ON A COCYCLE IN THE ADJOINT REPRESENTATION OF THE ORTHOGONAL FREE QUANTUM GROUPS

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ABSTRACT. We show that the orthogonal free quantum groups are not inner amenable and we construct an explicit proper cocycle weakly contained in the regular representation. This strengthens the result of Vaes and the second author, showing that the associated von Neumann algebras are full II_1-factors and Brannan's result showing that the orthogonal free quantum groups have Haagerup's approximation property. We also deduce Ozawa-Popa's property strong (HH) and give a new proof of Isono's result about strong solidity.

1. INTRODUCTION

The orthogonal free quantum groups $\mathbb{F}O_n$ are given by universal unital C^*-algebras $A_o(n) = C^*(\mathbb{F}O_n)$ which where introduced by Wang [Wan95] as follows:

$$A_o(n) = C^*\langle v_{ij}, 1 \leq i,j \leq n \mid v_{ij} = v_{ij}^* \text{ and } (v_{ij}) \text{ unitary}\rangle.$$

Equipped with the coproduct defined by the formula $\Delta(v_{ij}) = \sum v_{ik} \otimes v_{kj}$, this algebra becomes a Woronowicz C^*-algebra [Wor98] and hence corresponds to a compact and a discrete quantum group in duality [Wor98, PW90], denoted respectively O_n^+ and $\mathbb{F}O_n$.

These C^*-algebras have been extensively studied since their introduction, and it has been noticed that the discrete quantum groups $\mathbb{F}O_n$ share many analytical features with the usual free groups F_n from the operator algebraic point of view. Let us just quote two such results that will be of interest for this article:

(1) the von Neumann algebras $\mathcal{L}(\mathbb{F}O_n)$ associated to $\mathbb{F}O_n$, $n \geq 3$, are full II_1 factors [VV07] ;
(2) the von Neumann algebras $\mathcal{L}(\mathbb{F}O_n)$ have Haagerup's approximation property [Bra12].

On the other hand there is one result that yields an operator algebraic distinction between $\mathbb{F}O_n$ and F_n: it was shown in [Ver12] that the first L^2-cohomology group of $\mathbb{F}O_n$ vanishes.

The purpose of this article is to present slight reinforcements and alternate proofs of the results (1), (2) above. Namely we will:

(1) show that the discrete quantum groups $\mathbb{F}O_n$ are not inner amenable (see [MvN43, Eff75] for non abelian free groups);
(2) construct an explicit proper cocycle witnessing Haagerup's property [DFSW13].

Moreover, putting these constructions together we will obtain a new result, namely Property strong (HH) from [OP10b], which is a strengthening of Haagerup's property and corresponds to the existence of a proper cocycle in a representation which is weakly contained in the regular representation. This sheds interesting light on the result from [Ver12] mentioned above, according to which such a proper cocycle cannot live in a representation strongly contained in the regular representation (or multiples of it). Note that the cocycle corresponding to the original proof [Haa79] of Haagerup's property for F_n does live in (multiples of) the regular representation.

Finally we will mention applications to solidity: indeed the constructions of the article and Property strong (HH) allow for two new proofs of the strong solidity of $\mathcal{L}(\mathbb{F}O_n)$ using the tools of [OP10a, OP10b]. Notice that strong solidity has already been obtained in [Iso12] using Property AO^+ from [Ver05, VV07] and the tools of [PV12].

2. Notation

If ζ, ξ are vectors in a Hilbert space H, we denote $\omega_{\zeta,\xi} = (\zeta| \cdot \xi)$ the associated linear form on $B(H)$, and we put $\omega_\zeta = \omega_{\zeta,\zeta} \in B(H)_*^+$.

We will use at some places the q-numbers $[k]_q = U_{k-1}(q + q^{-1}) = (q^k - q^{-k})/(q - q^{-1})$, where $(U_k)_{k \in \mathbb{N}}$ are the dilated Chebyshev polynomials of the second kind, defined by $U_0 = 1$, $U_1 = X$ and $XU_k = U_{k-1} + U_{k+1}$.

