\mathscr{MF}(\delta^{-1})$ for some unitary character δ.

Recall (see section 2.5 of [30] for a summary) that Sen's theory associates to D a finite free $L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^\infty})$-module D_{Sen} endowed with a Sen operator Θ_{Sen}, whose eigenvalues are the generalized Hodge-Tate weights of D.

Proposition 3.16. *If* $D \in \mathscr{MF}(\delta^{-1})$ *is absolutely irreducible,* 2*-dimensional and if* Θ_{Sen} *is not a scalar operator on* D_{Sen}*, then* $\delta\delta_D^{-1}$ *is locally constant.*

Proof. By [23, chap. VI], the G-equivariant sheaf $U \mapsto D \boxtimes_\delta U$ induces a G-equivariant sheaf $U \mapsto D_{\mathrm{rig}} \boxtimes_\delta U$ on $\mathbf{P}^1(\mathbb{Q}_p)$. We let $D_{\mathrm{rig}} \boxtimes_\delta \mathbf{P}^1$ be the space of global sections of this sheaf. This is naturally an LF space and G acts continuously on it. Moreover, this action extends to a structure of topological $\mathscr{D}(\mathrm{GL}_2(\mathbb{Z}_p))$-module on $D_{\mathrm{rig}} \boxtimes_\delta \mathbf{P}^1$, where $\mathscr{D}(\mathrm{GL}_2(\mathbb{Z}_p))$ is the Fréchet-Stein algebra [58] of L-valued distributions on $\mathrm{GL}_2(\mathbb{Z}_p)$. In particular, the enveloping algebra of $\mathfrak{gl}_2 = \mathrm{Lie}(G)$ acts on $D_{\mathrm{rig}} \boxtimes_\delta \mathbf{P}^1$.

Consider the Casimir element

$$C = u^+u^- + u^-u^+ + \frac{1}{2}h^2 \in U(\mathfrak{gl}_2),$$

where $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $u^+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $u^- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. The action of C on $D_{\mathrm{rig}} \boxtimes_\delta \mathbf{P}^1$ preserves $D_{\mathrm{rig}} = D_{\mathrm{rig}} \boxtimes_\delta \mathbb{Z}_p$, viewed as a sub-module of $D_{\mathrm{rig}} \boxtimes_\delta \mathbf{P}^1$ via the extension by 0. By theorem 3.1 and remark 3.2 of [29] the operator C acts by a scalar c on D_{rig} and we have an equality of operators on D_{Sen}

$$(2\Theta_{\mathrm{Sen}} - (1 + \kappa(\delta)))^2 = 1 + 2c. \tag{12}$$

Let a and b be the generalized Hodge-Tate weights of D. By Cayley-Hamilton we have $(\Theta_{\mathrm{Sen}} - a)(\Theta_{\mathrm{Sen}} - b) = 0$ as endomorphisms of D_{Sen}. Combining this relation with (12) yields

$$4(a + b - 1 - \kappa(\delta))\Theta_{\mathrm{Sen}} + (1 + \kappa(\delta))^2 - 4ab = 1 + 2c.$$

Since Θ_{Sen} is not scalar, the previous relation forces $a + b - 1 = \kappa(\delta)$ and, as $a + b - 1 = \kappa(\delta_D)$, this gives $\kappa(\delta\delta_D^{-1}) = 0$. The result follows. □

Remark 3.17. If D is trianguline, 2-dimensional and irreducible, then Θ_{Sen} is not scalar, see proposition 4.6 of [17].

3.3.3. *Dévissage of* $D_{\mathrm{rig}} \boxtimes_\delta \mathbf{P}^1$. If $\eta_1, \eta_2 : \mathbb{Q}_p^\times \to L^\times$ are continuous characters, let

$$B^{\mathrm{an}}(\eta_1, \eta_2) = (\mathrm{Ind}_B^G \eta_2 \otimes \varepsilon^{-1}\eta_1)^{\mathrm{an}}$$

be the locally analytic parabolic induction of the character $\eta_2 \otimes \varepsilon^{-1}\eta_1$. The recipe giving rise to the sheaf $U \mapsto D \boxtimes_\delta U$ for $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ can be used [28, § 3.1] to create a G-equivariant sheaf $U \mapsto \mathscr{R}(\eta_1) \boxtimes_\delta U$ on $\mathbf{P}^1$, attached to the pair $(\mathscr{R}(\eta_1), \delta)$. We will only be interested in the space $\mathscr{R}(\eta_1) \boxtimes_\delta \mathbf{P}^1$ of its global sections, which is described by the following proposition, whose proof is easily deduced from remark 3.7 of [28]. Let

$$\mathscr{R}^+(\eta_1) \boxtimes_\delta \mathbf{P}^1 = \{z \in \mathscr{R}(\eta_1) \boxtimes_\delta \mathbf{P}^1,\ \mathrm{Res}_{\mathbb{Z}_p} z \in \mathscr{R}^+(\eta_1),\ \mathrm{Res}_{\mathbb{Z}_p} w \cdot z \in \mathscr{R}^+(\eta_1)\}.$$

Proposition 3.18. *If* $\varepsilon^{-1}\eta_1\eta_2 = \delta$ *for some continuous characters* η_1, η_2, δ*, then there is an exact sequence of topological* G*-modules*

$$0 \to B^{\mathrm{an}}(\eta_2, \eta_1)^* \otimes \delta \to \mathscr{R}(\eta_1) \boxtimes_\delta \mathbf{P}^1 \to B^{\mathrm{an}}(\eta_1, \eta_2) \to 0$$

and a G-equivariant isomorphism $B^{\mathrm{an}}(\eta_2,\eta_1)^*\otimes\delta\cong\mathscr{R}^+(\eta_1)\boxtimes_\delta\mathbf{P}^1$ of topological vector spaces.

