COMPOSITIO MATHEMATICA

PIERRE COLMEZ
LEILA SCHNEPS

p-adic interpolation of special values of Hecke L-functions

Compositio Mathematica, tome 82, nº 2 (1992), p. 143-187.
<http://www.numdam.org/item?id=CM_1992__82_2_143_0>

Compositio Mathematica **82**: 143–187, 1992.

p-Adic interpolation of special values of Hecke L-functions

PIERRE COLMEZ[1] and LEILA SCHNEPS[2]

[1]*C.N.R.S., Laboratoire de Mathématiques de l'Ecole Normale Supérieure, U.R.A. 762, Ecole Normale Supérieure, 45 rue d'Ulm, 75005 Paris.* [2]*C.N.R.S., Laboratoire de Mathématiques de la Faculté des Sciences de Besançon, U.R.A. 741, 25030 Besançon, France*

Received 22 February 1990; accepted 17 July 1991

0. Introduction

Let K be a quadratic imaginary field. Let $\bar{K}$ be its algebraic closure and fix an embedding of $\bar{K}$ into $\mathbf{C}$ and $\mathbf{C}_p$ for all primes p. Let F be an extension of degree n of K. A Hecke character ψ of F will be called K-admissible if there exist $k(\psi)\in\mathbf{N}$ and $j(\psi)\in\mathbf{N}-\{0\}$ such that $\psi((\alpha))=\overline{N_{F/K}(\alpha)}^{k(\psi)}N_{F/K}(\alpha)^{-j(\psi)}$ for all $\alpha\in F^*$ congruent to 1 modulo the conductor $\mathbf{m}_\psi$ of ψ. If ψ is a K-admissible Hecke character of F, we set $\Lambda(\psi)=\Gamma(j(\psi))^n(2\pi i)^{-nj(\psi)}L(\psi,0)$, where $L(\psi,s)$ is the Hecke L-function attached to ψ. A conjecture of Deligne [D] proved by Harder [H-S] predicts the value of $\Lambda(\psi)$ up to an algebraic number. The aim of this paper is the study of the p-adic behavior of $\Lambda(\psi)$ as ψ varies.

Let $p\neq 2,3$ be a prime splitting in K. Let $\mathbf{p}$ be the prime of K induced by the embedding of $\bar{K}$ into $\mathbf{C}_p$ and $\bar{\mathbf{p}}$ the other prime of K above p. As observed by Weil [W1], any Hecke character ψ of F of type A_0 (thus any K-admissible Hecke character of F) gives rise to a unique continuous character $\psi^{(p)}$ of $\mathrm{Gal}(F^{ab}/F)$ with values in $\mathbf{C}_p^*$. If $\mathbf{m}$ is an ideal of the ring of integers of F, let $|\mathbf{m}|$ be the set of places of F dividing $\mathbf{m}$, and if S is a finite set of places of F not dividing (p), let $\mathcal{G}_{F,S,p}$ (resp. $\mathcal{G}_{F,S,\mathbf{p}}$) be the Galois group over F of the union of all abelian extensions of level $\mathbf{m}$ such that $|\mathbf{m}|\subset S\cup|(p)|$ (resp. $|\mathbf{m}|\subset S\cup|\mathbf{p}|$). If ψ is a K-admissible Hecke character of F of conductor $\mathbf{m}_\psi$, then $\psi^{(p)}$ factors through $\mathcal{G}_{F,S,p}$ for all S such that $|\mathbf{m}_\psi|\subset S\cup|(p)|$ and even through $\mathcal{G}_{F,S,\mathbf{p}}$ if $k(\psi)=0$ and $|\mathbf{m}_\psi|\subset S\cup|\mathbf{p}|$. Finally, let $F^\vee$ be the complex conjugate of F and if ψ is a Hecke character of F, let $\psi^\vee$ be the Hecke character of $F^\vee$ defined by $\psi^\vee(\mathbf{a})=N(\mathbf{a})^{-1}\psi^{-1}(\bar{\mathbf{a}})$ for all fractional ideals $\mathbf{a}$ of $F^\vee$.

Our main result can be stated as follows:

THEOREM. (i) *There exists a unique measure μ_S on $\mathcal{G}_{F,S,p}$ such that for all K-admissible Hecke characters ψ of F such that $\psi^{(p)}$ factors through $\mathcal{G}_{F,S,p}$ (and with the additional assumption that $k(\psi)=0$ or $j(\psi)=1$ if $n\geqslant 3$), we have*:

$$\int_{\mathcal{G}_{F,S,p}}\psi^{(p)}\,\mathrm{d}\mu_S=E_{|\bar{\mathbf{p}}|}(\psi^\vee)E_{|\bar{\mathbf{p}}|}(\psi)W_{\mathbf{p}}(\psi)E_S(\psi)\Lambda(\psi).$$

(ii) *There exists a unique pseudo-measure λ_S (which is a measure if $S \neq \varnothing$) such that for all K-admissible Hecke characters ψ of F such that $\psi^{(p)}$ factors through $\mathscr{G}_{F,S,\mathbf{p}}$, we have:*

$$\int_{\mathscr{G}_{F,S,\mathbf{p}}} \psi^{(p)}\,\mathrm{d}\lambda_S = E_{|\bar{\mathbf{p}}|}(\psi^\vee)W_{\mathbf{p}}(\psi)E_S(\psi)\Lambda(\psi),$$

where if T is a finite set of places, $E_T(\psi)$ is the Euler factor above T (at $s=0$) of the L-function attached to ψ and $W_{\mathbf{p}}(\psi)$ is a local root number.

REMARK. Stated like this the theorem does not really make sense because in each equality, the left-hand side belongs to $\mathbf{C}_p$ and the right-hand side to $\mathbf{C}$. So, to make sense of these equalities, we choose an elliptic curve E defined over $\bar{K}$ with complex multiplication by K; then $H^1_{DR}(E)$ splits canonically as $H^0(E, \Omega^1_E) \oplus H^1(E, \mathcal{O}_E)$ where both terms are stable under the action of End E. Now, as we have fixed embeddings of $\bar{K}$ into $\mathbf{C}$ and $\mathbf{C}_p$, if we choose a generator η of $H^1(E, \mathcal{O}_E)$ and a generator γ of the 1-dimensional K vector space $H_1(E(\mathbf{C}), \mathbf{Q})$, we can define a p-adic period $\eta_p = \int_\gamma \eta$ and a complex period $\eta_\infty = \int_\gamma \eta$ (cf. III §2). The fields $\bar{K}(\eta_\infty)$ and $\bar{K}(\eta_p)$ as well as the isomorphism between them sending η_∞ to η_p are independent of the choices of E, η and γ and all equalities take place in $\bar{K}(\eta_\infty) \simeq \bar{K}(\eta_p)$.

Such measures have been previously constructed in the case $n=1$ by Manin–Vishik [M-V] and Katz [K]. Using ideas of Coates–Wiles [C-W], Yager [Ya1], [Ya2] and Tilouine [T] (see also de Shalit [d Sh]) obtained a much more elementary construction of this measure (still in the case $n=1$).

We obtain our theorem in the following way. Using a method developed in [Co1], similar to Shintani's method [Sh] in the totally real case, we can define a value $\Lambda^?(\psi)$ explicitly given as a polynomial in Kronecker–Eisenstein series attached to lattices in K and a priori depending on various auxiliary choices (mainly the choice of 'Shintani decomposition') which is formally (i.e. without worrying about convergence problems) equal to $\Lambda(\psi)$. To prove that $\Lambda^?(\psi) = \Lambda(\psi)$ in general turned out to be beyond our capacities, but by a suitable modification of the methods of [Co1], we were able to prove the desired equality whenever $n=1,2$ or $n \geqslant 3$ and $k(\psi)=0$ or $j(\psi)=1$. As is well-known, the existence of a measure is equivalent to the integrality of a certain power series and our explicit formulae for $\Lambda^?(\psi)$ in terms of Eisenstein–Kronecker series allowed us to deduce the necessary integrality results from the corresponding results for the case $n=1$, i.e. for the Eisenstein–Kronecker series themselves, which are more or less well-known (more or less because the results in the literature are not stated in a way that we can use, which means that we have to reprove them in a form more suitable for our purposes). A by-product of the existence of this measure is that $\Lambda^?(\psi)$ is independent of all choices.

If χ is a continuous $\mathbf{C}_p^*$-valued character of $\mathcal{G}_{F,S,p}$ (resp. $\mathcal{G}_{F,S,\mathbf{p}}$), we set $L_{p,S}(\chi)=\int_{\mathcal{G}_{F,S,p}} \chi \, d\mu_S$ (resp. $L_{\mathbf{p},S}(\chi)=\int_{\mathcal{G}_{F,S,\mathbf{p}}} \chi \, d\lambda_S$). We can then make the preceding theorem more precise as follows:

MAIN THEOREM. (i) $L_{p,s}(\chi)$ *is a holomorphic (and even Iwasawa) function of* χ.

(ii) *If* ψ *is an admissible Hecke character of* F *such that* $\psi^{(p)}$ *factors through* $\mathcal{G}_{F,S,p}$, *then*

$$L_{p,S}(\psi^{(p)}) = E_{|\bar{\mathbf{p}}|}(\psi)E_{|\bar{\mathbf{p}}|}(\psi^{\vee})W_{\mathbf{p}}(\psi)E_S(\psi)\Lambda^?(\psi).$$

(iii) *If the conductor of* χ *is divisible by all the elements of* S, *then there exists a* p-*adic unit* $W^{(p)}(\chi)$ *such that* $W^{(p)}(\chi)L_{p,S}(\chi)=L_{p,S}(\chi^{\vee})$ *where* $\chi^{\vee}$ *is the character of* $\mathcal{G}_{F^{\vee},\bar{S},p}$ *obtained from* χ *in the same way as* $\psi^{\vee}$ *was obtained from* ψ *for* ψ *a Hecke character of* F.

(iv) $L_{\mathbf{p},S}(\chi)$ *is a meromorphic function of* χ, *holomorphic except for a simple pole at* $\chi=1$ *if* $S=\varnothing$, *of residue* $hR_{\mathbf{p}}E_{\mathbf{p}}$, *where as usual* h *is the class number of* F, R_p *is the* $\mathbf{p}$-*adic regulator of the group of units of the ring of integers of* F *and* $E_{\mathbf{p}}$ *is a certain Euler factor.*

(v) *If* ψ *is an admissible Hecke character of* F *such that* $\psi^{(p)}$ *factors through* $\mathcal{G}_{F,S,\mathbf{p}}$ *then* $L_{\mathbf{p},S}(\psi^{(p)}) = E_{|\bar{\mathbf{p}}|}(\psi^{\vee})W_{\mathbf{p}}(\psi)E_S(\psi)\Lambda(\psi)$.

COROLLARY. $L_{\mathbf{p},\varnothing}$ *has a pole at* $\chi=1$, *or equivalently,* $\lambda_{\varnothing}$ *is not a measure, if and only if Leopoldt's conjecture is true for* $(F,\mathbf{p})$. *If this is the case then Leopoldt's conjecture is true for* (F,p).

This paper is organized as follows. In Section I we introduce the basic notations and recall some basic facts about Fourier transforms of functions on adeles. In Section II we present a slight modification of the Shintani-like method developed in [Co1]. In Section III, we prove the existence of p-adic measures attached to n-dimensional generalizations of Eisenstein–Kronecker series attached to lattices in K. As a consequence of the existence of these measures, we derive the fact that all choices that we had to make in Section II lead to the same result. In Section IV we prove a number of functional equations satisfied by $\Lambda(\psi)$ and apply the results of the two preceding sections to give a formula for $\Lambda(\psi)$ in terms of polynomials in Eisenstein–Kronecker series. Finally, Section V is devoted to the construction of μ_S and λ_S using the measures constructed in Section III and to the study of the p-adic L-functions $L_{p,S}$ and $L_{\mathbf{p},S}$.

I. Notations and Definitions

Let K be a quadratic imaginary field. Let $\alpha \to \bar{\alpha}$ denote the non-trivial automorphism of K. Let $F \simeq K[X]/P(X)$, for P an irreducible polynomial of degree n, be an extension of degree n of K. Let $F^{\vee}=K[X]/\bar{P}(X)$. We still write

$\alpha \to \bar{\alpha}$ for the antilinear isomorphism from F to $F^\vee$ sending X to X. We shall use H to denote either F or $F^\vee$ so $H^\vee$ will be $F^\vee$ (resp. F) if $H = F$ (resp. $H = F^\vee$). We write

O_H for the ring of integers of H,
U_H for the group of units of O_H,
$I(H)$ for the group of fractional ideals of H,
$I^+(H) \subset I(H)$ for the set of ideals of O_H,
$Cl(O_H)$ for the group of ideal classes,
$C(H) \subset I^+(H)$ for the set of ideals $\mathbf{a}$ of O_H such that $O_H/\mathbf{a}$ is cyclic as an abelian group,
$C^0(H)$ for the set of principal ideals of $C(H)$,
$P(H)$ for the set of prime ideals of O_H,
$\mathscr{P}(H)$ for the set of finite subsets of $P(H)$,
$\mathbf{A}_H$ for the ring of adeles of H,
$\mathbf{A}_H^f$ for the ring of finite adeles of H, and
$\mathbf{d}_H$ for the absolute different of O_H.

If V is a subgroup of U_H let $V^\vee = \{\bar{v} \mid v \in V\}$ be the corresponding subgroup of $U_{H^\vee}$.

If $\mathbf{a} \in I(H)$, let $\bar{\mathbf{a}} = \{\bar{\alpha} \mid \alpha \in \mathbf{a}\} \in I(H^\vee)$ and if $S \in \mathscr{P}(H)$, let $\bar{S} = \{\bar{\mathbf{p}} \mid \mathbf{p} \in S\} \in \mathscr{P}(H^\vee)$.

If $\mathbf{m} \in I(H)$, let $|\mathbf{m}| = \{\mathbf{q} \in P(H) \mid v_{\mathbf{q}}(\mathbf{m}) \neq 0\} \in \mathscr{P}(H)$ and if $S \in \mathscr{P}(H)$, let $I_S(H) = \{\mathbf{a} \in I(H) \mid |\mathbf{a}| \cap S = \varnothing\}$.

Let $O_{H,S}$ (resp. $O'_{H,S}$) be the subring of H defined by $x \in O_{H,S}$ (resp. $O'_{H,S}$) if and only if $v_{\mathbf{q}}(x) \geqslant 0$ if $\mathbf{q} \in S$ (resp. $\mathbf{q} \notin S$).

Fix an embedding of the algebraic closure $\bar{K}$ of K into $\mathbf{C}$. Let $Y_{H,\infty} = H \otimes_{\mathbf{Q}} \mathbf{C} \simeq Y_1 \times Y_2$, where $Y_1 = H \otimes_K \mathbf{C}$ and $Y_2 = H^\vee \otimes_K \mathbf{C}$. Let $\tau_1, \ldots, \tau_n$ be the n embeddings of H into $\bar{K}$; we obtain an isomorphism of Y_1 (resp. Y_2) with $\mathbf{C}^n$ sending $\alpha \otimes 1$ to $(\tau_1(\alpha), \ldots, \tau_n(\alpha))$ (resp. to $(\overline{\tau_1(\bar{\alpha})}, \ldots, \overline{\tau_n(\bar{\alpha})})$). With these identifications, H and $H^\vee$ become dense K-vector subspaces of $\mathbf{C}^n$ and $\mathbf{a} \in I(H)$ becomes a lattice in $\mathbf{C}^n$. If $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$ belong to $\mathbf{C}^n$, let

$$\mathrm{Tr}(y) = \sum_{i=1}^{n} y_i, \qquad N(y) = \prod_{i=1}^{n} y_i,$$

$$yz = (y_1 z_1, \ldots, y_n z_n), \qquad \langle y \mid z \rangle = \mathrm{Tr}(y\bar{z} + \bar{y}z),$$

and

$$(y \mid z)_\infty = \exp(-2\pi i \langle y \mid z \rangle).$$

If B is a basis of H over K, we let $B^\vee$ be the basis of $H^\vee$ over K dual to B with respect to $\langle \mid \rangle$ and if $\mathscr{B}$ is a finite set of bases of H over K, we let

$\mathscr{B}^\vee = \{B^\vee \mid B \in \mathscr{B}\}$. If $\mathbf{a} \in I(H)$, let $\mathbf{a}^\vee$ be the dual lattice of $\mathbf{a}$ with respect to $\langle\,|\,\rangle$. Then, $\mathbf{a}^\vee \in I(H^\vee)$ and we have $\mathbf{a}^\vee = \bar{\mathbf{a}}^{-1}\overline{{}_H^{-1}}{}_{\vee}^{1} = (\overline{\mathbf{a}\mathbf{d}_H})^{-1}$.

If $\mathbf{q} \in P(H)$, let $H_\mathbf{q}$ be its completion at $\mathbf{q}$ and $O_\mathbf{q}$ be the ring of integers of $H_\mathbf{q}$. If $S \in \mathscr{P}(H)$, let $H_S = \prod_{\mathbf{q}\in S} H_\mathbf{q}$ and $O_S = \prod_{\mathbf{q}\in S} O_\mathbf{q}$. We can describe $\mathbf{A}_H^f$ as the set of $x = (\ldots, x_\mathbf{q}, \ldots)$ such that $x_\mathbf{q} \in H_\mathbf{q}$ for all $\mathbf{q} \in P(H)$ and $x_\mathbf{q} \in O_\mathbf{q}$ for almost all $\mathbf{q} \in P(H)$. We can define a pairing $(\,|\,)_H$ on $\mathbf{A}_H^f \times \mathbf{A}_H^f$ with values in the group of roots of unity of $\bar{K}^* \subset \mathbf{C}^*$ in the following way. The above defined pairing $\langle\,|\,\rangle$ on $\mathbf{C}^n \times \mathbf{C}^n$ induces a pairing on $H \times H^\vee$ with values in $\mathbf{Q}$ which we can extend to a pairing on $\mathbf{A}_H^f \times \mathbf{A}_{H^\vee}^f$ with values in $\mathbf{A}_\mathbf{Q}^f$, and using the canonical isomorphism between $\mathbf{A}_\mathbf{Q}^f/\prod_p \mathbf{Z}_p$ and $\mathbf{Q}/\mathbf{Z}$, we set $(x \mid y)_H = \exp(-2\pi i \widetilde{\langle x \mid y\rangle})$ where $\widetilde{\langle x \mid y\rangle}$ is the image of $\langle x \mid y\rangle$ in $\mathbf{Q}/\mathbf{Z}$. This pairing induces local pairings $(\,|\,)_S$ on $H_S \times H_{\bar{S}}^\vee$ and we have $(x \mid y)_H = \prod_{\mathbf{q}\in P(H)} (x_\mathbf{q} \mid y_{\bar{\mathbf{q}}})_{|\mathbf{q}|}$.

Using these pairings, we can define the (local and global) Fourier transform. Let $\mathscr{S}_{S,H}$ be the space of $\bar{K}$-valued locally constant compactly supported functions on H_S. If $\mathbf{a} \subset \mathbf{b}$ are two fractional ideals of H_S and $\phi \in \mathscr{S}_{S,H}$ is constant modulo $\mathbf{a}$ and zero outside $\mathbf{b}$, we define its Fourier transform $\mathscr{F}_S(\phi) \in \mathscr{S}_{S,H}$ by:

$$\mathscr{F}_S(\phi)(y) = \begin{cases} \sqrt{\dfrac{N_{\bar{S}}(\mathbf{a}^\vee)}{N_S(\mathbf{a})}} \displaystyle\sum_{x\in\mathbf{b}/\mathbf{a}} \phi(x)(x \mid y)_S & \text{if } y \in \mathbf{a}^\vee \\ 0 & \text{if } y \notin \mathbf{a}^\vee, \end{cases}$$

where $\mathbf{a}^\vee$ is the ideal of $H_{\bar{S}}^\vee$ dual to $\mathbf{a}$ with respect to $(\,|\,)_S$ and $N_S(\mathbf{a})$ is the norm of $\mathbf{a}$ as a fractional ideal of H_S. It is an exercise to verify that this definition does not depend on the choices of $\mathbf{a}$ and $\mathbf{b}$ and that $\mathscr{F}_{\bar{S}}(\mathscr{F}_S(\phi))(y) = \phi(-y)$.

Let $\mathscr{S}(H)$ be the space of $\bar{K}$-valued locally constant compactly supported functions on $\mathbf{A}_H^f$. The fractional ideals of $\mathbf{A}_H^f$ are in 1-to-1 correspondence with elements of $I(H)$. So if $\mathbf{a} \subset \mathbf{b}$ are elements of $I(H)$ and $\phi \in \mathscr{S}(H)$ is constant modulo $\mathbf{a}$ and zero outside of $\mathbf{b}$, we define its Fourier transform $\mathscr{F}_H(\phi)$ by the same formula as before (with the subscript S replaced by H) and we have $\mathscr{F}_{H^\vee}(\mathscr{F}_H(\phi))(y) = \phi(-y)$.

If $S \in \mathscr{P}(H)$, let $\mathscr{S}_S(H)$ be the subspace of $\mathscr{S}(H)$ of functions of the form $\phi_S(x_S) \prod_{\mathbf{q}\notin S} 1_{O_\mathbf{q}}(x_\mathbf{q})$, where $\phi_S \in \mathscr{S}_{S,H}$ and $1_{O_\mathbf{q}}$ is the characteristic function of $O_\mathbf{q}$. There is an obvious isomorphism between $\mathscr{S}_{S,H}$ and $\mathscr{S}_S(H)$ and $\mathscr{S}(H) = \bigcup_{S\in\mathscr{P}(H)} \mathscr{S}_S(H)$. If $S \cap S' = \varnothing$ and $\phi = \phi_S(x_S) \prod_{\mathbf{q}\notin S} 1_{O_\mathbf{q}}(x_\mathbf{q}) \in \mathscr{S}_S(H)$ and $\phi' \in \mathscr{S}_{S',H}$, we define $\phi' * \phi \in \mathscr{S}_{S\cup S'}(H)$ by $\phi' * \phi(x) = \phi'(x_{S'})\phi_S(x_S) \prod_{\mathbf{q}\notin S\cup S'} 1_{O_\mathbf{q}}(x_\mathbf{q})$. Finally, if $\mathbf{b} \in I(H)$, define $\delta_\mathbf{b} \in \mathscr{S}_{|\mathbf{b}|,H}$ by $\delta_\mathbf{b} = 1_{O_{|\mathbf{b}|}} - 1_\mathbf{b}$ where $1_\mathbf{b}$ is the characteristic function of $\mathbf{b}$ considered as an $H_{|\mathbf{b}|}$ fractional ideal, and if $\mathbf{b} \in I(H^\vee)$, let $\delta_\mathbf{b}^\vee \in \mathscr{S}_{|\bar{\mathbf{b}}|,H}$ be defined by $\delta_\mathbf{b}^\vee = 1_{O_{|\bar{\mathbf{b}}|}} - N(\mathbf{b})^{-1} 1_{\bar{\mathbf{b}}^{-1}}$. Let γ be a generator of the fractional ideal of $H_{|\mathbf{b}|}$ generated by $\mathbf{d}_H$. Then we have

$$\mathscr{F}_{|\bar{\mathbf{b}}|}(\delta_{\bar{\mathbf{b}}})(x) = \frac{1}{\sqrt{N_{|\mathbf{b}|}(\mathbf{d}_H)}} \delta_\mathbf{b}^\vee(\gamma x).$$

II. Shintani's method

Let $k\in\mathbf{N}, j\in\mathbf{N}-\{0\}$, and let V be a subgroup of finite index in U_H. Let $\mathscr{S}_{k,j,V}(H)$ be the subspace of $\mathscr{S}(H)$ of functions satisfying:

$$\phi(vx)\overline{N_{H/K}(v)}^k N_{H/K}(v)^{-j}=\phi(x) \quad \forall x\in\mathbf{A}_H^f \quad \text{and} \quad v\in V. \tag{1}$$

If $\phi\in\mathscr{S}_{k,j,V}$, we set

$$\Lambda(k,j,\phi,s)=\frac{1}{[U_H:V]}\frac{\Gamma(j)^n}{(2i\pi)^{nj}}\sum_{\beta\in H^*/V}\phi(\beta)\frac{\overline{\beta_1\cdots\beta_n^k}}{(\beta_1\cdots\beta_n)^j}\frac{1}{|\beta_1\cdots\beta_n|^{2s}}. \tag{2}$$

This expression is independent of the choice of V and converges for $\mathrm{Re}(s)\gg 0$. By a theorem of Hecke, $\Lambda(k,j,\phi,s)$ admits an analytic continuation to the whole complex plane and a functional equation relating it to $\Lambda(j-1,k+1,\mathscr{F}_H(\phi),-s)$. We set

$$\Lambda(k,j,\phi)=\Lambda(k,j,\phi,0), \tag{3}$$

and the functional equation gives

$$\Lambda(k,j,\phi)=(-1)^{n(j-1)}i^n\Lambda(j-1,k+1,\mathscr{F}_H(\phi)). \tag{4}$$

The aim of this section is to obtain a finite expression for $\Lambda(k,j,\phi)$ in terms of elliptic functions attached to lattices in K. To this end, we shall (briefly) recall the methods developed in [Co1] and improve on them a little bit.

