ADDENDUM

par

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There has been a number of developpements concerning conjecture 0.4 of the paper and, in particular, its relation to Faltings heights of CM abelian varieties:

• Andrew Obus [**9**] managed to remove the indeterminacy at the prime 2 in th. 0.5; hence we now have a full proof of the conjecture in the abelian case.

• Vincent Maillot and Damian Rössler [**7**] have extended the conjecture to logarithmic derivatives of Artin L-functions at all negative integers where the L-function does not vanish.

• I showed [**3**] that one can deduce from the conjecture lower and upper bounds for Faltings heights of abelian varieties with CM by the ring of integers of the field of complex multiplication in terms related to the discrimant of this ring. This was extended to abelian varieties with CM by a subring of the ring of integers by Lucia Mocz [**8**].

• In his thesis [**17**], Florent Urfels has given a p-adic analog of conjecture 0.4.

• The monomial relations of Shimura-Deligne make it possible (cf. prop. II.2.7) to define a linear period map on all of $\mathcal{CM}$ and not on its subspace $\mathcal{CM}^0$ only (but with values in $\mathbb{C}^*/\overline{\mathbb{Q}}^*$; one can make the indeterminacy smaller than $\overline{\mathbb{Q}}^*$ but it seems difficult to get a well-defined map into $\mathbb{C}^*$). Hiroyuki Yoshida [**19, 20**] has made a conjecture expressing this period map in terms of the Shintani's Gamma functions mentioned in the first point of 0.7 Remarques diverses. With Tomokazu Kashio, he stated p-adic analogs of this conjecture [**4, 5**].

One case that looked clearly easier is the "average conjecture" where one averages the Faltings heights of all possible CM types of a given CM field E with maximal real subfield E_+: see fourth point of 0.7 Remarques diverses. Note that the formula given there is incorrect; it should be:

$$\frac{1}{2^{[E_+:\mathbb{Q}]}}\sum_{\Phi\in\Phi(E)} h_{\rm Fal}(X_\Phi) = -\frac{1}{2}\Big(\frac{L'(\chi,0)}{L(\chi,0)}+\frac{1}{2}\log \mathbf{f}_\chi\Big)-\frac{[E_+:\mathbb{Q}]}{2}\log 2\pi.$$

• In 2010 (arXiv:1008.1854 [math.NT]), Tonghai Yang [**18**] proved the average conjecture for degree 4 CM fields, not necessarily abelian.

After that things accelerated.

• At the Gan-Gross-Prasad summer school in Paris (June 2014), Howard announced the proof, in an ongoing work with Jan Bruinier, Steve Kudla, Michael Rapoport and Tonghai Yang [**2**], of special (but already in arbitrary degree) cases of the average conjecture (namely, when the CM field E is the compositum of E_+ and an imaginary quadratic extension of $\mathbb{Q}$).

• In december 2014, at the conference in honor of Michael Harris at MSRI, Benjamin Howard announced a proof of the full average conjecture (main result of [**1**]) but up to uncontrolled rational multiples of $\log p$'s, for prime numbers p dividing the discriminant of E.

• Shortly after, Jacob Tsimerman mentionned to the authors of [**1**] that proving the full conjecture (or giving sufficiently sharp bounds for the contributions of the bad primes) would provide [**14**] lower bounds for Galois orbits of CM points that would allow to remove the use of GRH in the o-minimality proofs of the André-Oort conjecture for $\mathcal{A}_g$ ([**11, 6**], culmination of a long series of works including [**13, 10, 15, 16**] which use the magical tools provided by [**12**]). By early-April 2015, only the prime 2 was resisting but its contribution was under control; hence the o-minimality proofs of the André-Oort conjecture for $\mathcal{A}_g$ had become unconditional.

• In mid-April, Shouwu Zhang announced that they could prove the full average conjecture with Xinyi Yuan [**21**], by methods quite different from those of [**1**].

• The papers [**1**] and [**21**], with full proofs of the average conjecture appeared on arXiv a little later in 2015.

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