KAZHDAN'S PROPERTY T FOR DISCRETE QUANTUM GROUPS

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Abstract

We give a simple definition of property T for discrete quantum groups. We prove the basic expected properties: discrete quantum groups with property T are finitely generated and unimodular. Moreover we show that, for "I.C.C." discrete quantum groups, property T is equivalent to Connes' property T for the dual von Neumann algebra. This allows us to give the first example of a property T discrete quantum group which is not a group using the twisting construction.

1 Introduction

In the 1980's, Woronowicz [19], [20], [21] introduced the notion of a compact quantum group and generalized the classical Peter-Weyl representation theory. Many interesting examples of compact quantum groups are available by now: Drinfel'd and Jimbo [5], [9] introduced q-deformations of compact semi-simple Lie groups, and Rosso [13] showed that they fit into the theory of Woronowicz. Free orthogonal and unitary quantum groups were introduced by Van Daele and Wang [18] and studied in detail by Banica [1], [2].

Some discrete group-like properties and proofs have been generalized to (the dual of) compact quantum groups. See, for example, the work of Tomatsu [14] on amenability, the work of Banica and Vergnioux [3] on growth and the work of Vergnioux and Vaes [15] on boundary.

The aim of this paper is to define property T for discrete quantum groups. We give a definition analogous to the group case using almost invariant vectors. We show that a discrete quantum group with property T is finitely generated, i.e. the dual is a compact quantum group of matrices. Recall that a locally compact group with property T is unimodular. We show that the same result holds for discrete quantum groups, i.e. every discrete quantum group with property T is a Kac algebra. In [4] Connes and Jones defined property T for arbitrary von Neumann algebras and showed that an I.C.C. group has property T if and only if its group von Neumann algebra (which is a II_1 factor) has property T. We show that if the group von Neumann algebra of a discrete quantum group $\widehat{\mathbb{G}}$ is

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an infinite dimensional factor (i.e. $\widehat{\mathbb{G}}$ is "I.C.C."), then $\widehat{\mathbb{G}}$ has property T if and only if its group von Neumann algebra is a II_1 factor with property T. This allows us to construct an example of a discrete quantum group with property T which is not a group by twisting an I.C.C. property T group. In addition we show that free quantum groups do not have property T.

This paper is organized as follows: in Section 2 we recall the notions of compact and discrete quantum groups and the main results of this theory. We introduce the notion of discrete quantum sub-groups and prove some basic properties of the quasi-regular representation. We also recall the definition of property T for von Neumann algebras. In Section 3 we introduce property T for discrete quantum groups, we give some basic properties and we show our main result.

2 Preliminaries

2.1 Notations

The scalar product of a Hilbert space H, which is denoted by $\langle .,. \rangle$, is supposed to be linear in the first variable. The von Neumann algebra of bounded operators on H will by denoted by $\mathcal{B}(H)$ and the C^* algebra of compact operators by $\mathcal{B}_0(H)$. We will use the same symbol $\otimes$ to denote the tensor product of Hilbert spaces, the minimal tensor product of C^* algebras and the spatial tensor product of von Neumann algebras. We will use freely the leg numbering notation.

2.2 Compact quantum groups

We briefly overview the theory of compact quantum groups developed by Woronowicz in [21]. We refer to the survey paper [12] for a smooth approach to these results.

Definition 1. A *compact quantum group is a pair* $\mathbb{G} = (A, \Delta)$, where A is a unital C^* algebra; Δ is unital *-homomorphism from A to $A \otimes A$ satisfying $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ and $\Delta(A)(A \otimes 1)$ and $\Delta(A)(1 \otimes A)$ are dense in $A \otimes A$.

Notation 1. We denote by $C(\mathbb{G})$ the C^* algebra A.

The major results in the general theory of compact quantum groups are the existence and uniqueness of the Haar state and the Peter-Weyl representation theory.

Theorem 1. *Let* $\mathbb{G}$ *be a compact quantum group. There exists a unique state* φ *on* $C(\mathbb{G})$ *such that* $(id \otimes \varphi)\Delta(a) = \varphi(a)1 = (\varphi \otimes id)\Delta(a)$ *for all* $a \in C(\mathbb{G})$. *The state* φ *is called the Haar state of* $\mathbb{G}$.

Notation 2. The Haar state need not be faithful. We denote by $\mathbb{G}_{\mathrm{red}}$ the *reduced quantum group* obtained by taking $C(\mathbb{G}_{\mathrm{red}}) = C(\mathbb{G})/I$ where $I = \{x \in A \,|\, \varphi(x^*x) = 0\}$. The Haar measure is faithful on $\mathbb{G}_{\mathrm{red}}$. We denote by $L^\infty(\mathbb{G})$ the von Neumann algebra generated by the G.N.S. representation of the Haar state of $\mathbb{G}$. Note that $L^\infty(\mathbb{G}_{\mathrm{red}}) = L^\infty(\mathbb{G})$.

Definition 2. A *unitary representation* u of a compact quantum group $\mathbb{G}$ on a Hilbert space H is a unitary element $u \in \mathrm{M}(\mathcal{B}_0(H) \otimes C(\mathbb{G}))$ satisfying

$$(\mathrm{id} \otimes \Delta)(u) = u_{12}u_{13}.$$

Let u^1 and u^2 be two unitary representations of $\mathbb{G}$ on the respective Hilbert spaces H_1 and H_2. We define the set of *intertwiners*

$$\mathrm{Mor}(u^1, u^2) = \{T \in \mathcal{B}(H_1, H_2) \,|\, (T \otimes 1)u^1 = u^2(T \otimes 1)\}.$$

A unitary representation u is said to be *irreducible* if $\mathrm{Mor}(u, u) = \mathbb{C}1$. Two unitary representations u^1 and u^2 are said to be *unitarily equivalent* if there is a unitary element in $\mathrm{Mor}(u^1, u^2)$.

Theorem 2. *Every irreducible representation is finite-dimensional. Every unitary representation is unitarily equivalent to a direct sum of irreducibles.*

Definition 3. Let u^1 and u^2 be unitary representations of $\mathbb{G}$ on the respective Hilbert spaces H_1 and H_2. We define the tensor product

$$u^1 \otimes u^2 = u^1_{13}u^2_{23} \in \mathrm{M}(\mathcal{B}_0(H_1 \otimes H_2) \otimes C(\mathbb{G})).$$

Notation 3. We denote by $\mathrm{Irred}(\mathbb{G})$ the set of (equivalence classes) of irreducible unitary representations of a compact quantum group $\mathbb{G}$. For every $x \in \mathrm{Irred}(\mathbb{G})$ we choose representatives u^x on the Hilbert space H_x. Whenever $x, y \in \mathrm{Irred}(\mathbb{G})$, we use $x \otimes y$ to denote the (class of the) unitary representation $u^x \otimes u^y$. The class of the trivial representation is denoted by 1.

The set $\mathrm{Irred}(\mathbb{G})$ is equipped with a natural involution $x \mapsto \bar{x}$ such that $u^{\bar{x}}$ is the unique (up to unitary equivalence) irreducible representation such that

$$\mathrm{Mor}(1, x \otimes \bar{x}) \neq 0 \neq \mathrm{Mor}(1, \bar{x} \otimes x).$$

This means that $x \otimes \bar{x}$ and $\bar{x} \otimes x$ contain a non-zero invariant vector. Let $E_x \in H_x \otimes H_{\bar{x}}$ be a non-zero invariant vector and J_x the invertible antilinear map from H_x to $H_{\bar{x}}$ defined by

$$\langle J_x\xi, \eta\rangle = \langle E_x, \xi \otimes \eta\rangle, \quad \text{for all } \xi \in H_x,\ \eta \in H_{\bar{x}}.$$

Let $Q_x = J_x^*J_x$. We will always choose E_x and $E_{\bar{x}}$ normalized such that $||E_x|| = ||E_{\bar{x}}||$ and $J_{\bar{x}} = J_x^{-1}$. Then Q_x is uniquely determined, $\mathrm{Tr}(Q_x) = ||E_x||^2 = \mathrm{Tr}(Q_x^{-1})$ and $Q_{\bar{x}} = (J_xJ_x^*)^{-1}$. $\mathrm{Tr}(Q_x)$ is called the *quantum dimension* of x and is denoted by $\dim_{\mathrm{q}}(x)$. The unitary representation $u^{\bar{x}}$ is called the *contragredient* of u^x.

The G.N.S. representation of the Haar state is given by $(L^2(\mathbb{G}), \Omega)$ where $L^2(\mathbb{G}) = \bigoplus_{x\in\mathrm{Irred}(\mathbb{G})} H_x \otimes H_{\bar{x}}$, $\Omega \in H_1 \otimes H_{\bar{1}}$ is the unique norm one vector, and

$$(\omega_{\xi,\eta} \otimes \mathrm{id})(u^x)\Omega = \frac{1}{||E_x||}\xi \otimes J_x(\eta), \quad \text{for all } \xi, \eta \in H_x.$$

It is easy to see that φ is a trace if and only if $Q_x = \mathrm{id}$ for all $x \in \mathrm{Irred}(\mathbb{G})$. In this case $||E_x|| = \sqrt{n_x}$ where n_x is the dimension of H_x and J_x is an anti-unitary operator.