From now on, V will be a torsion free subgroup of finite index of the subgroup of U_H of elements of norm 1 over K. Let $\mathscr{B}(V)$ be the set of finite sets of bases of H over K satisfying:

$$\frac{1}{z_1\cdots z_n}=\sum_{v\in V}\sum_{B\in\mathscr{B}}f_B(vz) \tag{5}$$

for all $z\in(\mathbf{C}^*)^n$ such that the right-hand side converges, where if $B=(f_{1,B},\ldots,f_{n,B})$ is a basis of H over K, we set

$$f_B(z)=\det(B)\prod_{i=1}^n\left(\mathrm{Tr}(f_{i,B}z)\right)^{-1}. \tag{6}$$

REMARK. This condition is an 'algebraic' version of Shintani's condition [Sh] (in the totally real case), that the union over $B\in\mathscr{B}$ of the cones generated by $f_{1,B},\ldots,f_{n,B}$ be a fundamental domain of $(\mathbf{R}_+^*)^n$ modulo the action of V.

LEMMA 1. (i) *$\mathscr{B}(V)$ is not empty.* (ii) *If $\mathscr{B} \in \mathscr{B}(V)$, then $\mathscr{B}^\vee \in \mathscr{B}(V^\vee)$.*

Proof. We shall use Theorem 1 of [Co1] to construct explicit elements of $\mathscr{B}(V)$. By a theorem of Dirichlet, V is of rank $n-1$. Let us choose a basis $\eta_1, \dots, \eta_{n-1}$ of V, and for each $\sigma \in S_{n-1}$, let $f_{1,\sigma} = 1$ and $f_{i,\sigma} = \Pi_{j<i}\eta_{\sigma(j)}$ for $2 \leqslant i \leqslant n$. Write $\varepsilon(\sigma)$ for the signature of σ and suppose that $(f_{1,\sigma}, \dots, f_{n,\sigma})$ is a basis of H over K for all $\sigma \in S_{n-1}$ (we can always find $\eta_1, \dots, \eta_{n-1}$ such that this is true). Then there exists a sign $\omega = \omega(\eta_1, \dots, \eta_{n-1})$ such that, if $B_\sigma = (f_{1,\sigma}, \dots, f_{n,\sigma})$ when $\omega\varepsilon(\sigma) = 1$ and $B_\sigma = (f_{n,\sigma}, f_{2,\sigma}, \dots, f_{n-1,\sigma}, f_{1,\sigma})$ when $\omega\varepsilon(\sigma) = -1$, then $\mathscr{B} = \{B_\sigma \mid \sigma \in S_{n-1}\} \in \mathscr{B}(V)$. Part (ii) of the lemma follows by taking the Fourier transform of both sides of (5) and using the fact that the Fourier transform of $f_B(z)$ with respect to $(\,|\,)_\infty$ is $i^n f_{B^\vee}(z)$.

Let $z_i = (z_{i,1}, \dots, z_{i,n})$ for $i = 1, 2$ be variables in $Y_i \simeq \mathbf{C}^n$. Let $\nabla_i = \Pi_{j=1}^n \left(-\dfrac{\partial}{\partial z_{i,j}}\right)$. We deduce from (5) and the fact that $\nabla_1 \circ v = \nabla_1$ if $v \in V$, that whenever the right-hand side converges and $\mathscr{B} \in \mathscr{B}(V)$, we have

$$\frac{\Gamma(j)^n}{(2i\pi)^{(nj)}} \frac{\overline{\beta_1 \cdots \beta_n^k}}{(\beta_1 \cdots \beta_n)^j}$$
$$= \frac{1}{(2i\pi)^{n(k+j)}} \nabla_1^{j-1} \nabla_2^k \left(\sum_{v \in V} \sum_{B \in \mathscr{B}} (v\beta + z_1 \mid z_2)_\infty f_B(v\beta + z_1) \right)_{z_1 = z_2 = 0}. \tag{7}$$

If $\mathscr{B}$ is a finite set of bases of H over K and $\phi \in \mathscr{S}(H)$, we set

$$\mathbf{K}(z_1, z_2, \phi, \mathscr{B}, s) = \sum_{\beta \in H} \sum_{B \in \mathscr{B}} \phi(\beta) f_B(\beta + z_1) |f_B(\beta + z_1)|^{2s} (\beta + z_1 \mid z_2)_\infty. \tag{8}$$

This series is absolutely convergent for $\mathrm{Re}(s) > 1/2$ and can be expressed as a polynomial in Kronecker–Eisenstein series attached to lattices in K (cf. [Co1] or III §3 of this paper). This implies that $\mathbf{K}(z_1, z_2, \phi, \mathscr{B}, s)$ can be analytically continued to the whole complex plane and we set:

$$\mathbf{K}(z_1, z_2, \phi, \mathscr{B}) = \mathbf{K}(z_1, z_2, \phi, \mathscr{B}, 0).$$

If $\phi \in \mathscr{S}_{k,j,V}$ and $\mathscr{B} \in \mathscr{B}(V)$, we set

$$\mathbf{F}(z_1, z_2, \phi, \mathscr{B}) = \frac{1}{[U_H : V]} \frac{1}{(2\pi i)^n} \mathbf{K}\left(\frac{z_1}{2\pi i}, \frac{z_2}{2i\pi}, \phi, \mathscr{B}\right). \tag{9}$$

Now, plugging (7) into (2) with $s = 0$ yields the following *formal* identity:

$$\Lambda(k, j, \phi) = \nabla_1^{j-1} \nabla_2^k (\mathbf{F}(z_1, z_2, \phi, \mathscr{B}))_{z_1 = z_2 = 0}. \tag{10}$$

The main problem with (10) is that $\mathbf{F}(z_1, z_2, \phi, \mathscr{B})$ is in general not regular at $z_1 = z_2 = 0$. In fact, we have the following lemma:

LEMMA 2. *The singularities of* $\mathbf{K}(z_1, z_2, \phi, \mathscr{B})$ *are simple poles situated on the hyperplanes* $\mathrm{Tr}(f_{i,B}(\beta + z_1)) = 0$ (*resp.* $\mathrm{Tr}(f_{i,B^\vee}(\beta + z_2)) = 0$ *where* β *runs through elements of* H (*resp.* $H^\vee$) *such that* $\phi(\beta) \neq 0$ (*resp.* $\mathscr{F}_H(\phi)(\beta) \neq 0$), B *runs through elements of* $\mathscr{B}$ *and* $1 \leqslant i \leqslant n$.

Proof. The proof results from the expression of $\mathbf{K}(z_1, z_2, \phi, \mathscr{B})$ in terms of elliptic functions.

REMARK. The poles on the hyperplanes of equation $\mathrm{Tr}(f_{i,B}(\beta + z_1))$ are already apparent in formula (8); the others appear if we use the following functional equation which is a direct consequence of the Poisson summation formula:

$$\mathbf{K}(z_1, z_2, \phi, \mathscr{B}) = i^n (z_1 \mid z_2)_\infty \mathbf{K}(z_2, -z_1, \mathscr{F}_H(\phi), \mathscr{B}^\vee). \tag{11}$$

We shall say that $(\phi, \mathscr{B})$ satisfies the condition (*) if $\mathbf{K}(z_1, z_2, \phi, \mathscr{B})$ has no singularity at $z_1 = z_2 = 0$. This is equivalent to

(1) $\phi(x) \neq 0 \Rightarrow \mathrm{Tr}(f_{i,B}\chi) \neq 0$ for all $x \in H$, $B \in \mathscr{B}$ and $1 \leqslant i \leqslant n$.
(2) $\mathscr{F}_H(\phi)(x) \neq 0 \Rightarrow \mathrm{Tr}(f_{i,B^\vee} x) \neq 0$ for all $x \in H^\vee$, $B \in \mathscr{B}$ and $1 \leqslant i \leqslant n$.

We shall say that $(\phi, \mathscr{B})$ satisfies (**) if it satisfies (*) and if we have moreover

(3) $\phi(x) \neq 0 \Rightarrow \mathrm{Tr}(fx) \neq 0$ for all $x \in H$ and $f \in \mathscr{E}(\mathscr{B})$,
(4) $\mathscr{F}_H(\phi)(x) \neq 0 \Rightarrow \mathrm{Tr}(fx) \neq 0$ for all $x \in H^\vee$ and $f \in \mathscr{E}(\mathscr{B}^\vee)$,

where $\mathscr{E}(\mathscr{B})$ (resp. $\mathscr{E}(\mathscr{B}^\vee)$) is a finite subset of H (resp. $H^\vee$) which will appear in the proof of Theorem 3.

If $(\phi, \mathscr{B}) \in \mathscr{S}_{k,j,V}(H) \times \mathscr{B}(V)$ satisfies condition (*), we set

$$\Lambda_{\mathscr{B}}(k, j, \phi) = \nabla_1^{j-1} \nabla_2^k (\mathbf{F}(z_1, z_2, \phi, \mathscr{B}))_{z_1 = z_2 = 0} \tag{12}$$

and

$$\mathbf{F}_j(z_2, \phi, \mathscr{B}) = \nabla_1^{j-1} (\mathbf{F}(z_1, z_2, \phi, \mathscr{B}))_{z_1 = 0}. \tag{13}$$

Let g be a C^∞ compactly supported function on $\mathbf{C}$ equal to 1 in a neighborhood of 0. Let $\varepsilon > 0$ and $\mu_k(s) = i^k \dfrac{\Gamma(k+1-s)}{\pi^{k+1-s}} \dfrac{\pi^s}{\Gamma(s)}$, and set

$$\Lambda_{\mathscr{B},\varepsilon}(k, j, \phi, s) = \int_{\mathbf{C}^n} \mathbf{F}_j(z_2, \phi, \mathscr{B}) \prod_{i=1}^n \left(g(\varepsilon z_{2,i}) \mu_k(s) \frac{\overline{z_{2,i}}^k}{|z_{2,i}|^{2(k+1-s)}} \right). \tag{14}$$

THEOREM 3. (i) $\Lambda_{\mathscr{B},\varepsilon}(k, j, \phi, s)$ *is a meromorphic function of* $s \in \mathbf{C}$ *and the locus of its poles is independent of* ε.

(ii) *When* ε *goes to* 0, $\Lambda_{\mathscr{B},\varepsilon}(k, j, \phi, s)$ *converges uniformly (outside the poles) to* $\Lambda(k, j, \phi, s)$ *on each compact subset of* $\mathrm{Re}(s) > \frac{k}{2}+1$.

(iii) *For all* $\varepsilon \in 0$, *we have* $\Lambda_{\mathscr{B},\varepsilon}(k, j, \phi, 0) = \Lambda_{\mathscr{B}}(k, j, \phi)$.

(iv) *There exists a constant* $\varepsilon_n > 0$ *such that, if* $(\phi, \mathscr{B})$ *satisfies condition* (**), *then* $\Lambda_{\mathscr{B},\varepsilon}(k, j, \phi, s)$ *converges uniformly (outside the poles) on each compact subset of* $\mathrm{Re}(s) > \frac{k}{2} - \varepsilon_n$ *(resp.* $\mathbf{C}$), *if* $n \geqslant 3$ *(resp.* $n = 1, 2$).

(v) $\Lambda_{\mathscr{B}}(k, j, \phi) = (-1)^{n(j-1)} i^n \Lambda_{\mathscr{B}^\vee}(j-1, k+1, \mathscr{F}_H(\phi))$.

COROLLARY. *If* $(\phi, \mathscr{B})$ *satisfies condition* (**), *we have* $\Lambda_{\mathscr{B}}(k, j, \phi) = \Lambda(k, j, \phi)$ *if* $n = 1, 2$ *or if* $n \geqslant 3$ *and* $k = 0$ *or* $j = 1$.

Proof of Theorem 3. (v) is an immediate consequence of formula (11). Using the same method as in [Co1, p. 198], we see that $\Lambda(k, j, \phi, s)$ is a finite combination of the functions studied in [Co1, II]. Granting this, (i) follows from [Co1, II Lemma 8], (ii) from [Co1, II Lemma 9] and (iii) from [Co1, II, §6]. The only thing which is new is (iv), which will allow us to remove from [Co1, Th. 5 and 6] the meaningless condition about embeddings of F into $\bar{K}$. This improvement is made possible by replacing Lemma 1 of [Co1, III] by the following stronger theorem of Schmidt:

LEMMA 4 (Schmidt's subspace theorem). *Let* $\delta > 0$ *and* $\{(L_{i,1}, \ldots, L_{i,n}) \mid i \in I\}$ *be a finite set of families of* n *linearly independent linear forms with algebraic coefficients. Then there exists a finite set* $\mathscr{E}$ *of elements of* $H^\vee$ *such that for all* $\phi \in \mathscr{S}(H^\vee)$, *the set of elements of* $H^\vee$ *satisfying*

(i) $\phi(x) \neq 0$,

(ii) *there exists* $i \in I$ *such that* $\prod_{j=1}^n |L_{i,j}(x)| \leqslant \|x\|^{-\delta}$

is contained in the union of the hyperplanes of equation $\mathrm{Tr}(fx) = 0$ *for* $f \in \mathscr{E}$ *up to a finite set.*

This lemma is a direct consequence of [Sch, Ch. VIII, Th. 7A]. Let us go back to the proof of (iv). A slight modification of the proof of [Co1, II, Lemma 10] shows that there exists a finite set $\mathscr{L}(\mathscr{B}^\vee) = \{(L_{i,1}, \ldots, L_{i,n}) \mid i \in I\}$ of families of n linearly independent linear forms with algebraic coefficients (they are the $N_{I,j}$ of [Co1, Th. 2]) and δ_n, $\varepsilon_n > 0$ such that if, for all $i \in I$, the set of $x \in H^\vee$ such that $\mathscr{F}_H(\phi)(x) \neq 0$ and $\prod_{j=1}^n |L_{i,j}(x)| \leqslant \|x\|^{-\delta_n}$ is finite, then $\Lambda_{\mathscr{B},\varepsilon}(k, j, \phi, s)$ converges uniformly (outside the poles) on each compact subset of $\mathrm{Re}(s) > \frac{k}{2} - \varepsilon_n$ (resp. $\mathbf{C}$) if $n \geqslant 3$ (resp. $n = 1, 2$). To finish with the proof, we just have to take $\mathscr{E}(\mathscr{B})$ (resp. $\mathscr{E}(\mathscr{B}^\vee)$) of condition (**) to be the set $\mathscr{E}$ associated to $\mathscr{L}(\mathscr{B}^\vee)$ (resp. $\mathscr{L}(\mathscr{B})$) and $\delta = \delta_n$ in Lemma 4.

When $(\phi, \mathscr{B})$ does not satisfy condition (*), we cannot define $\Lambda_{\mathscr{B}}(k, j, \phi)$ by formula (12). As the singularities of $\mathrm{F}(z_1, z_2, \phi, \mathscr{B})$ are simple enough, we could give a meaning to (12) by taking a suitable finite part as in [Co1, II, §6], but here we shall use the standard technique of replacing ϕ by a suitable linear combination to eliminate the poles (cf. [Ca]). If $S\in\mathscr{P}(H)$, let

$$S_K=\{\mathbf{q}\cap O_K\mid \mathbf{q}\in S\}\in\mathscr{P}(K)$$

and if $S\in\mathscr{P}(K)$, let $S^H=\{\mathbf{q}\in\mathscr{P}(H)\mid\exists\mathbf{p}\in S \text{ such that } \mathbf{q}\mid\mathbf{p}\}\in\mathscr{P}(H)$.

LEMMA 5. *Let $\mathscr{E}$ be a finite subset of H^*; then there exists $S(\mathscr{E})\in\mathscr{P}(H)$ such that for all $S\in\mathscr{P}(H)$, all $\mathbf{b}\in C(H)$ satisfying $|\mathbf{b}|\cap(S(\mathscr{E})\cup(S_K)^H)=\varnothing$ and all $f\in\mathscr{E}$, we have: if $x\in\mathbf{b}^{-1}O'_{H,S}-O'_{H,S}$, then $\mathrm{Tr}(fx)\notin O'_{K,S_K}$ and in particular $\mathrm{Tr}(fx)$ is non-zero.*

Proof. Let $S'=|\mathbf{d}_H|\bigcup_{f\in\mathscr{E}}|(f)|$ and $S(\mathscr{E})=(S'_K)^H$. Let $\mathbf{b}\in C(H)$ be such that $|\mathbf{b}|\cap(S(\mathscr{E})\cup(S_K)^H)=\varnothing$ and $x\in\mathbf{b}^{-1}O'_{H,S}-O'_{H,S}$. There exists $\mathbf{q}\in|\mathbf{b}|$ such that $v_\mathbf{q}(x)<0$. As $O_H/\mathbf{b}$ is cyclic, $\mathbf{q}$ is of degree 1 and if $\mathbf{p}=\mathbf{q}\cap O_K$ and $\mathbf{q}'\in|\mathbf{p}|-\{\mathbf{q}\}$, then $\mathbf{q}'\notin|\mathbf{b}|$, hence $v_{\mathbf{q}'}(x)\geqslant 0$; and this implies, as $\mathbf{q}\notin S'$, that $v_\mathbf{p}(\mathrm{Tr}(fx))=v_\mathbf{q}(x)$ which implies $\mathrm{Tr}(fx)\notin O'_{K,S_K}$.

If $S\in\mathscr{P}(H)$ and $S'\in\mathscr{P}(H^\vee)$, let

$$C(S,S')=\{(\mathbf{b}_1,\mathbf{b}_2)\in C(H)\times C(H^\vee)\mid|\mathbf{b}_1|\cap(S_K)^H=\varnothing,|\mathbf{b}_2|\cap(S'_K)^{H^\vee}=\varnothing$$
$$\text{and } |\mathbf{b}_1|_K\cap|\bar{\mathbf{b}}_2|_K=\varnothing\},$$

and if $T\in\mathscr{P}(H)$, let

$$C_T(S,S')=C(S\cup T,S'\cup\bar{T}).$$

Also let $C^0(S,S')$ (resp. $C^0_T(S,S')$) be the intersection of $C(S,S')$ (resp. $C_T(S,S')$) with $C^0(H)\times C^0(H^\vee)$. If $\phi\in\mathscr{S}(H)$, and $\mathbf{b}_1\in I(H)$, $\mathbf{b}_2\in I(H^\vee)$, set $\phi_{\mathbf{b}_1,\mathbf{b}_2}=\delta_{\mathbf{b}_1^{-1}}*\delta^\vee_{\mathbf{b}_2^{-1}}*\phi$, whenever this is defined.

LEMMA 6. *Let $\mathscr{B}$ be a finite set of bases of H over K. Then there exist $S=S_1(\mathscr{B})\in\mathscr{P}(H)$ and $S'=S'_1(\mathscr{B})\in\mathscr{P}(H^\vee)$ such that, for all $T\in\mathscr{P}(H)$, all $\phi\in\mathscr{S}_T(H)$ and all $(\mathbf{b}_1,\mathbf{b}_2)\in C_T(S,S')$, the conditions (*) and (**) are satisfied by $(\phi_{\mathbf{b}_1,\mathbf{b}_2},\mathscr{B})$.*

Proof. $\phi_{\mathbf{b}_1,\mathbf{b}_2}(x)\neq 0$ implies $x\in\mathbf{b}_1^{-1}O'_{H,T}-O'_{H,T}$ and $\mathscr{F}_H(\phi_{\mathbf{b}_1,\mathbf{b}_2})(x)\neq 0$ implies $x\in\mathbf{b}_2^{-1}O'_{H^\vee,\bar{T}}-O'_{H^\vee,\bar{T}}$. Hence, the result is an immediate consequence of Lemma 5.

Let O_T^* act on $\mathscr{S}_{H,T}$ by $\phi\to\phi\circ\gamma$ where $\phi\circ\gamma(x)=\phi(\gamma x)$. Any $\phi\in\mathscr{S}_{H,T}$ has a unique decomposition $\phi=\Sigma_\chi\phi_\chi$ where $\phi_\chi=0$ for almost all χ, χ running through the locally constant characters of O_T^*, and $\phi_\chi\circ\gamma=\chi(\gamma)\phi_\chi$ for all $\gamma\in O_S^*$. Now, using the identification between $\mathscr{S}_{H,T}$ and $\mathscr{S}_T(H)$, we can decompose any $\phi\in\mathscr{S}_T(H)$ as $\Sigma_\chi\phi_\chi$ and if ϕ belongs to $\mathscr{S}_{k,j,V}(H)$ then so does ϕ_χ. Let $(\mathbf{b}_1,\mathbf{b}_2)\in C_T^0(S_1(\mathscr{B}),S'_1(\mathscr{B}))$ and $\beta_1\in H$ be a generator of $\mathbf{b}_1$ and $\beta_2\in H^\vee$ be a generator of $\mathbf{b}_2$. If $\gamma\in H^*$ and $\phi\in\mathscr{S}(H)$, let $\phi\circ\gamma\in\mathscr{S}(H)$ be defined by

$(\phi \circ \gamma)(x) = \phi(\gamma x)$. Then we have

$$(\phi_\chi)_{\mathbf{b}_1,\mathbf{b}_2} = \phi_\chi - \chi(\beta_1)^{-1}\phi_\chi \circ \beta_1 - N(\mathbf{b}_2)\chi(\bar{\beta}_2)\phi_\chi \circ \bar{\beta}_2^{-1} + N(\mathbf{b}_2)\chi(\bar{\beta}_2\beta_1^{-1})\phi_\chi \circ (\beta_1\bar{\beta}_2^{-1}), \tag{15}$$

but as

$$\Lambda(k, j, \phi \circ \gamma) = \frac{N_{H/K}(\gamma)^j}{\overline{N_{H/K}(\gamma)}^k}\Lambda(k, j, \phi), \tag{16}$$

we obtain

$$\Lambda(k, j, \phi) = \sum_\chi \nu_{\beta_1,\beta_2}(k, j, \chi)\Lambda(k, j, (\phi_\chi)_{\mathbf{b}_1,\mathbf{b}_2}), \tag{17}$$

where

$$\nu_{\beta_1,\beta_2}(k, j, \chi) = \left(1 - \frac{\chi(\beta_1)^{-1}N_{H/K}(\beta_1)^j}{\overline{N_{H/K}(\beta_1)}^k}\right)^{-1}\left(1 - \frac{\chi(\bar{\beta}_2)N_{H^\vee/K}(\beta_2)^{k+1}}{\overline{N_{H^\vee/K}(\beta_2)}^{j-1}}\right)^{-1}. \tag{18}$$

To be coherent with formula (17), we set, if $\phi \in \mathscr{S}_T(H) \cap \mathscr{S}_{k,j,V}(H)$, $\mathscr{B} \in \mathscr{B}(V)$ and $(\mathbf{b}_1, \mathbf{b}_2) = ((\beta_1), (\beta_2)) \in C_T(S_1(\mathscr{B}), S_1'(\mathscr{B}))$,

$$\Lambda_{\mathscr{B},\beta_1,\beta_2}(k, j, \phi) = \sum_\chi \nu_{\beta_1,\beta_2}(k, j, \chi)\Lambda_{\mathscr{B}}(k, j, (\phi_\chi)_{\mathbf{b}_1,\mathbf{b}_2}), \tag{19}$$

and the right-hand side is well-defined by Lemma 6.

REMARK. We expect that $\Lambda_{\mathscr{B},\beta_1,\beta_2}(k, j, \phi) = \Lambda(k, j, \phi)$ and by the corollary to Theorem 3, this equality is true if $n = 1, 2$ or if $n \geqslant 3$ and $k = 0$ or $j = 1$. Moreover, we shall prove using p-adic methods (cf. III §4 of this paper) that, to a large extent, $\Lambda_{\mathscr{B},\beta_1,\beta_2}(k, j, \phi)$ does not depend on the auxiliary choices of $\mathscr{B}$, β_1 and β_2.

