Duality for the proétale p-adic cohomology of analytic spaces

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Let K be a finite extension of $\mathbf{Q}_p$ and let C be the completion of $\overline{K}$.

The geometric proétale cohomology of an open annulus

Let Y_K be an open annulus defined over K and set $Y_C := Y_K \times C$. Then, Kummer theory or syntomic computions give, for $k \in \mathbf{Z}$,

$$H^i_{\text{proét}}(Y_C, \mathbf{Q}_p(k)) \simeq \begin{cases} \mathbf{Q}_p(k) & \text{if } i = 0, \\ \mathbf{Q}_p(k-1) \oplus (\mathcal{O}(Y_C)/C)(k-1) & \text{if } i = 1, \\ 0 & \text{if } i \geq 2 \end{cases}$$

To define the cohomology with compact support, let us first define the boundary ∂Y_C of Y_C as $\varprojlim_X Y_C \backslash X$, where X runs over quasi-compact subspaces of Y_C (hence $\mathcal{O}(Y_C) = \varinjlim_X \mathcal{O}(Y_C \backslash X)$). This is the disjoint union of two "ghost circles". Then, define $\mathrm{R}\Gamma_{\text{proét},c}(Y_C, \mathbf{Q}_p(k))$ as the fiber of the map

$$\mathrm{R}\Gamma_{\text{proét}}(Y_C, \mathbf{Q}_p(k)) \to \mathrm{R}\Gamma_{\text{proét}}(\partial Y_C, \mathbf{Q}_p(k))$$

The usual long exact sequence gives isomorphisms

$$H^1_{\text{proét},c}(Y_C, \mathbf{Q}_p(k)) \simeq H^{i-1}_{\text{proét}}(\partial Y_C, \mathbf{Q}_p(k))/H^{i-1}_{\text{proét}}(Y_C, \mathbf{Q}_p(k))$$

from which one gets:

$$H^i_{\text{proét},c}(Y_C, \mathbf{Q}_p(k)) \simeq \begin{cases} \mathbf{Q}_p(k) & \text{if } i = 1, \\ \mathbf{Q}_p(k-1) \oplus \frac{\mathcal{O}(\partial Y_C)/C^2}{\mathcal{O}(Y_C)/C}(k-1) & \text{if } i = 2, \\ 0 & \text{if } i \neq 1, 2 \end{cases}$$

which does not look too good for a duality between cohomology and cohomology with compact support. Nevertheless, $(\mathcal{O}(\partial Y_C)/C^2)/(\mathcal{O}(Y_C)/C)$ is naturally the C-dual of $\mathcal{O}(Y_C)/C$ by Serre duality (but not its $\mathbf{Q}_p$-dual since C is infinite dimensional over $\mathbf{Q}_p$).

The arithmetic case

One can use Hochschild-Serre spectral sequence to compute the cohomology of Y_K from the cohomology of Y_C, and Tate's results that $H^i(G_K, C(j)) = K$ if $j = 0$ and $i = 0, 1$ and $H^i(G_K, C(j)) = 0$ if $j \neq 0$ or $i \neq 0, 1$ (here $G_K = \mathrm{Gal}(\overline{K}/K)$). A case by case inspection, using Poitou-Tate duality between $H^i(G_K, \mathbf{Q}_p(j))$ and $H^{2-i}(G_K, \mathbf{Q}_p(1-j))$ shows that

$$H^4_{\text{proét},c}(Y_K, \mathbf{Q}_p(2)) = \mathbf{Q}_p$$

and $H^{4-i}_{\text{proét},c}(Y_K, \mathbf{Q}_p(2-k))$ is the $\mathbf{Q}_p$-dual of $H^i_{\text{proét}}(Y_K, \mathbf{Q}_p(k))$.

This leads to the following conjecture:

Conjecture: *Let Y_K be a partially proper analytic space* (or a dagger space) *defined over K, smooth, connected, of dimension d. Then*:

(i) $H^{2d+2}_{\text{proét},c}(Y_K, \mathbf{Q}_p(d+1)) = \mathbf{Q}_p$.