Notation 4. Let $C(\mathbb{G})_s$ be the vector space spanned by the coefficients of all irreducible representations of $\mathbb{G}$. Then $C(\mathbb{G})_s$ is a dense unital *-subalgebra of $C(\mathbb{G})$. Let $C(\mathbb{G}_{\max})$ be the maximal C^* completion of the unital *-algebra $C(\mathbb{G})_s$. $C(\mathbb{G}_{\max})$ has a canonical structure of a compact quantum group. This quantum group is denoted by $\mathbb{G}_{\max}$ and it is called the *maximal quantum group*.

A *morphism of compact quantum groups* $\pi : \mathbb{G} \to \mathbb{H}$ is a unital *-homomorphism from $C(\mathbb{G}_{\max})$ to $C(\mathbb{H}_{\max})$ such that $\Delta_{\mathbb{H}} \circ \pi = (\pi \otimes \pi) \circ \Delta_{\mathbb{G}}$, where $\Delta_{\mathbb{G}}$ and $\Delta_{\mathbb{H}}$ denote the comultiplications for $\mathbb{G}_{\max}$ and $\mathbb{H}_{\max}$ respectively. We will need the following easy Lemma.

Lemma 1. *Let π be a surjective morphism of compact quantum group from $\mathbb{G}$ to $\mathbb{H}$ and $\tilde{\pi}$ be the surjective *-homomorphism from $C(\mathbb{G}_{max})$ to $C(\mathbb{H})$ obtained by composition of π with the canonical surjection $C(\mathbb{H}_{max}) \to C(\mathbb{H})$. Then for every irreducible unitary representation v of $\mathbb{H}$ there exists an irreducible unitary representation u of $\mathbb{G}$ such that v is contained in the unitary representation $(id \otimes \tilde{\pi})(u)$.*

Proof. Let φ be the Haar state of $\mathbb{H}$ and v be an irreducible unitary representation of $\mathbb{H}$ on the Hilbert space H_v. Because v is irreducible it is sufficient to show that there exists a unitary irreducible representation u of $\mathbb{G}$ such that $\mathrm{Mor}(w, v) \neq \{0\}$, where $w = (\mathrm{id} \otimes \tilde{\pi})(u)$. Suppose that the statement is false. Then for all irreducible unitary representations u of $\mathbb{G}$ on H_u, we have $\mathrm{Mor}(w, v) = \{0\}$. By [12], Lemma 6.3, for every operator $a : H_v \to H_u$ the operator $(\mathrm{id} \otimes \varphi)(v^*(a \otimes 1)w)$ is in $\mathrm{Mor}(w, v)$. It follows that for every irreducible unitary representation u of $\mathbb{G}$ and every operator $a : H_v \to H_u$ we have $(\mathrm{id} \otimes \varphi)(v^*(a \otimes 1)w) = 0$. Using the same techniques as in [12], Theorem 6.7, (because, by the surjectivity of π, $\tilde{\pi}(C(\mathbb{G})_s)$ is dense in $C(\mathbb{H})$) we find $(\mathrm{id} \otimes \varphi)(v^* v) = 0$. But this is a contradiction as $v^* v = 1$. □

The collection of all finite-dimensional unitary representations (given with the concrete Hilbert spaces) of a compact quantum group $\mathbb{G}$ is a *complete concrete monoidal W^*-category*. We denote this category by $\mathcal{R}(\mathbb{G})$. We say that $\mathcal{R}(\mathbb{G})$ is finitely generated if there exists a finite subset $E \subset \mathrm{Irred}(\mathbb{G})$ such that for all finite-dimensional unitary representations r there exists a finite family of morphisms $b_k \in \mathrm{Mor}(r_k, r)$, where r_k is a product of elements of E, and $\sum_k b_k b_k^* = I_r$. It is not difficult to show that $\mathcal{R}(\mathbb{G})$ is finitely generated if and only if $\mathbb{G}$ is a compact quantum group of matrices (see [20]).

2.3 Discrete quantum groups

A discrete quantum group is defined as the dual of a compact quantum group.

Definition 4. Let $\mathbb{G}$ be a compact quantum group. We define the dual *discrete quantum group* $\widehat{\mathbb{G}}$ as follows:

$$c_0(\widehat{\mathbb{G}}) = \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})}^{c_0} \mathcal{B}(H_x), \quad l^\infty(\widehat{\mathbb{G}}) = \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})}^{\infty} \mathcal{B}(H_x).$$

We denote the minimal central projection of $l^\infty(\widehat{\mathbb{G}})$ by p_x, $x \in \mathrm{Irred}(\mathbb{G})$. We have a natural unitary $\mathbb{V} \in \mathrm{M}(c_o(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}))$ given by

$$\mathbb{V} = \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})} u^x.$$

We have a natural comultiplication

$$\hat{\Delta} : l^\infty(\widehat{\mathbb{G}}) \to l^\infty(\widehat{\mathbb{G}}) \otimes l^\infty(\widehat{\mathbb{G}}) : (\hat{\Delta} \otimes id)(\mathbb{V}) = \mathbb{V}_{13}\mathbb{V}_{23}.$$

The comultiplication is given by the following formula

$$\hat{\Delta}(a)S = Sa, \quad \text{for all } a \in \mathcal{B}(H_x),\ S \in \mathrm{Mor}(x, yz),\ x, y, z \in \mathrm{Irred}(\mathbb{G}).$$

Remark 1. The maximal and reduced versions of a compact quantum group are different versions of the same underlying compact quantum group. This different versions give the same dual discrete quantum group, i.e. $\widehat{\mathbb{G}} = \widehat{\mathbb{G}_{\mathrm{red}}} = \widehat{\mathbb{G}_{\mathrm{max}}}$. This means that $\widehat{\mathbb{G}}$, $\widehat{\mathbb{G}_{\mathrm{red}}}$ and $\widehat{\mathbb{G}_{\mathrm{max}}}$ have the same C^* algebra, the same von Neumann algebra and the same comultiplication.

A *morphism of discrete quantum groups* $\hat{\pi} : \widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$ is a non-degenerate *-homomorphism from $c_0(\widehat{\mathbb{G}})$ to $\mathrm{M}(c_0(\widehat{\mathbb{H}}))$ such that $\hat{\Delta}_{\mathbb{H}} \circ \pi = (\pi \otimes \pi) \circ \hat{\Delta}_{\mathbb{G}}$, where $\hat{\Delta}_{\mathbb{G}}$ and $\hat{\Delta}_{\mathbb{H}}$ denote the comultiplication for $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ respectively. Every morphism of compact quantum groups $\pi : \mathbb{G} \to \mathbb{H}$ admits a canonical dual morphism of discrete quantum groups $\hat{\pi} : \widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$. Conversely, every morphism of discrete quantum groups $\hat{\pi} : \widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$ admits a canonical dual morphism of compact quantum groups $\pi : \mathbb{G} \to \mathbb{H}$. Moreover, π is surjective (resp. injective) if and only if $\hat{\pi}$ is injective (resp. surjective).

We say that a discrete quantum group $\widehat{\mathbb{G}}$ is *finitely generated* if the category $\mathcal{R}(\mathbb{G})$ is finitely generated.

We will work with representations in the von Neumann algebra setting.

Definition 5. Let $\widehat{\mathbb{G}}$ be a discrete quantum group. A *unitary representation* U of $\widehat{\mathbb{G}}$ on a Hilbert space H is a unitary $U \in l^\infty(\widehat{\mathbb{G}}) \otimes \mathcal{B}(H)$ such that :

$$(\hat{\Delta} \otimes \mathrm{id})(U) = U_{13}U_{23}.$$

Consider the following maximal version of the unitary $\mathbb{V}$:

$$\mathcal{V} = \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})} u^x \in \mathrm{M}(c_o(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}_{\mathrm{max}})).$$

For every unitary representation U of $\widehat{\mathbb{G}}$ on a Hilbert space H there exists a unique *-homomorphism $\rho : C(\mathbb{G}_{\mathrm{max}}) \to \mathcal{B}(H)$ such that $(\mathrm{id} \otimes \rho)(\mathcal{V}) = U$.

Notation 5. Whenever U is a unitary representation of $\widehat{\mathbb{G}}$ on a Hilbert space H we write $U = \sum_{x \in \mathrm{Irred}(\mathbb{G})} U^x$ where $U^x = Up_x$ is a unitary in $\mathcal{B}(H_x) \otimes \mathcal{B}(H)$.

The discrete quantum group $l^\infty(\widehat{\mathbb{G}})$ comes equipped with a natural modular structure. Let us define the following canonical states on $\mathcal{B}(H_x)$:

$$\varphi_x(A) = \frac{\mathrm{Tr}(Q_x A)}{\mathrm{Tr}(Q_x)}, \quad \text{and} \quad \psi_x(A) = \frac{\mathrm{Tr}(Q_x^{-1} A)}{\mathrm{Tr}(Q_x^{-1})}, \quad \text{for all } A \in \mathcal{B}(H_x).$$

The states φ_x and ψ_x provide a formula for the invariant normal semi-finite faithful (n.s.f.) weights on $l^\infty(\widehat{\mathbb{G}})$.

Proposition 1. *The left invariant weight $\hat{\varphi}$ and the right invariant weight $\hat{\psi}$ on $\widehat{\mathbb{G}}$ are given by*

$$\hat{\varphi}(a) = \sum_{x \in Irred(\mathbb{G})} dim_q(x)^2 \varphi_x(ap_x) \quad \text{and} \quad \hat{\psi}(a) = \sum_{x \in Irred(\mathbb{G})} dim_q(x)^2 \psi_x(ap_x),$$

for all $a \in l^\infty(\widehat{\mathbb{G}})$ whenever this formula makes sense.