III. Construction of the basic measure

1. p-Adic measures

Let $p \neq 2, 3$ be a prime which splits in K. Fix an embedding of $\bar{K}$ into $\mathbf{C}_p$ (and keep the previous embedding of $\bar{K}$ into $\mathbf{C}$). Let $\mathbf{p}$ be the prime ideal of O_K determined by this embedding, $O_{\mathbf{p}}$ be the completion of O_K at $\mathbf{p}$ and $\bar{\mathbf{p}}$ the other prime ideal of O_K above p. Let

$$Y_{H,\mathbf{p}} = O_H \otimes_{O_K} O_{\mathbf{p}} \simeq O_{H^\vee} \otimes_{O_K} O_{\bar{\mathbf{p}}} \quad \text{and} \quad Y_{H,p} = O_H \otimes \mathbf{Z}_p = Y_1 \times Y_2$$

where

$$Y_1 = Y_{H,\mathbf{p}} \quad \text{and} \quad Y_2 = Y_{H^\vee,\bar{\mathbf{p}}}.$$

We can also describe Y_1 (resp. Y_2) as the topological closure of O_H (resp. $O_{H^\vee}$) into $\mathbf{C}_p^n$ via the map $\alpha \rightarrow (\tau_1(\alpha), \ldots, \tau_n(\alpha))$ (resp. $(\overline{\tau_1(\bar{\alpha})}, \ldots, \overline{\tau_n(\bar{\alpha})})$). With this description, we can write $y_i \in Y_i$ as $(y_{i,1}, \ldots, y_{i,n})$. If $z \in \mathbf{C}_p^n$, we set $\mathrm{Tr}(z) = \Sigma_{i=1}^n z_i$ and $N(z) = \Pi_{i=1}^n z_i$. If l is a prime ideal of O_K, let $\mathbf{d}_{H,l}$ be the part of $\mathbf{d}_H$ above l. Fix a basis $B = (f_1, \ldots, f_n)$ of $\mathbf{d}_{\bar{H},\bar{\mathbf{p}}}^{-1} O_{H,p}$ over $O_{K,p}$. Let $B^* = (g_1, \ldots, g_n)$ be the basis of H over K dual to B with respect to the bilinear form $\mathrm{Tr}_{H/K}(xy)$ and $B^\vee = (f_1^\vee, \ldots, f_n^\vee)$ and $(B^*)^\vee = (g_1^\vee, \ldots, g_n^\vee)$ be the bases of $H^\vee$ over K dual to B and B^* with respect to $\langle \mid \rangle$. Then B^* is a basis of $\mathbf{d}_{H,\mathbf{p}}^{-1} O_{H,p}$ over $O_{K,p}$, $B^\vee$ is a basis of $\mathbf{d}_{H^\vee,\bar{\mathbf{p}}}^{-1} O_{H^\vee,p}$ over $O_{K,p}$ and $(B^*)^\vee$ is a basis of $\mathbf{d}_{H^\vee,\mathbf{p}}^{-1} O_{H^\vee,p}$ over $O_{K,p}$.

If $y_i \in Y_i$, we set $x_i = (x_{i,1}, \ldots, x_{i,n})$, where $x_{1,j} = \mathrm{Tr}(g_j y_1)$ and $x_{2,j} = \mathrm{Tr}(g_j^\vee y_2)$. The map $y_i \rightarrow x_i$ induces an isomorphism of $O_{\mathbf{p}}$-modules between Y_i and $O_{\mathbf{p}}^n \simeq \mathbf{Z}_p^n$. If $z_i = (z_{i,1}, \ldots, z_{i,n})$ for $i = 1, 2$ is sufficiently close to zero in $\mathbf{C}_p^n$, we set $w_i = (w_{i,1}, \ldots, w_{i,n})$, where $w_{1,j} = \exp(-\mathrm{Tr}(f_j z_1)) - 1$ and $w_{2,j} = \exp(-\mathrm{Tr}(f_j^\vee z_2)) - 1$.

Let Λ be a closed subring of $\hat{O}$ the ring of integers of $\mathbf{C}_p$. A Λ-valued measure on a compact and totally disconnected topological space X is a continuous (for the supremum norm) linear map on the space of continuous functions on X with values in C_p whose values on characteristic functions of compact open subsets of X are in Λ. If μ is a Λ-valued measure on $Y_{H,p}$, we define its Fourier–Laplace transform by

$$F_\mu(z_1, z_2) = \int_{Y_{H,p}} \exp(-\mathrm{Tr}(y_1 z_1 + y_2 z_2))\, \mathrm{d}\mu = \int_{\mathbf{Z}_p^{2n}} \prod_{i=1}^{2} \prod_{j=1}^{n} (1 + w_{i,j})^{x_{i,j}}\, \mathrm{d}\lambda_B,$$

where λ_B is the measure on $\mathbf{Z}_p^{2n}$ deduced from $\mu\sigma$ via the map $(y_1, y_2) \rightarrow (x_1, x_2)$.

LEMMA 7. *If μ is a Λ-valued measure on $Y_{H,p}$, then $F_\mu(z_1, z_2)$ is given by a power series in a neighborhood of zero, and reciprocally, if $F(z_1, z_2)$ is a power series, then for $F(z_1, z_2)$ to be the Fourier–Laplace transform of a Λ-valued measure, it is necessary and sufficient that $F(z_1, z_2)$ expressed in w_1, w_2 is a power series with coefficients in Λ.*

Proof. The general case reduces easily to the case $n = 1$ which is well-known.

We shall write $W_{B,\mu}(w_1, w_2)$ for the Fourier–Laplace transform of μ expressed in w_1, w_2. If $\gamma \in H/\mathbf{d}_{\bar{H},\bar{\mathbf{p}}}^{-1} O_{H,\mathbf{p}}$, we define a locally constant character χ_γ of Y_1 identified with $O_{H^\vee} \otimes_{O_K} O_{\bar{\mathbf{p}}}$ by the formula $\chi_\gamma(y_1) = (\gamma \mid y_1)_{|\mathbf{p}|}$ (cf. I), and if $\gamma \in H^\vee/\mathbf{d}_{H^\vee,\mathbf{p}}^{-1} O_{H^\vee,\mathbf{p}}$, we define a locally constant character χ_γ of $Y_2 \simeq O_H \otimes_{O_K} O_{\bar{\mathbf{p}}}$ by the formula $\chi_\gamma(y_2) = (y_2 \mid \gamma)_{|\bar{\mathbf{p}}|}$. The map $\gamma \rightarrow \chi_\gamma$ induces an isomorphism from

$H/\mathbf{d}_{H,\mathbf{p}}^{-1}O_{H,\mathbf{p}}$ (resp. $H^\vee/\mathbf{d}_{H^\vee,\mathbf{p}}^{-1}O_{H^\vee,\mathbf{p}}$) to the group of locally constant characters on Y_1 (resp. Y_2).

LEMMA 8. *Let $j, k \in \mathbf{N}$ and $\gamma_1 \in H/\mathbf{d}_{H,\mathbf{p}}^{-1}O_{H,\mathbf{p}}$ and $\gamma_2 \in H^\vee/\mathbf{d}_{H^\vee,\mathbf{p}}^{-1}O_{H^\vee,\mathbf{p}}$. Then*

(i) $\int_{Y_{H,p}} \chi_{\gamma_1}(y_1)\chi_{\gamma_2}(y_2)N(y_1)^j N(y_2)^k \,d\mu = \nabla_1^j \nabla_2^k (F_{\chi_{\gamma_1}\chi_{\gamma_2}\mu}(z_1, z_2))_{z_1=z_2=0}$, *where, if ϕ is a continuous function on $Y_{H,p}$ then $\phi\mu$ is the measure defined by $\int_{Y_{H,p}} \psi \,d(\phi\mu) = \int_{Y_{H,p}} \phi\psi \,d\mu$, and*

(ii) $F_{\chi_{\gamma_1}\chi_{\gamma_2}\mu}(z_1, z_2) = W_{B,\mu}(\ldots, \varepsilon_{i,j}(1 + w_{i,j}) - 1, \ldots)$,

where the $\varepsilon_{i,j}$ are p^∞th roots of unity defined by $\varepsilon_{1,j} = \chi_{\gamma_1}(\bar{f}_j)$ and $\varepsilon_{2,j} = \chi_{\gamma_2}(f_j^\vee)$.

Proof. (i) follows by developing $\exp(-\mathrm{Tr}(y_1 z_1 + y_2 z_2))$ as a power series and (ii) is evident if we remark that $\chi_{\gamma_i}(y_i) = \prod_{j=1}^n \varepsilon_{i,j}^{x_{i,j}}$, which gives

$$F_{\chi_{\gamma_1}\chi_{\gamma_2}\mu}(z_1, z_2) = \int_{Y_{H,p}} \prod_{i=1}^{2} \prod_{j=1}^{n} (\varepsilon_{i,j}(1 + w_{i,j}))^{x_{i,j}} \,d\lambda_B.$$

Our aim in the rest of this section will be to prove that under suitable conditions, the holomorphic part of $\mathbf{K}(z_1, z_2, \phi, \mathscr{B})$ is the Fourier–Laplace transform of a measure on $Y_{H,p}$. We shall first consider the case $H = K$, and this will involve the study of the p-adic behavior of Eisenstein–Kronecker series. This is the aim of the next paragraph, and in the paragraph after that we shall reduce the general case to the case $H = K$.

2. *p-Adic properties of Eisenstein–Kronecker series*

As stated at the end of the last paragraph, this paragraph deals with the case $F = K$. Our aim is to obtain integrality results for Eisenstein–Kronecker series attached to lattices in K. Although these results are more or less equivalent to those obtained by Yager or de Shalit, there does not seem to exist in the literature a formulation of them suitable for our purposes. Therefore, we develop a method giving naturally the desired formulation.

Let us begin by recalling the definitions and some basic facts about Eisenstein–Kronecker series. We refer to [W2] for the proofs. Let L be a lattice in $\mathbf{C}$ and $A(L) = \pi^{-1}\mathrm{Vol}(L)$. If $u, z \in \mathbf{C}$, we set

$$\langle z, u\rangle_L = \exp(A(L)^{-1}(z\bar{u} - u\bar{z})). \tag{20}$$

If $k \geqslant 1$ is an integer, we define for $\mathrm{Re}(s) \gg 1$ the function $H_k(s, z, u, L)$ by the formula

$$H_k(s, z, u, L) = \Gamma(s)A(L)^{s-k} \sum_{\omega\in L}{}' \langle \omega, u\rangle_L \frac{(\bar{z} + \bar{\omega})^k}{|z + \omega|^{2s}}. \tag{21}$$

This function has an analytic continuation to the whole complex plane and satisfies the functional equations

$$H_k(s, z, u, L) = \langle u, z\rangle_L H_k(k+1-s, u, z, L), \tag{22}$$

$$H_k(s, z, u, L') = [L' : L]^{k-s} \sum_{\gamma\in L'/L} \langle \gamma, u\rangle_{L'} H_k(s, z+\gamma, [L' : L]u, L) \tag{23}$$

if L is a sublattice of L', and

$$H_k(s, \lambda z, \lambda u, \lambda L) = \lambda^{-k} H_k(s, z, u, L) \quad \text{for } \lambda\in\mathbf{C}. \tag{24}$$

From (23) and (24) one deduces that if $u\in\mathbf{Q}L$ and $b\in\mathbf{C}$ is an endomorphism of L such that $\bar{b}u\in L$,

$$H_k(s, z, u, L) = \frac{\bar{b}^k}{|b|^{2s}} \sum_{\gamma\in b^{-1}L/L} \langle \gamma, \bar{b}u\rangle_L H_k(s, \gamma+b^{-1}z, 0, L). \tag{25}$$

If j is an integer such that $1 \leqslant j$ and $k\in\mathbf{N}$, we define

$$E_{k,j}(z, L) = H_{k+j}(j, z, 0, L) \quad \text{and } E_j(z, L) = E_{0,j}(z, L), \tag{26}$$

$$a_j(L) = E_j(0, L) = E_{j-1,1}(0, L), \tag{27}$$

and

$$\wp(z, L) = E_2(z, L) - a_2(L) \text{ (so } \wp'(z, L) = -E_3(z, L)). \tag{28}$$

$E_1(z, L)$ has the following Laurent expansion in a neighborhood of 0:

$$E_1(z, L) = -\frac{\bar{z}}{A(L)} + z^{-1} + \sum_{n=1}^{\infty} a_{n+1}(L)\frac{(-z)^n}{n!}, \tag{29}$$

and $E_{k,j}(z, L) - \left(\frac{\bar{z}}{A(L)}\right)^k \frac{\Gamma(j)}{z^j}$ is real analytic in a neighborhood of 0.

PROPOSITION 9. *There exists a (non-unique) polynomial $P_{k,j}$ with rational coefficients in the variables $E(z, L) = \{E_1(z, L), \ldots, E_j(z, L), \ldots\}$ and $a(L) = \{a_1(L), \ldots, a_j(L), \ldots\}$ such that $P_{k,j}(E(z, L), a(L)) = E_{k,j}(z, L)$ for $z\notin L$.*

REMARK. If $k < j$, there is a stronger statement proved in [W2, Ch. VI (11)]: the variables $a(L)$ are unnecessary and $P_{k,j}$ has coefficients in $\mathbf{Z}[\frac{1}{2}]$.

Proof. The proof is by induction. The statement is trivial for $k=0$ and $j \geqslant 1$. Moreover, as $\frac{d}{dz} E_{k,j}(z, L) = -E_{k,j+1}(z, L)$, if the statement is true for (k, j) it is true for $(k, j+1)$. Thus the problem is to show the existence of $P_{n+1,1}$ assuming the existence of $P_{k,j}$ for $k \leqslant n$ and $j \geqslant 1$. If we write down a Laurent expansion for $E_{n+1,1}(z, L) + \frac{1}{n+2} E_1(z, L)^{n+2}$ in a neighborhood of 0, we obtain

$$E_{n+1,1}(z, L) + \frac{1}{n+2} E_1(z, L)^{n+2} = \sum_{k=0}^{n} \sum_{j=1}^{n+2-k} Q_{n,k,j}(a(L)) \left(\frac{\bar{z}}{A(L)} \right)^k \frac{\Gamma(j)}{z^j} + R_n(z),$$

where the $Q_{n,k,j}$ are polynomials with rational coefficients and $R_n(z)$ is real analytic in a neighborhood of 0. From this we deduce that

$$E_{n+1,1}(z, L) + \left(\frac{1}{n+2} E_1(z, L)^{n+2} - \sum_{k=0}^{n} \sum_{j=1}^{n+2-k} P_{k,j}(E(z, L), a(L)) Q_{n,k,j}(a(L)) \right)$$

is a doubly periodic real analytic function annihilated by a power of $\left(\frac{\partial}{\partial \bar{z}} \right)$ and hence a constant. Using the fact that $E_{n+1,1}(0, L) = a_{n+2}(L)$ we find that this constant can be expressed as a polynomial in the $a_j(L)$ with rational coefficients, which concludes the proof.

Let E be an elliptic curve with Weierstrass model $y^2 = 4x^3 - g_2 x - g_3$, defined over $O_{\bar{K}}$ with complex multiplication by O_K and with good ordinary reduction at p. Let L be the period lattice of $\omega = \mathrm{d}x/y$. Choose a basis (γ_1, γ_2) of $H_1(E(\mathbf{C}), \mathbf{Z})$: then $\int_{\gamma_2} \omega = \tau \int_{\gamma_1} \omega$ for some $\tau \in K$, and $\mathbf{a} = \mathbf{Z} + \mathbf{Z}\tau$ is a fractional ideal of K. We assume that we have chosen our basis (γ_1, γ_2) in such a way that $v_{\mathbf{p}}(\mathbf{a}) = v_{\bar{\mathbf{p}}}(\mathbf{a}) = 0$. Let $\eta = (x + a_2(L))\omega$. Then (ω, η) is a basis of $H^1_{DR}(E)$ and if $\alpha \in O_K$, then $\alpha^* \omega = \alpha\omega$ and $\alpha^* \eta = \bar{\alpha}\eta$ in $H^1_{DR}(E)$. Set $\omega_\infty = \int_{\gamma_1} \omega$ and $\eta_\infty = \int_{\gamma_1} \eta$. Using Legendre's relation, we obtain $A(L) = -\bar{\omega}_\infty / \eta_\infty$. If $\alpha \in K$, we let $\tilde{\alpha} = \alpha\omega_\infty \in \mathbf{Q}L$, and if P is a torsion point on E, we let $z(P) \in K$ be any element such that $\tilde{z}(P) = \omega_\infty z(P)$ corresponds to P via the isomorphism $\mathbf{C}/L \simeq E(\mathbf{C})$. Of course, $z(P)$ is only determined up to an element in $\mathbf{a}$.

Let $t = -2x/y = -2\wp(z)/\wp'(z) = (z + \cdots)$ be the parameter of the formal group $\hat{E}$ which is the kernel of reduction mod p, $\lambda(t)$ be the power series giving z in terms of t (it is the logarithm of $\hat{E}$ and we have $\mathrm{d}\lambda(t) = \omega(t)$), and $\oplus$ denote the formal group law on $\hat{E}$. Let $I_p \subset \hat{O}$ be the ring of integers of the completion of the maximal unramified extension of $\mathbf{Q}_p$ and $M = \mathbf{Q}_p(g_2, g_3)$. The formal groups $\hat{E}$ and $\mathbf{G}_m$ are then isomorphic over $I_{p,E} \stackrel{\text{def}}{=} I_p(g_2, g_3)$. We shall fix an isomorphism $\imath$ from $\hat{E}$ to $\mathbf{G}_m$ by requiring that the following condition holds. Let Q be a point of $\mathbf{p}^\infty$-division on E. Then we want $1 + \imath(t(Q)) = \langle \tilde{z}(Q), \tilde{1} \rangle_L$ where the left-hand side is a p^∞th root of unity in $\mathbf{C}_p$ and the right-hand side is a p^∞th root of unity in $\mathbf{C}$. We will write $\varepsilon(Q)$ for this p^∞th root of unity. For reasons to become obvious later, we write $-\eta_p$ for the coefficient of t in $\imath \in I_{p,E}[\![t]\!]$ ($\imath$ has no constant term), and extend the isomorphism from $\bar{K} \subset \mathbf{C}$ to $\bar{K} \subset \mathbf{C}_p$ to an isomorphism from $\bar{K}(\eta_\infty)$ to $\bar{K}(\eta_p)$ sending η_∞ to η_p. Note that this is possible because η_∞ is transcendent due to a theorem of Čudnovskiĭ (cf. [Wa]) and η_p also in a more trivial way.

Suppose $G(z_1, \dots, z_n)$ is locally real analytic around 0. We define the holomorphic part of G to be $\mathscr{H}(G(z_1, \dots, z_n))$, the power series in $z_1, \dots, z_n$ obtained by equating $\bar{z}_1, \dots, \bar{z}_n$ to 0 in the formal Taylor series expansion of G in $z_1, \dots, z_n, \bar{z}_1, \dots, z_n$. If $H(z_1, \dots, z_n)$ is locally of the form $F(z_1, \dots, z_n)/G(z_1, \dots, z_n)$, where F is real analytic around 0 and G holomorphic, we define the holomorphic part of H, $\mathscr{H}(H(z_1, \dots, z_n)) \in \mathbf{C}((z_1, \dots, z_n))$, by $\mathscr{H}(H) = \mathscr{H}(F)/G$. If moreover $\mathscr{H}(F)$ and G have coefficients in $\bar{K}(\eta_\infty)$, we shall also view $\mathscr{H}(H)$ as an element of $\mathbf{C}_p((z_1, \dots, z_n))$.

PROPOSITION 10. *Let $\alpha \in (K - \mathbf{p}^{-\infty}\mathbf{a}) \cup \mathbf{a}$, which means that the division point $P(\alpha)$ corresponding to α is either 0 or does not belong to $\hat{E}$. Then if $1_{\mathbf{a}}$ is the characteristic function of* $\mathbf{a}$, *we have*:

(i) $\mathscr{H}(E_1(\tilde{\alpha} + \lambda(t), L)) = 1_{\mathbf{a}}(\alpha)t^{-1} + E_1(\tilde{\alpha}, L) + \Sigma_{n=1}^{\infty} b_n(\alpha)t^n \overset{\text{def}}{=} G_1(\alpha, t)$, *where $b_n(P)$ is in the ring of integers of $M(P(\alpha))$.*

(ii) $E_1(\tilde{\alpha}, L) \equiv \bar{\alpha}\eta_p \pmod{\hat{O}}$.

(iii) *If Q is a $\mathbf{p}^\infty$-division point, then $G_1(\alpha, t(Q))$ (which converges by* (i)) *is equal to $E_1(\tilde{\alpha} + \tilde{z}(Q), L)$.*

Proof. Let $\phi(z, u) = E_1(z+u, L) - E_1(z, L) - E_1(u, L)$. Then ϕ is a meromorphic function in u and z and hence an algebraic function on $E \times E$. Moreover, it is easily seen to belong to $M(E \times E)$ and to have a well-defined reduction $\bmod p$ $\left(\text{in fact } \phi(z, u) = \dfrac{(\wp'(u) - \wp'(z))}{2(\wp(u) - \wp(z))}\right)$. Now, if $\alpha \in K - \mathbf{p}^{-\infty}\mathbf{a}$, then $\mathscr{H}(E_1(\lambda(t) + \tilde{\alpha}, L) - E_1(\lambda(t), L) - E_1(\tilde{\alpha}, L)) + t^{-1}$ is an algebraic function on E without singularities on $\hat{E}$ and whose reduction mod p is defined, and so is given on $\hat{E}$ by a power series in t with coefficients in the ring of integers of $M(P(\alpha))$. Hence, to prove (i) and (iii) for any α, it suffices to prove them for $\alpha = 0$.

Let $\beta \in O_K$ such that β is prime to p. By the same arguments as before, one sees that $\mathscr{H}(E_1(\beta\lambda(t), L) - \bar{\beta}E_1(\lambda(t), L)) + \beta^{-1}(N(\beta) - 1)t^{-1}$ is an algebraic function on E with no singularities on $\hat{E}$, and so is given on $\hat{E}$ by a power series $G_\beta(t)$ with coefficients in the ring of integers of M. Now take $\beta_n \in O_K$ satisfying $\beta_n \equiv 1 \pmod{\mathbf{p}^n}$ and $\beta_n \equiv 0 \pmod{\bar{\mathbf{p}}^n}$. Let n tend to $+\infty$. Then $G_{\beta_n}(t)$ obviously tends to $E_1(\lambda(t)) - t^{-1}$ which concludes the proof of (i). To prove (iii), suppose Q is a $\mathbf{p}^m$-torsion point. Then if $n \geqslant m$, we have $\beta_n Q = Q$ and so $G_{\beta_n}(t(Q)) = (1 - \bar{\beta}_n)E_1(\tilde{z}(Q), L) - \beta_n^{-1}(N(\beta_n) - 1)t(Q)^{-1}$ (as G_{β_n} is an algebraic function, one can evaluate it at a point defined over $\bar{K}$ using complex arguments). But when n

tends to $+\infty$, $G_{\beta_n}(t(Q))$ tends to $G_1(0, t(Q)) - t(Q)^{-1}$ and the right-hand side tends to $E_1(\tilde{z}(Q), L) - t(Q)^{-1}$ which concludes the proof of (iii).

It remains to prove (ii). First note that if $\alpha \in \mathbf{a}$, there is nothing to prove as $E_1(\tilde{\alpha}, L) = 0$. So suppose $\alpha \notin \mathbf{a}$ and write $\alpha = \alpha_0 + \alpha_1$ where $\alpha_1 \in \mathbf{p}^{-\infty}\mathbf{a}$ and $v_{\mathbf{p}}(\alpha_0) \geqslant 0$. Then, using (i) and (iii) with $\alpha = \alpha_0$ and Q corresponding to $\tilde{\alpha}_1$, we deduce that if (ii) is true for α_0 then it is true for α and we are reduced to the case when $\alpha \notin \mathbf{a}$ and $v_{\mathbf{p}}(\alpha) \geqslant 0$. Now, if $\beta \in O_K$, then $F_\beta(z) = E_1(\beta z, L) - \bar{\beta} E_1(z, L)$ is an algebraic function on E whose reduction mod p is defined, so if z corresponds to a point defined over $\bar{K}$ which does not reduce to a β-division point mod p, then $F_\beta(z) \in \hat{O}$. One deduces from this that if (ii) is true for α it is true for $\beta\alpha$, and if β is prime to p and (ii) is true for α then it is true for $\beta^{-1}\alpha$. Now let h be the class number of K and let π be a generator of $\mathbf{p}^h$. By the previous reductions, it suffices to verify (ii) for $\alpha = \bar{\pi}^{-n}$ and $n \geqslant 1$. Let $k \in \mathbf{Z}$ and $\alpha_n = \bar{\pi}^{-n}$. Then

$$E_1(k\tilde{\alpha}_n, L) = H_1(1, k\tilde{\alpha}_n, 0, L) = H_1(1, 0, k\tilde{\alpha}_n, L) = \frac{1}{\pi^n} \sum_{\substack{\gamma \in \pi^{-n}L/L \\ \gamma \neq 0}} \langle \gamma, \tilde{k} \rangle_L E_1(\gamma, L).$$

Let $\varepsilon = \langle \gamma, \tilde{1} \rangle_L$. Then using the isomorphism ι, we see that

$$E_1(k\tilde{\alpha}_n, L) = \frac{1}{\pi^n} \sum_{\substack{\varepsilon^{p^{nh}} = 1 \\ \varepsilon \neq 1}} \varepsilon^k G_1(0, \iota^{-1}(\varepsilon - 1)).$$

So

$$E_1(\tilde{\alpha}_n, L) = E_1(\tilde{\alpha}_n, L) - E_1(0, L) = \frac{1}{\pi^n} \sum_{\substack{\varepsilon^{p^{nh}} = 1 \\ \varepsilon \neq 1}} (\varepsilon - 1) G_1(0, \iota^{-1}(\varepsilon - 1)).$$

But as $tG_1(0, \iota^{-1}(t)) \in -\eta_p + tI_{p,E}[\![t]\!]$, we obtain the desired result by applying the following obvious identities:

$$\sum_{\substack{\varepsilon^{p^{nh}} = 1 \\ \varepsilon \neq 1}} (\varepsilon - 1)^i \equiv \begin{cases} -1 \pmod{\pi^n} & \text{if } i = 0 \\ \ \ 0 \pmod{\pi^n} & \text{if } i \geqslant 1. \end{cases}$$

COROLLARY. $\eta_p = \lim_{n \to \infty} p^n E_1(p^{-n}\omega_\infty, L)$.

