Titel: Semi-stable Galois representations

Autor: Jean-Marc Fontaine

Adresse: I.H.E.S., 91440 Bures-sur-Yvette, France

The purpose of this talk is to discuss a possible characterisation of the ℓ-adic representations of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ that we get from the ℓ-adic cohomology of proper and smooth varieties defined over $\mathbb{Q}$.

1) p-adic semi-stable representations of $G_{\mathbb{Q}_p}$.

Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ (the field of p-adic numbers) and $G_p = G_{\mathbb{Q}_p} = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Let K be the maximal unramified extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ and σ the absolute Frobenius acting on K.

Definition: A (φ, N, G_p)-module is a finite dimensional K vector space D equipped with

a) A «Frobenius», i.e, a σ-semi-linear, bijective map
$$\varphi : D \to D;$$

b) A «monodromy operator», i.e. a K-linear endomorphism N of D satisfying
$$N\varphi = p\varphi N \quad (\Rightarrow N \text{ is nilpotent}),$$

c) A discrete semi-linear action of G_p ($\Rightarrow$ the action of the inertia sub-group factors through a finite quotient), commuting with φ and N.

If D is such a module and if $\Delta = (\overline{\mathbb{Q}}_p \otimes_K D)^{G_p}$, then $\dim_{\mathbb{Q}_p} \Delta = \dim_K D$.

<u>Definition</u>: <u>A</u> (φ, N, G_p)-<u>filtered module</u> consists of a (φ, N, G_p)-module D together with a decreasing filtration $(\mathrm{Fil}^i\Delta)_{i\in\mathbb{Z}}$ of Δ by sub-$\mathbb{Q}_p$-vector spaces satisfying $\mathrm{Fil}^i\Delta = 0$ if $i \gg 0$ and $\mathrm{Fil}^i\Delta = \Delta$ if $i \ll 0$.

<u>Fact</u>: One can define a functor

D_p : (p-adic representations of G_p) $\to$ ((φ, N, G_p)-filtered modules) :

We say that a p-adic representation V is <u>potentially semi-stable</u> if $\dim_K D_p(V) = \dim_{\mathbb{Q}_p} V$ ($\Leftrightarrow \dim_{\mathbb{Q}_p} \Delta_p(V) = \dim_{\mathbb{Q}_p} V$ if $\Delta_p(V) = (\overline{\mathbb{Q}}_p \otimes_K D_p(V))^{G_p}$).

The functor D_p induces an equivalence between the category of pot. semi-stable p-adic rep'ns of G_p and the category of «admissible» (φ, N, G_p) ~~modules~~ filtered modules.

<u>Conjecture</u>: Let X be a proper and smooth variety over $\mathbb{Q}_p$. Then $V = H^m_{\acute{e}t}(X \times \overline{\mathbb{Q}}_p, \mathbb{Q}_p)$ is pot. semi-stable. Moreover

(i) $\Delta_p(V)$ can be identified to $H^m_{DR}(X)$;

(ii) if L is a finite Galois extension of $\mathbb{Q}_p$, contained in $\overline{\mathbb{Q}}_p$, on which $X \times L$ has good reduction, $D_p(V)$ can be identified to the crystalline cohomology of the special fiber of a smooth model of $X \times L^{nr}$ over the integers (where L^{nr} is the maximal unramified extension of L in $\overline{\mathbb{Q}}_p$);

(iii) if L is a finite Galois extension of $\mathbb{Q}_p$, on which $X \times L$ has semi-stable reduction, $D_p(V)$ can be identified to a new cohomology, the «crystalline with log poles cohomology» of $X \times L^{ur}$.

There is a lot of partial results in this direction and this conjecture is known for a wild class of varieties, and also for some other «motivic representations» (Tate, Raynaud, Bloch, Kato, Faltings, Messing, Hyodo, Scholl, Illusie, ... and the autor). The new cohomology theory seems to work (according to a quite recent work of Kato).

The definition of D_p uses the construction of three rings

$$B_{cris} \subset B_{st} \subset B_{DR}.$$

I constructed B_{cris} and B_{DR} a few years ago; the idea that something like B_{st} should exist is due to Uwe Jannsen.

2) ℓ-adic semi-stable representations of $G_{\mathbb{Q}}$ (joint work with B. Mazur).

Definition: Let ℓ be a prime number and V be a ℓ-adic representation of $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. We say that V is *geometric* if

(i) V is unramified almost everywhere;

(ii) V is potentially semi-stable at $p = \ell$.

Assume we have such a V, plus an embedding of a finite extension E_λ of $\mathbb{Q}_\ell$ into $End_{G_{\mathbb{Q}}}(V)$. Let $d = \dim_{E_\lambda} V$. For each prime p, one can associate to V a d-dimensional linear representation of the Weil-Deligne group $W'_p = W'_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ (for $p \neq \ell$, this is the usual construction and for $p = \ell$, one uses prop. (ii) and $D_p(V)$).

Using also $D_p(V)$, for $p=\ell$, one can define the Hodge numbers of V, its weight, whenever V is simple, and (under a mild assumption) an isomorphism class of a d-dimensional linear representation of the Weil group $W_{\mathbb{R}}$.

Therefore, if we choose an embedding of E_λ into $\mathbb{C}$, we can define the conductor N_V of V, the L-function $L(V,s)$ (with or without the Γ-factor), the ε-factor $\varepsilon(V,s)$.

We then have a lot of natural questions (which are not unrelated), e.g.:

① If V is a semi-simple ℓ-adic geometric rep'n, does it exist $i \in \mathbb{Z}$ and a proper and smooth variety X over $\mathbb{Q}$ s.t. $V(i)$ is isomorphic to a direct summand of $H^{*}_{et}(X \times \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$?

② If V is a geometric representation, is it true that $L(V,s)$ has a meromorphic continuation in the whole complex plane? Does it satisfy the functional equation

$$L(V,s) = \varepsilon(V,s) \cdot L(V^{*}, 1-s) \qquad ?$$

③ Are there only finitely many isomorphism classes of semi-simple geometric E_λ-representations with a given conductor and given Hodge numbers?

④ If $\dim_{E_\lambda} V = 2$, and if V is simple and geometric, with conductor N and $h^{0,k-1} = h^{k-1,0} = 1$ for a suitable integer $k \geq 1$, is it true that V is the representation associated to a modular form of weight k and level N (of course we already know the Fourier coefficients of the modular form).

⑤ If V is a simple geometric representation of weight m, does V satisfy the Weil's conjecture (i.e. is it true that, for all p such that V is unramified at p, all the absolute values of the eigenvalues of the geometric Frobenius at p are equal to $p^{m/2}$) ?