A discrete quantum is unimodular (i.e. the left and right invariant weights are equal) if and only if the Haar state φ on the dual is a trace. In general, a discrete quantum group is not unimodular, and it is easy to check that the Radon-Nikodym derivative is given by

$$[D\hat{\psi} : D\hat{\varphi}]_t = \hat{\delta}^{it} \quad \text{where} \quad \hat{\delta} = \sum_{x \in \mathrm{Irred}(\mathbb{G})} Q_x^{-2} p_x.$$

The positive self-adjoint operator $\hat{\delta}$ is called the *modular element*: it is affiliated with $c_0(\widehat{\mathbb{G}})$ and satisfies $\hat{\Delta}(\hat{\delta}) = \hat{\delta} \otimes \hat{\delta}$.

The following Proposition is very easy to prove.

Proposition 2. *Let Γ be the subset of $\mathbb{R}_+^*$ consisting of all the eigenvalues of the operators Q_x^{-2} for $x \in Irred(\mathbb{G})$. Then Γ is a subgroup of $\mathbb{R}_+^*$ and $Sp(\widehat{\delta}) = \Gamma \cup \{0\}$.*

Proof. Note that, because $J_{\bar{x}} = J_x^{-1}$, the eigenvalues of $Q_{\bar{x}}$ are the inverse of the eigenvalues of Q_x. Using the formula $SQ_z = Q_x \otimes Q_y S$, when $z \subset x \otimes y$ and $S \in \mathrm{Mor}(z, x \otimes y)$ is an isometry, the Proposition follows immediately. □

2.4 Discrete quantum subgroups

Let $\mathbb{G}$ be a compact quantum group with representation category $\mathcal{C}$. Let $\mathcal{D}$ be a full subcategory such that $1_{\mathcal{C}} \in \mathcal{D}$, $\mathcal{D} \otimes \mathcal{D} \subset \mathcal{D}$ and $\overline{\mathcal{D}} = \mathcal{D}$. By the Tannaka-Krein Reconstruction Theorem of Woronowicz [20] we know that there exists a compact quantum group $\mathbb{H}$ such that the representation category of $\mathbb{H}$ is $\mathcal{D}$. We say that $\widehat{\mathbb{H}}$ is a *discrete quantum subgroup* of $\widehat{\mathbb{G}}$. We have $\mathrm{Irred}(\mathbb{H}) \subset \mathrm{Irred}(\mathbb{G})$. We collect some easy observations in the next proposition. We denote by a subscript $\mathbb{H}$ the objects associated to $\mathbb{H}$.

Proposition 3. *Let* $p = \sum_{x \in Irred(\mathbb{H})} p_x$. *We have:*

1. $\hat{\Delta}(p)(p \otimes 1) = p \otimes p$;
2. $l^\infty(\widehat{\mathbb{H}}) = p(l^\infty(\widehat{\mathbb{G}}))$;
3. $\hat{\Delta}_{\mathbb{H}}(a) = \hat{\Delta}(a)(p \otimes p)$ *for all* $a \in l^\infty(\widehat{\mathbb{H}})$;
4. $\hat{\varphi}(p.) = \hat{\varphi}_{\mathbb{H}}$ *and* $\hat{\delta}_{\mathbb{H}} = p\hat{\delta}$.

Proof. For $x, y, z \in \mathrm{Irred}(\mathbb{G})$ such that $y \subset z \otimes x$, we denote by $p_y^{z \otimes x} \in \mathrm{End}(x \otimes y)$ the projection on the sum of all sub-representations equivalent to y. Note that

$$\hat{\Delta}(p_y)(p_z \otimes p_x) = \begin{cases} p_y^{z \otimes x} & \text{if } y \subset z \otimes x, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus:

$$\hat{\Delta}(p)(p_z \otimes p_x) = \sum_{y \in \mathrm{Irred}(\mathbb{H}),\, y \subset z \otimes x} p_y^{z \otimes x}.$$

Note that if $y \subset z \otimes x$ and $y,\, z \in \mathrm{Irred}(\mathbb{H})$ then $x \in \mathrm{Irred}(\mathbb{H})$. It follows that:

$$\hat{\Delta}(p)(p \otimes p_x) = \begin{cases} p \otimes p_x & \text{if } x \in \mathrm{Irred}(\mathbb{H}), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\hat{\Delta}(p)(p \otimes 1) = p \otimes p$. The other assertions are obvious. □

We introduce the following equivalence relation on $\mathrm{Irred}(\mathbb{G})$ (see [17]): if $x, y \in \mathrm{Irred}(\mathbb{G})$ then $x \sim y$ if and only if there exists $t \in \mathrm{Irred}(\mathbb{H})$ such that $x \subset y \otimes t$. We define the right action of $\widehat{\mathbb{H}}$ on $l^\infty(\widehat{\mathbb{G}})$ by translation:

$$\alpha \,:\, l^\infty(\widehat{\mathbb{G}}) \to l^\infty(\widehat{\mathbb{G}}) \otimes l^\infty(\widehat{\mathbb{H}}), \quad \alpha(a) = \hat{\Delta}(a)(1 \otimes p).$$

Using $\hat{\Delta}(p)(p \otimes 1) = p \otimes p$ and $\hat{\Delta}_{\mathbb{H}} = \hat{\Delta}(.)(p \otimes p)$ it is easy to see that α satisfies the following equations:

$$(\alpha \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \hat{\Delta}_{\mathbb{H}})\alpha \quad \text{and} \quad (\hat{\Delta} \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \alpha)\hat{\Delta}.$$

The first equality means that α is a right action of $\widehat{\mathbb{H}}$ on $l^\infty(\widehat{\mathbb{G}})$. Let $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ be the set of fixed points of the action α:

$$l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}) := \{a \in l^\infty(\widehat{\mathbb{G}}), \,|\, \alpha(a) = a \otimes 1\}.$$

Using the second equality for α it is easy to see that:

$$\hat{\Delta}(l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})) \subset l^\infty(\widehat{\mathbb{G}}) \otimes l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}).$$

Thus the restriction of $\hat{\Delta}$ to $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ gives an action of $\widehat{\mathbb{G}}$ on $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$. We denote this action by β.

Proposition 4. *Let* $T_\alpha = (id \otimes \hat{\varphi}_{\mathbb{H}})\alpha$ *be the normal faithful operator valued weight from* $l^\infty(\widehat{\mathbb{G}})$ *to* $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ *associated to* α. T_α *is semi-finite and there exists a unique n.s.f. weight* θ *on* $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ *such that* $\hat{\varphi} = \theta \circ T_\alpha$.

Proof. It follows from Eq. (1) that $T_\alpha(p_y)p_z = 0$ if $z \nsim y$. Take $z \sim y$, we have:

$$\begin{aligned} T_\alpha(p_y)p_z &= \sum_{x \in \mathrm{Irred}(\mathbb{H})} \dim_q(x)^2 (\mathrm{id} \otimes \varphi_x)(p_y^{z \otimes x}) \\ &\leq \sum_{x \in \mathrm{Irred}(\mathbb{G})} \dim_q(x)^2 (\mathrm{id} \otimes \varphi_x)(p_y^{z \otimes x}) \\ &= (\mathrm{id} \otimes \hat{\varphi})(\hat{\Delta}(p_y))p_z = \hat{\varphi}(p_y)p_z \\ &= \dim_q(y)^2 p_z. \end{aligned}$$

It follows that $T_\alpha(p_y) < \infty$ for all y. This implies that T_α is semi-finite. Note that $\alpha(\delta^{-it}) = \delta^{-it} \otimes \delta_{\mathbb{H}}^{-it}$. It follows from [10], Proposition 8.7, that there exists a unique n.s.f. weight θ on $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ such that $\hat{\varphi} = \theta \circ T_\alpha$. □

Denote by $l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ the G.N.S. space of θ and suppose that $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}) \subset \mathcal{B}(l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}))$. Let $U^* \in l^\infty(\widehat{\mathbb{G}}) \otimes \mathcal{B}(l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}))$ be the unitary implementation of β associated to θ in the sense of [16]. Then U is a unitary representation of $\widehat{\mathbb{G}}$ on $l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ and $\beta(x) = U^*(1 \otimes x)U$. We call U the *quasi-regular* representation of $\widehat{\mathbb{G}}$ modulo $\widehat{\mathbb{H}}$.

Lemma 2. *We have* $p \in l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}) \cap \mathcal{N}_\theta$. *Put* $\xi = \Lambda_\theta(p)$. *If* $\widehat{\mathbb{G}}$ *is unimodular then* $U^x \eta \otimes \xi = \eta \otimes \xi$ *for all* $x \in$ *Irred*$(\mathbb{H})$ *and all* $\eta \in H_x$.