Thus η_p appears as the p-adic period of the differential form $\eta = (x + a_2(L))(\mathrm{d}x/y)$ integrated along the cycle γ_1 viewed in $T_p(E)$ in the obvious way (cf. [P-R], [de S]). Using this remark, it is easy to show that the isomorphism between $\bar{K}(\eta_\infty)$ and $\bar{K}(\eta)$ does not depend on the choice of E or γ_1; it depends only on the embeddings of $\bar{K}$ into $\mathbf{C}$ and $\mathbf{C}_p$.

PROPOSITION 11. *Let* $\alpha \in (K - \mathbf{p}^{-\infty}\mathbf{a})$ *and let* $G_{k,j}(\alpha, t) = \mathscr{H}(E_{k,j}(\tilde{\alpha} + \lambda(t), L))$. *Then*

(i) $G_{k,j}(\alpha, t) \in \hat{O}[\![t]\!] \otimes \mathbf{Q}_p$.

(ii) *If* Q *is a* $\mathbf{p}^\infty$*-division point, then* $G_{k,j}(\alpha, t(Q)) = E_{k,j}(\tilde{\alpha} + \tilde{z}(Q), L)$.

Proof. If $k = 0$, then (i) follows from Proposition 10 and the fact that

$$E_{0,j} = -\frac{\mathrm{d}}{\mathrm{d}z} E_{0,j-1} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z} = \frac{\mathrm{d}t}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}t}$$

where

$$\frac{\mathrm{d}t}{\mathrm{d}z} \in 1 + t\hat{O}[\![t]\!],$$

and (ii) follows from the fact that $E_{0,j}$ is a rational function on E. The general case follows then from the existence of $P_{k,j}$ (Proposition 9).

PROPOSITION 12. *Let* $\alpha \in K$, $v_{\mathbf{p}}(\alpha) \geqslant 0$. *Let* $\Delta_\alpha(z) = \langle z, \tilde{\alpha} \rangle_L$. *Then*

(i) $\mathscr{H}(\Delta_\alpha(\lambda(t))) \in \hat{O}[\![t]\!]$.

(ii) *If* Q *is a* $\mathbf{p}^\infty$*-division point, then* $\mathscr{H}(\Delta_\alpha(\lambda(t)))$ *evaluated at* $t = t(Q)$ *is equal to* $\Delta_\alpha(\tilde{z}(Q))$ *where* $z(Q)$ *has to be chosen so that* $\bar{\alpha}z(Q) \in \mathbf{p}^{-\infty}\mathbf{a}$ *(this restriction being due to the fact that* $\Delta_\alpha(z)$ *is not periodic of period* L *in* z*).*

Proof. Everything is obvious once we have proved that $\mathscr{H}(\Delta_\alpha(\lambda(t))) = (1 + \imath(t))^{\bar{\alpha}}$. But we have $\Delta_\alpha(z) = \exp(A(L)^{-1}(\overline{z\alpha\omega_\infty} - \alpha\omega_\infty\bar{z}))$. So using the identity $A(L) = -\bar{\omega}_\infty \eta_\infty^{-1}$ we obtain: $\mathscr{H}(\Delta_\alpha(z)) = \exp(-\eta_\infty \bar{\alpha} z)$, and p-adically, $\mathscr{H}(\Delta_\alpha(\lambda(t))) = \exp(-\eta_p \bar{\alpha}\lambda(t))$. As λ is an isomorphism from $\hat{E}$ to $\mathbf{G}_a$, we find that $\imath(t) = \exp(u\lambda(t)) - 1$ for some $u \in \mathbf{C}_p$. Equating terms of degree 1 in t gives $u = -\eta_p$ which allows us to conclude.

PROPOSITION 13. *Let* $\alpha \in K - \mathbf{p}^{-\infty}\mathbf{a}$ *and* $\beta \in K$ *such that* $v_{\mathbf{p}}(\beta) \geqslant 0$. *Then for* $1 \leqslant j \leqslant k$ *we have*:

(i) $\mathscr{H}(H_k(j, \tilde{\alpha} + \lambda(t), \bar{\beta}, L)) \in \hat{O}[\![t]\!] \otimes \mathbf{Q}_p$.

(ii) *If* Q *is a* $\mathbf{p}^\infty$*-division point then the previous series evaluated at* $t = t(Q)$ *is equal to* $H_k(j, \tilde{\alpha} + \tilde{z}(Q), \tilde{\beta}, L)$, *where* $z(Q)$ *has to be chosen in such a way that* $\bar{\beta}z(Q) \in \mathbf{p}^{-\infty}\mathbf{a}$.

Proof. Choose $b \in O_K$ satisfying $(b, \mathbf{p}) = 1$ and $\bar{b}\beta \in \mathbf{a}$. Then formula (25) gives:

$$H_k(j, \tilde{\alpha} + \lambda(t), \tilde{\beta}, L) = \frac{\bar{b}^{k-j}}{b^j} \sum_{\gamma \in b^{-1}L/L} \langle \gamma, \bar{b}\tilde{\beta} \rangle_L E_{k-j,j}\left(\gamma + \frac{\tilde{\alpha}}{b} + \frac{\lambda(t)}{b}, L\right).$$

Since b is prime to $\mathbf{p}$, b^{-1} is an endomorphism of $\hat{E}$ and $b^{-1}\lambda(t)=\lambda([b^{-1}]t)$. Then (i) follows directly from Proposition 11(i).

Now let Q be a $\mathbf{p}^n$-division point and let $b^*\in O_K$ be such that $b^*b\equiv 1 \pmod{\mathbf{p}^n}$. Then $[b^{-1}]t(Q)=t(b^*Q)$ and so by Proposition 11(ii), we obtain that $\mathscr{H}(H_k(j,\tilde{\alpha}+\lambda(t),\tilde{\beta},L))$ evaluated at $t=t(Q)$ is equal to

$$\frac{\bar{b}^{k-j}}{b^j}\sum_{\gamma\in b^{-1}L/L}\langle\gamma,\bar{b}\tilde{\beta}\rangle_L E_{k-j,j}(\gamma+\tilde{\alpha}b^{-1}+b^*\tilde{z}(Q),L)=H_k(j,\tilde{\alpha}+bb^*\tilde{z}(Q),\tilde{\beta},L),$$

which allows us to conclude.

PROPOSITION 14. *Let $\alpha\in K$ be such that $v_{\bar{\mathbf{p}}}(\alpha)\geqslant 0$ and $\beta\in K-\mathbf{p}^{-\infty}\mathbf{a}$. Then for* $1\leqslant j\leqslant k$

(i) $\mathscr{H}(H_k(j,\tilde{\alpha},\tilde{\beta}+\lambda(t),L))\in\hat{O}[\![t]\!]\otimes\mathbf{Q}_p$.

(ii) *If Q is a $\mathbf{p}^\infty$-division point, then the previous series evaluated at $t=t(Q)$ is equal to $H_k(j,\tilde{\alpha},\tilde{\beta}+\tilde{z}(Q),L)$.*

Proof. Everything follows easily from the previous proposition and the functional equation for $H_k(j,u,z,L)$ which says that

$$\mathscr{H}(H_k(j,\tilde{\alpha},\tilde{\beta}+\lambda(t),L))=(1+\iota(t))^{\tilde{\alpha}}\mathscr{H}(H_k(k+1-j,\tilde{\beta}+\lambda(t),\tilde{\alpha},L)).$$

Note however that Proposition 13(ii) would give some restrictions as to the possible value of $z(Q)$ which makes (ii) work, but since $H_k(j,u,z,L)$ is periodic of period L in z this restriction is unnecessary.

PROPOSITION 15. *Let $\alpha,\beta\in K-\mathbf{a}$. Let $k,l\in\mathbf{N}$ and $G_{k,l,\alpha,\beta}(t_1,t_2)$ be the power series defined by*

$$G_{k,l,\alpha,\beta}(t_1,t_2)=\mathscr{H}(H_{k+l}(l,\tilde{\alpha}+\lambda(t_1),\tilde{\beta}+\lambda(t_2),L)).$$

If α, $\beta\in O_{K,p}$, then

(i) $G_{k,l,\alpha,\beta}(t_1,t_2)\in I_{p,E}[\![t_1,t_2]\!]$.

(ii) *If Q_1, Q_2 are $\mathbf{p}^\infty$-division points, then*

$$G_{k,l,\alpha,\beta}(t(Q_1),t(Q_2))=H_{k+l}(l,\tilde{\alpha}+\tilde{z}(Q_1),\tilde{\beta}+\tilde{z}(Q_2),L),$$

where $z(Q_1)$ has been chosen so that $z(Q_1)\overline{(\beta+z(Q_2))}\in\mathbf{p}^{-\infty}\mathbf{a}$.

The proof of this proposition will need several lemmas (as well as the preceding propositions). First, call a power series $H(t_1,t_2)=\Sigma_{i,j}a_{i,j}t_1^it_2^j$ 'almost bounded' if, when i is fixed, $a_{i,j}$ is bounded as j varies and if Q is a $\mathbf{p}^\infty$-division point, then $H(t_1, t(Q))$, which converges because of what precedes, is a bounded power series in t_1. If H is almost bounded, then if Q_1 and Q_2 are $\mathbf{p}^\infty$-division points we can define $H(t(Q_1), t(Q_2))$ as the value of $H(t_1, t(Q_2))$ at $t_1 = t(Q_1)$.

LEMMA 16. *If H is an almost bounded power series satisfying $H(t(Q_1), t(Q_2)) = 0$ whenever Q_1 and Q_2 are $\mathbf{p}^\infty$-division points, then H is identically equal to 0.*

Proof. If you fix Q_2, then the series $H(t_1, t(Q_2))$ is bounded and is equal to 0 if $t_1 = t(Q)$ where Q is a $\mathbf{p}^\infty$-division point. This implies that $H(t_1, t(Q_2))$ is equal to 0 as a power series in t_1, hence for all $i \geqslant 0$, $\Sigma_{j=0}^{\infty} a_{i,j}(t(Q_2))^j = 0$. But this is true for all $\mathbf{p}^\infty$-division points Q_2, so $a_{i,j} = 0$ for all i and j.

LEMMA 17. *$G_{k,l,\alpha,\beta}$ is almost bounded.*

Proof. We have

$$G_{k,l,\alpha,\beta}(t_1, t_2) = \sum_{i=0}^{\infty} \frac{(-\lambda(t_1))^i}{i!} \mathscr{H}(H_{k+l+i}(l+i, \tilde{\alpha}, \tilde{\beta} + \lambda(t_2), L)) = \sum_{i,j} a_{i,j} t_1^i t_2^j.$$

By Proposition 14(i) and the fact that $\lambda(t)$ has no constant term, we obtain that when i is fixed, $a_{i,j}$ is bounded as j varies. Moreover, by Proposition 14(ii), if Q_2 is a $\mathbf{p}^\infty$-division point then:

$$\begin{aligned} G_{k,l,\alpha,\beta}(t_1, t(Q_2)) &= \sum_{i=0}^{\infty} \frac{(-\lambda(t_1))^i}{i!} H_{k+l+i}(l+i, \tilde{\alpha}, \tilde{\beta} + \tilde{z}(Q_2), L) \\ &= \mathscr{H}(H_{k+l}(l, \tilde{\alpha} + \lambda(t_1), \tilde{\beta} + \tilde{z}(Q_2), L)). \end{aligned}$$

Then Proposition 13(i) allows us to conclude. But in addition, Proposition 13(ii) gives (ii) of Proposition 15.

LEMMA 18. *Let $\delta \in O_{K,p}$, $d \in O_K$ satisfy $\bar{d}\delta \in \mathbf{a}$ and $(d, p) = 1$. Let π be a generator of $\mathbf{p}^h$ and $\delta_0 \in \mathbf{a}$ satisfy $\delta_0 \equiv \delta \pmod{\bar{\pi}^n}$. Then*

$$S = \sum_{\gamma \in d\pi^{-n}L/dL} \langle \tilde{\delta}, \gamma \rangle_L H_1(1, z, u+\gamma, L) = \pi^n \langle u, \tilde{\delta}_0 \rangle_L H_1\left(1, \frac{z - \tilde{\delta}_0}{\bar{\pi}^n}, \pi^n u, L\right).$$

Proof. We have

$$S = \sum_{\gamma \in d\pi^{-n}L/dL} \sum_{\omega \in L} \frac{\langle \tilde{\delta} + \omega, \gamma \rangle_L}{\omega + z} \langle \omega, u \rangle_L.$$

But

$$\sum_{\gamma \in d\pi^{-n}L/dL} \langle \tilde{\delta} + \omega, \gamma \rangle_L = \begin{cases} p^{nh} & \text{if } \tilde{\delta}_0 + \omega \in \bar{\pi}^n L \\ 0 & \text{otherwise.} \end{cases}$$

The result follows easily using formula (24). Note that the above formal computations can be justified by analytic continuation.

LEMMA 19. *Let a, $b \in \mathbf{Z}$, and set $G_{\alpha,\beta} = G_{0,1,\alpha,\beta}$. Then we have, for $\alpha, \beta \in O_{K,p}$:*

$$\Sigma_{a,b,n,m} \stackrel{\text{def}}{=} \frac{1}{p^{h(n+m)}} \sum_{Q_1 \in E_{\pi^m}} \sum_{Q_2 \in E_{\bar{\pi}^n}} \varepsilon(Q_1)^{-a} \varepsilon(Q_2)^{-b} G_{\alpha,\beta}(t(Q_1), t(Q_2))$$
$$= \langle \tilde{\beta}, \tilde{b} \rangle_L \bar{\pi}^{-(n+m)} H_1(1, (\pi^m/\bar{\pi}^n)(\tilde{\alpha} - \tilde{b}), (\pi^n/\bar{\pi}^m)(\tilde{\beta} - \tilde{\beta}_0 - \tilde{a}), L),$$

where $\beta_0 \in \mathbf{a}$ and $\beta_0 \equiv \beta \pmod{\bar{\pi}^m}$, and $\varepsilon(Q)$ is the p^∞th root of unity defined together with η_p.

Proof. Using Lemma 18 with $\delta = \delta_0 = b$, $d = 1$ and the value of $G_{\alpha,\beta}(t(Q_1), t(Q_2))$, we obtain

$$\frac{1}{p^{hn}} \sum_{Q_2 \in E_{p^n}} \varepsilon(Q_2)^{-b} G_{\alpha,\beta}(t(Q_1), t(Q_2))$$
$$= \langle \tilde{\beta}, \tilde{b} \rangle_L \bar{\pi}^{-n} H_1(1, (\tilde{\alpha} + \tilde{z}(Q_1) - \tilde{b})/\bar{\pi}^n, \pi^n \tilde{\beta}, L)$$
$$= \bar{\pi}^{-n} \langle \tilde{\beta}, \tilde{\alpha} + \tilde{z}(Q_1) \rangle_L H_1(1, \pi^n \tilde{\beta}, (\tilde{\alpha} + \tilde{z}(Q_1) - \tilde{b})/\bar{\pi}^n, L).$$

Let $d' \in O_K$ verify $(d', p) = 1$ and $\overline{d'\beta} \in \mathbf{a}$. By Proposition 15(ii), we have to choose $z(Q_1)$ for Q_1 in E_{π^m} so that $z(Q_1)\overline{(\beta + z(Q_2))} \in \mathbf{p}^{-\infty}\mathbf{a}$. This means that we can take the $z(Q_1)$ in $\pi^{-m}\bar{\pi}^n d'\mathbf{a}$ and the $\bar{\pi}^{-n} z(Q_1)$ will run through a set of representatives of $\pi^{-m} d'\mathbf{a}/d'\mathbf{a}$. So, writing $\langle \tilde{\beta}, \tilde{z}(Q_1) \rangle_L = \langle \pi^n \tilde{\beta}, \bar{\pi}^{-n} \tilde{z}(Q_1) \rangle_L$ and $\varepsilon(Q_1)^{-a} = \langle \pi^n \tilde{a}, \bar{\pi}^{-n} \tilde{z}(Q_1) \rangle_L$, we can apply Lemma 18 again with $\delta = \pi^n(a + \beta)$, $\delta_0 = \pi^n(a + \beta_0)$, $d = d'$, $u = \bar{\pi}^{-n}(\tilde{\alpha} - \tilde{b})$ and $z = \pi^n \tilde{\beta}$, to find that $\Sigma_{a,b,n,m}$ is equal to

$$\bar{\pi}^{-(n+m)} \langle \tilde{\beta}, \tilde{\alpha} \rangle_L \langle \tilde{\alpha} - \tilde{b}, \tilde{\beta}_0 + \tilde{\alpha} \rangle_L H_1(1, (\pi^n/\bar{\pi}^m)(\tilde{\beta} - \tilde{\beta}_0 - \tilde{\alpha}), (\pi^m/\bar{\pi}^n)(\tilde{\alpha} - \tilde{b}), L).$$

The result follows from the functional equation of H_1.

LEMMA 20. *Let $\gamma, \delta \in K - \mathbf{a}$ verify $v_\mathbf{p}(\gamma) \geqslant 0$, $v_\mathbf{p}(\delta) \geqslant 0$, $v_\mathbf{p}(\gamma) + v_{\bar{\mathbf{p}}}(\delta) \geqslant p$. Then $H_1(1, \tilde{\gamma}, \tilde{\delta}, L) \in I_{p,E}$.*

Proof. Choose $d \in O_K$ such that $v_\mathbf{p}(d) = \sup(0, -v_{\bar{\mathbf{p}}}(\delta))$, $v_{\bar{\mathbf{p}}}(d) = 0$ and $\bar{d}\delta \in \mathbf{a}$. By formula (25), $H_1(1, \tilde{\gamma}, \tilde{\delta}, L) = (1/d) \Sigma_{y \in d^{-1}L/L} \langle y, \bar{d}\tilde{\delta} \rangle_L E_1((\tilde{\gamma}/d) + y, L)$. Writing $(d) = \mathbf{p}^k \mathbf{d}$ where $\mathbf{d}$ is prime to p, we can write $y \in d^{-1}L/L$ in a unique way as $\tilde{z}_0 + \tilde{z}_1$ where $z_0 \in \mathbf{d}^{-1}\mathbf{a}/\mathbf{a}$ and $z_1 \in \mathbf{p}^{-k}\mathbf{a}/\mathbf{a}$. Set $\delta' = \bar{d}\delta$ and $\gamma' = d^{-1}\gamma$. We obtain

$$H_1(1, \tilde{\gamma}, \tilde{\delta}, L) = d^{-1} \sum_{z_0 \in \mathbf{d}^{-1}\mathbf{a}/\mathbf{a}} \langle \tilde{z}_0, \tilde{\delta}' \rangle_L \sum_{z_1 \in \mathbf{p}^{-k}\mathbf{a}/\mathbf{a}} \langle \tilde{z}_1, \tilde{\delta}' \rangle_L E_1(\tilde{\gamma}' + \tilde{z}_0 + \tilde{z}_1, L).$$

By Proposition 10, we can write

$$d^{-1} \sum_{z_1 \in p^{-k}\mathbf{a}/\mathbf{a}} \langle \tilde{z}_1, \tilde{\delta}' \rangle_L E_1(\tilde{\gamma}' + \tilde{z}_0 + \tilde{z}_1, L) = d^{-1} \sum_{\varepsilon^{p^k}=1} \varepsilon^{\bar{\delta}'} G_1(\gamma' + z_0, \iota^{-1}(\varepsilon - 1)).$$

But $G_1(\gamma' + z_0, t) \in M(P(\gamma' + z_0))\,[\![t]\!]$ and with the exception of the constant term has integral coefficients. As $\gamma' + z_0 \in O_{K,p}$, $P(\gamma' + z_0)$ is defined over the maximal unramified extension of M which implies that

$$G_1(\gamma' + z_0, \iota^{-1}(w)) \in E_1(\tilde{\gamma}' + \tilde{z}_0, L) + wI_{p,E}[\![w]\!].$$

But we also have

$$E_1(\tilde{\gamma}' + \tilde{z}_0, L) \equiv \eta_p(\bar{\gamma}' + \bar{z}_0) \equiv \eta_p \bar{\gamma}' \pmod{I_{p,E}},$$

so we see that

$$G_1(\gamma' + z_0, \iota^{-1}(w)) - \eta_p \bar{\gamma}' \in I_{p,E}[\![w]\!].$$

As $\Sigma_{\varepsilon^{p^k}=1} (\varepsilon - 1)^i \varepsilon^{\bar{\delta}'} \in \mathbf{Z}$ and is congruent to $0 \,(\mathrm{mod}\, p^k)$, we finally obtain

$$H_1(1, \gamma, \delta, L) - \eta_p \bar{\gamma}' \sum_{y \in d^{-1}L/L} \langle y, \tilde{\delta}' \rangle_L \in (p^k/d) I_{p,E},$$

which gives the result, since $(p^k/d) \in I_{p,E}$ and $\Sigma_{y \in d^{-1}L/L} \langle y, \delta' \rangle_L = 0$ when $\delta \notin \mathbf{a}$.

COROLLARY. $\Sigma_{a,b,n,m} \in I_{p,E}$.

Define a measure $\mu_{\alpha,\beta}$ on $Y_{K,p} = O_{\mathbf{p}} \times O_{\bar{\mathbf{p}}}$ by

$$\mu_{\alpha,\beta}((a + \mathbf{p}^{mh}) \times (b + \bar{\mathbf{p}}^{nh})) = \Sigma_{a,b,n,m}.$$

This is an $I_{p,E}$-valued measure. Let

$$H(t_1, t_2) = \int_{Y_{K,p}} (1 + \iota(t_1))^x (1 + \iota(t_2))^y \, \mathrm{d}\mu_{\alpha,\beta}(x, y);$$

then $H(t_1, t_2) \in I_{p,E}[\![t_1, t_2]\!]$ and if Q_1, Q_2 are p^∞-division points, then by construction of $\mu_{\alpha,\beta}$, $H(t(Q_1), t(Q_2)) = G_{\alpha,\beta}(t(Q_1), t(Q_2))$ and so $H = G_{\alpha,\beta}$ by virtue of Lemma 16.

This concludes the proof of Proposition 15 for $k = 0$ and $l = 1$. The general case follows from the following identity:

$$H_{k+j}(j,\, \tilde{\alpha}+z,\, \tilde{\beta}+u,\, L)$$
$$=\langle u,\, \tilde{\alpha}+z\rangle_L\left(-\frac{\partial}{\partial u}\right)^k\left[\langle\tilde{\alpha}+z,\, u\rangle_L\left(-\frac{\partial}{\partial z}\right)^{j-1} H_1(1,\, \tilde{\alpha}+z,\, \tilde{\beta}+u,\, L)\right],$$

which yields

$$G_{k,j,\alpha,\beta}(t_1,\, t_2)=(1+\imath(t_2))^{\bar{\alpha}}\left(-\frac{\partial}{\partial\lambda(t_2)}\right)^k\left[(1+\imath(t_2))^{-\bar{\alpha}}\left(-\frac{\partial}{\partial\lambda(t_1)}\right)^{j-1} G_{\alpha,\beta}(t_1,t_2)\right]. \tag{30}$$

PROPOSITION 21. *If Q_1 and Q_2 are $\mathbf{p}^\infty$-division points, then*

$$G_{\alpha,\beta}(t_1 \oplus t(Q_1),\, t_2 \oplus t(Q_2)) = (1+\imath(t_2))^{-\overline{z(Q_1)}}G_{\alpha+z(Q_1),\beta+z(Q_2)}(t_1,t_2).$$

where $z(Q_1)$ has to be chosen so that $z(Q_1)\overline{(\beta+z(Q_2))}\in\mathbf{p}^{-\infty}\mathbf{a}$.