Following [KV00], a locally compact quantum group $\mathbb{G}$ is given by a Hopf-C^*-algebra $C_0(\mathbb{G})$, with coproduct denoted Δ, satisfying certain axioms including the existence of left and right invariant weights φ, $\psi : C_0(\mathbb{G})_+ \to [0, +\infty]$:

$$\varphi((\omega \otimes \mathrm{id})\Delta(f)) = \omega(1)\varphi(f) \quad \text{and} \quad \psi((\mathrm{id} \otimes \omega)\Delta(f)) = \omega(1)\psi(f),$$

for all $\omega \in C_0(\mathbb{G})_+^*$ and $f \in C_0(\mathbb{G})_+$ such that $\varphi(f) < \infty$. When there is no risk of confusion, we will denote $\|f\|_2^2 = \varphi(f^*f) \in [0, +\infty]$ the hilbertian norm associated with φ. Using a GNS construction (H, Λ_φ) for φ, one defines the fundamental multiplicative unitary $V \in B(H \otimes H)$, denoted W in [KV00], by putting $V^*(\Lambda_\varphi \otimes \Lambda_\varphi)(f \otimes g) = (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(g)(f \otimes 1))$. Recall that $\mathbb{G}$ is called unimodular, or of Kac type, if $\varphi = \psi$.

Now assume that $\mathbb{G}$ is discrete, meaning for instance that there exists a fixed vector $\xi_0 \in H$ such that $V(\xi_0 \otimes \zeta) = \xi_0 \otimes \zeta$ for all $\zeta \in H$. In that case one can always identify $C_0(\mathbb{G})$ with its image by the GNS representation on H, moreover it can be reconstructed from V as the norm closure of $(\mathrm{id} \otimes B(H)_*)(V)$, and the coproduct is given by the formula $\Delta(f) = V^*(1 \otimes f)V$.

From V one can also construct the dual algebra $C_{\mathrm{red}}^*(\mathbb{G})$ as the norm closure of $(B(H)_* \otimes \mathrm{id})(V)$, equipped with the coproduct $\hat{\Delta}(x) = V(x \otimes 1)V^*$ (opposite to the dual coproduct of [KV00]). Since $(\omega_{\xi_0} \otimes \mathrm{id})(V) = 1$, this C^*-algebra is unital, and it is in fact a Woronowicz C^*-algebra [Wor98]. Moreover the restriction of ω_{ξ_0} is the Haar state h of $C_{\mathrm{red}}^*(\mathbb{G})$. When there is no risk of confusion, we will denote $\|x\|_2^2 = \varphi(x^*x)$ the hilbertian norm associated with h. Using $\Lambda_h : x \mapsto x\xi_0$ as a GNS construction for h we have $(\Lambda_h \otimes \Lambda_h)(\hat{\Delta}(x)(1 \otimes y)) = \hat{\Delta}(x)V(\xi_0 \otimes y\xi_0) = V(x\xi_0 \otimes y\xi_0)$, so that V coincides with the multiplicative unitary associated to $(C_{\mathrm{red}}^*(\mathbb{G}), \hat{\Delta})$ as constructed in [BS93].

The von Neumann algebras associated to $\mathbb{G}$ are denoted $L^\infty(\mathbb{G}) = C_0(\mathbb{G})''$ and $M = \mathcal{L}(\mathbb{G}) = C_{\mathrm{red}}^*(\mathbb{G})''$, they carry natural extensions of the respective coproducts and Haar weights.