Proof. Using $\hat{\Delta}(p_1)(1 \otimes p_x) = p_1^{\bar{x} \otimes x}$ it is easy to see that $T_\alpha(p_1) = p$. It follows that $p \in l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ and $\theta(p) = \hat{\varphi}(p_1) = 1$. Thus $p \in \mathcal{N}_\theta$. Let $x \in M^+$ such that $T_\alpha(x) < \infty$, $\omega \in l^\infty(\widehat{\mathbb{G}})_*^+$ and μ a n.s.f. weight on $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$. Using $(\hat{\Delta} \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \alpha)\hat{\Delta}$ we find:

$$\begin{aligned} (\omega \otimes \mu)\beta(T_\alpha(x)) &= (\omega \otimes \mu)\hat{\Delta}(T_\alpha(x)) = (\omega \otimes \mu)\hat{\Delta}((\mathrm{id} \otimes \hat{\varphi}_{\mathbb{H}})(\alpha(x))) \\ &= (\omega \otimes \mu \otimes \hat{\varphi}_{\mathbb{H}})((\hat{\Delta} \otimes \mathrm{id})\alpha(x)) \\ &= (\omega \otimes \mu \otimes \hat{\varphi}_{\mathbb{H}})((\mathrm{id} \otimes \alpha)\hat{\Delta}(x)) \\ &= (\omega \otimes \mu \circ T_\alpha)\hat{\Delta}(x). \end{aligned} \tag{2}$$

It follows that, for all $\omega \in l^\infty(\widehat{\mathbb{G}})_*^+$ and all $y \in l^\infty(\widehat{\mathbb{G}})^+$ such that $T_\alpha(y) < \infty$, we have:

$$(\omega \otimes \theta)\beta(T_\alpha(y)) = (\omega \otimes \hat{\varphi})(\hat{\Delta}(y)) = \hat{\varphi}(y)\omega(1) = \theta(T_\alpha(y))\omega(1).$$

Let $x \in l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})^+$. Because T_α is a faithful and semi-finite, there exists an increasing net of positive elements y_i in $l^\infty(\widehat{\mathbb{G}})^+$ such that $T_\alpha(y_i) < \infty$ for all i and $\mathrm{Sup}_i(T_\alpha(y_i)) = x$. It follows that:

$$(\omega \otimes \theta)\beta(x) = \mathrm{Sup}((\omega \otimes \theta)\beta(T_\alpha(y_i))) = \mathrm{Sup}(\theta(T_\alpha(y_i)))\omega(1) = \theta(x)\omega(1),$$

for all $\omega \in l^\infty(\widehat{\mathbb{G}})_*^+$. This means that θ is β-invariant. Using this invariance we define the following isometry:

$$V^*(\hat{\Lambda}(x) \otimes \Lambda_\theta(y)) = (\hat{\Lambda} \otimes \Lambda_\theta)(\beta(y)(x \otimes 1)).$$

Because $\widehat{\mathbb{G}}$ is unimodular we know from [16], Proposition 4.3, that V^* is the unitary implementation of β associated to θ i.e. $V = U$. Using $\hat{\Delta}(p)(p \otimes 1) = p \otimes p$, it follows that, for all $x \in \mathcal{N}_{\hat{\varphi}}$, we have:

$$U^*(p\hat{\Lambda}(x) \otimes \Lambda_\theta(p)) = (\hat{\Lambda} \otimes \Lambda_\theta)(\hat{\Delta}(p)(px \otimes 1)) = p\hat{\Lambda}(x) \otimes \Lambda_\theta(p).$$

This concludes the proof. □

Remark 2. For general discrete quantum groups it can be proved, as in [6], Théorème 2.9, that V^* is a unitary implementing the action β and, as in [16], Proposition 4.3, that V^* is the unitary implementation of β associated to θ. Thus the previous lemma is also true for general discrete quantum groups.

Lemma 3. *Suppose that U has a non-zero invariant vector $\xi \in l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$. Then Irred$(\mathbb{G})$/Irred$(\mathbb{H})$ is a finite set.*

Proof. Let $\xi \in l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ be a normalized U-invariant vector. Using $\beta(x) = U^*(1 \otimes x)U$ it is easy to see that ω_ξ is a β-invariant normal state on $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$, i.e. $(\mathrm{id} \otimes \omega_\xi)\beta(x) = \omega_\xi(x)1$ for all $x \in l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$. Let s be the support of ω_ξ and $e = 1 - s$. Let ω be a faithful normal state on $l^\infty(\widehat{\mathbb{G}})$. Because the support of $\omega \otimes \omega_\xi$ is $1 \otimes s$ and $(\omega \otimes \omega_\xi)\beta(e) = \omega_\xi(e) = 0$ we find $\hat{\Delta}(e) = \beta(e) \leq 1 \otimes e$. It follows from [11], Lemma 6.4, that $e = 0$ or $e = 1$. Because ξ is a non-zero vector we have $e = 0$. Thus ω_ξ is faithful. Let $x \in M^+$ such that $T_\alpha(x) < \infty$. By Eq. (2) we have:

$$(\omega \otimes \omega_\xi \circ T_\alpha)(\hat{\Delta}(x)) = (\omega \otimes \omega_\xi)\beta(T_\alpha(x)) = \omega_\xi(T_\alpha(x))\omega(1),$$

for all $\omega \in l^\infty(\widehat{\mathbb{G}})_*^+$. Because T_α is n.s.f., it follows easily that $\omega_\xi \circ T_\alpha$ is a left invariant n.s.f. weight on $\widehat{\mathbb{G}}$. Thus, up to a positive constant, we have $\omega_\xi \circ T_\alpha = \hat{\varphi}$.

Suppose that Irred$(\mathbb{G})$/Irred$(\mathbb{H})$ is infinite, and let $x_i \in$ Irred$(\mathbb{G})$, $i \in \mathbb{N}$ be a complete set of representatives of Irred$(\mathbb{G})$/Irred$(\mathbb{H})$. Let a be the positive element of $l^\infty(\widehat{\mathbb{G}})$ defined by $a = \sum_{i \geq 0} \frac{1}{\dim_q(x_i)^2} p_{x_i}$. Then we have $\hat{\varphi}(a) = +\infty$ and $T_\alpha(a) = \sum_i \sum_{x \simeq x_i} p_x = 1 < \infty$, which is a contradiction. □

2.5 Property T for von Neumann algebras

Here we recall several facts from [4]. If M and N are von Neumann algebras then a correspondence from M to N is a Hilbert space H which is both a left M-module and a right N-module, with commuting normal actions π_l and π_r respectively. The triple (H, π_l, π_r) is simply denoted by H and we shall write $a\xi b$ instead of $\pi_l(a)\pi_r(b)\xi$ for $a \in M$, $b \in N$ and $\xi \in H$. We shall denote by

$\mathcal{C}(M)$ the set of unitary equivalence classes of correspondences from M to M. The standard representation of M defines an element $L^2(M)$ of $\mathcal{C}(M)$, called the identity correspondence.

Given $H \in \mathcal{C}(M)$, $\epsilon > 0$, $\xi_1, \ldots, \xi_n \in H$, $a_1, \ldots, a_p \in M$, let $\mathcal{V}_H(\epsilon, \xi_i, a_i)$ be the set of $K \in \mathcal{C}(M)$ for which there exist $\eta_1, \ldots, \eta_n \in K$ with

$$|\langle a_j \eta_i a_k, \eta_{i'} \rangle - \langle a_j \xi_i a_k, \xi_{i'} \rangle| < \epsilon, \quad \text{for all } i, i^{'}, j, k.$$

Such sets form a basis of a topology on $\mathcal{C}(M)$ and, following [4], M is said to have property T if there is a neighbourhood of the identity correspondence, each member of which contains $L^2(M)$ as a direct summand.

When M is a II_1 factor the property T is easier to understand. A II_1 factor M has property T if we can find $\epsilon > 0$ and $a_1, \ldots, a_p \in M$ satisfying the following condition: every $H \in \mathcal{C}(M)$ such that there exists $\xi \in H$, $||\xi|| = 1$, with $||a_i\xi - \xi a_i|| < \epsilon$ for all i, contains a non-zero central vector η (i.e. $a\eta = \eta a$ for all $a \in M$). We recall the following Proposition from [4].

Proposition 5. *If M is a II_1 factor with property T then there exist $\epsilon > 0$, $b_1, \ldots, b_m \in M$ and $C > 0$ with the following property: for any $\delta \leq \epsilon$, if $H \in \mathcal{C}(M)$ and $\xi \in H$ is a unit vector satisfying $||b_i\xi - \xi b_i|| < \delta$ for all $1 \leq i \leq m$, then there exists a unit central vector $\eta \in H$ such that $||\xi - \eta|| < C\delta$.*

It is proved in [4] that a discrete I.C.C. group has property T if and only if the group von Neumann algebra $\mathcal{L}(G)$ has property T.

3 Property T for Discrete Quantum Groups

Definition 6. Let $\widehat{\mathbb{G}}$ be a discrete quantum group.

- Let $E \subset \mathrm{Irred}(\mathbb{G})$ be a finite subset, $\epsilon > 0$ and U a unitary representation of $\widehat{\mathbb{G}}$ on a Hilbert space K. We say that U has an (E, ϵ)-*invariant vector* if there exists a unit vector $\xi \in K$ such that for all $x \in E$ and $\eta \in H_x$ we have:
$$||U^x \eta \otimes \xi - \eta \otimes \xi|| < \epsilon ||\eta||.$$

- We say that U has *almost invariant vectors* if, for all finite subsets $E \subset \mathrm{Irred}(\mathbb{G})$ and all $\epsilon > 0$, U has an (E, ϵ)-invariant vector.