Proof. Set $G'_{k,j,\alpha,\beta}(t_1,t_2)=(1+\imath(t_2))^{-\bar{\alpha}}G_{k,j,\alpha,\alpha}(t_1,t_2)$. Then formula (30) becomes:

$$\left(-\frac{\partial}{\partial\lambda(t_1)}\right)^{j-1}\left(-\frac{\partial}{\partial\lambda(t_2)}\right)^k G'_{\alpha,\beta}(t_1,t_2)=G'_{k,j,\alpha,\beta}(t_1,t_2),$$

so that, as power series, we get

$$G'_{\alpha,\beta}(t_1 \oplus w_1,\, t_2 \oplus w_2) = \sum_{j,k}\frac{(-\lambda(t_1))^j}{j!}\frac{(-\lambda(t_2))^k}{k!}G'_{k,j+1,\alpha,\beta}(w_1,w_2).$$

Now, let $w_1=t(Q_1)$ and $w_2=t(Q_2)$. Using Proposition 15(ii) and Proposition 12, we obtain:

$$G_{\alpha,\beta}(t_1 \oplus t(Q_1),\, t_2 \oplus t(Q_2))$$
$$=(1+\imath(t_2))^{\bar{\alpha}}\sum_{j,k}\frac{(-\lambda(t_1))^j}{j!}\frac{(-\lambda(t_2))^k}{k!}H_{k+j+1}(j+1,\, \tilde{\alpha}+\tilde{z}(Q_1),\, \tilde{\beta}+\tilde{z}(Q_2),\, L).$$

But using (30) replacing t_1, t_2 by 0, α by $\alpha+z(Q_1)$ and β by $\beta+z(Q_2)$, we find that:

$$\sum_{j,k}\frac{(-\lambda(t_1)^j)}{j!}\frac{(-\lambda(t_2)^k)}{k!}H_{k+j+1}(j+1,\, \tilde{\alpha}+\tilde{z}(Q_1),\, \tilde{\beta}+\tilde{z}(Q_2),\, L)$$

is the Taylor expansion in $-\lambda(t_1)$, $-\lambda(t_2)$ of

$$(1+\iota(t_2))^{-\bar{\alpha}-\bar{z}(Q_1)}G_{\alpha+z(Q_1),\beta+z(Q_2)}(t_1,t_2),$$

which concludes the proof.

If $\mathbf{a}$ is a fractional ideal of K, let $\mathbf{K}(z_1,z_2,\mathbf{a})=\Sigma_{\omega\in\mathbf{a}}\dfrac{1}{\omega+z_1}(\omega+z_1\,|\,z_2)_\infty$. If $1_{\mathbf{a}}$ is the characteristic function of $\mathbf{a}$, then we have $\mathbf{K}(z_1,z_2,\mathbf{a})=\mathbf{K}(z_1,z_2,1_{\mathbf{a}},(1))$ in the notations of Section II. Let $\tau\in\mathbf{C}$ with $\mathrm{Im}(\tau)>0$ be such that $O_K=\mathbf{Z}\oplus\mathbf{Z}\tau$. Then, we have $(y\,|\,z)_\infty=\langle y,(\tau-\bar{\tau})N(\mathbf{a})z\rangle_{\mathbf{a}}$ and formally:

$$\begin{aligned}\mathbf{K}(z_1,\,z_2,\,\mathbf{a}) &= \langle z_1,\,(\tau-\bar{\tau})N(\mathbf{a})z_2\rangle_{\mathbf{a}}H_1(1,\,z_1,\,(\tau-\bar{\tau})N(\mathbf{a})z_2,\,\mathbf{a})\\ &= H_1(1,\,(\tau-\bar{\tau})N(\mathbf{a})z_2,\,z_1,\,\mathbf{a}).\end{aligned}$$

This formal computation can be justified, as usual, using analytic continuation. Set $w_i=\exp(-z_i)-1$ for $i=1,2$.

PROPOSITION 22. *Let $\delta\in K$, $\beta_1\in K-\delta O_K$ and $\beta_2\in K-(\delta O_K)^\vee$. Then*

(i) $\mathscr{H}\left(\dfrac{1}{2\pi i}\mathbf{K}\left(\beta_1+\dfrac{z_1}{2\pi i},\,\beta_2+\dfrac{z_2}{2\pi i},\,\delta O_K\right)\right)\in\bar{K}(\eta_\infty)[\![z_1,z_2]\!]$.

(ii) *If moreover δ is a unit in $O_{K,p}$, β_1 and β_2 belong to $O_{K,p}$ and $W_{\delta,\beta_1,\beta_2}(w_1,w_2)\in\mathbf{C}_p[\![w_1,w_2]\!]$ is equal to $\mathscr{H}\left(\dfrac{1}{2\pi i}\mathbf{K}\left(\beta_1+\dfrac{z_1}{2\pi i},\,\beta_2+\dfrac{z_2}{2\pi i},\delta O_K\right)\right)$ expressed in w_1,w_2, then $W_{\delta,\beta_1,\beta_2}\in I_p[\![w_1,w_2]\!]$.*

(iii) *If $\gamma_1,\gamma_2\in K/O_{K,\mathbf{p}}$ and $\varepsilon_i=\chi_{\gamma_i}(1)$, then*

$$\begin{aligned}&W_{\delta,\beta_1,\beta_2}((1+w_1)\varepsilon_1-1,\,(1+w_2)\varepsilon_2-1)\\ &=\mathscr{H}\left(\frac{1}{2\pi i}\mathbf{K}\left(\beta_1+\hat{\gamma}_1+\frac{z_1}{2\pi i},\,\beta_2+\hat{\gamma}_2+\frac{z_2}{2\pi i},\,\delta O_K\right)\left(-\beta_2\left|\frac{z_1}{2\pi i}\right.\right)_\infty\right),\end{aligned}$$

where $\hat{\gamma}_1$ is a representative of γ_1 in $\mathbf{p}^{-\infty}\delta O_K$ and $\hat{\gamma}_2$ a representative of γ_2 in $\mathbf{p}^{-\infty}(\delta O_K+(\beta_1)+(\gamma_1))^\vee$.

(iv) *If $\beta\in K$, then* $\mathscr{H}\left(\left(\beta\left|\dfrac{z_1}{2\pi i}\right.\right)_\infty\right)=(1+w_1)^{\bar{\beta}}$.

Proof. Part (iv) is obvious. To prove (i), (ii) and (iii), let us introduce an elliptic curve E with Weierstrass model defined over the ring of integers of the Hilbert class field of K with good reduction at all places above p and j-invariant equal to $j(O_K)$. This implies that the period lattice of E has the form $\omega_\infty(E)O_K$ for some

$\omega_\infty(E) \in \mathbf{C}^*$. Then, we have $\eta_\infty(E) = \dfrac{2\pi i}{\omega_\infty(E)} \dfrac{1}{\bar{\tau}-\tau}$ and $\eta_p(E) \in I_{\mathfrak{p}}^*$. Now, straightforward computation yields

$$\begin{aligned}\frac{1}{2\pi i}\mathbf{K}\left(\frac{z_1}{2\pi i}, \frac{z_2}{2\pi i}, \delta O_K\right) &= \frac{1}{2\pi i} H_1\left(1, (\tau-\bar{\tau})|\delta|^2 \frac{z_2}{2\pi i}, \frac{z_1}{2\pi i}, \delta O_K\right) \\ &= \frac{\omega_\infty}{2\pi i \delta} H_1\left(1, \frac{(\tau-\bar{\tau})\omega_\infty}{2\pi i} \bar{\delta} z_2, \frac{z_1}{\delta} \frac{\omega_\infty}{2\pi i}, \omega_\infty O_K\right) \\ &= \frac{1}{\delta(\bar{\tau}-\tau)\eta_\infty} H_1\left(1, -\bar{\delta}\frac{z_2}{\eta_\infty}, \frac{z_1}{\eta_\infty} \frac{1}{\delta(\bar{\tau}-\tau)}, \omega_\infty O_K\right).\end{aligned}$$

Hence, as $\iota^{-1}(w_i) = \lambda^{-1}(-\eta_p^{-1}\log(1+w_i)) = \lambda^{-1}(\eta_p^{-1} z_i)$, we obtain

$$W_{\delta,\beta_1,\beta_2}(w_1, w_2) = \frac{1}{\delta(\bar{\tau}-\tau)\eta_{\mathfrak{p}}} G_{\alpha_1,\alpha_2}([-\bar{\delta}]\cdot\iota^{-1}(w_2), [\delta(\bar{\tau}-\tau)]^{-1}\cdot\iota^{-1}(w_1)),$$

where ι is the isomorphism between $\hat{E}$ and $\mathbf{G}_m$ and $[\beta]$ is the endomorphism of $\hat{E}$ associated to β, $\alpha_1 = -\bar{\delta}\beta_2$ and $\alpha_2 = \dfrac{1}{\delta(\bar{\tau}-\tau)}\beta_1$. Now (i), (ii) and (iii) are just reinterpretations of Propositions 15 and 21.

3. Construction of p-adic measures attached to generalized Eisenstein–Kronecker series

If $\phi \in \mathscr{S}_T(H)$ where $T \cap |(p)| = \varnothing$, set $\tilde{\phi} = \phi_{\mathfrak{p}} * \phi$ where

$$\phi_{\mathfrak{p}} = N(\mathbf{d}_{H,\mathfrak{p}})^{-1/2} 1_{\mathbf{d}_H^{-1}} \in \mathscr{S}_{|\mathfrak{p}|,H}$$

is the Fourier transform of the characteristic function of $O_{|\bar{\mathfrak{p}}|}$ considered as an element of $\mathscr{S}_{|\bar{\mathfrak{p}}|,H^\vee}$. Let $I_{p,H}$ be the ring of integers of the completion of the maximal unramified extension of the field generated over $\mathbf{Q}_p$ by all conjugates of H and $\sqrt{N(\mathbf{d}_{H,\mathfrak{p}})}$. The aim of this paragraph is to prove the following theorem:

THEOREM 23. *Let $\mathscr{B}$ be a finite set of bases of H over K. Then there exists $S = S_2(\mathscr{B}) \in \mathscr{P}(H)$ and $S' = S_2'(\mathscr{B}) \in \mathscr{P}(H^\vee)$ such that for all $T \in \mathscr{P}(H)$ satisfying $T \cap |(p)| = \varnothing$, all $\phi \in \mathscr{S}_T(H)$, all $(\mathbf{b}_1, \mathbf{b}_2) \in C_{T\cup|(p)|}(S, S')$, we have:*

(i) $\mathscr{H}\left((2\pi i)^{-n}\mathbf{K}\left(\dfrac{z_1}{2\pi i}, \dfrac{z_2}{2\pi i}, \tilde{\phi}_{\mathbf{b}_1,\mathbf{b}_2}, \mathscr{B}\right)\right) \in \bar{K}(\eta_\infty)[\![z_1, z_2]\!]$ *and is the Fourier–Laplace transform of an $I_{p,H}$-valued measure $\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi,\mathscr{B}}$ on $Y_{H,p}$.*

(ii) *Let ϕ_1 be a locally constant function on Y_1 which can also be considered as an element of $\mathscr{S}_{|\bar{\mathbf{p}}|,H^\vee}$, and ϕ_2 a locally constant function on Y_2 also considered as an element of $\mathscr{S}_{|\bar{\mathbf{p}}|,H}$. Then the Fourier–Laplace transform of $\phi_1\phi_2\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi,\mathscr{B}}$ is*

$$\mathscr{H}\left((2\pi i)^{-n}\mathbf{K}\left(\frac{z_1}{2\pi i},\frac{z_2}{2\pi i},\mathscr{F}_{|\bar{\mathbf{p}}|}(\phi_1)*\phi_2*\phi_{\mathbf{b}_1,\mathbf{b}_2},\mathscr{B}\right)\right).$$

LEMMA 24. *Let A be a principal ideal domain having only a finite number of prime ideals and let K be its field of fractions. Let $v_1,\dots,v_n\in A^n$ be a basis of K^n over K but not of A^n over A; then there exists $w\in A^n$ such that for all $1\leqslant i\leqslant n$, $\det(v_1,\dots,v_{i-1},w,v_{i+1},\dots,v_n)$ is either 0 or a strict divisor of $\det(v_1,\dots,v_n)$.*

Proof. Choose $w_1=\Sigma_{i=1}^n a_iv_i$ with $a_i\in K$, belonging to A^n but not to the submodule of A^n spanned by the v_i's. If $a_i\in A$, set $b_i=a_i$. If $a_i\notin A$, we can write $a_i=c_i/d_i$, with $c_i,d_i\in A$ relatively prime; let $b_i\in A$ be relatively prime to c_i and be divisible by all prime ideals of A not dividing c_i. Then $e_i=c_i-b_id_i$ is a unit in A and $w=\Sigma_{i=1}^n(a_i-b_i)v_i$ obviously answers the question.

If $M\in M_n(K)$, we set

$$M\begin{pmatrix}z_1\\ \vdots\\ z_n\end{pmatrix}=\begin{pmatrix}M_1(z)\\ \vdots\\ M_n(z)\end{pmatrix}\quad\text{and}\quad f_M(z)=\det(M)\prod_{i=1}^n M_i(z)^{-1}.$$

LEMMA 25. *Let A be as in Lemma 24 and $M\in GL_n(K)$. We can find a finite family $\mathcal{N}$ of elements of $GL_n(A)$ such that*

$$f_M(z)=\sum_{N\in\mathcal{N}}f_N(z).$$

Proof. First note that $f_M(z)$ does not change if M is multiplied by a scalar; so we may suppose that $M\in M_n(A)$. Let $v_1,\dots,v_n$ be the rows of this matrix. Then either $v_1,\dots,v_n$ generate A^n in which case $M\in GL_n(A)$ and there is nothing to prove, or we can find w as in the preceding lemma. Let $v_{i,j}$ (resp. w_j) for $1\leqslant j\leqslant n$ be the coordinates of v_i (resp. of w) and $M_{n+1}(z)=\Sigma_{i=1}^n w_jz_j$. Let N_i be the matrix whose jth row is equal to v_j if $j\neq i$ and w if $j=i$. We obtain:

$$0=\det\begin{pmatrix}v_{1,1}&\cdots&v_{1,n}&M_1(z)\\ \vdots&&\vdots&\vdots\\ v_{n,1}&\cdots&v_{n,n}&M_n(z)\\ w_1&\cdots&w_n&M_{n+1}(z)\end{pmatrix}=\left(f_M(z)-\sum_{i=1}^n f_{N_i}(z)\right)\prod_{j=1}^{n+1}M_j(z),$$

where the first equality is obtained remarking that the last column is a linear combination of the others, and the second is obtained by developing the determinant with respect to the last column. Now, removing from the N_i those with determinant 0, we obtain $f_M(z)=\Sigma_N f_N(z)$, where the $\det(N)$ are strict divisors of $\det(M)$. We just go on with this process until we reach the desired result.

COROLLARY. *Let B be a basis of H over K. We can find a finite family $\mathscr{C}(B)$ of bases of $\mathbf{d}_{H,\mathfrak{p}}^{-1}O_{H,p}$ over $O_{K,p}$ such that, for all $\phi\in\mathscr{S}(H)$, we have*

$$\mathbf{K}(z_1, z_2, \phi, B) = \sum_{C\in\mathscr{C}(B)} \mathbf{K}(z_1, z_2, \phi, C).$$

Proof. Choose a basis C_0 of $\mathbf{d}_{H,\mathfrak{p}}^{-1}O_{H,p}$ over $O_{K,p}$; then there exists $M\in GL_n(K)$ such that $B=MC_0$. We just apply Lemma 25 to this M and $A=O_{K,p}$ (which has only two prime ideals) to conclude.

REMARK 1. The above computation should be first performed for $\mathbf{K}(z_1, z_2, \phi, \mathscr{B}, s)$ and then evaluated at $s=0$. As this does not create any problem, we shall content ourselves with formal computations in the rest of this paragraph.

REMARK 2. Replacing $\mathscr{B}$ in Theorem 23 by $\bigcup_{B\in\mathscr{B}}\mathscr{C}(B)$, we see that we can suppose that all elements of $\mathscr{B}$ are bases of $\mathbf{d}_{H,\mathfrak{p}}^{-1}O_{H,p}$ over $O_{K,p}$. On the other hand if $\mathscr{B}_1$ and $\mathscr{B}_2$ are finite sets of bases of H over K satisfying Theorem 23, then setting $S_2(\mathscr{B}_1\cup\mathscr{B}_2)=S_2(\mathscr{B}_1)\cup S_2(\mathscr{B}_2)$ and $S_2'(\mathscr{B}_1\cup\mathscr{B}_2)=S_2'(\mathscr{B}_1)\cup S_2'(\mathscr{B}_2)$, we see that $\mathscr{B}_1\cup\mathscr{B}_2$ also satisfies Theorem 23. Hence, it is enough to treat the case where $\mathscr{B}=B$ and $B=(f_1,\dots,f_n)$ is a basis of $\mathbf{d}_{H,\mathfrak{p}}^{-1}O_{H,p}$ over $O_{K,p}$ which we can take to be the B used in III, §1.

If $\mathbf{a}\in I(H)$, set $\tilde{\mathbf{a}}=\mathbf{d}_{H,\mathfrak{p}}^{-1}\mathbf{a}$. If ϕ belongs to $\mathscr{S}_T(H)$ with $T\cap|(p)|=\varnothing$, then $\tilde{\phi}$ is constant modulo $\tilde{\mathbf{a}}$ for some $\mathbf{a}\in I(H)$ satisfying $|\mathbf{a}|\subset T$; so by linearity, we are reduced to the case where $\tilde{\phi}$ is the characteristic function of $\alpha+\tilde{\mathbf{a}}$, where $|\mathbf{a}|\subset T$ and $\alpha\in\mathbf{d}_{H,\mathfrak{p}}^{-1}O'_{H,T}$. Let $g_B\colon \mathbf{C}^n\to\mathbf{C}^n$ be defined by $g_B(z)=(\mathrm{Tr}(f_1z),\dots,\mathrm{Tr}(f_nz))$. As B is a basis of $\mathbf{d}_{H,\mathfrak{p}}^{-1}O_{H,p}$ over $O_{K,p}$, the image of $\mathbf{d}_{H,p}^{-1}$ by g_B is a lattice L contained in $(O_{K,p})^n$ such that $O_{K,p}L=(O_{K,p})^n$ and so contains $(\delta_BO_K)^n$ for some $\delta_B\in O_K$ relatively prime to p. There exists $\delta_{\mathbf{a}}\in K^*$ with $|(\delta_{\mathbf{a}})|\subset T_K$ such that $\mathbf{a}$ contains $\delta_{\mathbf{a}}O_H$. Hence, if we set $\delta_{\mathbf{a},B}=\delta_{\mathbf{a}}\delta_B$, we have $|(\delta_{\mathbf{a},B})|\subset T_K\cup|(\delta_B)|$ and $g_B(\tilde{\mathbf{a}})$ contains $(\delta_{\mathbf{a},B}O_K)^n$. Let Y be a set of representatives of $g_B(\tilde{\mathbf{a}})$ modulo $(\delta_{\mathbf{a},B}O_K)^n$. Using the identity $(z_1\mid z_2)_\infty=\Pi_{i=1}^n(\mathrm{Tr}\, f_iz_1\mid \mathrm{Tr}\, f_i^\vee z_2)_\infty$, we obtain

$$\mathbf{K}(z_1, z_2, \tilde{\phi}, B)=\frac{\det B}{\sqrt{N(\mathbf{d}_{H,\mathfrak{p}})}} \sum_{\substack{y\in Y\\ y=(y_1,\dots,y_n)}} \prod_{j=1}^{n} \mathbf{K}(y_j+\mathrm{Tr}(f_j(z_1+\alpha)),\ \mathrm{Tr}(f_j^\vee z_2),\ \delta_{\mathbf{a},B}O_K). \tag{31}$$

On the other hand, a straightforward computation yields

$$\mathbf{K}(z_1, z_2, \tilde{\phi}_{\mathbf{b}_1,\mathbf{b}_2}, B) = \sum_{\substack{\beta_1 \in \mathbf{b}_1^{-1}\tilde{\mathbf{a}}/\tilde{\mathbf{a}} \\ \beta_1 \notin \tilde{\mathbf{a}}}} \sum_{\substack{\beta_2 \in \mathbf{b}_2^{-1}\bar{\mathbf{b}}_1\tilde{\mathbf{a}}^{\vee}/\bar{\mathbf{b}}_1\tilde{\mathbf{a}}^{\vee} \\ \beta_2 \notin \bar{\mathbf{b}}_1\tilde{\mathbf{a}}^{\vee}}} (-\beta_2 \,|\, \alpha + z_1)_\infty \mathbf{K}(z_1+\beta_1, z_2+\beta_2, \tilde{\phi}, B). \tag{32}$$

Now, by Lemma 5, we can find $S(B) \in \mathscr{P}(H)$ and $S'(B) \in \mathscr{P}(H^{\vee})$ such that if $(\mathbf{b}_1, \mathbf{b}_2) \in C_{T \cup |(p)|}(S(B), S'(B))$, then for all $\mathbf{a} \in I(H)$ with $|\mathbf{a}| \subset T$, all α in $\mathbf{d}_{\bar{H},\mathbf{p}}^{-1} O'_{H,T}$, all β_1, β_2 as above, we have

$$\mathrm{Tr}(f_i(\alpha+\beta_1)) \notin \delta_{\mathbf{a},B} O_K \quad \text{and} \quad \mathrm{Tr}(f_i^{\vee}\beta_2) \notin (\delta_{\mathbf{a},B} O_K)^{\vee}.$$

On the other hand $\mathrm{Tr}(f_i(\alpha+\beta_1))$ and $\mathrm{Tr}\, f_i^{\vee}\beta_2$ belong to $O_{K,p}$, so putting together formulae (31) and (32) we see that $\mathbf{K}(z_1, z_2, \tilde{\phi}_{\mathbf{b}_1,\mathbf{b}_2}, B)$ can be expressed in terms of the functions studied in Proposition 22. Thus part (i) of Theorem 23 is a direct consequence of (i) and (ii) of this proposition. To prove (ii), we can restrict ourselves to the case $\phi_1 = \chi_{\gamma_1}$ and $\phi_2 = \chi_{\gamma_2}$, since the χ_γ form a basis of the space of locally constant functions. Now, using Proposition 22(iii) along with formulae (31) and (32), we obtain that the Fourier–Laplace transform of $\chi_{\gamma_1}\chi_{\gamma_2}\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi,\mathscr{B}}$ is:

$$\mathscr{H}\left(\frac{1}{(2\pi i)^n}\left(-\hat{\gamma}_2 \left| \frac{z_1}{2\pi i}\right)_\infty \mathbf{K}\left(\hat{\gamma}_1 + \frac{z_1}{2\pi i}, \hat{\gamma}_2 + \frac{z_2}{2\pi i}, \tilde{\phi}_{\mathbf{b}_1\mathbf{b}_2}, \mathscr{B}\right)\right)\right), \tag{33}$$

where $\hat{\gamma}_1$ is a representative of γ_1 in $\mathbf{p}^{-\infty}\mathbf{a}\bar{\mathbf{b}}_2$ and $\hat{\gamma}_2$ is a representative of γ_2 in $\mathbf{p}^{-\infty}\bar{\mathbf{b}}_1(\mathbf{a}+(\alpha)+(\gamma_1)+d_{\bar{H},\bar{\mathbf{p}}}^{-1})^{\vee}$, from which we can deduce the result after a straightforward computation (the main ingredient being the fact that if $\omega \in \hat{\gamma}_1 + \alpha + \mathbf{b}_1^{-1}\tilde{\mathbf{a}}$, then $\chi_{\gamma_2}(\omega) = (\hat{\gamma}_2 \,|\, \omega)_\infty$).

4. *Complements to Shintani's method*

In this paragraph, we shall use the results of the preceding paragraph to prove that $\Lambda_{\mathscr{B},\beta_1\beta_2}(k, j, \phi)$ does not really depend on the choice of $\mathscr{B}$, β_1 or β_2.

THEOREM 26. *Let $\phi \in \mathscr{S}_{k,j,V}(H)$. We can define a number $\Lambda^?(k, j, \phi)$ such that*

(i) *For all $\mathscr{B} \in \mathscr{B}(V)$, there exists $S(\mathscr{B}) \in \mathscr{P}(H)$ and $S'(\mathscr{B}) \in \mathscr{P}(H^{\vee})$ such that $\Lambda_{\mathscr{B},\beta_1,\beta_2}(k,j,\phi) = \Lambda^?(k,j,\phi)$ for all $\phi \in \mathscr{S}_T(H)$ and all $((\beta_1),(\beta_2)) \in C_T^0(S(\mathscr{B}), S'(\mathscr{B}))$,*
(ii) $\Lambda^?(k, j, \phi) = \Lambda(k, j, \phi)$ *if either* $n = 1, 2$ or $n \geqslant 3$ *and* $k = 0$ *or* $j = 1$,
(iii) $\Lambda^?(k, j, \phi \circ \gamma) = N_{H/K}(\gamma)^j \overline{N_{H/K}(\gamma)}^{-k} \Lambda^?(k, j, \phi)$,
(iv) $\Lambda^?(k, j, \phi) = (-1)^{n(j-1)} i^n \Lambda^?(j-1, k+1, \mathscr{F}_H(\phi))$.

REMARK. Of course, we expect that $\Lambda^?(k, j, \phi)$ is always equal to $\Lambda(k, j, \phi)$. In this direction, (iii) and (iv) are functional equations also satisfied by $\Lambda(k, j, \phi)$ (formulae (4) and (16)).