From now on we suppose that $D=D(s)\in\mathscr{MF}(\delta^{-1})$ for some $s=(\delta_1,\delta_2,\mathscr{L})\in\mathscr{S}_{\mathrm{irr}}$ and some unitary character $\delta:\mathbb{Q}_p^\times\to\mathscr{O}_L^*$, and we let $\eta=\delta\delta_D^{-1}$. By proposition 3.3 we have $\check{D}\in\mathscr{MF}(\delta)$ and we let $\check{\Pi}=\Pi_{\delta^{-1}}(\check{D})$. Since $\dim D=2$, there is a natural isomorphism $\check{D}\cong D\otimes\delta_D^{-1}$. Combining these observations with part (i) of proposition 3.3 and with corollary VI.12 of [23], we obtain the following result.

Lemma 3.19. *We have $D\in\mathscr{MF}(\eta\delta_D^{-1})$ and there is a natural isomorphism $\check{\Pi}\cong\Pi_{\delta_D\eta^{-1}}(D)\otimes\delta_D^{-1}$ as well as an exact sequence of topological G-modules*

$$0\to(\check{\Pi}^{\mathrm{an}})^*\to D_{\mathrm{rig}}\boxtimes_\delta\mathbf{P}^1\to\Pi^{\mathrm{an}}\to 0.$$

Lemma 3.20. *There is an exact sequence of topological G-modules*

$$0\to\mathscr{R}(\delta_1)\boxtimes_\delta\mathbf{P}^1\to D_{\mathrm{rig}}\boxtimes_\delta\mathbf{P}^1\to\mathscr{R}(\delta_2)\boxtimes_\delta\mathbf{P}^1\to 0.$$

Proof. Since $\kappa(\delta)=\kappa(\delta_D)$ by remark 3.17, the desired result is proved in the same way as corollary 3.6 of [28]. For the reader's convenience, we sketch the argument. Let a,b be the Hodge-Tate weights of D. Extension by 0 allows to view $D_{\mathrm{rig}}=D_{\mathrm{rig}}\boxtimes_\delta\mathbb{Z}_p$ as a subspace of $D_{\mathrm{rig}}\boxtimes_\delta\mathbf{P}^1$ and any element of $D_{\mathrm{rig}}\boxtimes_\delta\mathbf{P}^1$ can be written as $z_1+w\cdot z_2$ with $z_1,z_2\in D_{\mathrm{rig}}$. The equality $\kappa(\delta)=\kappa(\delta_D)$ combined with theorem 3.1 in [29] yields

$$u^+(z_1)=tz_1,\quad u^+(w\cdot z_2)=-w\cdot\frac{(\nabla-a)(\nabla-b)z_2}{t},\quad\text{where } t=\log(1+T).$$

We deduce that $X:=(D_{\mathrm{rig}}\boxtimes_\delta\mathbf{P}^1)^{u^+=0}$ is isomorphic as an L-vector space to the space of solutions of the equation $(\nabla-a)(\nabla-b)z=0$ on D_{rig}. Proposition 2.1 of [28] shows that X is finite dimensional and lemma 2.6 of loc.cit implies that all elements of X are invariant under the action of the upper unipotent subgroup U of G. In particular, if e_1 is a basis of $\mathscr{R}(\delta_1)$ which is an eigenvector of φ and Γ, then $(0,e_1)\in X$ is U-invariant. Then the arguments in § 3.2 of [28] go through by replacing δ_D by δ. The result follows. ☐

Lemma 3.21. *There is a morphism $B^{\mathrm{an}}(\delta_1,\eta\delta_2)\to\Pi^{\mathrm{an}}$ with a finite dimensional kernel.*

Proof. Consider the inclusions $\mathscr{R}^+(\delta_1)\boxtimes_\delta\mathbf{P}^1\subset D_{\mathrm{rig}}\boxtimes_\delta\mathbf{P}^1$ given by lemma 3.20 and $(\check{\Pi}^{\mathrm{an}})^*\subset D_{\mathrm{rig}}\boxtimes_\delta\mathbf{P}^1$ given by lemma 3.19. It follows from [23, cor. VI.14] that $\mathscr{R}^+(\delta_1)\boxtimes_\delta\mathbf{P}^1\subset(\check{\Pi}^{\mathrm{an}})^*$, hence there is a morphism

$$(\mathscr{R}(\delta_1)\boxtimes_\delta\mathbf{P}^1)/(\mathscr{R}^+(\delta_1)\boxtimes_\delta\mathbf{P}^1)\to(D_{\mathrm{rig}}\boxtimes_\delta\mathbf{P}^1)/(\check{\Pi}^{\mathrm{an}})^*.$$

The left hand-side is isomorphic to $B^{\mathrm{an}}(\delta_1,\eta\delta_2)$ by proposition 3.18 and the right hand-side is isomorphic to Π^{an} by lemma 3.19. Consequently, we obtain a morphism $B^{\mathrm{an}}(\delta_1,\eta\delta_2)\to\Pi^{\mathrm{an}}$, whose kernel is a closed subspace of $B^{\mathrm{an}}(\delta_1,\eta\delta_2)$, thus a space of compact type. On the other hand, the kernel is isomorphic to the quotient of $(\check{\Pi}^{\mathrm{an}})^*\cap(\mathscr{R}(\delta_1)\boxtimes_\delta\mathbf{P}^1)$ by the closed subspace $\mathscr{R}^+(\delta_1)\boxtimes_\delta\mathbf{P}^1$. Let $\sigma:D_{\mathrm{rig}}\to\mathscr{R}(\delta_2)$ be the natural projection. Then

$$(\check{\Pi}^{\mathrm{an}})^*\cap(\mathscr{R}(\delta_1)\boxtimes_\delta\mathbf{P}^1)=\{z\in(\check{\Pi}^{\mathrm{an}})^*|\sigma(\mathrm{Res}_{\mathbb{Z}_p}(z))=\sigma(\mathrm{Res}_{\mathbb{Z}_p}(wz))=0\}$$

is closed in the Fréchet space $(\check{\Pi}^{\mathrm{an}})^*$, thus it is itself a Fréchet space. Since a Fréchet space which is also a space of compact type is finite dimensional, the result follows. ☐

Remark 3.22. Combining lemmas 3.19 and 3.21, we also obtain the existence of a morphism $B^{\rm an}(\delta_1,\eta^{-1}\delta_2)\to\Pi_{\delta_D\eta^{-1}}(D)^{\rm an}$ with a finite dimensional kernel.