- We say that $\widehat{\mathbb{G}}$ has *property T* if every unitary representation of $\hat{\mathbb{G}}$ having almost invariant vectors has a non-zero invariant vector.

Remark 3. Let $\mathbb{G} = (C^*(\Gamma), \Delta)$, where Γ is a discrete group and $\Delta(g) = g \otimes g$ for $g \in \Gamma$. It follows from the definition that $\widehat{\mathbb{G}}$ has property T if and only if Γ has property T.

The next proposition will be useful to show that the dual of a free quantum group does not have property T.

Proposition 6. *Let $\mathbb{G}$ and $\mathbb{H}$ be compact quantum groups. Suppose that there is a surjective morphism of compact quantum groups from $\mathbb{G}$ to $\mathbb{H}$ (or an injective morphism of discrete quantum groups from $\widehat{\mathbb{H}}$ to $\widehat{\mathbb{G}}$). If $\widehat{\mathbb{G}}$ has property T then $\widehat{\mathbb{H}}$ has property T.*

Proof. We can suppose that $\mathbb{G} = \mathbb{G}_{\max}$ and $\mathbb{H} = \mathbb{H}_{\max}$. We will denote by a subscript $\mathbb{G}$ (resp. $\mathbb{H}$) the object associated to $\mathbb{G}$ (resp. $\mathbb{H}$). Let π be the surjective morphism from $C(\mathbb{G})$ to $C(\mathbb{H})$ which intertwines the comultiplications. Let U be a unitary representation of $\widehat{\mathbb{H}}$ on a Hilbert space K and suppose that U has almost invariant vectors. Let ρ be the unique morphism from $C(\mathbb{H})$ to $\mathcal{B}(K)$ such that $(\mathrm{id}\otimes\rho)(\mathcal{V}_{\mathbb{H}}) = U$. Consider the following unitary representation of $\widehat{\mathbb{G}}$ on K: $V = (\mathrm{id}\otimes(\rho\circ\pi))(\mathcal{V}_{\mathbb{G}})$. We will show that V has almost invariant vectors. Let $E \subset \mathrm{Irred}(\mathbb{G})$ be a finite subset and $\epsilon > 0$. For $x \in \mathrm{Irred}(\mathbb{G})$ and $y \in \mathrm{Irred}(\mathbb{H})$ denote by $u^x \in \mathcal{B}(H_x)\otimes C(\mathbb{G})$ and $v^y \in \mathcal{B}(H_y)\otimes C(\mathbb{H})$ a representative of x and y respectively. Note that $w^x = (\mathrm{id}\otimes\pi)(u^x)$ is a finite dimensional unitary representation of $\mathbb{H}$, thus we can suppose that $w^x = \oplus\, n_{x,y}v^y$. Let $L = \{y \in \mathrm{Irred}(\mathbb{H}) \,|\, \exists x \in E,\, n_{x,y} \neq 0\}$. Because U has almost invariant vectors, there exists a norm one vector $\xi \in K$ such that $||U^y\eta\otimes\xi - \eta\otimes\xi|| < \epsilon||\eta||$ for all $y \in L$ and all $\eta \in H_y$. Using the isomorphism

$$H_x = \bigoplus_{y\in\mathrm{Irred}(\mathbb{H}),\ n_{x,y}\neq 0} \underbrace{(H_y\oplus\ldots\oplus H_y)}_{n_{x,y}},$$

we can identify V^x with $\oplus\, n_{x,y}U^y$ in $\bigoplus_y \mathcal{B}(H_y)\oplus\mathcal{B}(H_y)\oplus\ldots\oplus\mathcal{B}(H_y)\otimes\mathcal{B}(K)$. With this identification it is easy to see that, for all $x \in E$ and all η in H_x, we have $||V^x\eta\otimes\xi - \eta\otimes\xi|| < \epsilon||\eta||$. It follows that V has almost invariant vectors and thus there is a non-zero V-invariant vector, say l, in K. To show that l is also U-invariant it is sufficient to show that for every $y \in \mathrm{Irred}(\mathbb{H})$ there exists $x \in \mathrm{Irred}(\mathbb{G})$ such that $n_{x,y} \neq 0$. This follows from Lemma 1. □

Corollary 1. *The discrete quantum groups $\widehat{A_o(n)}$, $\widehat{A_u(n)}$ and $\widehat{A_s(n)}$ do not have property T for $n \geq 2$.*

Proof. It follows directly from the preceding proposition and the following surjective morphisms:

$$A_o(n) \to C^*(\star_{i=1}^n\mathbb{Z}_2),\ A_u(n) \to C^*(\mathbb{F}_n),\ A_s(n) \to C^*(\star_{i=1}^n\mathbb{Z}_{n_i}),$$

where $\sum n_i = n$. □

In the next Proposition we show that discrete quantum groups with property T are unimodular.

Proposition 7. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group. If $\widehat{\mathbb{G}}$ has property T then it is a Kac algebra, i.e. the Haar state φ on $\mathbb{G}$ is a trace.*

Proof. Suppose $\widehat{\mathbb{G}}$ has property T and let Γ be the discrete group introduced in Proposition 2. Because $\mathrm{Sp}(\widehat{\delta}) = \Gamma \cup \{0\}$ and $\hat{\Delta}\widehat{\delta} = \widehat{\delta} \otimes \widehat{\delta}$, we have an injective *-homomorphism

$$\alpha : c_0(\Gamma) \to c_0(\widehat{\mathbb{G}}), \quad \alpha(f) = f(\widehat{\delta})$$

satisfying $\Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_\Gamma$. By Proposition 6, Γ has property T. It follows that $\Gamma = \{1\}$ and $\widehat{\delta} = 1$. Thus $Q_x = 1$ for all $x \in \mathrm{Irred}(\mathbb{G})$. This means that φ is a trace. □

Proposition 8. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group. If $\widehat{\mathbb{G}}$ has property T then it is finitely generated.*

Proof. Let $\mathrm{Irred}(\mathbb{G}) = \{x_n \,|\, n \in \mathbb{N}\}$ and $\mathcal{C}$ be the category of finite dimensional unitary representations of $\mathbb{G}$. For $i \in \mathbb{N}$ let $\mathcal{D}_i$ be the full subcategory of $\mathcal{C}$ generated by $(x_0, \ldots, x_i)$. This means that the irreducibles of $\mathcal{D}_i$ are the irreducible representations u of $\mathbb{G}$ such that u is equivalent to a sub-representation of $x_{k_1}^{\epsilon_1} \otimes \ldots \otimes x_{k_l}^{\epsilon_l}$ for $l \geq 1$, $0 \leq k_j \leq n$, and ϵ_j is nothing or the contragredient. The Hilbert spaces and the morphisms are the same in $\mathcal{D}_i$ or in $\mathcal{D}$. Thus we have $1_\mathcal{C} \in \mathcal{D}_i$, $\mathcal{D}_i \otimes \mathcal{D}_i \subset \mathcal{D}_i$ and $\overline{\mathcal{D}_i} = \mathcal{D}_i$. Let $\mathbb{H}_i$ be the compact quantum group such that $\mathcal{D}_i$ is the category of representation of $\mathbb{H}_i$. Let $U_i \in l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}_i) \otimes \mathcal{B}(l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}_i))$ be the quasi-regular representation of $\widehat{\mathbb{G}}$ modulo $\widehat{\mathbb{H}}_i$. Let U be the direct sum of the U_i; this a unitary representation on $K = \bigoplus l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}_i)$. Let us show that U has almost invariant vectors. Let $E \subset \mathrm{Irred}(\mathbb{G})$ be a finite subset. There exists i_0 such that $E \subset \mathrm{Irred}(\mathbb{H}_i)$ for all $i \geq i_0$. By Lemma 2 we have a unit vector ξ in $l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}_i)$ such that $U_{i_0}^x \eta \otimes \xi = \eta \otimes \xi$ for all $x \in E$ and all $\eta \in H_x$. Let $\tilde{\xi} = (\xi_i) \in K$ where $\xi_i = 0$ if $i \neq i_0$ and $\xi_{i_0} = \xi$. Then $\tilde{\xi}$ is a unit vector in K such that $U^x \eta \otimes \tilde{\xi} = \eta \otimes \tilde{\xi}$ for all $x \in E$. It follows that U has an almost invariant vector. By property T there exists a non-zero invariant vector $l = (l_i) \in K$. There exists m such that $l_m \neq 0$. Then l_m is an invariant vector for U_m. By Lemma 3, $\mathrm{Irred}(\mathbb{G})/\mathrm{Irred}(\mathbb{H}_m)$ is a finite set. Let $y_1, \ldots, y_l$ be a complete set of representatives of $\mathrm{Irred}(\mathbb{G})/\mathrm{Irred}(\mathbb{H}_m)$. Then $\mathcal{C}$ is generated by $\{y_1, \ldots, y_l, x_0, \ldots, x_m, \bar{x}_0, \ldots, \bar{x}_m\}$. □

As in the classical case, we can show that property T is equivalent to the existence of a Kazhdan pair.