Proof. Suppose $\phi \in \mathscr{S}_T(H)$. By linearity, we can restrict ourselves to the case $\phi = \phi_\chi$ for some locally constant character χ of O_T^*. Choose a prime p splitting in K such that $T \cap |(p)| = \varnothing$ and $|\mathbf{d}_H| \cap |(p)| = \varnothing$. Let $\mathscr{B} \in \mathscr{B}(V)$ and let

$$S(\mathscr{B}) = S_1(\mathscr{B}) \cup S_2(\mathscr{B}) \cup |(p)| \quad \text{and} \quad S'(\mathscr{B}) = S'_1(\mathscr{B}) \cup S'_2(\mathscr{B}) \cup |(p)|,$$

where $S_1(\mathscr{B})$ and $S'_1(\mathscr{B})$ are defined in Lemma 6 and $S_2(\mathscr{B})$ and $S'_2(\mathscr{B})$ are defined in Theorem 23.

If μ is a measure on $Y_{H,p}$ and $\gamma \in O_{H,p}^*$, we define a measure $\mu \circ \gamma$ on $Y_{H,p}$ and a measure $\pi(\mu)$ on $Y_{K,p} = O_\mathbf{p} \times O_{\bar{\mathbf{p}}}$ by the following formulae:

$$\int_{Y_{H,p}} f(y_1, y_2)\, \mathrm{d}(\mu \circ \gamma) = \int_{Y_{H,p}} f(\gamma y_1, \gamma^{-1} y_2)\, \mathrm{d}\mu, \tag{34}$$

$$\int_{Y_{K,p}} f(x_1, x_2)\, \mathrm{d}\pi(\mu) = \int_{Y_{H,p}} f(N(y_1), N(y_2))\, \mathrm{d}\mu. \tag{35}$$

LEMMA 27. *If* $(\mathbf{b}_1, \mathbf{b}_2) \in C_T^0(S(\mathscr{B}), S'(\mathscr{B}))$ *and* $\gamma \in O_{H,p}^+$ *satisfies*

$$|(\gamma)|_K \cap (|\mathbf{b}_1| \cup |\bar{\mathbf{b}}_2|) = \varnothing,$$

then

$$\pi(\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi\circ\gamma,\mathscr{B}}) = N_{H/K}(\gamma)(\pi(\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi,\mathscr{B}} \circ \gamma)).$$

Proof. To prove that two measures μ_1 and μ_2 on $Y_{K,p}$ are equal, it is sufficient to verify that $\int_{Y_{K,p}} x_1^i \psi(x_2)\, \mathrm{d}\mu_1 = \int_{Y_{K,p}} x_1^i \psi(x_2)\, \mathrm{d}\mu_2$ for all $i \in \mathbf{N}$ and all locally constant functions ψ on $O_{\bar{\mathbf{p}}}$. But we have

$$\int_{Y_{K,p}} x_1^i \psi(x_2)\, \mathrm{d}\pi(\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi\circ\gamma,\mathscr{B}}) = \int_{Y_{H,p}} N(y_1)^i \psi \circ N(y_2)\, \mathrm{d}\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi\circ\gamma,\mathscr{B}}, \tag{36}$$

and by Lemma 8, this is equal to ∇_1^i applied to the Fourier–Laplace transform of $\psi \circ N(y_2)\, \mathrm{d}\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi\circ\gamma,\mathscr{B}}$ and evaluated at $z_1 = z_2 = 0$. Now, as $\psi \circ N$ is locally constant, we can use Theorem 23(ii) to obtain (cf. formula (12))

$$\int_{Y_{K,p}} x_1^i \psi(x_2)\, \mathrm{d}\pi(\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi\circ\gamma,\mathscr{B}}) = \Lambda_{\mathscr{B}}(0, i+1, \psi \circ N * (\phi \circ \gamma)_{\mathbf{b}_1,\mathbf{b}_2}). \tag{37}$$

The same computation gives

$$N_{H/K}(\gamma)\int_{Y_{K,p}} x_1^i\psi(x)\,\mathrm{d}(\pi(\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi,\mathscr{B}}\circ\gamma))$$

$$= N_{H/K}(\gamma)^{i+1}\int_{Y_{H,p}} N(y_1)^i\psi(N(\gamma^{-1}y_2))\,\mathrm{d}\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi,\mathscr{B}}$$

$$= N_{H/K}(\gamma)^{i+1}\Lambda_{\mathscr{B}}(0,\, i+1,\, (\psi'*\phi)_{\mathbf{b}_1,\mathbf{b}_2}), \tag{38}$$

where $\psi'(y_2)=\psi(N(\gamma^{-1}y_2))$.

Let $\phi'=(\psi'*\phi)_{\mathbf{b}_1,\mathbf{b}_2}$. Then $\psi\circ N*(\phi\circ\gamma)_{\mathbf{b}_1,\mathbf{b}_2}$ is neither more nor less than $\phi'\circ\gamma$. Now, using the corollary to Theorem 3, we obtain $\Lambda_{\mathscr{B}}(0,i+1,\phi')=\Lambda(0,i+1,\phi')$ and $\Lambda_{\mathscr{B}}(0,i+1,\phi'\circ\gamma)=\Lambda(0,i+1,\phi\circ\gamma)$, and the desired equality follows from formula (16).

COROLLARY 1. *Under the same hypothesis as in Lemma* 27, we have

$$\Lambda_{\mathscr{B},\beta_1,\beta_2}(k,j,\,\phi\circ\gamma) = N_{H/K}(\gamma)^j\overline{N_{H/K}(\gamma)}^{\,-k}\Lambda_{\mathscr{B},\beta_1,\beta_2}(k,j,\,\phi).$$

Proof. By the very definition of $\Lambda_{\mathscr{B},\beta_1,\beta_2}(k,j,\phi)$ (cf. (19)) and of $\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi,\mathscr{B}}$, we obtain, using Lemma 8,

$$\Lambda_{\mathscr{B},\beta_1,\beta_2}(k,j,\,\phi)=v_{\beta_1,\beta_2}(k,j,\,\chi)\int_{Y_{K,p}} x_1^{j-1}x_2^k\,\mathrm{d}\pi(\mu_{(\beta_1),(\beta_2),\phi,\mathscr{B}}), \tag{39}$$

and the result is an immediate consequence of Lemma 27.

COROLLARY 2. *Let* $((\beta_1),(\beta_2))$ *and* $((\beta_1'),(\beta_2'))$ *belong to* $C_T^0(S(\mathscr{B}),S'(\mathscr{B}))$. *Then*

$$\Lambda_{\mathscr{B},\beta_1,\beta_2}(k,j,\,\phi) = \Lambda_{\mathscr{B},\beta_1',\beta_2'}(k,j,\,\phi).$$

Proof. Up to introducing an auxiliary $((\beta_1''),(\beta_2''))\in C_T^0(S(\mathscr{B}),S'(\mathscr{B}))$, we may suppose

$$(|(\beta_1)|_K\cup|(\bar\beta_2)|_K)\cap(|(\beta_1')|_K\cup|(\bar\beta_2')|_K=\varnothing.$$

As

$$(\phi_{(\beta_1),(\beta_2)})_{(\beta_1'),(\beta_2')}=(\phi_{(\beta_1'),(\beta_2')})_{(\beta_1),(\beta_2)},$$

we have

$$\mu_{(\beta_1),(\beta_2),\phi_{(\beta_1'),(\beta_2')},\mathscr{B}}=\mu_{(\beta_1'),(\beta_2'),\phi_{(\beta_1),(\beta_2)},\mathscr{B}},$$

hence by formula (39) we have

$$\nu_{\beta_1,\beta_2}(k, j, \chi)^{-1}\Lambda_{\mathscr{B},\beta_1,\beta_2}(k, j, \phi_{(\beta'_1),(\beta'_2)})$$
$$= \nu_{\beta'_1,\beta'_2}(k, j, \chi)^{-1}\Lambda_{\mathscr{B},\beta'_1,\beta'_2}(k, j, \phi_{(\beta_1),(\beta_2)}).$$

We obtain the result using formula (15) and the previous corollary.

COROLLARY 3. $\Lambda^?(k,j,\phi)$ *does not depend on the choice of* $((\beta_1),(\beta_2))\in C^0_T(S(\mathscr{B}), S'(\mathscr{B}))$.

It remains to check that $\Lambda^?(k, j, \phi)$ is independent of the choice of $\mathscr{B}$ and this follows from the following lemma whose proof is identical to that of Lemma 27.

LEMMA 28. *Let* $\mathscr{B}_1, \mathscr{B}_2\in\mathscr{B}(V)$ *and* $S=S(\mathscr{B}_1)\cup S(\mathscr{B}_2)$, $S'=S'(\mathscr{B}_1)\cup S'(\mathscr{B}_2)$. *If* $\phi\in\mathscr{S}_T(H)$ *and* $((\beta_1),(\beta_2))\in C^0_T(S,S')$ *then* $\pi(\mu_{(\beta_1),(\beta_2),\phi,\mathscr{B}_1})=\pi(\mu_{(\beta_1),(\beta_2),\phi,\mathscr{B}_2})$.

This concludes the proof of (i). Now (ii) is a consequence of the corollary of Theorem 3, while (iii) follows from Corollary 1 of Lemma 27 and (iv) from Theorem 3(v).

IV. Special values of Hecke *L*-functions

Let ψ be a Hecke character of H (i.e. a continuous $\mathbf{C}^*$-valued character of $\mathbf{A}^*_H/H^*$). Let $\mathbf{m}_\psi$ be the conductor of ψ. We can associate to ψ a character of $I_{\mathbf{m}_\psi}(H)$, still denoted by ψ, by the formula: if $\mathbf{q}\in P(H)-|\mathbf{m}_\psi|$, then $\psi(\mathbf{q})=\psi((1,\ldots,1,w_{\mathbf{q}}^{-1},1,\ldots,1))$, where $w_{\mathbf{q}}$ is a uniformizing parameter of $O_{\mathbf{q}}$. If ψ is a Hecke character of H, let $\psi^\vee$ be the Hecke character of $H^\vee$ defined by $\psi^\vee(\mathbf{a})=N(\mathbf{a})^{-1}\psi(\bar{\mathbf{a}}^{-1})$ if $\mathbf{a}\in I_{\bar{\mathbf{m}}_\psi}(H^\vee)$.

A Hecke character of H will be called K-admissible if there exists $k(\psi)\in\mathbf{N}$ and $j(\psi)\in\mathbf{N}-\{0\}$ such that for all $\alpha\equiv 1 \pmod{\mathbf{m}_\psi}$,

$$\psi((\alpha)) = \overline{N_{H/K}(\alpha)}^{k(\psi)}N_{H/K}(\alpha)^{-j(\psi)},$$

In particular, a K-admissible Hecke character is of type A_0 and critical in the sense of Deligne (cf. [D]). If ψ is K-admissible, so is $\psi^\vee$ and we have $k(\psi^\vee)=j(\psi)-1$ and $j(\psi^\vee)=k(\psi)+1$.

If ψ is a Hecke character of H and $S\in\mathscr{P}(H)$ contains $|\mathbf{m}_\psi|$, we set

$$L_S(\psi, s) = \sum_{\mathbf{b}\in I_S^+(H)} \frac{\psi(\mathbf{b})}{N(\mathbf{b})^s}, \tag{40}$$

and if $\dot{\mathbf{a}} \in Cl(O_H)$, we set

$$L_S(\psi, \dot{\mathbf{a}}, s) = \sum_{\mathbf{b} \in I_S^+(H) \cap \dot{\mathbf{a}}} \frac{\psi(\mathbf{b})}{N(\mathbf{b})^s}. \tag{41}$$

These two series converge for $\mathrm{Re}(s) \gg 0$ and define functions of s possessing meromorphic continuations to the whole s-plane, holomorphic if ψ is K-admissible. If ψ is a K-admissible Hecke character, we set

$$\Lambda_S(\psi) = \frac{\Gamma(j(\psi))^n}{(2\pi i)^{nj(\psi)}} L_S(\psi, 0) \quad \text{and} \quad \Lambda_S(\psi, \dot{\mathbf{a}}) = \frac{\Gamma(j(\psi))^n}{(2\pi i)^{nj(\psi)}} L_S(\psi, \dot{\mathbf{a}}, 0), \tag{42}$$

and if $S = |\mathbf{m}_\psi|$, we drop it from the notations.

If $\mathbf{q} \in P(H)$, let $\psi_{\mathbf{q}} \in \mathscr{S}_{|\mathbf{q}|,H}$ be defined by

$$\psi_{\mathbf{q}}(x_{\mathbf{q}}) = \begin{cases} \psi((1, \ldots, 1, x_{\mathbf{q}}, 1, \ldots, 1)) & \text{if } x_{\mathbf{q}} \in O_{\mathbf{q}}^* \\ 0 & \text{if } x_{\mathbf{q}} \notin O_{\mathbf{q}}^*. \end{cases} \tag{43}$$

Hence, if $\mathbf{q} \notin |\mathbf{m}_\psi|$, we have $\psi_{\mathbf{q}} = \delta_{\mathbf{q}}$. Let $\omega_{\mathbf{q}}$ be a uniformizing parameter of $O_{\mathbf{q}}$ and $a_{\mathbf{q}} = v_{\mathbf{q}}(\mathbf{m}_\psi \mathbf{d}_H)$. Let us view $H_{\mathbf{q}}^*$ as a subgroup of $\mathbf{A}_H^*$ in the obvious way, so that $\psi(\omega_{\mathbf{q}})$ has a well defined meaning.

LEMMA 29. (i) *There exists a constant $W_{\mathbf{q}}(\psi)$ (the local root number of ψ at $\mathbf{q}$) independent of the choice of $\omega_{\mathbf{q}}$ such that*

$$\mathscr{F}_{\bar{\mathbf{q}}}(\psi_{\bar{\mathbf{q}}}^\vee)(x) = \begin{cases} W_{\mathbf{q}}(\psi)\psi(\omega_{\mathbf{q}}^{-a_{\mathbf{q}}})\psi_{\mathbf{q}}(\omega_{\mathbf{q}}^{a_{\mathbf{q}}}x) & \text{if } \mathbf{q} \in |\mathbf{m}_\psi| \\ W_{\mathbf{q}}(\psi)\psi(\omega_{\mathbf{q}}^{-a_{\mathbf{q}}})\delta_{\mathbf{q}}^\vee(\omega_{\mathbf{q}}^{a_{\mathbf{q}}}x) & \text{if } \mathbf{q} \notin |\mathbf{m}_\psi|. \end{cases}$$

Moreover, we have

(ii) $W_{\mathbf{q}}(\psi) = 1$ *if* $\mathbf{q} \notin |\mathbf{m}_\psi \mathbf{d}_H|$.
(iii) $W_{\mathbf{q}}(\psi) W_{\bar{\mathbf{q}}}(\psi^\vee) = \psi_{\mathbf{q}}(-1)$.

Proof. Everything follows from standard computations (cf. [L]).

The following observation is an easy consequence of the weak approximation theorem for the multiplicative group.

OBSERVATION. Let $\Theta(k, j, \cdot)$: $\mathscr{S}_H \to \mathbf{C}$ be a map satisfying formula (16). Let $S \in \mathscr{P}(\mathscr{H})$ contain $|\mathbf{m}_\psi|$ and $\phi_S \in \mathscr{S}_{S,H}$ satisfy $\phi_S \circ b = \psi(b)\phi_S$ for all $b \in O_S^*$. For $\mathbf{a} \in I_S(H)$ set $\phi_{S,\mathbf{a}}(x) = \phi_S(x_S)\Pi_{\mathbf{q} \notin S} 1_{\mathbf{a}_{\mathbf{q}}^{-1}}(x_{\mathbf{q}})$ where $1_{\mathbf{a}_{\mathbf{q}}^{-1}}$ is the characteristic function of the fractional ideal of $H_{\mathbf{q}}$ generated by $\mathbf{a}^{-1}$. Finally, if $A \subset I_S(H)$ is a set of representatives for $Cl(O_H)$ set $\phi_{S,A} = \Sigma_{\mathbf{a} \in \mathbf{A}} \psi(\mathbf{a})\phi_{S,\mathbf{a}}$. Then we have:

(i) $\Theta(k(\psi), j(\psi), \phi_{S,A})$ is independent of the choice of A.
(ii) $\Theta(k(\psi), j(\psi), \phi_{S,A} \circ b) = \psi(b)\Theta(k(\psi), j(\psi), \phi_{S,A})$ for all $b \in (\mathbf{A}_H^f)^*$.

Whenever it is defined, we have, with obvious notations

$$\delta_{\mathbf{b}} * \phi_{S,A} = \phi_{S,A} - \phi_{S,A\mathbf{b}^{-1}} \quad \text{and} \quad \delta_{\mathbf{c}}^{\vee} * \phi_{S,A} = \phi_{S,A} - N(\mathbf{c})^{-1}\phi_{S,A\bar{\mathbf{c}}}, \tag{44}$$

from which we deduce, using the fact that multiplication by an ideal induces a bijection on $Cl(O_H)$, that we have

$$\begin{aligned} &\Theta(k(\psi), j(\psi), \mathop{*}_{i=1}^{k} \delta_{\mathbf{b}_i} \mathop{*}_{j=1}^{l} \delta_{\mathbf{c}_j}^{\vee} * \phi_{S,A}) \\ &= \prod_{i=1}^{k} (1-\psi(\mathbf{b}_i)) \prod_{j=1}^{l} (1-\psi^{\vee}(\mathbf{c}_j))\Theta(k(\psi), j(\psi), \phi_{S,A}), \end{aligned} \tag{45}$$

whenever everything is defined.

If $T \subset \mathscr{P}(H)$, we define the Euler factor of ψ above T by

$$E_T(\psi) = \prod_{\mathbf{q} \in T - |\mathbf{m}_\psi|} (1-\psi(\mathbf{q})), \tag{46}$$

the local root number of ψ above T by

$$W_T(\psi) = \prod_{\mathbf{q} \in T} W_{\mathbf{q}}(\psi) \tag{47}$$

and

$$\omega_T = \prod_{\mathbf{q} \in T} \omega_{\mathbf{q}}^{a_{\mathbf{q}}} \in H_T^*. \tag{48}$$

If $S \in \mathscr{P}(H)$ contains $|\mathbf{m}_\psi|$, let $\psi_S \in \mathscr{S}_{S,H}$ be defined by $\psi_S(x) = \Pi_{\mathbf{q} \in S}\psi_{\mathbf{q}}(x_{\mathbf{q}})$. We have $\psi_S \circ b = \psi(b)\psi_S$ for all $b \in O_S^*$, so we choose a set $A \in I_S(H)$ of representatives for $Cl(O_H)$ and we set $\Theta_S(\psi) = \Theta(k(\psi), j(\psi), \psi_{S,A})$, $\Theta(\psi) = \Theta_{|\mathbf{m}_\psi|}(\psi)$. As everything we said about $\Theta(k, j, \cdot)$ applies to the map $\Lambda(k, j, \cdot)$, we get two different definitions for $\Lambda_S(\psi)$ (cf. formula (42)). But, if $\mathbf{a} \in I_S(H)$ is in the ideal class $\dot{\mathbf{a}}$, writing $\mathbf{b} \in I_S^+(H) \cap \dot{\mathbf{a}}$ in the form $\mathbf{b} = (\beta)\mathbf{a}$, where $\beta \in \mathbf{a}^{-1}$ is uniquely determined modulo U_H, we see that $\Lambda_S(\psi, \dot{\mathbf{a}})$ is neither more nor less than $\psi(\mathbf{a})\Lambda(k(\psi), j(\psi), \psi_{S,\mathbf{a}})$, which means that the two definitions coincide.

If $S_0 \subset S$, we define $\psi_{S,S_0} \in \mathscr{S}_{S,H}$ by

$$\psi_{S,S_0}(x) = \prod_{\mathbf{q} \in S - S_0} \psi_{\mathbf{q}}(x_{\mathbf{q}}) \prod_{\mathbf{q} \in S_0} \mathscr{F}_{\bar{\mathbf{q}}}(\psi_{\bar{\mathbf{q}}}^{\vee})(x_{\mathbf{q}}). \tag{49}$$

LEMMA 30. (i) $\Theta_S(\psi) = E_S(\psi)\Theta(\psi)$.

(ii) *We have* $\psi_{S,S_0} \circ b = \psi(b)\psi_{S,S_0}$ *for all* $b \in O_S^*$ *and if* $A \in I_S(H)$ *is a set of representatives for* $Cl(O_H)$, *then*

$$\Theta(k(\psi), j(\psi), \psi_{S,S_0,A}) = W_{S_0}(\psi)E_{\bar{S}_0}(\psi^\vee)E_{S-S_0}(\psi)\Theta(\psi).$$

Proof. (i) We have $\psi_{S,A} = *_{\mathbf{q} \in S - |\mathbf{m}_\psi|}\, \delta_\mathbf{q} * \psi_{\mathbf{m}_\psi, A}$, so the result is an immediate consequence of formula (45).

(ii) Using Lemma 29, we obtain

$$\psi_{S,S_0,A} = W_{S_0}(\psi)\psi(\omega_{S_0}^{-1})\left(\underset{\mathbf{q} \in S-(S_0 \cup |\mathbf{m}_\psi|)}{*} \delta_\mathbf{q} \underset{\mathbf{q} \in S_0 - |\mathbf{m}_\psi|}{*} \delta_{\bar{\mathbf{q}}}^\vee * \psi_{\mathbf{m}_\psi, A}\right) \circ \omega_{S_0}, \tag{50}$$

from which everything follows easily using formula (45).

Let us define the global root number $W(\psi)$ by

$$W(\psi) = (-1)^{nk(\psi)} \prod_{\mathbf{q} \in |\mathbf{m}_\psi \mathbf{d}_H|} W_\mathbf{q}(\psi).$$

Suppose that $\Theta(k, j, \cdot)$ satisfies the following functional equation:

$$\Theta(k, j, \phi) = (-1)^{n(j-1)} i^n \Theta(j-1, k+1, \mathscr{F}_H(\phi)). \tag{51}$$

for all $j \geqslant 1$, $k \geqslant 0$ and all $\phi \in \mathscr{S}(H)$. Then we have

LEMMA 31. $W(\psi)\Theta(\psi) = i^{-n}\Theta(\psi^\vee)$.

Proof. Let $\mathbf{a} \in I_{|\mathbf{m}_\psi \mathbf{d}_H|}(H)$. Using Lemma 29, we obtain

$$\mathscr{F}_{H^\vee}(\psi_{\bar{m}_\psi, \bar{\mathbf{a}}}^\vee) = N(\mathbf{a}) \,.\, W_{|\mathbf{m}_\psi \mathbf{d}_H|}(\psi) \,.\, \psi(\omega_{|\mathbf{m}_\psi \mathbf{d}_H|}^{-1}) \,.\, \psi_{\mathbf{m}_\psi, \mathbf{a}^{-1}} \circ \omega_{|\mathbf{m}_\psi \mathbf{d}_H|}, \tag{52}$$

from which we obtain, with obvious notations

$$\mathscr{F}_{H^\vee}(\psi_{\bar{\mathbf{m}}_\psi, \bar{A}}^\vee) = W_{|\mathbf{m}_\psi \mathbf{d}_H|}(\psi) \,.\, \psi(\omega_{|\mathbf{m}_\psi \mathbf{d}_H|}^{-1}) \,.\, \psi_{\mathbf{m}_\psi, A^{-1}} \circ \omega_{|\mathbf{m}_\psi \mathbf{d}_H|}. \tag{53}$$

The lemma follows easily, using formula (51).

This lemma applied to $\Lambda(k, j, \cdot)$ is nothing else than Hecke's functional equation. But, we can also apply everything to $\Lambda^?(k, j, \cdot)$. The results that we obtain in this way are summarized in the following proposition.

PROPOSITION 32. (i) *If* $\mathbf{a} \in I_S(H)$, *we set* $\Lambda_S^?(\psi, \mathbf{a}) = \psi(\mathbf{a})\Lambda^?(k(\psi), j(\psi), \psi_{S,\mathbf{a}})$. *Then* $\Lambda_S^?(\psi, \mathbf{a})$ *depends only on the image of* $\mathbf{a}$ *in* $Cl(O_H)$.

If $A \subset I_S(H)$ *is a set of representatives of* $Cl(O_H)$, *set* $\Lambda_S^?(\psi) = \Sigma_{\mathbf{a} \in A} \Lambda_S^?(\psi, \mathbf{a})$. *Then we have*:

(ii) $\Lambda^?_S(\psi)=E_S(\psi)\Lambda^?(\psi)$.
(iii) $\Sigma_{\mathbf{a}\in A}\,\psi(\mathbf{a})\Lambda^?(k(\psi),j(\psi),\psi_{S,S_0,\mathbf{a}})=(\Pi_{\mathbf{q}\in S_0}W_{\mathbf{q}}(\psi))E_{\bar{S}_0}(\psi^\vee)E_{S-S_0}(\psi)\Lambda^?(\psi)$.
(iv) $W(\psi)\Lambda^?(\psi)=i^{-n}\Lambda^?(\psi^\vee)$.
(v) $\Lambda^?(\psi)=\Lambda(\psi)$ *if* $n=1,2$ *or* $n\geqslant 3$ *and* $k(\psi)=0$ *or* $j(\psi)=1$.