3.3.4. *Universal unitary completions and completion of the proof.* The next theorem requires some preliminaries. We say that $s=(\delta_1,\delta_2,\mathscr{L})\in\mathscr{S}_{\rm irr}$ is

- exceptional if $\kappa(s)\in\mathbf{N}^*$ and $\delta_1=x^{\kappa(s)}\delta_2$ (in particular, $s\in\mathscr{S}_*^{\rm cris}$).
- special if $\kappa(s)\in\mathbf{N}^*$ and $\delta_1=x^{\kappa(s)-1}\varepsilon\delta_2$ (this includes $s\in\mathscr{S}_*^{\rm st}$).

If s is special, then setting $W(\delta_1,\delta_2)={\rm Sym}^{\kappa(s)-1}(L^2)\otimes_L\delta_2$ there is [21, th. 2.7, rem. 2.11] a natural isomorphism

$$\mathrm{Ext}^1_G(W(\delta_1,\delta_2),B^{\rm an}(\delta_1,\delta_2)/W(\delta_1,\delta_2))\cong\mathrm{Ext}^1(\mathscr{R}(\delta_2),\mathscr{R}(\delta_1)).$$

The extension $D(s)_{\rm rig}$ of $\mathscr{R}(\delta_2)$ by $\mathscr{R}(\delta_1)$ associated to s gives therefore rise to an extension $E_{\mathscr{L}}$ of $W(\delta_1,\delta_2)$ by $B^{\rm an}(\delta_1,\delta_2)/W(\delta_1,\delta_2)$ (these extensions were introduced and studied by Breuil [9, 11]).

If $s=(\delta_1,\delta_2,\mathscr{L})\in\mathscr{S}_{\rm irr}$ we have $D(s)\in\mathscr{MF}(\delta_{D(s)}^{-1})$ by proposition 3.4. We write $\Pi(s)$ instead of $\Pi_{\delta_{D(s)}}(D(s))$. Propositions 3.15 and 3.3(iv) imply that $\Pi(s)$ is in ${\rm Rep}_L(\delta_{D(s)})$ and is absolutely irreducible.

If π is a representation of G on a locally convex L-vector space, we let $\widehat{\pi}$ be the universal unitary completion of π (if it exists).

Theorem 3.23. *If $s=(\delta_1,\delta_2,\mathscr{L})\in\mathscr{S}_{\rm irr}$ then the following hold:*

a) If $s\in\mathscr{S}_*^{\rm cris}$ is not special, then[19] $\Pi(s)=\widehat{B^{\rm alg}(\delta_1,\delta_2)}=\widehat{B^{\rm an}(\delta_1,\delta_2)}$.

b) If $s\in\mathscr{S}_*^{\rm ng}$, then $\Pi(s)=\widehat{B^{\rm an}(\delta_1,\delta_2)}$.

c) If s is special (which includes the case $s\in\mathscr{S}_*^{\rm st}$), then $\Pi(s)=\widehat{E_{\mathscr{L}}}$.

Proof. Assume first that s is not exceptional. Let $B(s)$ be the space of functions $\phi:\mathbb{Q}_p\to L$ of class $\mathscr{C}^{u(s)}$, such that $x\mapsto(\delta_1\delta_2^{-1}\varepsilon^{-1})(x)\phi(1/x)$ extends to a function of class $\mathscr{C}^{u(s)}$. By results of Berger, Breuil and Emerton[20] one can express $\widehat{B^{\rm alg}(\delta_1,\delta_2)}$, $\widehat{B^{\rm an}(\delta_1,\delta_2)}$ and $\widehat{E_{\mathscr{L}}}$ (according to whether $s\in\mathscr{S}_*^{\rm cris},\mathscr{S}_*^{\rm ng}$ or s special) as a quotient $\Pi_{\rm aut}(s)$ of $B(s)$. Theorem IV.4.12 of [20] (which builds on [5], [18], [17]) shows that $D(s)\in\mathscr{MF}(\delta_{D(s)}^{-1})$ and that $\Pi_{\rm aut}(s)=\Pi(s)$, which finishes the proof in this case.

It remains to deal with the exceptional case[21]. Let $\Pi=\Pi(s)$. The description of $\Pi(s)^{\rm an}$ given by [21, prop. 4.11] shows that there is an injection $B^{\rm alg}(\delta_1,\delta_2)\to\Pi$. If $X=\widehat{B^{\rm alg}(\delta_1,\delta_2)}$, we obtain a morphism $X\to\Pi$ and an injection $B^{\rm alg}(\delta_1,\delta_2)\to X$. In particular $X\neq 0$, and then the second paragraph of the proof of prop. 2.10 in [52] shows that we can find a non-exceptional point $s'\in\mathscr{S}_*^{\rm cris}$ and lattices Θ_1,Θ_2 in $B^{\rm alg}(\delta_1,\delta_2)$ and $\Pi(s')^{\rm alg}$, both finitely generated as $\mathscr{O}[G]$-modules and such that $\Theta_1/\varpi\cong\Theta_2/\varpi$.

Since Θ_1,Θ_2 are finitely generated over $\mathscr{O}[G]$, their p-adic completions are open, bounded, G-stable lattices in X and $\Pi(s')=\widehat{\Pi(s')^{\rm alg}}$, respectively. As s' is not exceptional, we know (by the first paragraph) that Θ_2/ϖ is admissible, of finite length, thus X is admissible, of finite length as Banach representation and $\overline{X}^{\rm ss}\cong\overline{\Pi(s')}^{\rm ss}$. In particular, the image of the morphism $X\to\Pi$ is closed [57]. Since Π is irreducible we obtain an exact sequence $0\to K\to X\to\Pi\to 0$ in ${\rm Ban}_G^{\rm adm}(L)$. It

[19] $B^{\rm alg}(\delta_1,\delta_2)$ is the space of ${\rm SL}_2(\mathbb{Q}_p)$ locally algebraic vectors in $B^{\rm an}(\delta_1,\delta_2)$.