Proposition 9. *Let $\widehat{\mathbb{G}}$ be a finitely generated discrete quantum group. Let $E \subset \mathrm{Irred}(\mathbb{G})$ be a finite subset with $1 \in E$ such that $\mathcal{R}(\mathbb{G})$ is generated by E. The following assertions are equivalent:*

1. *$\widehat{\mathbb{G}}$ has property T.*
2. *There exists $\epsilon > 0$ such that every unitary representation of $\widehat{\mathbb{G}}$ having an (E, ϵ)-invariant vector has a non-zero invariant vector.*

Proof. It is sufficient to show that 1 implies 2. Let $n \in \mathbb{N}^*$ and $E_n = \{y \in \mathrm{Irred}(\mathbb{G}) \,|\, y \subset x_1 \ldots x_n,\ x_i \in E\}$. Because $1 \in E$, the sequence $(E_n)_{n \in \mathbb{N}^*}$ is increasing. Let us show that $\mathrm{Irred}(\mathbb{G}) = \bigcup E_n$. Let $r \in \mathrm{Irred}(\mathbb{G})$. Because $\mathcal{R}(\mathbb{G})$

is generated by E, there exists a finite family of morphisms $b_k \in \mathrm{Mor}(r_k, r)$, where r_k is a product of elements of E and $\sum_k b_k b_k^* = I_r$. Let L be the maximum of the length of the elements r_k. Because $1 \in E$, we can suppose that all the r_k are of the form $x_1 \dots x_L$ with $x_i \in E$. Put $t_k = b_k^*$. Note that $t_k^* t_k \in \mathrm{Mor}(r, r)$. Because r is irreducible and $\sum_k t_k^* t_k = I_r$, there exists a unique k such that $t_k^* t_k = I_r$ and $t_l^* t_l = 0$ if $l \neq k$. Thus $t_k \in \mathrm{Mor}(r, r_k)$ is an isometry. This means that $r \subset r_k = x_1 \dots x_L$, i.e. $r \in E_L$.

Suppose that $\widehat{\mathbb{G}}$ has property T and 2 is false. Let $N = \mathrm{Max}\{n_x \,|\, x \in E\}$ and $\epsilon_n = \frac{1}{n^2 \sqrt{N^n}}$. For all $n \in \mathbb{N}^*$ there exists a unitary representation U_n of $\widehat{\mathbb{G}}$ on a Hilbert space K_n with an (E, ϵ_n)-invariant vector but without a non-zero invariant vector. Let ξ_n be a unit vector in K_n which is (E, ϵ_n)-invariant. Write $U_n = \sum_{y \in \mathrm{Irred}(\mathbb{G})} U^{n,y}$ where $U^{n,y}$ is a unitary element in $\mathcal{B}(H_y) \otimes \mathcal{B}(K_n)$. Let us show the following:

$$||U^{n,y}\eta \otimes \xi_n - \eta \otimes \xi_n||_{H_y \otimes K_n} < \frac{1}{n}||\eta||_{H_y},\ \forall n \in \mathbb{N}^*,\ \forall y \in E_n,\ \forall \eta \in H_y. \quad (3)$$

Let $y \in E_n$ and $t_y \in \mathrm{Mor}(y, x_1 \dots x_n)$ such that $t_y^* t_y = I_y$. Note that, by the definition of a representation and using the description of the coproduct on $\widehat{\mathbb{G}}$, we have $(t_y \otimes 1)U^{n,y} = U^{n,x_1}_{1,n+1} U^{n,x_2}_{2,n+1} \dots U^{n,x_n}_{n,n+1}(t_y \otimes 1)$ where the subscripts are used for the leg numbering notation. It follows that, for all $\eta \in H_y$, we have:

$$\begin{aligned} ||U^{n,y}\eta \otimes \xi_n - \eta \otimes \xi_n|| &= ||(t_y \otimes 1)U^{n,y}\eta \otimes \xi_n - (t_y \otimes 1)\eta \otimes \xi_n|| \\ &= ||U^{n,x_1}_{1,n+1} U^{n,x_2}_{2,n+1} \dots U^{n,x_n}_{n,n+1} t_y\eta \otimes \xi_n - t_y\eta \otimes \xi_n|| \\ &\leq \sum_{k=1}^{n} ||U^{n,x_k}_{k,n+1} t_y\eta \otimes \xi_n - t_y\eta \otimes \xi_n||. \end{aligned}$$

Let $(e^{x_i}_j)_{1 \leq j \leq n_{x_i}}$ be an orthonormal basis of H_{x_i} and put

$$t_y\eta = \sum \lambda_{i_1 \dots i_n} e^{x_1}_{i_1} \otimes \dots \otimes e^{x_n}_{i_n}.$$

Then we have, for all $y \in E_n$ and $\eta \in H_y$,

$$\begin{aligned} ||U^{n,y}\eta \otimes \xi_n - \eta \otimes \xi_n|| &\leq \sum_k || \sum_{i_1 \dots i_n} \lambda_{i_1 \dots i_n} (U^{n,x_k}_{k,n+1} e^{x_1}_{i_1} \otimes \dots \otimes e^{x_n}_{i_n} \otimes \xi_n - e^{x_1}_{i_1} \otimes \dots \otimes e^{x_n}_{i_n} \otimes \xi_n)|| \\ &\leq \sum_k \sum_{i_1 \dots i_n} |\lambda_{i_1 \dots i_n}| ||U^{n,x_k} e^{x_k}_{i_k} \otimes \xi_n - e^{x_k}_{i_k} \otimes \xi_n|| \\ &\leq n\epsilon_n ||t_y\eta||_1, \end{aligned}$$

where $||t_y\eta||_1 = \sum |\lambda_{i_1 \dots i_n}|$. Note that $||t_y\eta||_1 \leq \sqrt{N^n}||\eta||$, thus we have

$$\begin{aligned} ||U^{n,y}\eta \otimes \xi_n - \eta \otimes \xi_n|| &\leq n\epsilon_n \sqrt{N^n}||\eta|| \\ &\leq \frac{1}{n}||\eta||. \end{aligned}$$

This proves Eq. (3). It is now easy to finish the proof. Let U be the direct sum of the U_n. It is a unitary representation of $\widehat{\mathbb{G}}$ on $K = \bigoplus K_n$. Let $\delta > 0$ and $L \subset \text{Irred}(\mathbb{G})$ a finite subset. Because $\text{Irred}(\mathbb{G}) = \bigcup^{\uparrow} E_n$ there exists n_1 such that $L \subset E_n$ for all $n \geq n_1$. Choose $n \geq n_1$ such that $\frac{1}{n} < \delta$. Put $\xi = (0, \ldots, 0, \xi_n, 0, \ldots)$ where ξ_n appears in the n-th place. Let $x \in L$ and $\eta \in H_x$. We have:

$$\begin{aligned} ||U^x \eta \otimes \xi - \eta \otimes \xi|| &= ||U^{n,x} \eta \otimes \xi_n - \eta \otimes \xi_n|| \\ &\leq \frac{1}{n} ||\eta|| < \delta ||\eta||. \end{aligned}$$

Thus U has almost invariant vectors. It follows from property T that U has a non-zero invariant vector, say $l = (l_n)$. There is a n such that $l_n \neq 0$ and from the U-invariance of l we conclude that l_n is U_n-invariant. This is a contradiction. □

Such a pair (E, ϵ) as defined Proposition 9 is called a *Kazhdan pair* for $\widehat{\mathbb{G}}$. Let us give an obvious example of a Kazhdan pair.

Proposition 10. *Let $\widehat{\mathbb{G}}$ be a finite-dimensional discrete quantum group. Then $(\mathit{Irred}(\mathbb{G}), \sqrt{2})$ is a Kazhdan pair for $\widehat{\mathbb{G}}$.*

Proof. If $\widehat{\mathbb{G}}$ is finite-dimensional then it is compact, φ is a trace and $\hat{\varphi}$ is a normal functional. For $x \in \text{Irred}(\mathbb{G})$ let (e_i^x) be an orthonormal basis of H_x and e_{ij}^x the associated matrix units. As $Q_x = 1$, we have $\hat{\varphi}(e_{ij}^x) = \frac{\dim_q(x)^2}{n_x} \delta_{ij}$. Let $U \in l^\infty(\widehat{\mathbb{G}}) \otimes \mathcal{B}(K)$ be a unitary representation of $\widehat{\mathbb{G}}$ with a unit vector $\xi \in K$ such that:

$$\text{Sup}_{x \in \text{Irred}(\mathbb{G}), 1 \leq j \leq n_x} ||U^x e_j^x \otimes \xi - e_j^x \otimes \xi|| < \sqrt{2}.$$

Because $\hat{\varphi}(1)^{-1}(\hat{\varphi} \otimes \text{id})(U)$ is the projection on the U-invariant vectors, $\tilde{\xi} = (\hat{\varphi} \otimes \text{id})(U)\xi \in K$ is invariant. Let us show that $\tilde{\xi}$ is non-zero. Writing $U^x = \sum e_{ij}^x \otimes U_{ij}^x$ with $U_{ij}^x \in \mathcal{B}(K)$, we have:

$$||U^x e_j^x \otimes \xi - e_j^x \otimes \xi||^2 = 2 - 2\text{Re}\langle U_{jj}^x \xi, \xi \rangle, \quad \text{for all } x \in \text{Irred}(\mathbb{G}),\ 1 \leq j \leq n_x.$$

It follows that $\text{Re}\langle U_{jj}^x \xi, \xi \rangle > 0$ for all $x \in \text{Irred}(\mathbb{G})$ and all $1 \leq j \leq n_x$. Thus,

$$\text{Re}\langle \tilde{\xi}, \xi \rangle = \sum_{x,i,j} \text{Re}(\hat{\varphi}(e_{ij}^x) \langle U_{ij}^x \xi, \xi \rangle) = \sum_{x,i} \frac{\dim_q(x)^2}{n_x} \text{Re}(\langle U_{ii}^x \xi, \xi \rangle) > 0.$$

□

Remark 4. It is easy to see that a discrete quantum group is amenable and has property T if and only if it is finite-dimensional. Indeed, the existence of almost invariant vectors for the regular representation is equivalent with amenability and it is well known that a discrete quantum group is finite dimensional if and only if the regular representation has a non-zero invariant vector. Moreover the previous proposition implies that all finite-dimensional discrete quantum groups have property T.