V. p-Adic measures on Galois groups and p-adic L-functions

1. Preliminary constructions

Let ψ be a Hecke character of H of type A_0 and conductor $\mathbf{m}_\psi$. We can associate to ψ a unique continuous character $\psi^{(p)}$ with values in $\mathbf{C}_p^*$ satisfying $\psi^{(p)}(\mathbf{a})=\psi(\mathbf{a})$ for any $\mathbf{a}\in I_{\mathbf{m}_\psi(p)}(H)$ (cf. [W1]). But, as $\psi^{(p)}$ is trivial on the connected component of 1 of $\mathbf{A}_H^*/H^*$, it can be interpreted as a character of $\mathrm{Gal}(H^{ab}/H)$. In fact $\psi^{(p)}$ factors through $\mathscr{G}_{H,\mathbf{m},p}=\mathrm{Gal}(H_{\mathbf{m}(p)^\infty}/H)$, where $\mathbf{m}$ is the prime-to-p part of $\mathbf{m}_\psi$ and $H_{\mathbf{m}(p)^\infty}$ is the union of all abelian extensions of H of level $\mathbf{m}(p)^k$ for $k\geqslant 0$. We shall say that ψ is $\mathbf{p}$-admissible if it is K-admissible and $\psi^{(p)}$ factors through $\mathscr{G}_{H,\mathbf{m},\mathbf{p}}=\mathrm{Gal}(H_{\mathbf{mp}^\infty}/H)$ (note that this is equivalent to $k(\psi)=0$ and $\psi_{\bar{\mathbf{p}}}=1$ on $O^*_{|\bar{\mathbf{p}}|}$). Let us choose a set $A\subset I_{|\mathbf{m}(p)|}(H)$ of representatives of $Cl(O_H)$. We have the following isomorphisms of topological spaces:

$$\mathscr{G}_{H,\mathbf{m},p}\simeq A\times((O_H/\mathbf{m})^*\times Y^*_{H,p})/\overline{U_H}$$

and

$$\mathscr{G}_{H,\mathbf{m},\mathbf{p}}\simeq A\times((O_H/\mathbf{m})^*\times Y^*_{H,\mathbf{p}})/\overline{U_H},$$

where $\overline{U_H}$ denotes the topological closure of U_H in the space considered. If f is a function on $\mathscr{G}_{H,\mathbf{m},p}$ (resp. on $\mathscr{G}_{H,\mathbf{m},\mathbf{p}}$), let $\tilde{f}$ be the function on $A\times(O_H/\mathbf{m})^*\times Y^*_{H,p}$ (resp. $A\times(O_H/\mathbf{m})^*\times Y^*_{H,\mathbf{p}}$) obtained by composing with the projection modulo $\overline{U_H}$.

Choose a torsion free subgroup V of finite index of the subgroup of U_H of elements of norm 1 over K and $\mathscr{B}\in\mathscr{B}(V)$. Let $T\in\mathscr{P}(H)$ contain $|\mathbf{m}|$, $|(p)|$ and $|\mathbf{a}|$ for all $\mathbf{a}\in A$. If $\alpha\in(O_{H,|\mathbf{m}|})^*$ and $\mathbf{a}\in A$, let $\phi_{\alpha,\mathbf{a}}\in\mathscr{S}_T(H)$ be the function defined by

$$\phi_{\alpha,\mathbf{a}}(x)=\phi_{\alpha,|\mathbf{m}|}(x_{|\mathbf{m}|})\cdot\phi_{\mathbf{p}}(x_{\mathbf{p}})\prod_{\mathbf{q}\notin|\mathbf{mp}|}1_{\mathbf{a}^{-1}}(x_{\mathbf{q}}),\tag{61}$$

where $\phi_{\alpha,|\mathbf{m}|}(x_{|\mathbf{m}|})=1$ if $x_{|\mathbf{m}|}\in\alpha+\mathbf{m}O_{|\mathbf{m}|}$ and 0 otherwise, and $\phi_{\mathbf{p}}$ is the function defined in III §3.

For all $(\mathbf{b}_1,\mathbf{b}_2)\in C_T(S(\mathscr{B}),S'(\mathscr{B}))$, where $S(\mathscr{B})$ and $S'(\mathscr{B})$ are as defined in Theorem 26 we define a measure $\lambda_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}}$ on $\mathscr{G}_{H,\mathbf{m},\mathbf{p}}$ and a measure $\mu_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}}$ on

$\mathscr{G}_{H,\mathbf{m},p}$ by the formulae:

$$\int_{\mathscr{G}_{H,\mathbf{m},\mathbf{p}}} f\,\mathrm{d}\lambda_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}} = \frac{1}{[U_H : V]} \sum_{\mathbf{a}\in A} \sum_{\alpha\in(O_H/\mathbf{m})^*} \int_{Y_1^* \times Y_2} \tilde{f}(\mathbf{a}, \alpha, y_1)\,\mathrm{d}\tilde{\mu}_{\mathbf{b}_1,\mathbf{b}_2,\phi_{\alpha,\mathbf{a}},\mathscr{B}}, \tag{62}$$

$$\int_{\mathscr{G}_{H,\mathbf{m},p}} f\,\mathrm{d}\mu_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}} = \frac{1}{[U_H : V]} \sum_{\mathbf{a}\in A} \sum_{\alpha\in(O_H/\mathbf{m})^*} \int_{Y_1^* \times Y_2^*} \tilde{f}(\mathbf{a}, \alpha, y_1, y_2)\,\mathrm{d}\tilde{\mu}_{\mathbf{b}_1,\mathbf{b}_2,\phi_{\alpha,\mathbf{a}},\mathscr{B}}, \tag{63}$$

where $\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi,\mathscr{B}}$ is the measure constructed in Theorem 23, and if μ is a measure on $Y_1^* \times Y_2$, then $\tilde{\mu}$ is the measure defined by

$$\int_{Y_1^* \times Y_2} f(y_1, y_2)\,\mathrm{d}\tilde{\mu} = \int_{Y_1^* \times Y_2} N(y_1)^{-1} f(y_1^{-1}, y_2)\,\mathrm{d}\mu. \tag{64}$$

Let $v_{\mathbf{b}_1,\mathbf{b}_2}(\psi) = (1 - \psi(\mathbf{b}_1^{-1}))(1 - \psi^\vee(\mathbf{b}_2^{-1}))$.

PROPOSITION 33. (i) *$\lambda_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}}$ is the unique measure on $\mathscr{G}_{H,\mathbf{m},\mathbf{p}}$ such that*

$$\int_{\mathscr{G}_{H,\mathbf{m},\mathbf{p}}} \psi^{(p)}\,\mathrm{d}\lambda_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}} = v_{\mathbf{b}_1,\mathbf{b}_2}(\psi) E_{|\bar{\mathbf{p}}|}(\psi^\vee) W_{|\mathbf{p}|}(\psi) E_{|\mathbf{m}|}(\psi) \Lambda(\psi) \tag{65}$$

for all **p***-admissible Hecke characters of H of conductor dividing* $\mathbf{mp}^\infty$.

(ii) *$\mu_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}}$ is the unique measure on $\mathscr{G}_{H,\mathbf{m},p}$ such that*

$$\int_{\mathscr{G}_{H,\mathbf{m},\mathbf{p}}} \psi^{(p)}\,\mathrm{d}\mu_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}} = v_{\mathbf{b}_1,\mathbf{b}_2}(\psi) E_{|\bar{\mathbf{p}}|}(\psi^\vee) W_{|\mathbf{p}|}(\psi) E_{|\mathbf{m}\bar{\mathbf{p}}|}(\psi) \Lambda(\psi) \tag{66}$$

for all K-admissible Hecke characters of H of conductor dividing $\mathbf{m}(p)^\infty$ *satisfying* $k(\psi)=0$ *or* $j(\psi)=1$.

(iii) *Moreover, if we do not assume* $k(\psi)=0$ *or* $j(\psi)=1$, *then*

$$\int_{\mathscr{G}_{H,\mathbf{m},p}} \psi^{(p)}\,\mathrm{d}\mu_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}} = v_{\mathbf{b}_1,\mathbf{b}_2}(\psi) E_{|\bar{\mathbf{p}}|}(\psi^\vee) W_{|\mathbf{p}|}(\psi) E_{|\mathbf{m}\bar{\mathbf{p}}|}(\psi) \Lambda^?(\psi). \tag{67}$$

COROLLARY. *If one can prove by any other method* (*for example using refinements of Harder's proof*) *that there exists a measure satisfying* (ii) *for all K-admissible* ψ, *then* $\Lambda(\psi)=\Lambda^?(\psi)$ *in all cases.* (*If H* is a *CM field,* Katz[K] *has constructed such a measure, but we have not checked the compatibility of his results with ours.*)

Proof. Let ψ be a K-admissible Hecke character. By definition of $\mu_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}}$, we have

$$\int_{\mathscr{G}_{H,\mathbf{m},p}} \psi^{(p)} \, d\mu_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}} = \frac{1}{[U_H : V]} \sum_{\mathbf{a}\in A} \psi(\mathbf{a}) \sum_{\alpha\in(O_H/m)^*} G(\alpha) \tag{68}$$

where

$$G(\alpha) = \psi_{|\mathbf{m}|}(\alpha) \int_{Y_{H,p}} \psi_{\bar{\mathbf{p}}}^{\vee}(y_1)\psi_{\bar{\mathbf{p}}}(y_2) N(y_1)^{j(\psi)-1} N(y_2)^{k(\psi)} \, d\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi_{\alpha,\mathbf{a}},\mathscr{B}}. \tag{69}$$

Now, using Theorem 23(ii) and formula (48), we obtain that the Fourier–Laplace transform of

$$\frac{1}{[U_H : V]} \sum_{\alpha\in(O_H/\mathbf{m})^*} \psi_{|\mathbf{m}|}(\alpha)\psi_{\bar{\mathbf{p}}}^{\vee}(y_1)\psi_{\bar{\mathbf{p}}}(y_2)\mu_{\mathbf{b}_1,\mathbf{b}_2,\phi_{\alpha,\mathbf{a}},\mathscr{B}}$$

is

$$\frac{1}{[U_H : V]} \mathscr{H}\left(\frac{1}{2\pi i}\mathbf{K}\left(\frac{z_1}{2\pi i}, \frac{z_2}{2\pi i}, (\psi_{|\mathbf{m}(p)|,|\mathbf{p}|,\mathbf{a}})_{\mathbf{b}_1,\mathbf{b}_2,\mathscr{B}}\right)\right), \tag{70}$$

and we can deduce (iii) from Lemma 8 and Proposition 32(ii). Then (ii) follows from the fact that $\Lambda^?(\psi) = \Lambda(\psi)$ if $k(\psi) = 0$ or $j(\psi) = 1$ and (i) is obtained in exactly the same way as (iii). The unicity of $\lambda_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}}$ and $\mu_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}}$ is due to the fact that the subspace of the space of continuous functions on $\mathscr{G}_{H,\mathbf{m},p}$ (resp. $\mathscr{G}_{H,\mathbf{m},\mathbf{p}}$) generated by the $\psi^{(p)}$ with $k(\psi) = 0$ and $j(\psi) = 1$ is dense (we are allowed to multiply by any locally constant character).

2. *Measures and pseudo-measures on profinite abelian groups*

In order to put the results of the preceding paragraph in a more satisfactory form, we shall shift to the language of pseudo-measures. In this paragraph, we shall collect from [[*Se*]] the definitions and some basic facts about pseudo-measures.

Let G be a profinite abelian group and Λ be a closed subring of $\hat{O}$. We define the Iwasawa algebra $\Lambda[[G]]$ of G as $\varprojlim \Lambda[[G/H]]$ where H runs through the open subgroups of G. Then $\Lambda[[G]]$ is a dense subalgebra of $\Lambda[[G]]$ and we have a canonical isomorphism between $\Lambda[[G]]$ and the algebra of Λ-valued measures on G, the multiplication in $\Lambda[[G]]$ corresponding to convolution of measures. This will enable us to view a Λ-valued measure on G as an element of $\Lambda[[G]]$. For example, the measure associated to $g \in G$ is the Dirac measure at g.

Let $X(G)$ be the group of continuous $\mathbf{C}_p^*$-valued homomorphisms of G endowed with the topology of uniform convergence. If $\chi \in X(G)$ and $\mu \in \Lambda[\![G]\!]$, we write $\langle \chi, \mu \rangle$ instead of $\int_G \chi \, \mathrm{d}\mu$ and let $\chi\mu \in \Lambda[\![G]\!]$ be defined by $\langle \psi, \chi\mu \rangle = \langle \psi\chi, \mu \rangle$. Then we have $\langle \chi, \mu\lambda \rangle = \langle \chi, \mu \rangle \langle \chi, \lambda \rangle$ and $\chi(\mu\lambda) = (\chi\mu)(\chi\lambda)$.

Suppose from now on that G has a quotient isomorphic to $\mathbf{Z}_p$ and let $\Gamma \subset G$ be a lifting of $\mathbf{Z}_p$. Let $\Lambda'[\![G]\!]$ be the total fraction ring of $\Lambda[\![G]\!]$ (i.e. the ring of $\alpha^{-1}\beta$ where α, β are elements of $\Lambda[\![G]\!]$ and α is not a zero divisor). If $\lambda = \alpha^{-1}\beta \in \Lambda'[\![G]\!]$ and $\chi \in X(G)$ satisfies $\langle \chi, \alpha \rangle \neq 0$, we set $\langle \chi, \lambda \rangle = \langle \chi, \alpha \rangle^{-1} \langle \chi, \beta \rangle$ and this depends only on λ, not on the particular decomposition of λ in the form $\alpha^{-1}\beta$. The map $\chi \to \langle \chi, \lambda \rangle$ is defined on a dense open subset of $X(G)$. If $\lambda \in \Lambda'[\![G]\!]$ and $\chi \in X(G)$ we can still define $\chi\lambda \in \Lambda'[\![G]\!]$ and we still have $\chi(\lambda\mu) = (\chi\lambda)(\chi\mu)$. An element $\lambda \in \Lambda'[\![G]\!]$ will be called a 'pseudo-measure' if $(1-g)\lambda \in \Lambda[\![G]\!]$ for all $g \in G$. We shall write $\tilde{\Lambda}[\![G]\!]$ for the space of pseudo-measures.

Let $\pi \colon G' \to G$ be a surjective morphism of profinite abelian groups. Then π induces a surjective morphism from $\Lambda[\![G']\!]$ to $\Lambda[\![G]\!]$ which can be prolonged in a unique way to a morphism from $\tilde{\Lambda}[\![G']\!]$ to $\tilde{\Lambda}[\![G]\!]$ in the following way. If $g \in G$, then $g-1$ is a zero divisor if and only if the topological closure of the subgroup generated by g in G has a finite p-Sylow subgroup; in particular if the image of g in $\mathbf{Z}_p$ is non-zero, then $g-1$ is not a zero divisor and the set of $g \in G$ such that $g-1$ is a zero divisor is contained in a closed subset with empty interior. So take $\lambda \in \tilde{\Lambda}[\![G]\!]$ and $g \in G'$ such that $\pi(g)-1$ is not a zero divisor and set $\pi(\lambda) = (\pi(g)-1)^{-1}\pi((g-1)\lambda)$. This clearly does not depend on the choice of g and defines a pseudo-measure on G.

LEMMA 34. (i) *If the p-Sylow subgroup of* G/Γ *is infinite, then* $\tilde{\Lambda}[\![G]\!] = \Lambda[\![G]\!]$, *or otherwise stated, all pseudo-measures are measures.*

(ii) *If* $\pi \colon G' \to G$ *is a surjective morphism of profinite abelian groups and* λ *is a pseudo-measure on* G' *such that* $\pi(\lambda)$ *is a measure, then* λ *itself is a measure.*

Proof. This follows easily from the structure of $\tilde{\Lambda}[\![G]\!]$ given in Th. 1.15 of [[*Se*]].

COROLLARY. *Suppose* G *has a quotient isomorphic to* $\mathbf{Z}_p^2$. *Let* $\chi_1, \ldots, \chi_n \in X(G)$ *and* $\lambda \in \Lambda'[\![G]\!]$ *be such that* $\forall (g_1, \ldots, g_n) \in G^n$, $\lambda \prod_{i=1}^n (1 - \chi_i(g_i)g_i) \in \Lambda[\![G]\!]$, *then* λ *is a measure.*

Proof. An immediate induction reduces the study to the case $n=1$. So let $\chi \in X(G)$ and $\lambda \in \Lambda'[\![G]\!]$ be such that $(1-\chi(g)g)\lambda \in \Lambda[\![G]\!]$ for all $g \in G$. We then find that $\chi^{-1}((1-\chi(g)g)\lambda) = (1-g)(\chi^{-1}\lambda)$ is a measure for all $g \in G$. As G has a quotient isomorphic to $\mathbf{Z}_p^2$ this implies by (i) that $\chi^{-1}\lambda$ is a measure, hence λ also.

3. *p-Adic L-functions*

If $S \in \mathscr{P}(H)$ satisfies $S \cap |(p)| = \varnothing$, let $\mathscr{G}_{H,S,p}$ (resp. $\mathscr{G}_{H,S,\mathbf{p}}$) be the Galois group over H of the union of all abelian extensions of H of level $\mathbf{m}$ with $|\mathbf{m}| \subset S \cup |(p)|$

(resp. $|\mathbf{m}| \subset S \cup |\mathbf{p}|$). If ρ denotes the complex conjugation on $\bar{K}$ induced by the embedding of $\bar{K}$ into $\mathbf{C}$, the map $\sigma \to \bar{\sigma}$ defined by $\bar{\sigma}(x) = \rho(\sigma(\rho(x)))$ induces a (canonical) isomorphism between $\mathscr{G}_{H,S,p}$ and $\mathscr{G}_{H^\vee,\bar{S},p}$. If $x \in A_H^*/H^*$, let $\sigma_x \in \mathrm{Gal}(H^{ab}/H)$ be its Artin symbol. If $\mathbf{b} \in I_{S \cup |(p)|}(H)$, let $\sigma_\mathbf{b} \in \mathscr{G}_{H,S,p}$ (resp. $\mathscr{G}_{H,S,\mathbf{p}}$) be the Artin symbol of the idele $(\ldots, x_\mathbf{q}, \ldots)$, where $v_\mathbf{q}(x_\mathbf{q}) = -v_\mathbf{q}(\mathbf{b})$, $x_\mathbf{q} = 1$, if $\mathbf{q} \in S \cup |(p)|$ and $\sigma_{-1} \in \mathscr{G}_{H,S,p}$ be the Artin symbol of $(\ldots, x_\mathbf{q}, \ldots)$ where $x_\mathbf{q} = -1$ if $\mathbf{q} \in |\bar{\mathbf{p}}|$ and $x_\mathbf{q} = 1$ otherwise. If $\mathbf{b} \in I_{S \cup |(p)|}(H)$, we have $\sigma_{\bar{\mathbf{b}}} = \overline{\sigma_\mathbf{b}}$ in $\mathscr{G}_{H^\vee,\bar{S},p}$. Let N be the cyclotomic character of $\mathscr{G}_{H,S,p}$ defined by $N(\sigma_\mathbf{b}) = N(\mathbf{b})$ and if χ is a $\mathbf{C}_p^*$-valued continuous character of $\mathscr{G}_{H,S,p}$, let $\chi^\vee$ be the character of $\mathscr{G}_{H^\vee,\bar{S},p}$ defined by $\chi^\vee(\sigma) = N(\sigma)^{-1}\chi(\bar{\sigma}^{-1})$.

If $(\mathbf{b}_1, \mathbf{b}_2) \in C_T(S(\mathscr{B}), S'(\mathscr{B}))$, we let $\mu_{\mathbf{b}_1,\mathbf{b}_2,S} \in I_{p,H}[\![\mathscr{G}_{H,S,p}]\!]$ and $\lambda_{\mathbf{b}_1,\mathbf{b}_2,S} \in I_{p,H}[\![\mathscr{G}_{H,S,\mathbf{p}}]\!]$ *be the* respective projective limits of the $\mu_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}}$ and $\lambda_{\mathbf{b}_1,\mathbf{b}_2,\mathbf{m}}$ defined in Proposition 33. If χ is a continuous $\mathbf{C}_p^*$-valued character of $\mathscr{G}_{H,S,\mathbf{p}}$ (resp. $\mathscr{G}_{H,S,p}$), we set

$$L_{\mathbf{p},S}(\chi) = [(1-\chi(\sigma_{\mathbf{b}_1})^{-1})(1-N(\mathbf{b}_2)\chi(\sigma_{\bar{\mathbf{b}}_2}))]^{-1} \int_{\mathscr{G}_{H,S,\mathbf{p}}} \chi \,\mathrm{d}\lambda_{\mathbf{b}_1,\mathbf{b}_2,S},$$

and

$$L_{p,S}(\chi) = [(1-\chi(\sigma_{\mathbf{b}_1})^{-1})(1-\chi^\vee(\sigma_{\mathbf{b}_2})^{-1})]^{-1} \int_{\mathscr{G}_{H,S,p}} \chi \,\mathrm{d}\mu_{\mathbf{b}_1,\mathbf{b}_2,S}.$$

$L_{\mathbf{p},S}$ and $L_{p,S}$ are independent of the choice of $(\mathbf{b}_1, \mathbf{b}_2)$ as can easily be deduced from Proposition 33. We can now state our main result:

THEOREM 35. (i) *$L_{p,S}(\chi)$ is an Iwasawa function of χ, i.e. there exists a (unique) measure μ_S on $\mathscr{G}_{H,S,p}$ such that $L_{p,S}(\chi) = \int_{\mathscr{G}_{H,S,p}} \chi\,\mathrm{d}\mu_S$.*

(ii) *If ψ is a K-admissible Hecke character of conductor $\mathbf{m}_\psi$ satisfying $|\mathbf{m}_\psi| \subset S \cup |(p)|$, then $L_{p,S}(\psi^{(p)}) = E_{|\bar{\mathbf{p}}|}(\psi^\vee)E_{S\cup|\bar{\mathbf{p}}|}(\psi)W_{|\mathbf{p}|}(\psi)\Lambda^?(\psi)$.*

(iii) *If the conductor of χ is divisible by all elements of S, then there exists a p-adic unit $W^{(p)}(\chi)$ such that*

$$W^{(p)}(\chi)L_{p,S}(\chi) = \chi(\sigma_{-1})L_{p,\bar{S}}(\chi^\vee).$$

Moreover, if ψ is an admissible Hecke character, then

$$W^{(p)}(\psi^{(p)}) = i^n \prod_{\mathbf{q} \in S \cup |\mathbf{d}_H| - |(p)|} W_\mathbf{q}(\psi).$$

(iv) *There exists a (unique) pseudo-measure λ_S on $\mathscr{G}_{H,S,\mathbf{p}}$ such that*

$$L_{\mathbf{p},S}(\chi) = \int_{\mathscr{G}_{H,S,\mathbf{p}}} \chi\,\mathrm{d}\lambda_S,$$

and λ_S is a measure if $S \neq \varnothing$ or if the $\mathbf{p}$*-adic regulator $R_{\mathbf{p}}$ of U_H is equal to* 0. *If* $R_{\mathbf{p}} \neq 0$, $L_{\mathbf{p},\varnothing}$ *has a simple pole at $\chi = 1$ of residue* $-\dfrac{hR_{\mathbf{p}}}{\sqrt{N(\mathbf{d}_{H,\mathbf{p}})}}\Pi_{\mathbf{q}\in|\mathbf{p}|}\left(1-\dfrac{1}{N(\mathbf{q})}\right)$, *where* $h = \mathrm{card}(Cl(O_H))$ *is the class number of* H.

(v) *If ψ is a* $\mathbf{p}$*-admissible Hecke character of H of conductor $\mathbf{m}_\psi$ satisfying* $|\mathbf{m}_\psi| \subset S \cup |\mathbf{p}|$, *then* $L_{\mathbf{p},S}(\psi^{(p)}) = E_{|\bar{\mathbf{p}}|}(\psi^\vee)W_{|\mathbf{p}|}(\psi)E_S(\psi)\Lambda(\psi)$.