Recall that a corepresentation X of V is a unitary $X \in B(H \otimes H_X)$ such that $V_{12}X_{13}X_{23} = X_{23}V_{12}$. There exists a so-called maximal C^*-algebra $C^*(\mathbb{G})$ together with a corepresentation $\mathbb{V} \in M(C_0(\mathbb{G}) \otimes C^*(\mathbb{G}))$ such that any corepresentation X corresponds to a unique $*$-representation $\pi : C^*(\mathbb{G}) \to B(H_X)$ via the formula $X = (\mathrm{id} \otimes \pi)(\mathbb{V})$. We say that X, π are unitary representations of $\mathbb{G}$. When $X = V$, we obtain the regular representation $\pi = \lambda : C^*(\mathbb{G}) \to C_{\mathrm{red}}^*(\mathbb{G})$. Moreover $C^*(\mathbb{G})$ admits a unique Woronowicz C^*-algebra structure given by $(\mathrm{id} \otimes \Delta)(\mathbb{V}) = \mathbb{V}_{12}\mathbb{V}_{13}$, and λ is then the GNS representation associated with the Haar state h of $C^*(\mathbb{G})$.

On the other hand, a unitary corepresentation of $\mathbb{G}$, or representation of V, is a unitary $Y \in M(K(H_Y) \otimes C^*(\mathbb{G}))$ such that $(\mathrm{id} \otimes \Delta)(Y) = Y_{12}Y_{13}$, corresponding to a $*$-representation of $C_0(\mathbb{G})$ as above. We denote $\mathrm{Corep}(\mathbb{G})$ the category of finite dimensional unitary corepresentations of $\mathbb{G}$, and we choose a complete set $\mathrm{Irr}\,\mathbb{G}$ of representatives of irreducible ones. In the discrete case, it is known that any corepresentation decomposes in $\mathrm{Irr}\,\mathbb{G}$. We have in particular a $*$-isomorphism $C_0(\mathbb{G}) \simeq c_0\text{-}\bigoplus_{r \in \mathrm{Irr}\,\mathbb{G}} B(H_r)$ such that V corresponds to $\bigoplus_{r \in \mathrm{Irr}\,\mathbb{G}} r$. We denote $p_r \in C_0(\mathbb{G})$ the minimal central projection corresponding to $B(H_r)$, and $C_c(\mathbb{G}) \subset C_0(\mathbb{G})$ the algebraic direct sum of the blocks $B(H_r)$. The $*$-algebra $C_c(\mathbb{G})$, together with the restriction of Δ, is a multiplier Hopf algebra in the sense of [VD94], and we denote its antipode by $\hat{S}$.

On the other hand we denote $C^*(\mathbb{G})_r \subset C^*(\mathbb{G})$ the subspace of coefficients $(\omega \otimes \mathrm{id})(r)$ of an irreducible corepresentation r, and $\mathbb{C}[\mathbb{G}]$ the algebraic direct sum of these subspaces. The coproduct on $C^*(\mathbb{G})$ restricts to an algebraic coproduct on $\mathbb{C}[\mathbb{G}]$, which becomes then a plain Hopf $*$-algebra. Recall also that λ restricts to an injective map on $\mathbb{C}[\mathbb{G}]$, hence we shall sometimes consider $\mathbb{C}[\mathbb{G}]$ as a subspace of $C_{\mathrm{red}}^*(\mathbb{G})$. We will denote by ϵ and S the co-unit and the antipode of $\mathbb{C}[\mathbb{G}]$, and we note that ϵ extends to $C^*(\mathbb{G})$: it is indeed the $*$-homomorphism corresponding to $X = \mathrm{id}_H \otimes \mathrm{id}_\mathbb{C}$.

We denote $(f_z)_z$ the Woronowicz characters of $C^*(\mathbb{G})$, satisfying in particular $h(xy) = h(y(f_1 * x * f_1))$, where $\phi * x = (\mathrm{id} \otimes \phi)\Delta(x)$ and $x * \phi = (\phi \otimes \mathrm{id})\Delta(x)$. We put as well $F_v = (\mathrm{id} \otimes f_1)(v) \in$