The main result of this paper is the following.

Theorem 3. *Let $\widehat{\mathbb{G}}$ be discrete quantum group such that $L^\infty(\mathbb{G})$ is an infinite dimensional factor. The following assertions are equivalent :*

1. *$\widehat{\mathbb{G}}$ has property T.*
2. *$L^\infty(\mathbb{G})$ is a II_1 factor with property T.*

Proof. We can suppose that $\mathbb{G}$ is reduced, $C(\mathbb{G}) \subset \mathcal{B}(L^2(\mathbb{G}))$ and $\mathbb{V} \in l^\infty(\widehat{\mathbb{G}}) \otimes$ Ł$^\infty(\mathbb{G})$. We denote by M the von Neumann algebra $L^\infty(\mathbb{G})$. For each $x \in \mathrm{Irred}(\mathbb{G})$ we choose an orthonormal basis $(e_i^x)_{1 \leq i \leq n_x}$ of H_x. When φ is a trace we take $e_i^{\bar{x}} = J_x(e_i^x)$. We put $u_{ij}^x = (\omega_{e_j^x, e_i^x} \otimes \mathrm{id})(u^x)$.

$1 \Rightarrow 2$: Suppose that $\widehat{\mathbb{G}}$ has property T. By Proposition 7, M is finite factor. Thus, it is a II_1 factor. Let (E, ϵ) be a Kazhdan pair for $\widehat{\mathbb{G}}$. Let $K \in \mathcal{C}(M)$ with morphisms $\pi_l : M \to \mathcal{B}(K)$ and $\pi_r : M^{op} \to \mathcal{B}(K)$. Let $\delta = \frac{\epsilon}{\mathrm{Max}\{n_x\sqrt{n_x},\, x \in E\}}$. Suppose that there exists a unit vector $\xi' \in K$ such that:

$$||u_{ij}^x \xi' - \xi' u_{ij}^x|| < \delta, \quad \forall x \in E,\ \forall 1 \leq i, j \leq n_x.$$

Define $U = (\mathrm{id} \otimes \pi_r)(\mathbb{V}^*)(\mathrm{id} \otimes \pi_l)(\mathbb{V})$. Because $\mathbb{V}$ is a unitary representation of $\widehat{\mathbb{G}}$ and π_r is an anti-homomorphism, it is easy to check that U is a unitary representation of $\widehat{\mathbb{G}}$ on K. Moreover, for all $x \in E$, we have:

$$\begin{aligned}
||U^x e_i^x \otimes \xi' - e_i^x \otimes \xi'|| &= ||(\mathrm{id} \otimes \pi_l)(u^x) e_i^x \otimes \xi' - (\mathrm{id} \otimes \pi_r)(u^x) e_i^x \otimes \xi'|| \\
&= ||\sum_{k=1}^{n_x} e_k^x \otimes (u_{ki}^x \xi' - \xi' u_{ki}^x)|| \\
&\leq \sum_{k=1}^{n_x} ||e_k^x \otimes (u_{ki}^x \xi' - \xi' u_{ki}^x)|| \\
&< n_x \delta \leq \frac{\epsilon}{\sqrt{n_x}}.
\end{aligned}$$

It follows easily that for all $x \in E$ and all $\eta \in H_x$ we have $||U^x \eta \otimes \xi' - \eta \otimes \xi'|| < \epsilon ||\eta||$. Thus there exists a non-zero U-invariant vector $\xi \in K$. It is easy to check that ξ is a central vector.

$2 \Rightarrow 1$: Suppose that M is a II_1 factor with property T and let $\epsilon > 0$ and $b_1, \ldots, b_n \in M$ be as in Proposition 5. Let φ be the Haar state on $\mathbb{G}$. By [7], Theorem 8, φ is the unique tracial state on M. We can suppose that $||b_i||_2 = 1$. Using the classical G.N.S. construction $(L^2(\mathbb{G}), \Omega)$ for φ we have, for all $a \in M$,

$$a\Omega = \sum_{x,k,l} n_x \varphi((u_{kl}^x)^* a) u_{kl}^x \Omega.$$

In particular, $||b_i||_2^2 = \sum n_x |\varphi((u_{kl}^x)^* b_i)|^2 = 1$. Fix $\delta > 0$ then there exists a finite subset $E \subset \mathrm{Irred}(\mathbb{G})$ such that, for all $1 \leq i \leq n$,

$$\sum_{x \notin E, k, l} n_x |\varphi((u_{kl}^x)^* b_i)|^2 < \delta^2.$$

Let U be a unitary representation of $\widehat{\mathbb{G}}$ on K having almost invariant vectors and $\xi \in K$ an (E,δ)-invariant unit vector. Turn $L^2(\mathbb{G}) \otimes K$ into a correspondence from M to M using the morphisms $\pi_l : M \to \mathcal{B}(L^2(\mathbb{G}) \otimes K)$, $\pi_l(a) = U(a \otimes 1)U^*$ and $\pi_r : M^{op} \to \mathcal{B}(L^2(\mathbb{G}) \otimes K)$, $\pi_r(a) = Ja^*J \otimes 1$, where J is the modular conjugation of φ. Let $\widehat{\xi} = \Omega \otimes \xi$. It is easy to see that $\pi_l(u^x_{kl}) = \sum_s u^x_{ks} \otimes U^x_{sl}$ and, for all $a \in M$,

$$a\widehat{\xi} = \sum n_x \varphi((u^x_{kl})^* a) u^x_{ks}\Omega \otimes U^x_{sl}\xi.$$

Note that, because φ is a trace, Ω is a central vector in $L^2(\mathbb{G})$ and we have, for all $a \in M$, $\widehat{\xi}a = a\Omega \otimes \xi$. It follows that, for all $1 \leq i \leq n$, we have

$$\begin{aligned}
\|b_i\widehat{\xi} - \widehat{\xi}b_i\|^2 &= \|\sum_{x,k,l,s} n_x \varphi((u^x_{kl})^* b_i) u^x_{ks}\Omega \otimes U^x_{sl}\xi - \sum_{x,k,l} n_x \varphi((u^x_{kl})^* b_i) u^x_{kl}\Omega \otimes \xi\|^2 \\
&= \|\sum_{x,k,l} n_x \varphi((u^x_{kl})^* b_i) \left(\sum_s u^x_{ks}\Omega \otimes U^x_{sl}\xi - u^x_{kl}\Omega \otimes \xi \right) \|^2 \\
&= \|\sum_{x,k,l} \sqrt{n_x} \varphi((u^x_{kl})^* b_i) \left(\sum_s e^x_s \otimes J_x(e^x_k) \otimes U^x_{sl}\xi - e^x_l \otimes J_x(e^x_k) \otimes \xi \right) \|^2 \\
&= \|\sum_{x,k,l} \sqrt{n_x} \varphi((u^x_{kl})^* b_i) J_x(e^x_k) \otimes \left(\sum_s e^x_s \otimes U^x_{sl}\xi - e^x_l \otimes \xi \right) \|^2 \\
&= \|\sum_{x,k,l} \sqrt{n_x} \varphi((u^x_{kl})^* b_i) J_x(e^x_k) \otimes (U^x e^x_l \otimes \xi - e^x_l \otimes \xi)\|^2 \\
&= \sum_{x,k} n_x \|\sum_l \varphi((u^x_{kl})^* b_i)(U^x e^x_l \otimes \xi - e^x_l \otimes \xi)\|^2 \\
&= \sum_{x,k} n_x \|U^x \eta^x_k \otimes \xi - \eta^x_k \otimes \xi)\|^2, \text{ where } \eta^x_k = \sum_l \varphi((u^x_{kl})^* b_i) e^x_l \\
&= \sum_{x \in E,k} n_x \|U^x \eta^x_k \otimes \xi - \eta^x_k \otimes \xi)\|^2 + \sum_{x \notin E,k} n_x \|U^x \eta^x_k \otimes \xi - \eta^x_k \otimes \xi)\|^2 \\
&< \delta^2 \sum_{x \in E,k} n_x \|\eta^x_k\|^2 + 4 \sum_{x \notin E,k} n_x \|\eta^x_k\|^2 \\
&< \delta^2 \sum_{x \in E,k,l} n_x |\varphi((u^x_{kl})^* b_i)|^2 + 4 \sum_{x \notin E,k,l} n_x |\varphi((u^x_{kl})^* b_i)|^2 \\
&< \delta^2 + 4\delta^2 = 5\delta^2.
\end{aligned}$$