Proof. (i) First note that $\mathscr{G}_{H,S,p}$ has a quotient isomorphic to $\mathbf{Z}_p^2$, namely $\mathrm{Gal}(HK_\infty/H)$, where K_∞ is the union of all $\mathbf{Z}_p$-extensions of K, and that the image of $C_T(S(\mathscr{B}), S'(\mathscr{B}))$ by the Artin map is dense in $\mathscr{G}_{H,S,p} \times \mathscr{G}_{H^\vee,\bar{S},p}$ by Tchebotarev's density theorem. Hence, there exists a subset C of $C_T(S(\mathscr{B}), S'(\mathscr{B}))$ dense in $\mathscr{G}_{H,S,p} \times \mathscr{G}_{H^\vee,\bar{S},p}$ such that the quotient of $\mu_{\mathbf{b}_1,\mathbf{b}_2,S}$ by $(1-\sigma_{\mathbf{b}_1}^{-1})(1-N(\mathbf{b}_2)\bar{\sigma}_{\mathbf{b}_2})$ is well-defined. An immediate consequence of Proposition 33 is that this quotient is independent of the choice of $(\mathbf{b}_1, \mathbf{b}_2) \in C$. We shall denote it by μ_S. We see that $(1-\sigma_{\mathbf{b}_1}^{-1})(1-N(\mathbf{b}_2)\bar{\sigma}_{\mathbf{b}_2})\mu_S$ is a measure on $\mathscr{G}_{H,S,p}$ for all $(\mathbf{b}_1, \mathbf{b}_2) \in C$. As C is dense in $\mathscr{G}_{H,S,p} \times \mathscr{G}_{H^\vee,\bar{S},p}$, this implies that $(1-\sigma_1)(1-N(\sigma_2)\sigma_2)\mu_S$ is a measure for all $\sigma_1, \sigma_2 \in \mathscr{G}_{H,S,p}$; hence, μ_S is a measure by virtue of the corollary of Lemma 34.

(ii) and (v) These are immediate consequences of Proposition 33.

(iii) Let ψ be an admissible Hecke character of conductor $\mathbf{m}_\psi$ satisfying $S \subset |\mathbf{m}_\psi| \subset S \cup |(p)|$. We have:

$$
\begin{aligned}
&L_{p,S}(\psi^{(p)}) = E_{|\bar{\mathbf{p}}|}(\psi)E_{|\bar{\mathbf{p}}|}(\psi^\vee)W_{|\mathbf{p}|}(\psi)\Lambda^?(\psi),\\
&L_{p,\bar{S}}((\psi^\vee)^{(p)}) = E_{|\bar{\mathbf{p}}|}(\psi^\vee)E_{|\bar{\mathbf{p}}|}(\psi)W_{|\mathbf{p}|}(\psi^\vee)\Lambda^?(\psi^\vee),\\
&W(\psi)\Lambda^?(\psi) = i^{-n}\Lambda^?(\psi^\vee),\\
&W_{|\mathbf{p}|}(\psi^\vee)W_{|\bar{\mathbf{p}}|}(\psi) = \psi_{|\bar{\mathbf{p}}|}(-1),\\
&W(\psi) = (-1)^{nk}W_{|\mathbf{p}|}(\psi)W_{|\bar{\mathbf{p}}|}(\psi)\prod_{\mathbf{q}\in|\mathbf{m}_\psi\mathbf{d}_H|-|(p)|}W_{\mathbf{q}}(\psi),\\
&\psi^{(p)}(\sigma_{-1}) = (-1)^{nk}\psi_{|\bar{\mathbf{p}}|}(-1),
\end{aligned}
$$

from which the formula for $W^{(p)}(\psi^{(p)})$ follows immediately. The fact that $W^{(p)}(\psi^{(p)})$ is a p-adic unit is a consequence of the fact that $W_{\mathbf{q}}(\psi)$ is a unit at all places prime to $N(\mathbf{q})$. The general case can be deduced from this case as in [d Sh, II, §6].

(iv) The definition of λ_S is about the same as that of μ_S. The quotient of $\lambda_{\mathbf{b}_1,\mathbf{b}_2,S}$ by $(1-\sigma_{\mathbf{b}_1}^{-1})(1-N(\mathbf{b}_2)\sigma_{\bar{\mathbf{b}}_2})$ does not depend on the choice of $(\mathbf{b}_1, \mathbf{b}_2) \in C_T(S(\mathscr{B}), S'(\mathscr{B}))$ and will be denoted by λ_S. The difference with (a) is that now, $N(\mathbf{b}_2)$ is not a continuous function of $\sigma_{\bar{\mathbf{b}}_2}$ and the image of $C_T(S(\mathscr{B}), S'(\mathscr{B}))$ in $(\mathscr{G}_{H,S,\mathbf{p}})^2 \times O_{\mathbf{p}}^*$ by the map $(\mathbf{b}_1, \mathbf{b}_2) \to (\sigma_{\mathbf{b}_1^{-1}}, \sigma_{\bar{\mathbf{b}}_2}, N(\mathbf{b}_2))$ is dense. This implies that

$(1-\sigma_1)(1-\alpha\sigma_2)\lambda_S$ is a measure for all $\sigma_1, \sigma_2 \in \mathscr{G}_{H,S,p}$ and $\alpha \in O_{\mathbf{p}}^*$. Hence

$$(1-\sigma_1)(1-p\sigma_2)\lambda_S = 2(1-\sigma_1)\left(1-\frac{1+p}{2}\sigma_2\right)\lambda_S-(1-\sigma_1)(1-\sigma_2)\lambda_S$$

is a measure. But $(1-p\sigma_2)^{-1}=\Sigma_{k=0}^{\infty} p^k\sigma_2^k$ is a measure and so $(1-\sigma_1)\lambda_S$ is a measure for all $\sigma_1 \in \mathscr{G}_{H,S,\mathbf{p}}$, which means that λ_S is a pseudo-measure.

Now, if $S \neq \emptyset$, take $\mathbf{q} \in S$ and let $S' = S-\mathbf{q}$. Let π be the projection from $\mathscr{G}_{H,S,\mathbf{p}}$ to $\mathscr{G}_{H,S',\mathbf{p}}$. Then we have $\pi(\lambda_S)=(1-\sigma_{\mathbf{q}})\lambda_{S'}$ and thus $\pi(\lambda_S)$ is a measure which implies by Lemma 34(b) that λ_S is a measure if $S \neq \emptyset$. The fact that $\lambda_\emptyset$ is a measure if $R_{\mathbf{p}}=0$ could be obtained by the same method as in [Se], but we shall deduce it from the formula giving the residue.

4. *Calculation of the residue*

If $F(z_1)$ is a function of $z_1=(z_{1,1},\ldots,z_{1,n}) \in \mathbf{C}^n$ with reasonable singularities (e.g. simple poles situated on hyperplanes), we define $\nabla_1^j(F)_{z_1=0}$ to be:

$$\lim_{s\to -j}\frac{1}{\Gamma(s)^n}\int_{\mathbf{C}^n}\phi(z_1)F(z_1)\prod_{k=1}^{n}\left(\overline{z_{1,k}}^{\,j}|z_{1,k}|^{2(s-1)}\frac{\mathrm{d}z_{1,k}\wedge \mathrm{d}\overline{z_{1,k}}}{2\pi i}\right),$$

where ϕ is any C^∞ compactly supported function on $\mathbf{C}^n$ equal to 1 in a neighborhood of 0. Of course, if F is C^∞ in a neighborhood of $z_1=0$, the two definitions of $\nabla_1^j(F)_{z_1=0}$ coincide.

Let V be a subgroup of finite index in the subgroup U_H of elements of norm 1 over K, $\mathscr{B} \in \mathscr{B}(V)$ and $\phi \in \mathscr{S}_{0,j,V}(H)$. Write $\Lambda(j,\phi)$ instead of $\Lambda(0,j,\phi)$ and suppose that ϕ satisfies conditions (2) and (4) of conditions (*) and (**) (cf. II), so that in particular, $\mathbf{F}(z_1, z_2, \phi, \mathscr{B})$ is regular at $z_2=0$.

LEMMA 36. $\Lambda(j,\phi)=\nabla_1^{(j-1)}(\mathbf{F}(z_1, 0, \phi, \mathscr{B}))_{z_1=0}$.

Proof. First note that this definition of $\nabla_1^{(j-1)}$ allows us to extend formula (7) to the case where β belongs to some of the hyperplanes of equation $\mathrm{Tr}(f_{i,B}vz)=0$, as can be seen by rearranging the terms as in [Co1, I, Lemma 4]. Thus the proof of Theorem 3 can be applied unchanged (but with the new meaning of $\nabla_1^{(j-1)}$ in formula (13)).

REMARK. The advantage of using this new $\nabla_1^{(j)}$ is that it dispenses us from introducing an auxiliary $\mathbf{b}_1$ which has the disagreeable effect of multiplying all the results by $1-\psi(\mathbf{b}_1^{-1})$ and removing the pole at $\psi=1$. The drawback is that the formulae become much more complicated.

If $\mathbf{a} \in I_{|(p)|}(H)$ and $\gamma \in (O_{H,|(p)|})^*$, let $\phi_{\mathbf{a},\gamma} \in \mathscr{S}(H)$ be the function defined by

$$\phi_{\mathbf{a},\gamma}(x) = \mathscr{F}_{|\bar{\mathbf{p}}|}(1_{\bar{\gamma}})(x_{|\mathbf{p}|}) \prod_{\mathbf{q} \notin |\mathbf{p}|} 1_{\mathbf{a}^{-1}}(x_{\mathbf{q}}),$$

where $1_{\bar{\gamma}} \in \mathscr{S}_{|\bar{\mathbf{p}}|,H^\vee}$ is the characteristic function of $\bar{\gamma} + \bar{\mathbf{p}} O_{|\bar{\mathbf{p}}|}$.

LEMMA 37. *Let ψ be an unramified $\mathbf{p}$-admissible Hecke character of H, $A \subset I_{(p)}(H)$ be a set of representatives of $Cl(O_H)$ and $C \subset (O_{H,|(p)|})^*$ be a set of representatives of $(O_H/\mathbf{p})^*$. Then*

$$\sum_{\mathbf{a} \in A} \sum_{\gamma \in C} \psi(\mathbf{a}(\gamma)) \Lambda(j(\psi), \phi_{\mathbf{a}(\gamma),1}) = L_{\mathbf{p},\varnothing}(\psi^{(p)}).$$

Proof. Note that $\Sigma_{\gamma \in C} 1_{\bar{\gamma}}(x_{|\bar{\mathbf{p}}|}) = \Pi_{\mathbf{q} \in |\bar{\mathbf{p}}|} \delta_{\mathbf{q}}(x_{\mathbf{q}})$. Thus, $\Sigma_{\gamma \in C} \phi_{\mathbf{a},\gamma^{-1}} = \psi_{|\mathbf{p}|,|\mathbf{p}|,\mathbf{a}}$ (see formula (48)). Since $\phi_{\mathbf{a}(\gamma),1}(x) = \phi_{\mathbf{a},\gamma^{-1}}(\gamma x)$, we apply formula (16) to obtain:

$$\begin{aligned} \sum_{\mathbf{a} \in A} \sum_{\gamma \in C} \psi(\mathbf{a}(\gamma)) \Lambda(j(\psi), \phi_{\mathbf{a}(\gamma),1}) &= \sum_{\mathbf{a} \in A} \psi(\mathbf{a}) \sum_{\gamma \in C} \Lambda(j(\psi), \phi_{\mathbf{a},\gamma^{-1}}) \\ &= \sum_{\mathbf{a} \in A} \psi(\mathbf{a}) \Lambda(j(\psi), \psi_{|\mathbf{p}|,|\mathbf{p}|,\mathbf{a}}), \end{aligned}$$

and the result follows from Lemma 30 and Theorem 35(v).

LEMMA 38. (i) *Let T, $\mathscr{B}$, $S_2'(\mathscr{B})$ and $\tilde{\phi}$ be as in Theorem 23. Then for all $B = (f_{1,B}, \ldots, f_{n,B}) \in \mathscr{B}$, all $\phi \in \mathscr{S}_T(H)$ and all $\mathbf{b}_2 \in C(H^\vee)$ satisfying $|\mathbf{b}_2| \cap (S_2'(\mathscr{B}) \cup T \cup |(p)|) = \varnothing$,*

$$\prod_{i=1}^{n} (\mathrm{Tr}(f_{i,B} z_1)) \mathscr{H}\left(\frac{1}{(2\pi i)^n} \mathbf{K}\left(\frac{z_1}{2\pi i}, 0, \delta_{\mathbf{b}_2^{-1}}^{\vee} * \tilde{\phi}, B\right)\right)$$

is the Fourier–Laplace transform of a p-adic distribution $T_{\mathbf{b}_2,\phi,B}$ on $Y_{H,\mathbf{p}}$ (cf. [Co2, §4]).

(ii) *If ϕ_1 is a locally constant function on $Y_{H,\mathbf{p}}$ viewed as an element of $\mathscr{S}_{|\bar{\mathbf{p}}|,H^\vee}$, the Fourier–Laplace transform of $\phi_1 T_{\mathbf{b}_2,\phi,B}$ is given by*

$$\prod_{i=1}^{n} (\mathrm{Tr}(f_{i,B} z_1)) \mathscr{H}\left(\frac{1}{(2\pi i)^n} \mathbf{K}\left(\frac{z_1}{2\pi i}, 0, \delta_{\mathbf{b}_2^{-1}}^{\vee} * \mathscr{F}_{|\bar{\mathbf{p}}|}(\phi_1) * \phi, B\right)\right).$$

Proof. The proof is about the same as that of Theorem 23 except that we need Proposition 10 to understand what happens at $z_1 = 0$.

COROLLARY. (i) *Let $\mathscr{B} \in \mathscr{B}(V)$, A and C be as in Lemma* 37 *and let $T \in \mathscr{P}(H)$ contain $|(p)|$, all $|\mathbf{a}|$ for $\mathbf{a} \in A$ and all $|(\gamma)|$ for $\gamma \in C$. Then for all $|\mathbf{b}_2| \in C(H^\vee)$*

satisfying $|\mathbf{b}_2|\cap(S_2'(\mathscr{B})\cup T)=\varnothing$, *all* $\mathbf{a}\in A$, *all* $\gamma\in C$ *and all* $B\in\mathscr{B}$,

$$\prod_{i=1}^{n}(\mathrm{Tr}(f_{i,B}z_1))\mathscr{H}\left(\frac{1}{(2\pi i)^n}\mathbf{K}\left(\frac{z_1}{2\pi i},0,\quad \delta_{\mathbf{b}_2^{-1}}^{\vee}*\phi_{\mathbf{a}(\gamma),1},B\right)\right)$$

is the Fourier–Laplace transform of a p-adic distribution $T_{\mathbf{b}_2,\mathbf{a},\gamma,B}$ *on* $Y_{H,\mathbf{p}}$ *whose support is in* $1+\mathbf{p}Y_{H,\mathbf{p}}$.

(ii) *We have*

$$(1-\psi^{\vee}(\mathbf{b}_2^{-1}))L_{\mathbf{p},\varnothing}(\psi)=\frac{(-1)^n}{[U_H:V]}\sum_{\mathbf{a}\in A}\sum_{\gamma\in C}\sum_{B\in\mathscr{B}}\psi(\mathbf{a}(\gamma))\int_{Y_{H,\mathbf{p}}}P_{j(\psi)-1,B}(y)\,\mathrm{d}T_{\mathbf{b}_2,\mathbf{a},\gamma,B},\tag{71}$$

where if $\mathscr{L}_B$ *is the family of linear forms* $(\mathrm{Tr}(f_{1,B}z),\dots,\mathrm{Tr}(f_{n,B}z))$ *and* $k\in\mathbf{N}$, $P_{k,B}$ *is the polynomial which was called* $P_{k,\mathscr{L}_B}$ *in* [Co2, Corollaire du Lemme 4.8].

Proof. Using Lemma 38, we see that $T_{\mathbf{b}_2,\mathbf{a},\gamma,B}=\phi_1 T_{\mathbf{b}_2,1_{\mathbf{a}^{-1}(\gamma^{-1})},B}$, where ϕ_1 is the characteristic function of $1+\mathbf{p}Y_{H,\mathbf{p}}$. This proves (i). The proof of (ii) is a consequence of the definition of $P_{k,B}$ of [Co2]. The $(-1)^n$ appears because of the difference between this article and [Co2] in sign convention for the Fourier–Laplace transform.

If $t\in\mathbf{N}$, let ψ_t be the unramified $\mathbf{p}$-admissible Hecke character of H defined by $\psi_t(\mathbf{a})=N_{H/K}(\alpha)^{(p-1)p^t}$ if $\mathbf{a}\in I(H)$ and α is a generator of the principal ideal $\mathbf{a}^h$. We have $j(\psi_t)=h(p-1)p^t$ and $\psi_t^{(p)}$ tends to 1 as t tends to $+\infty$. When we say that $L_{\mathbf{p},\varnothing}(\chi)$ has a simple pole at $\chi=1$ of residue R we mean explicitly: $\forall\sigma\in\mathscr{G}_{H,\varnothing,\mathbf{p}}$,

$$\lim_{\chi\to 1}(\chi(\sigma)-1)L_{\mathbf{p},\varnothing}(\chi)=(h(p-1))^{-1}\log_p(\psi_0^{(p)}(\sigma))R,$$

where $\log_p$ is the p-adic logarithm. Note that this limit exists and is equal to $\int_{\mathscr{G}_{H,\varnothing,\mathbf{p}}}\mathrm{d}((\sigma-1)\lambda_{\varnothing})$. Since the limit exists, we can compute it by using ψ_t and letting t tend to $+\infty$. Then the fact that $L_{\mathbf{p},\varnothing}(\chi)$ has a simple pole of residue R at $\chi=1$ is equivalent to

$$\lim_{t\to\infty}j(\psi_t)L_{\mathbf{p},\varnothing}(\psi_t)=R.$$

We indicate briefly how to compute this limit using the results of [Co2, §5]. Let $\eta_1,\dots,\eta_{n-1}$ be elements of U_H satisfying $\eta_i\equiv 1 \pmod{\mathbf{p}}$, $N_{H/K}(\eta_i)=1$ and generating a subgroup of finite index of U_H. If $r\in\mathbf{N}$, let $\eta_{i,r}=\eta_i^{p^r}$. Let V_r be the subgroup of U_H generated by $\eta_{1,r},\dots,\eta_{n-1,r}$ and if $\sigma\in S_{n-1}$, let $f_{1,\sigma,r}=1$ and $f_{i,\sigma,r}=\prod_{j<i}\eta_{\sigma(j),r}$. Let $\omega=\pm 1$ be the sign of $\det(1,\log|\eta_1|,\dots,\log|\eta_{n-1}|)$ where $\log|\eta_i|$ is the vector of R^n whose jth coordinate is $\log|\tau_j(\eta_i)|$. Set

$R_{\mathfrak{p}} = R_{\mathfrak{p}}(U_H) = \frac{1}{[U_H : V]} \frac{1}{n} \omega \det(1, \log_p \eta_1, \ldots, \log_p \eta_{n-1})$, where $\log_p \eta_i$ is the vector of $\mathbf{C}_p^n$ whose jth coordinate is $\log_p \tau_j(\eta_i)$. Also let $\mathscr{B}_r = \{B_{\sigma,r} \mid \sigma \in S_{n-1}\} \in \mathscr{B}(V_r)$, where $B_{\sigma,r} = (f_{1,\sigma,r}, \ldots, f_{n,\sigma,r})$ if $\omega.\varepsilon(\sigma) = 1$ and $B_{\sigma,r} = (f_{n,\sigma,r}, f_{2,\sigma,r} \ldots, f_{n-1,\sigma,r}, f_{1,\sigma,r})$ if $\omega.\varepsilon(\sigma) = -1$ and choose $\mathbf{b}_{2,r} \in C(H^\vee)$ satisfying $|\mathbf{b}_{2,r}| \cap (T \cup S_2'(\mathscr{B}_r)) = \varnothing$.

LEMMA 39. *The constant term of the Fourier–Laplace transform of* $T_{\mathbf{b}_{2,r},\mathbf{a},\gamma,B_{\sigma,r}}$ *is equal to* $\frac{1}{p^n} \frac{1}{\sqrt{N(\mathbf{d}_{H,\mathfrak{p}})}} (1 - N(\mathbf{b}_{2,r})) \det B_{\sigma,r}$.

Proof. Straightforward.

Thus, using [Co2, Corollaire du Lemme 4.8] while letting t tend to $+\infty$ in formula (71) and then letting r tend to $+\infty$ as in [Co2, §5], we obtain

$$\lim_{t \to \infty} j(\psi_t) L_{\mathfrak{p},\varnothing}(\psi_t) = -\frac{hR_{\mathfrak{p}}}{\sqrt{N(\mathbf{d}_{H,\mathfrak{p}})}} \prod_{\mathfrak{q} \in |\mathfrak{p}|} \left(1 - \frac{1}{N(\mathfrak{q})}\right),$$

where $R_{\mathfrak{p}}$ appears as

$$\frac{1}{n} \lim_{r \to \infty} [U_H : V_r]^{-1} \det(B_{\sigma,r}); \quad h \prod_{\mathfrak{q} \in |\mathfrak{p}|} \left(1 - \frac{1}{N(\mathfrak{q})}\right) \quad \text{as} \quad \frac{\text{card}(A \times C)}{p^n}$$

and

$$(-1) = (-1)^n \times (-1)^{n-1}$$

where the $(-1)^n$ comes from formula (71) and $(-1)^{n-1}$ from [Co2, Corollaire du Lemme 4.8]. The term $(1 - N(\mathbf{b}_{2,r}))$ disappears because $\lim_{t \to \infty} (1 - \psi_t^\vee(\mathbf{b}_{2,r}^{-1})) = 1 - N(\mathbf{b}_{2,r})$. The extra terms in [Co2, Corollaire du Lemme 4.8] disappear in the same way as in [Co2, §5]; namely, $A_i = A_i(\mathscr{L}_{B_{\sigma,r}})$ tends to 1 and $F_i(\mathscr{L}_{B_{\sigma,r}})$ to $1/(n-1)!$ This concludes the proof of Theorem 35.

Acknowledgements

We wish to thank the referee for a thorough reading of the manuscript and useful suggestions

References

[Ca] Cassou-Noguès, P.: Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques. *Invent Math.* 51, 29–59 (1979).

[C-W] Coates, J., Wiles, A.: On p-adic L-functions and elliptic units. *J. Austral. Math. Soc.* A26 (1978), 1–25.

[Co1] Colmez, P.: Algébricité de valeurs spéciales de fonctions L. *Inv. Math.* 95 (1989), 161–205.

[Co2] Colmez, P.: Résidu en $s=1$ des fonctions zêta p-adiques. *Inv. Math.* 91 (1988), 371–389.

[D] Deligne, P.: Valeurs de fonctions L et périodes d'intégrales. *Proc. Symp. Pure Math.* 33 (1979), 313–346.

[d Sh] de Shalit, E.: *Iwasawa Theory of Elliptic Curves with Complex Multiplication. Perspectives in Mathematics*, Vol. 3, Academic Press, 1987.

[H-S] Harder, G., Schappacher, N.: Special values of Hecke L-functions and Abelian integrals (*Lecture Notes in Math.*, Vol. 1111, pp. 17–49). Berlin-Heidelberg-New York: Springer, 1985.

[K] Katz, N.: p-Adic interpolation of real analytic Eisenstein series. *Ann. Math.* 104 (1976), 459–571.

[L] Lang, S.: Algebraic number theory. Addison-Wesley (1970).

[M-V] Manin, J., Višik, *M.*: p-adic Hecke series of imaginary quadratic fields. Math. Sbornik (N.S.) 95 (1974), 357–383. English trans.: *Math. USSR-Sb.* 24 (1974), 345–371.

[P-R] Perrin-Riou, B.: Périodes p-adiques. *C. R. Acad. Sci.* tôme 300, série 1 (1985), p. 455–457.

[Sch] Schmidt, W.: *Diophantine Approximation.* Lecture Notes in Math. 785, Berlin-Heidelberg-New York: Springer, 1980.

[Se] Serre, J.-P.: Sur le résidu de la fonction zêta p-adique d'un corps de nombres, *C. R. Acad. Sci. Paris* A287, 83–126 (1978).

[Sh] Shintani, T.: An evaluation of zeta-functions of totally real algebraic fields at non-positive integers. *J. Fac. Sci.*, Univ. Tokyo, Sect. IA 23 (1976), 393–417.

[T] Tilouine, J.: Fonctions L p-adiques à deux variables et $\mathbf{Z}_p^2$-extensions. *Bull. Soc. Math. France* 114 (1986), 3–66.

[Wa] Waldschmidt, M.: Les travaux de C. V. Čudnovskiĭ sur les Nombres Transcendents. *Sém. Bourbaki* 488 (1975/76), *Lecture Notes in Math.* 567, Berlin-Heidelberg-New York: Springer, 1977.

[W1] Weil, A.: On a certain type of characters of the idèle class group of an algebraic number field. In: *Proc. Int. Symp. on Alg. Number Theory Tokyo* (1955), p. 1–7.

[W2] Weil, A.: Elliptic functions according to Eisenstein and Kronecker. Springer-Verlag, 1976.

[Y1] Yager, R.: On two variable p-adic L functions. *Ann. Math.* 115 (1982), 411–449.

[Y2] Yager, R.: p-adic measures on Galois groups. *Inv. Math.* 76 (1984), 331–343.