$B(H_v)$ for any f.-d. unitary corepresentation $v \in B(H_v) \otimes C^*(\mathbb{G})$ and $\operatorname{qdim} v = \operatorname{Tr} F_v$. Denoting $v_{\zeta,\xi} = (\omega_{\zeta,\xi} \otimes \operatorname{id})(v)$, the Schur orthogonality relations [Wor87, (5.15)] read, for v, w irreducible:

$$h(v_{\zeta,\xi}^* w_{\zeta',\xi'}) = \frac{\delta_{v,w}}{\operatorname{qdim} v}(\zeta'|F_v^{-1}\zeta)(\xi|\xi').$$

Note that in Woronowicz' notation we have $v_{kl} = v_{e_k,e_l}$ if $(e_i)_i$ is a fixed ONB of H_v. Similarly, we have the formula $\varphi(a) = \operatorname{qdim}(v)\operatorname{Tr}(F_v a)$ for the left Haar weight φ and $a \in B(H_r) \subset C_0(\mathbb{G})$ [PW90, (2.13)] — note that the coproduct on $C_0(\mathbb{G})$ constructed from V in [PW90] is opposite to our, so that $\varphi = h_{dR}$ with their notation.

Finally we will use the following analogue of the Fourier transform:

$$\mathcal{F} : C_c(\mathbb{G}) \to C_{\mathrm{red}}^*(\mathbb{G}), \ a \mapsto (\varphi \otimes \operatorname{id})(V(a \otimes 1)).$$

On can show that it is isometric w.r.t. the scalar products $(a|b) = \varphi(a^*b)$, $(x|y) = h(x^*y)$ for a, $b \in C_c(\mathbb{G})$, x, $y \in C_{\mathrm{red}}^*(\mathbb{G})$ — see also Section 3.1.

In this article we will be mainly concerned with orthogonal free quantum groups. For $Q \in GL_n(\mathbb{C})$ such that $Q\bar{Q} \in \mathbb{C}I_N$, we denote $A_o(Q)$ the universal unital C^*-algebra generated by n^2 elements v_{ij} subject to the relations $Q\bar{v}Q^{-1} = v$ and $v^*v = vv^* = I_n \otimes 1$, where $v = (v_{ij})_{ij} \in M_n(A_o(n))$ and $\bar{v} = (v_{ij}^*)_{ij}$. Equipped with the coproduct Δ given by $\Delta(v_{ij}) = \sum v_{ik} \otimes v_{kj}$, it is a full Woronowicz C^*-algebra, and we denote $\mathbb{G} = \mathbb{F}O(Q)$ the associated discrete quantum group, such that $C^*(\mathbb{F}O(Q)) = A_o(Q)$. In the particular case $Q = I_n$ we denote $A_o(n) = A_o(I_n)$ and $\mathbb{F}O(n) = \mathbb{F}O(I_n)$.

It is known from [Ban96] that the irreducible corepresentations of $\mathbb{F}O(Q)$ can be indexed v^k, $k \in \mathbb{N}$ up to equivalence, in such a way that $v^0 = \operatorname{id}_{\mathbb{C}} \otimes 1$, $v^1 = v$, and the following fusion rules hold

$$v^k \otimes v^l \simeq v^{|k-l|} \oplus v^{|k-l|+2} \oplus \cdots \oplus v^{k+l}.$$

Moreover the contragredient of v^k is equivalent to v^k for all k. The classical dimensions $\dim v^k$ satisfy $\dim v^0 = 1$, $\dim v^1 = n$ and $n \dim v^k = \dim v^{k-1} + \dim v^{k+1}$. We denote $\rho \geq 1$ the greatest root of $X^2 - nX + 1$, so that $\dim v^k = [k+1]_\rho$. Similarly we have $\operatorname{qdim} v^0 = 1$ and $\operatorname{qdim} v^1 \operatorname{qdim} v^k = \operatorname{qdim} v^{k-1} + \operatorname{qdim} v^{k+1}$. Moreover, if Q is normalized in such a way that $Q\bar{Q} = \pm I_n$, we have $\operatorname{qdim} v = \operatorname{Tr}(Q^*Q)$. We denote $q \in]0,1]$ the smallest root of $X^2 - (\operatorname{qdim} v)X + 1$, so that $\operatorname{qdim} v^k = [k+1]_q$.