[20] See [5, th. 4.3.1], [32, prop. 2.5], [9, cor. 3.2.3, 3.3.4].

[21] This problem is solved in [48] for $p>2$.

follows from [51, lem.5.5] that this induces an exact sequence $0 \to \overline{K}^{\mathrm{ss}} \to \overline{X}^{\mathrm{ss}} \to \overline{\Pi}^{\mathrm{ss}} \to 0$. Thus we have a surjection $\overline{\Pi(s')}^{\mathrm{ss}} \to \overline{\Pi(s)}^{\mathrm{ss}}$. Compatibility of p-adic and mod p local Langlands ([4] or proposition 3.10) implies that this surjection must be an isomorphism, which in turn shows that $\overline{K}^{\mathrm{ss}} = 0$, hence $K = 0$. We conclude that $X \cong \Pi$ and we are done. □

Proposition 3.24. *If $D \in \mathscr{S}_{\mathrm{irr}}$ and $D \in \mathscr{MF}(\delta^{-1})$ for some unitary character δ, then $\delta = \delta_D$.*

Proof. Write $s = (\delta_1, \delta_2, \mathscr{L}) \in \mathscr{S}_{\mathrm{irr}}$ and $\delta = \delta_D \eta$ (note that $\delta_D = \varepsilon^{-1}\delta_1\delta_2$). We will prove that $\eta = 1$.

We start by proving that $\eta = \eta^{-1}$. Suppose that this is not the case and let $s' = (\delta_1, \eta\delta_2, \mathscr{L})$ and $s'' = (\delta_1, \eta^{-1}\delta_2, \mathscr{L})$. Since η is locally constant and unitary, we have $s', s'' \in \mathscr{S}_{\mathrm{irr}}$ and s, s', s'' are pairwise distinct. At least one of s', s'' is not special, and replacing[22] η by η^{-1} we may assume that s' has this property. Lemma 3.21 gives a nonzero morphism $B^{\mathrm{an}}(\delta_1, \eta\delta_2) \to \Pi$. Applying theorem 3.23 (to this morphism or to its restriction to $B^{\mathrm{alg}}(\delta_1, \eta\delta_2)$) yields a nonzero morphism $\Pi(s') \to \Pi$. This must be an isomorphism since both the source and target are topologically irreducible and admissible by proposition 3.3. Applying the functor $\Pi \mapsto \mathbf{V}(\Pi)$ and using proposition 3.3 again yields $D(s) \cong D(s')$, contradicting proposition 3.15. Thus $\eta = \eta^{-1}$, and the proof also shows that if s' is not special, then $\eta = 1$.

Assume that s' is special. Since $\eta^2 = 1$, we have $\Pi_{\delta_D \eta^{-1}}(D) = \Pi$ and the exact sequence in proposition 3.19 becomes

$$0 \to (\Pi^{\mathrm{an}})^* \otimes \delta_D \to D_{\mathrm{rig}} \boxtimes_\delta \mathbf{P}^1 \to \Pi^{\mathrm{an}} \to 0.$$

Proposition 3.18 also gives exact sequences

$$0 \to B^{\mathrm{an}}(\delta_2, \eta\delta_1)^* \otimes \delta_D \to \mathscr{R}(\delta_1) \boxtimes_\delta \mathbf{P}^1 \to B^{\mathrm{an}}(\delta_1, \eta\delta_2) \to 0,$$

$$0 \to B^{\mathrm{an}}(\delta_1, \eta\delta_2)^* \otimes \delta_D \to \mathscr{R}(\delta_2) \boxtimes_\delta \mathbf{P}^1 \to B^{\mathrm{an}}(\delta_2, \eta\delta_1) \to 0$$

We are now exactly in the context of the proof of prop. 4.11, part ii) of [21], which shows that Π^{an} contains an extension $E_{\mathscr{L}'}$ of $W(\delta_1, \eta\delta_2)$ by $B^{\mathrm{an}}(\delta_1, \eta\delta_2)/W(\delta_1, \eta\delta_2)$. This extension is necessarily non split since Π^{an} does not contain any finite dimensional G-invariant subspace. If $s'' = (\delta_1, \eta\delta_2, \mathscr{L}')$, then the inclusion $E_{\mathscr{L}'} \to \Pi^{\mathrm{an}}$ induces via theorem 3.23 a nonzero morphism $\Pi(s'') \to \Pi$. Arguing as in the previous paragraph, we obtain $D(s'') \cong D(s)$ and we conclude using proposition 3.15. □

3.4. **Representations of infinite height.**

3.4.1. (φ, Γ)-*modules of infinite height.* In this § we fix a character $\delta : \mathbb{Q}_p^\times \to \mathscr{O}^\times$ and an absolutely irreducible $D \in \mathscr{MF}(\delta^{-1})$ such that $D^+ = \{0\}$. Let $\Pi = \Pi_\delta(D)$ and $\check{\Pi} = \Pi_{\delta^{-1}}(\check{D})$. By proposition 3.3 we have an inclusion $\check{\Pi}^* \subset D \boxtimes_\delta \mathbf{P}^1$. We will use several times the inclusion $D^{\psi=\alpha} \subset \mathrm{Res}_{\mathbb{Z}_p}(\check{\Pi}^*)$ for all $\alpha \in \mathscr{O}^\times$, see the discussion in remark V.14 of [23]. Recall that $\mathscr{C}^\alpha = (1 - \alpha\varphi)D^{\psi=\alpha}$.

Proposition 3.25. a) *$\mathrm{Res}_{a+p^n\mathbb{Z}_p} : \check{\Pi}^* \to D$ is injective for $a \in \mathbb{Z}_p$ and $n \geq 0$.*

b) *$\mathscr{C}^\alpha \cap \mathscr{C}^\beta = \{0\}$ for all distinct $\alpha, \beta \in \mathscr{O}^\times$.*

[22] This uses lemma 3.19.