(ii) $H^i_{\text{proét}}(Y_K, \mathbf{Q}_p(k)) \times H^{2d+2-i}_{\text{proét},c}(Y_K, \mathbf{Q}_p(d+1-k)) \to \mathbf{Q}_p$ *is a duality of topological $\mathbf{Q}_p$-vector spaces.*

Remark: (i) The conjecture is true if Y_K is proper thanks to the recently proved Poincaré duality for proper analytic spaces over C (proofs by L. Mann and by B. Zavyalov).

(ii) The conjecture is perhaps overoptimistic for topological reasons in the general case (one should probably phrase it using condensed mathematics of Clausen and Scholze), but we have the following result.

Theorem: *The conjecture is true in dimension* 1.

The geometric case

For a proper curve,

$$H^i_{\text{proét}}(Y_C, \mathbf{Q}_p(1)) \times H^{2-i}_{\text{proét},c}(Y_C, \mathbf{Q}_p(1)) \to \mathbf{Q}_p(1)$$

is a perfect duality. In the case of the open annulus appearing above there seems to be a mixing of $\mathbf{Q}_p$-duality and C-duality as well as a shift of cohomological degrees.

Now, $\mathbf{Q}_p$ and C are C-points of BC's (which are sums of copies of $\mathbf{G}_a$ "up to finite dimensional $\mathbf{Q}_p$-vector spaces"), namely $\mathbf{Q}_p$ is the C-points of $\mathbf{Q}_p$ and C is the C points of $\mathbf{G}_a$. In this category (or even in the bigger category of VS's in which finite dimensionality is dropped), one has

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbf{Q}_p, \mathbf{Q}_p(1)) = \mathbf{Q}_p(1) \qquad \mathrm{Ext}^1_{\mathrm{VS}}(\mathbf{Q}_p, \mathbf{Q}_p(1)) = 0$$
$$\mathrm{Hom}_{\mathrm{VS}}(\mathbf{G}_a, \mathbf{Q}_p(1)) = 0 \qquad \mathrm{Ext}^1_{\mathrm{VS}}(\mathbf{G}_a, \mathbf{Q}_p(1)) = C$$

This leads to the following conjecture:

Conjecture: *Let Y be a partially proper analytic space* (or a dagger space) *defined over C, smooth, connected, of dimension d. Then*:

$$\mathrm{R}\Gamma_{\text{proét}}(Y, \mathbf{Q}_p(j)) \simeq \mathrm{RHom}_{\mathrm{VS}}(\mathrm{R}\Gamma_{\text{proét},c}(Y, \mathbf{Q}_p(d+1-j)), \mathbf{Q}_p(1))[+2d]$$

Theorem: *The conjecture is true in dimension* 1.

Remark: (i) If Y is proper, all spaces are finite dimensional over $\mathbf{Q}_p$ and the statement is true (with usual duality) by the above mentionned results of Mann and of Zavyalov.

(ii) In order to make sense of the conjecture, one has to "geometrize" p-adic proétale cohomology to turn it into VS's. This is done in [2] where a "geometrized" comparison theorem with syntomic cohomology is also established.

(iii) Using the point of view on BC's developped in Le Bras's thesis, one can express everything that comes out of syntomic cohomology in terms of coherent cohomology of complexes of sheaves on the Fargues-Fontaine curve X_{FF}. This suggests an ad hoc definition for $\mathrm{RHom}_{\mathrm{VS}}$ which, actually, was proved to be correct recently by Anschütz and Le Bras [1].

(iv) Using these tools, the proof for curves is rather straightforward, and probably extends, at least locally, to higher dimension.

References

[1] J. Anschütz, A.-C. Le Bras, *A Fourier Transform for Banach-Colmez spaces*, arXiv:2111.11116 [math.AG] (2021).

[2] P. Colmez, W. Nizioł, *On the cohomology of p-adic analytic spaces, I: The basic comparison theorem*, arXiv:2104.13448 [math.NT] (2021).