Cohomology of p-adic analytic spaces

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(joint work with Wiesława Nizioł)

Let K be a field complete for v_p (discrete), and let C be the completion of its algebraic closure, $\breve{C} = W(k_C)[\frac{1}{p}]$, where k_C is the residue field of C. Let G_K be the absolute Galois group of K.

Theorem 1. *Let Y/K be a smooth, geometrically connected, dagger analytic space. If $i \le r$, we have a bicartesian diagram of G_K-modules:*

$$\begin{array}{ccc} H^i_{\mathrm{proet}}(Y_C, \mathbf{Q}_p(r)) & \longrightarrow & (\mathbf{B}^+_{\mathrm{st}} \otimes H^i_{\mathrm{HK}}(Y))^{N=0,\varphi=p^r} \\ \downarrow & & \downarrow \\ H^i(\mathrm{Fil}^r(\mathbf{B}^+_{\mathrm{dR}} \otimes \mathrm{R}\Gamma_{\mathrm{dR}})) & \longrightarrow & \mathbf{B}^+_{\mathrm{dR}} \otimes H^i_{\mathrm{dR}}(Y) \end{array}$$

• Dagger analytic spaces include analytifications of algebraic varieties, overconvergent affinoids, étale coverings of Drinfeld's symmetric spaces or, more generally, analytic spaces with no boundary.

• The theorem gives a description of the proétale cohomology in terms of de Rham data which is more amenable to computations. In the realm of algebraic varieties (and with étale cohomology in place of proétale), theorems of this type have a long history, starting with the formulation of the conjectures C_{cris}, C_{st}, C_{dR} of Fontaine (refining the conjecture of Tate on the existence of a Hodge-like decomposition for étale cohomology), and their ensuing proofs by Fontaine-Messing, Kato, Hyodo-Kato, Faltings, Tsuji, Nizioł, Yamashita, Scholze, etc., the most general result being that of Beilinson [1] with no asumption on the existence of good models nor, even, smoothness.

• In the theorem, the Hyodo-Kato cohomology group $H^i_{\mathrm{HK}}(Y)$ is a $\breve{C}$-module with a frobenius φ, a monodromy operator N, a (pro)smooth action of G_K, and an isomorphism $\iota_{\mathrm{HK}} : C \otimes_{\breve{C}} H^i_{\mathrm{HK}}(Y) \xrightarrow{\sim} H^i_{\mathrm{dR}}(Y)$. The definition [4, 5] of $H^i_{\mathrm{HK}}(Y)$ and ι_{HK} is a big part of the theorem; it is adapted from Beilinson's and uses the alterations of Hartl and Temkin to produce good models (locally).

• In the case of algebraic varieties (or proper analytic ones), all cohomology groups in the diagram are finite dimensional (as in [3]) and the kernels of the horizontal arrows are 0. This is not the case for a general analytic variety and the tensor products are (derived) completed tensor products. Even if $H^i_{\mathrm{dR}}(Y)$ is finite dimensional, $H^i(\mathrm{Fil}^r(\mathbf{B}^+_{\mathrm{dR}} \otimes \mathrm{R}\Gamma_{\mathrm{dR}}))$ surjects onto $C \otimes \Omega^r(Y)^{d=0}$ and hence can be huge (and so is $H^i_{\mathrm{proet}}(Y_C, \mathbf{Q}_p(r))$).

• In the Stein case or for an overconvergent affinoid, the horizontal arrows are surjective and their kernels are isomorphic to $(C \otimes \Omega^{r-1}(Y))/\mathrm{Ker}\, d$ (as in [2]).

• The general case reduces to the quasi-compact case, and then uses an induction on the number of affinoids needed to cover the space. This induction uses fine properties of the category of BC's (BC stands for Banach-Colmez).

• Using results of Fontaine of the type $\mathrm{Hom}_{G_K}(\mathbf{B}_{\mathrm{dR}}^+/t^N, \mathbf{B}_{\mathrm{dR}}) = 0$, one can recover $H^i_{\mathrm{HK}}(Y)$ and $H^i_{\mathrm{dR}}(Y)$ from $H^i_{\mathrm{proet}}(Y_C, \mathbf{Q}_p)$. For example, we have

$$H^i_{\mathrm{dR}}(Y)^* = \mathrm{Hom}_{G_K}(H^i_{\mathrm{proet}}(Y_C, \mathbf{Q}_p), \mathbf{B}_{\mathrm{dR}}).$$

• If we don't assume Y to be defined over K, one can still prove the above results but this requires (in progress) to promote the diagram to a diagram of BC's (slightly generalized since the spaces involved do not satisfy the finiteness conditions required in the definition of BC's).

References

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