By Proposition 5, for δ small enough, there exists a central unit vector $\widehat{\eta} \in L^2(\mathbb{G}) \otimes K$ with $\|\widehat{\eta} - \widehat{\xi}\| < \sqrt{5}C\delta$. Let P be the orthogonal projection on $\mathbb{C}\Omega$. If δ is small enough then there is a non-zero $\eta \in K$ such that $(P \otimes 1)\widehat{\eta} = \Omega \otimes \eta$. Write $\widehat{\eta} = \sum_{y,s,t} e^y_t \otimes e^{\bar{y}}_s \otimes \eta^y_{s,t}$ where $\eta^y_{s,t} \in K$ and $\eta^1 = \eta$. We have, for all

$x \in \mathrm{Irred}(\mathbb{G})$ and all $1 \leq i, j \leq n_x$, $\pi_l(u_{ij}^x)\widehat{\eta} = \pi_r(u_{ij}^x)\widehat{\eta}$. This means:

$$\sum_{k,y,t,s} u_{ik}^x(e_t^y \otimes e_s^{\bar{y}}) \otimes U_{kj}^x \eta_{s,t}^y = \sum_{y,t,s} J(u_{ij}^x)^* J(e_t^y \otimes e_s^{\bar{y}}) \otimes \eta_{s,t}^y. \tag{4}$$

Let Q be the orthogonal projection on $H_x \otimes H_{\bar{x}}$. Using

$$u_{ik}^x(e_t^y \otimes e_s^{\bar{y}}) \subset \bigoplus_{z \subset x \otimes y} H_z \otimes H_{\bar{z}},$$

and $x \subset x \otimes y$ if and only if $y = 1$, we find:

$$Qu_{ik}^x(e_t^y \otimes e_s^{\bar{y}}) = \delta_{y,1} \frac{1}{\sqrt{n_x}} e_k^x \otimes e_i^{\bar{x}}.$$

Using the same arguments and the fact that $J = \bigoplus (J_x \otimes J_{\bar{x}})$ we find:

$$QJ(u_{ij}^x)^* J(e_t^y \otimes e_s^{\bar{y}}) = \delta_{y,1} \frac{1}{\sqrt{n_x}} e_j^x \otimes e_i^{\bar{x}}.$$

Applying $Q \otimes 1$ to Eq. (4) we obtain:

$$\sum_k e_k^x \otimes e_i^{\bar{x}} \otimes U_{kj}^x \eta = e_j^x \otimes e_i^{\bar{x}} \otimes \eta, \quad \text{for all } x \in \mathrm{Irred}(\mathbb{G}),\ 1 \leq i, j \leq n_x.$$

Thus, for all $x \in \mathrm{Irred}(\mathbb{G})$ and all $1 \leq j \leq n_x$, we have:

$$U^x(e_j^x \otimes \eta) = \sum_k e_k^x \otimes U_{kj}^x \eta = e_j^x \otimes \eta.$$

Thus η is a non-zero U-invariant vector. □

The preceding theorem admits the following corollary about the persistance of property T by twisting.

Corollary 2. *Let $\mathbb{G}$ be a compact quantum group such that $L^\infty(\mathbb{G})$ is an infinite dimensional factor. Suppose that K is an abelian co-subgroup of $\mathbb{G}$ (see [8]). Let σ be a continuous bicharacter on $\widehat{K}$ and denote by $\mathbb{G}^\sigma$ the twisted quantum group. If $\widehat{\mathbb{G}}$ has property T then $\widehat{\mathbb{G}^\sigma}$ is a discrete quantum group with property T.*

Proof. If $\widehat{\mathbb{G}}$ has property T then the Haar state φ on $\mathbb{G}$ is a trace. Thus the co-subgroup K is stable (in the sense of [8]) and the Haar state φ_σ on $\mathbb{G}^\sigma$ is the same, i.e. $\varphi = \varphi_\sigma$. It follows that $\mathbb{G}^\sigma$ is a compact quantum group with $L^\infty(\mathbb{G}^\sigma) = L^\infty(\mathbb{G})$. Thus $L^\infty(\mathbb{G}^\sigma)$ is a II_1 factor with property T and $\widehat{\mathbb{G}_\sigma}$ has property T. □

Example 1. The group $SL_{2n+1}(\mathbb{Z})$ is I.C.C. and has property T for all $n \geq 1$. Let K_n be the subgroup of diagonal matrices in $SL_{2n+1}(\mathbb{Z})$. We have $K_n = \mathbb{Z}_2^{2n} = \langle t_1, \ldots, t_{2n} \,|\, t_i^2 = 1\ \forall i,\ t_i t_j = t_j t_i\ \forall i, j \rangle$ and K_n is an abelian co-subgroup of $\mathbb{G}_{2n+1} = (C^*(SL_{2n+1}(\mathbb{Z})), \Delta)$. Consider the following bicharacter on $\widehat{K_n} = K_n$: σ is the unique bicharacter such that $\sigma(t_i, t_j) = -1$ if $i \leq j$ and $\sigma(t_i, t_j) = 1$ if $i > j$. By the preceding Corollary, the twisted quantum group $\widehat{\mathbb{G}^\sigma_{2n+1}}$ has property T for all $n \geq 1$. When n is even, $SL_n(\mathbb{Z})$ is not I.C.C. and I and $-I$ lie in the centre of $SL_n(\mathbb{Z})$. We consider the group $PSL_n(\mathbb{Z}) = SL_n(\mathbb{Z})/\{I, -I\}$ in place of $SL_n(\mathbb{Z})$ in the even case. It is well known that $PSL_{2n}(\mathbb{Z})$ is I.C.C. and has property T for $n \geq 2$. The group of diagonal matrices in $SL_{2n}(\mathbb{Z})$ is $\mathbb{Z}_2^{2n-1}$ which contains $\{I, -I\}$. We consider the following abelian subgroup of $PSL_{2n}(\mathbb{Z})$: $L_n = \mathbb{Z}_2^{2n-1}/\{I, -I\} = \mathbb{Z}_2^{2n-2} = K_{n-1}$ and the same bicharacter σ on K_{n-1}. Let $\mathbb{G}_{2n} = (C^*(PSL_{2n}(\mathbb{Z})), \Delta)$. By the preceding Corollary, the twisted quantum group $\widehat{\mathbb{G}^\sigma_{2n}}$ has property T for all $n \geq 2$.

References

[1] T. Banica, Théorie des représentations du groupe quantique compact libre O(n), *C. R. Acad. Sci. Paris Sér. I Math.* **322** no. 3 (1996), 241–244.

[2] T. Banica, Le groupe quantique compact libre U(n), *Comm. Math. Phys.* **190** no. 1 (1997), 143–172.

[3] T. Banica and R. Vergnioux, Growth estimates for discrete quantum groups, *J. Noncommut. Geom.*, to appear.

[4] A. Connes and V. Jones, Property T for von Neumann algebras, *Bull. London Math. Soc.* **17** no. 1 (1985), 57–62.

[5] V.G. Drinfeld, Quantum groups, *Proceedings of the International Congress of Mathematicians*, **1, 2** Berkeley, Calif. (1986), pp. 798–820, *Amer. Math. Soc.*, Providence, RI (1987).

[6] M. Enock, Sous-facteurs intermédiaires et groupes quantiques mesurés, *J. Operator Theory* **42** no. 2 (1999), 305–330.

[7] P. Fima, On locally compact quantum groups whose algebras are factors. *J. Funct. Anal.* **244** no. 1 (2007), 78–94.

[8] P. Fima and L. Vainerman, Twisting and Rieffel's deformation of locally compact quantum groups. Deformation of the Haar measure, *Comm. Math. Phys.*, to appear.

[9] M. Jimbo, A q-difference analogue of U(g) and the Yang-Baxter equation, *Lett. Math. Phys.* **10** no. 1 (1985), 63–69.

[10] J. Kustermans, Induced corepresentations of locally compact quantum groups, *J. Funct. Anal.* **194** no. 2 (2002), 410–459.

[11] J. Kustermans and S. Vaes, Locally compact quantum groups, *Ann. Sci. École Norm. Sup.* (4) **33** no. 6 (2000), 837–934.

[12] A. Maes and A. Van Daele, Notes on Compact Quantum Groups, *Nieuw Arch. Wisk.* (4) **16** no. 1-2 (1998), 73–112.

[13] M. Rosso, Algèbres enveloppantes quantifiées, groupes quantiques compacts de matrices et calcul différentiel non commutatif., *Duke Math. J.* **61** no. 1 (1990), 11–40.

[14] R.Tomatsu, Amenable discrete quantum groups, *J. Math. Soc. Japan* **58** no. 4 (2006), 949–964.

[15] S. Vaes and R. Vergnioux, The boundary of universal discrete quantum groups, exactness and factoriality, *Duke Math. J.* **140** no. 1 (2007), 35–84.

[16] S. Vaes, The unitary implementation of a locally compact quantum group action, *J. Funct. Anal.* **180** no. 2 (2001),426–480.

[17] R. Vergnioux, K-amenability for amalgamated free products of amenable discrete quantum groups, *J. Funct. Anal.* **212** no. 1 (2004), 206–221.

[18] A. Van Daele and S. Wang, Universal quantum groups, *Internat. J. Math.* **7** no. 2 (1996), 255–263.

[19] S.L. Woronowicz, Compact matrix pseudogroup, *Comm. Math. Phys.* **111** no. 4 (1987), 613–665.

[20] S.L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups, *Invent. Math.* **93** no. 1 (1988), 35–76.

[21] S.L. Woronowicz, Compact quantum groups, *Symétries quantiques*, Les Houches (1995), pp. 845–884, North-Holland, Amsterdam (1998).