Proof. a) $\mathrm{Res}_{a+p^n\mathbb{Z}_p}(z) = 0$ is equivalent to $\mathrm{Res}_{\mathbb{Z}_p}(\left(\begin{smallmatrix} p^{-n} & 0 \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -a \\ 0 & 1\end{smallmatrix}\right)z) = 0$, so it suffices to prove that $\mathrm{Res}_{\mathbb{Z}_p} : \check{\Pi}^* \to D$ is injective. Let D_0 be a stable lattice in D and let $\Pi_0 = \Pi_\delta(D_0)$ and $\check{\Pi}_0 = \Pi_{\delta^{-1}}(\check{D}_0)$. Then Π_0 and $\check{\Pi}_0$ are open, bounded and G-invariant lattices in Π and $\check{\Pi}$. Suppose that $z \in \check{\Pi}^*$ satisfies $\mathrm{Res}_{\mathbb{Z}_p}(z) = 0$. Multiplying z by a power of p, we may assume that $z \in \check{\Pi}_0^*$. If $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$, then $\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1\end{smallmatrix}\right)wz \in D_0 \cap \check{\Pi}_0^*$ for all $n \geq 1$, thus $\varphi^n(wz) = \mathrm{Res}_{\mathbb{Z}_p}(\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1\end{smallmatrix}\right)wz) \in \mathrm{Res}_{\mathbb{Z}_p}(\check{\Pi}_0^*)$. Since $\mathrm{Res}_{\mathbb{Z}_p}(\check{\Pi}_0^*)$ is compact (because $\check{\Pi}_0^*$ is compact and $\mathrm{Res}_{\mathbb{Z}_p}$ is continuous), we deduce that $wz \in D_0^+ = \{0\}$ and so $z = 0$.

b) Let $x \in D^{\psi=\alpha}$ and $y \in D^{\psi=\beta}$ be such that $(1-\alpha\varphi)x = (1-\beta\varphi)y$. Then $x - y = \varphi(\alpha x - \beta y)$, so $\mathrm{Res}_{1+p\mathbb{Z}_p}(x-y) = 0$. Since $D^{\psi=\alpha}, D^{\psi=\beta} \subset \mathrm{Res}_{\mathbb{Z}_p}(\check{\Pi}^*)$, we can write $x - y = \mathrm{Res}_{\mathbb{Z}_p}(z)$ for some $z \in \check{\Pi}^*$. Then $\mathrm{Res}_{1+p\mathbb{Z}_p}(z) = 0$, and part a) shows that $z = 0$, thus $x = y$. But then $\alpha x = \beta y$ and so $x = y = 0$. The result follows. □

Corollary 3.26. *Let $\eta : \mathbb{Q}_p^\times \to \mathscr{O}^\times$ be a locally constant character and let $\alpha, \beta \in \mathscr{O}^\times$. If $m_\eta(\mathscr{C}^\alpha) \cap \mathscr{C}^\beta \neq \{0\}$, then $\eta|_{\mathbb{Z}_p^\times} = 1$ and $\alpha = \beta$.*

Proof. Suppose that $z \in \mathscr{C}^\alpha$ and $y \in \mathscr{C}^\beta$ are nonzero and satisfy $m_\eta(z) = y$. Choose $\tilde{z}, \tilde{y} \in \check{\Pi}^*$ such that $z = \mathrm{Res}_{\mathbb{Z}_p^\times}(\tilde{z})$ and $y = \mathrm{Res}_{\mathbb{Z}_p^\times}(\tilde{y})$ (this uses the fact that $D^{\psi=\alpha}, D^{\psi=\beta} \subset \mathrm{Res}_{\mathbb{Z}_p}(\check{\Pi}^*)$). The hypothesis and part e) of proposition 3.1 yield the existence of $n \geq 1$ such that

$$m_\eta = \sum_{i \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \eta(i)\mathrm{Res}_{i+p^n\mathbb{Z}_p}.$$

For $i \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, applying $\mathrm{Res}_{i+p^n\mathbb{Z}_p}$ to the equality $m_\eta(z) = y$ (and using part d) of proposition 3.1) gives

$$\eta(i)\mathrm{Res}_{i+p^n\mathbb{Z}_p}(\tilde{z}) = \mathrm{Res}_{i+p^n\mathbb{Z}_p}(\tilde{y}),$$

hence (proposition 3.25) $\eta(i)\tilde{z} = \tilde{y}$. Since this holds for all $i \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ and since $\tilde{z} \neq 0$, we infer that $\eta|_{\mathbb{Z}_p^\times} = 1$. But then m_η is the identity map and so the hypothesis becomes $\mathscr{C}^\alpha \cap \mathscr{C}^\beta \neq \{0\}$. Proposition 3.25 shows that $\alpha = \beta$ and finishes the proof. □

3.4.2. *A family of unramified twists of D.* In this § we let V be any absolutely irreducible L-representation of $\mathscr{G}_{\mathbb{Q}_p}$ of dimension ≥ 2 and we let V_0 be a $\mathscr{G}_{\mathbb{Q}_p}$-stable $\mathscr{O}$-lattice in V. Let $S = \mathscr{O}[[X]]$ and let $\delta^{\mathrm{nr}} : \mathscr{G}_{\mathbb{Q}_p} \to S^\times$ be the unramified character sending a geometric Frobenius to $1 + X$. Then $V_{0,\mathrm{un}} = S \otimes_{\mathscr{O}} V_0$ becomes a $\mathscr{G}_{\mathbb{Q}_p}$ module for the diagonal action.

Let D_0 (resp. $D_{0,\mathrm{un}}$) be the étale (φ, Γ)-module associated to V_0 (resp. $V_{0,\mathrm{un}}$) by Fontaine's [41] equivalence of categories and its version for families [25]. Concretely, $D_{0,\mathrm{un}} = \mathscr{O}_{\mathscr{E},S} \otimes_{\mathscr{O}_{\mathscr{E}}} D_0$, where [23]

$$\mathscr{O}_{\mathscr{E},S} = S\widehat{\otimes}_{\mathscr{O}}\mathscr{O}_{\mathscr{E}} = \{\sum_{n\in\mathbb{Z}} a_nT^n, a_n \in S, \lim_{n\to-\infty} a_n = 0\},$$

$\gamma \in \Gamma$ and φ acting by $\gamma \otimes \gamma$ and $\varphi(\lambda \otimes z) = ((1+X)\varphi(\lambda)) \otimes \varphi(z)$.

[23] The limit is taken for the $\mathfrak{m}_S = (\varpi, X)$-adic topology.

For $\alpha \in 1+\mathfrak{m}_L$, there is a surjective specialization map $\mathrm{sp}_\alpha : S \to \mathscr{O}$, sending X to $\alpha-1$, with kernel $\wp_\alpha = (X-\alpha+1)$. The induced map

$$\mathrm{sp}_\alpha : \mathscr{O}_{\mathscr{E},S} \to \mathscr{O}_{\mathscr{E}}, \quad s_\alpha(\sum_{n\in\mathbb{Z}} a_n T^n) = \sum_{n\in\mathbb{Z}} a_n(\alpha-1)T^n$$

gives rise to a specialization map $\mathrm{sp}_\alpha : D_{0,\mathrm{un}} \to D_0$, which in turn induces an isomorphism of (φ,Γ)-modules $D_{0,\mathrm{un}}/\wp_\alpha \cong D_0 \otimes \alpha^{v_p}$. In particular, $\mathrm{sp}_\alpha : D_{0,\mathrm{un}} \to D_0$ induces a Γ-equivariant morphism $\mathrm{sp}_\alpha : D_{0,\mathrm{un}}^{\psi=1} \to D_0^{\psi=\alpha}$. Let $D_{\mathrm{un}} = L \otimes_{\mathscr{O}} D_{0,\mathrm{un}}$ and $D = L \otimes_{\mathscr{O}} D_0$.

Proposition 3.27. *For all $\alpha \in 1+\mathfrak{m}_L$ the map $\mathrm{sp}_\alpha : D_{\mathrm{un}}^{\psi=1} \to D^{\psi=\alpha}$ is surjective.*

Proof. Let $D_n = D_{0,\mathrm{un}}/\wp_\alpha^n$. It suffices to prove that the cokernel of the natural map $D_{0,\mathrm{un}}^{\psi=1} \to D_1^{\psi=1}$ is $\mathscr{O}$-torsion. The snake lemma applied to the sequence $0 \to D_{n-1} \to D_n \to D_1 \to 0$ mapped to itself by $\psi-1$ yields an exact sequence of $\mathscr{O}$-modules

$$0 \to D_{n-1}^{\psi=1} \to D_n^{\psi=1} \to D_1^{\psi=1} \to \frac{D_{n-1}}{\psi-1} \to \frac{D_n}{\psi-1} \to \frac{D_1}{\psi-1} \to 0.$$

All modules appearing in the exact sequence are compact [19, prop. II.5.5, II.5.6] and we have a natural isomorphism $\varprojlim_n D_n^{\psi=1} = D_{0,\mathrm{un}}^{\psi=1}$ (as $D_{0,\mathrm{un}} = \varprojlim_n D_n$). Passing to the limit we obtain therefore an exact sequence

$$0 \to D_{0,\mathrm{un}}^{\psi=1} \to D_{0,\mathrm{un}}^{\psi=1} \to D_1^{\psi=1} \to M \to M \to \frac{D_1}{\psi-1} \to 0,$$

where $M = \varprojlim_n \frac{D_n}{\psi-1}$. It is thus enough to prove that M is a torsion $\mathscr{O}$-module.

Let $\check{W}_n$ be the Galois representation associated to $\check{D}_n$, namely the Cartier dual of $W_n := V_{0,\mathrm{un}}/\wp_\alpha^n$. It follows from [19, remarque II.5.10] that there is an isomorphism[24]

$$\frac{D_n}{\psi-1} \cong [(\mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \check{D}_n)^{\varphi=1}]^\vee = [(\mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \check{W}_n)^H]^\vee,$$

hence it suffices to check that $(\mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \check{W}_n)^H$ are $\mathscr{O}$-torsion modules of bounded exponent (as n varies).

Let $\mathscr{H} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p^{\mathrm{ab}})$. Since $\check{V}_0$ is absolutely irreducible of dimension ≥ 2, there is $N \geq 1$ such that p^N kills $(\mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \check{V}_0)^{\mathscr{H}}$. As $V_{0,\mathrm{un}} \cong S \otimes_{\mathscr{O}} V_0$, with $\mathscr{H}$ acting trivially on S, and since $S/\wp_\alpha^n$ is a finite free $\mathscr{O}$-module, we have

$$\begin{aligned}(\mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \check{W}_n)^H \subset (\mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \check{W}_n)^{\mathscr{H}} &= (S/\wp_\alpha^n \otimes_{\mathscr{O}} (L/\mathscr{O} \otimes_{\mathscr{O}} \check{V}_0))^{\mathscr{H}} \\ &= S/\wp_\alpha^n \otimes_{\mathscr{O}} (L/\mathscr{O} \otimes_{\mathscr{O}} \check{V}_0)^{\mathscr{H}}\end{aligned}$$

and the last module is killed by p^N. The result follows. $\square$

3.4.3. *Analytic variation in the universal family.* Recall that for $\alpha \in \mathscr{O}^\times$ we denote $\mathscr{C}^\alpha = (1-\alpha\varphi)D^{\psi=\alpha}$. We recall that there is [19, prop. I.2.3] a φ and Γ-invariant perfect pairing $\{\ ,\ \} : \check{D} \times D \to L$, under which φ and ψ are adjoints. The following result follows from the proof of [19, lemme VI.1.1].

Lemma 3.28. *$\mathscr{C}^\alpha$ is the orthogonal (for the pairing $\{\ ,\ \}$) of $\check{D}^{\psi=1/\alpha}$ inside $D^{\psi=0}$.*

[24] Here $H = \mathrm{Ker}(\varepsilon) = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p(\mu_{p^\infty}))$ and $X^\vee$ is the Pontryagin dual of X.

Let $q = p$ if $p > 2$ and $q = 4$ if $p = 2$. Fix a topological generator γ of $1 + q\mathbb{Z}_p$ and define a map $\ell : \mathbb{Z}_p^\times \to \mathbb{Z}_p$ by

$$\ell : \mathbb{Z}_p^\times \cong \mu(\mathbb{Q}_p) \times (1 + q\mathbb{Z}_p) \to 1 + q\mathbb{Z}_p = \gamma^{\mathbb{Z}_p} \cong \mathbb{Z}_p,$$

the second map being the natural projection and the last map sending γ^x to x.

Lemma 3.29. *For all $\eta \in \hat{\mathscr{T}}^0(L)$ there is an equality of operators on $D \boxtimes \mathbb{Z}_p^\times$*

$$m_\eta = \sum_{n\geq 0} (\eta(\gamma) - 1)^n m_{\binom{\ell}{n}}$$

and $m_{\binom{\ell}{n}}(D_0 \boxtimes \mathbb{Z}_p^\times) \subset D_0 \boxtimes \mathbb{Z}_p^\times$.

Proof. For all $\eta \in \hat{\mathscr{T}}^0(L)$ and $x \in \mathbb{Z}_p^\times$ we have

$$\sum_{n\geq 0} (\eta(\gamma) - 1)^n \binom{\ell(x)}{n} = \eta(\gamma)^{\ell(x)} = \eta(\gamma^{\ell(x)}) = \eta(x),$$

the last equality being a consequence of the fact that $x^{-1} \cdot \gamma^{\ell(x)} \in \mu(\mathbb{Q}_p)$. Hence

$$\eta|_{\mathbb{Z}_p^\times} = \sum_{n\geq 0} (\eta(\gamma) - 1)^n \binom{\ell}{n},$$

the series being uniformly convergent on $\mathbb{Z}_p^\times$. This yields the first part. The second part is a consequence of the fact that $\binom{\ell}{n} \in \mathbb{Z}_p$. $\square$

We are now ready to prove a key technical ingredient in the proof of theorem 3.5. We identify $\hat{\mathscr{T}}^0(L)$ and $(1 + \mathfrak{m}_L) \times (1 + \mathfrak{m}_L)$ via the map $\eta \mapsto (\eta(\gamma), \eta(p))$.

Definition 3.30. A subset S of $(1+\mathfrak{m}_L) \times (1+\mathfrak{m}_L)$ is called *Zariski closed* if it is defined by a system of equations of the form $f(x-1, y-1) = 0$, with $f \in \mathscr{O}[[X, Y]]$.

Proposition 3.31. *The set*

$$H = \{\eta \in \hat{\mathscr{T}}^0(L) | \quad m_\eta(\mathscr{C}^\alpha) = \mathscr{C}^{\alpha\eta(p)} \quad \forall \alpha \in 1 + \mathfrak{m}_L\}$$

is a Zariski closed subgroup of $\hat{\mathscr{T}}^0(L)$.

Proof. Since $\hat{\mathscr{T}}^0(L) \to \mathrm{Aut}_L(D^{\psi=0})$, $\eta \mapsto m_\eta$ is a morphism of groups, H is a subgroup of $\hat{\mathscr{T}}^0(L)$. To conclude, it suffices to check that

$$H_\alpha = \{\eta \in \hat{\mathscr{T}}^0(L) | m_\eta(\mathscr{C}^\alpha) \subset \mathscr{C}^{\alpha\eta(p)}\}$$

is Zariski closed for all $\alpha \in 1 + \mathfrak{m}_L$.

Let us fix $\alpha \in 1 + \mathfrak{m}_L$ and denote $\mathscr{C}_{\mathrm{un}} = (1 - \varphi)D_{0,\mathrm{un}}^{\psi=1}$ and $\check{\mathscr{C}}_{\mathrm{un}} = (1 - \varphi)\check{D}_{0,\mathrm{un}}^{\psi=1}$, where $D_{0,\mathrm{un}}$ and $\check{D}_{0,\mathrm{un}}$ were defined in n° 3.4.2. If $\eta \in \hat{\mathscr{T}}^0(L)$, it follows from proposition 3.27 that the specialization maps induce surjections $\mathscr{C}_{\mathrm{un}} \otimes_{\mathscr{O}} L \to \mathscr{C}^\alpha$ and $\check{\mathscr{C}}_{\mathrm{un}} \otimes_{\mathscr{O}} L \to \check{\mathscr{C}}^{1/\alpha\eta(p)}$. Since $\mathscr{C}^{\alpha\eta(p)}$ is the orthogonal of $\check{\mathscr{C}}^{1/\alpha\eta(p)}$ in $D^{\psi=0}$ (lemma 3.28), it follows that

$$H_\alpha = \{\eta \in \hat{\mathscr{T}}^0(L) | \quad \{\mathrm{sp}_{1/\alpha\eta(p)}(\check{z}), m_\eta(\mathrm{sp}_\alpha(z))\} = 0 \quad \forall \check{z} \in \check{\mathscr{C}}_{\mathrm{un}}, z \in \mathscr{C}_{\mathrm{un}}\}.$$

Fix $\check{z} \in \check{\mathscr{C}}_{\mathrm{un}}$ and $z \in \mathscr{C}_{\mathrm{un}}$. We can write $\check{z} = \sum_{k\geq 0} X^k \check{z}_k$ with $\check{z}_k \in \check{D}_0$. By definition

$$\mathrm{sp}_{1/\alpha\eta(p)}(\check{z}) = \sum_{k\geq 0} (\frac{1}{\alpha\eta(p)} - 1)^k \check{z}_k.$$

Combining this relation and lemma 3.29, we obtain

$$\{\mathrm{sp}_{1/\alpha\eta(p)}(\check{z}), m_\eta(\mathrm{sp}_\alpha(z))\} = \sum_{k,n\geq 0}(\frac{1}{\alpha\eta(p)}-1)^k(\eta(\gamma)-1)^n\{\check{z}_k, m_{\binom{\ell}{n}}(\mathrm{sp}_\alpha(z))\}$$

and the last expression is the evaluation at $(\eta(\gamma)-1, \eta(p)-1)$ of an element of $\mathscr{O}[[X,Y]]$. Thus H_α is a Zariski closed subset of $(1+\mathfrak{m}_L)^2$, which finishes the proof of proposition 3.31. □

3.4.4. *The Zariski closure of* $(a^n, b^n)_{n\geq 1}$. We refer the reader to definition 3.30 for the notion of Zariski closed subset of $(1+\mathfrak{m}_L)\times(1+\mathfrak{m}_L)$.

Proposition 3.32. *Let* $a, b \in 1+\mathfrak{m}_L$*. The Zariski closure of* $\{(a^n, b^n) | n \geq 1\}$ *is*

- *A finite subgroup of* $\mu_{p^\infty}\times\mu_{p^\infty}$ *if* $\log a = \log b = 0$.
- *The set* $\{(x, x^s) | x \in 1+\mathfrak{m}_L\}$ *(respectively* $\{(x^s, x) | x \in 1+\mathfrak{m}_L\}$*) if* $\log b = s\log a$ *(respectively* $\log a = s \log b$*),* $s \in \mathbb{Z}_p$ *and* $(\log a, \log b) \neq (0,0)$.
- $(1+\mathfrak{m}_L)\times(1+\mathfrak{m}_L)$ *if* $\log a$ *and* $\log b$ *are linearly independent over* $\mathbb{Q}_p$.

Proof. The first two cases are immediate, so assume that $\log a$ and $\log b$ are linearly independent over $\mathbb{Q}_p$. Suppose that $f \in \mathscr{O}[[X,Y]]$ satisfies $f(a^n-1, b^n-1) = 0$ for all $n \geq 1$. We will prove that $f = 0$. We may assume that $b = a^s$, with $s \in \mathscr{O} - \mathbb{Z}_p$. For $n \geq 1$ we have

$$v_p\left(\binom{s}{n}\right) \geq -v_p(n!) > -\frac{n}{p-1},$$

hence $x^s = \sum_{n\geq 0}\binom{s}{n}(x-1)^n$ is well-defined for $v_p(x-1) > \frac{1}{p-1}$ and $x \mapsto x^s$ is analytic in this ball, with values in $1+\mathfrak{m}_L$. Thus $x \mapsto f(x, (1+x)^s - 1)$ is analytic on the ball $v_p(x) > \frac{1}{p-1}$ and vanishes at $a^n - 1$ for all $n \geq 1$. Since $\log a \neq 0$, it follows that $f(x, (1+x)^s-1) = 0$ for all $v_p(x) > \frac{1}{p-1}$, consequently $f(T, (1+T)^s - 1) = 0$ in $L[[T]]$. Proposition 3.33 below combined with the formal Weierstrass preparation theorem yield $f = 0$, which is the desired result. □

Proposition 3.33. *If* $s \in \mathscr{O} - \mathbb{Z}_p$*, then* $(1+T)^s$ *is transcendental over* $\mathrm{Frac}(\mathscr{E}^+)$.

Proof. Denote $f = (1+T)^s$ and assume that f is algebraic over $K := \mathrm{Frac}(\mathscr{E}^+)$. We need the following elementary result [31, prop. 7.3].

Lemma 3.34. $(1+T)^s \in \mathscr{R}^+$ *if and only if* $s \in \mathbb{Z}_p$.

We start with the case $f \in K$. Since a nonzero element of $\mathscr{E}^+$ generates the same ideal in $\mathscr{E}^+$ as a nonzero polynomial, we have $K \subset \mathscr{R}$, thus $f \in \mathscr{R} \cap L[[T]] = \mathscr{R}^+$ and we are done by the previous lemma.

Next, assume that f is algebraic over K and $f \notin K$. Let $P = X^n + a_{n-1}X^{n-1} + ... + a_0 \in K[X]$ be its minimal polynomial over K, with $n > 1$. Consider the differential operator $\partial = (1+T)\frac{d}{dT}$ and observe that $\partial f = sf$. The equality $\partial(P(f)) - nsP(f) = 0$ can also be written as

$$\sum_{k=0}^{n-1}(\partial a_k + s(k-n)a_k)f^k = 0.$$

By minimality of n we deduce that $\partial a_k + s(k-n)a_k = 0$ for all $k < n$, in particular $\partial(a_0 \cdot (1+T)^{-sn}) = 0$, hence $a_0 = c\cdot(1+T)^{sn}$ for some $c \in L^*$. Thus $(1+T)^{sn} \in K$ and by the previous paragraph this gives $sn \in \mathbb{Z}_p$, which combined with $s \in \mathscr{O}$ yields $s \in \mathbb{Z}_p$, a contradiction. The result follows. □

The following result follows immediately from proposition 3.32.

Corollary 3.35. *If $\mu_p \subset L$, then any nontrivial Zariski closed subgroup of $\hat{\mathscr{T}}^0(L)$ contains a nontrivial character of finite order.*

Remark 3.36. The conclusion of proposition 3.32 fails if we work with unbounded analytic functions instead of elements of $L \otimes_{\mathscr{O}} \mathscr{O}[[X,Y]]$ when defining the Zariski closure: if $a, b \in 1 + \mathfrak{m}_L$ satisfy $(\log a, \log b) \neq (0,0)$, then $\log b \cdot \log(1+X) - \log a \cdot \log(1+Y)$ vanishes at $(a^n - 1, b^n - 1)$ for all $n \geq 1$.

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