3. The adjoint representation of $\mathbb{F}O_n$

The aim of this section is to prove that the non-trivial part ad° of the adjoint representation of $\mathbb{F}O_n$ factors through the regular representation. Equivalently, we will prove that all states of the form $\omega_\xi \circ \operatorname{ad}^\circ$, with $\xi \in H$, factor through $C_{\mathrm{red}}^*(FO_n)$. For this we will use the criterion given by Lemma 3.5 below, which is analogous to Theorem 3.1 of [Haa79].

3.1. **Weak containment in the regular representation.** In this section we gather some useful results which are well-known in the classical case. We say that an element $f \in L^\infty(\mathbb{G})$ is a (normalized) positive type function if there exists a state ϕ on $C^*(\mathbb{G})$ such that $f = (\operatorname{id} \otimes \phi)(\mathbb{V})$. The associated multiplier is $M_\phi = (\operatorname{id} \otimes \phi)\Delta : \mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{G}]$, it extends to a bounded map on $C_{\mathrm{red}}^*(\mathbb{G})$ characterized by the identity $(\operatorname{id} \otimes M)(V) = V(f \otimes 1)$.

Lemma 3.1. *Let ϕ be a unital linear form on $\mathbb{C}[\mathbb{G}]$ and consider $M = (\operatorname{id} \otimes \phi)\Delta : \mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{G}]$. Then M extends to a completely positive map on $C_{\mathrm{red}}^*(\mathbb{G})$ iff ϕ extends to a state of $C^*(\mathbb{G})$.*

Proof. By Fell's absorption principle $V_{12}\mathbb{V}_{13} = \mathbb{V}_{23}V_{12}\mathbb{V}_{23}^*$, Δ extends to a $*$-homomorphism $\Delta' : C_{\mathrm{red}}^*(\mathbb{G}) \to C_{\mathrm{red}}^*(\mathbb{G}) \otimes C^*(\mathbb{G})$. Hence if ϕ extends to a state of $C^*(\mathbb{G})$, M extends to a CP map $M = (\operatorname{id} \otimes \phi)\Delta'$ on $C_{\mathrm{red}}^*(\mathbb{G})$. For the reverse implication, since $C^*(\mathbb{G})$ is the enveloping C^*-algebra of $\mathbb{C}[\mathbb{G}]$, it suffices to prove that $\phi(xx^*) \geq 0$ for all $x \in \mathbb{C}[\mathbb{G}]$

We choose a corepresentation $v \in B(H_v) \otimes C_{\mathrm{red}}^*(\mathbb{G})$. Let L be a GNS space for $B(H_v)$, with respect to an arbitrary given state. For $a \in B(H_v)$, resp. $\omega \in B(H_v)^*$, we denote $\hat{a}$, $\hat{\omega}$ the corresponding elements of $L = B(\mathbb{C}, L)$ resp. $L^* = B(L, \mathbb{C})$, and we identify $B(H_v)$ with a subspace of $B(L)$ via left multiplication. Similarly we denote $\hat{v}$ the element of $B(\mathbb{C}, L) \otimes C_{\mathrm{red}}^*(\mathbb{G})$ corresponding to v. We have then $\hat{v}\hat{v}^* \in B(L) \otimes C_{\mathrm{red}}^*(\mathbb{G})$ and $(\hat{\omega} \otimes 1)\hat{v}\hat{v}^*(\hat{\omega}^* \otimes 1) = xx^*$ if $x = (\omega \otimes \operatorname{id})(v)$.

If M is CP, the following element of $B(L) \otimes C^*_{\mathrm{red}}(\mathbb{G})$ is positive:

$$(\mathrm{id} \otimes M)(\hat{v}\hat{v}^*) = (\mathrm{id} \otimes \mathrm{id} \otimes \phi)(\mathrm{id} \otimes \Delta)(\hat{v}\hat{v}^*)$$
$$= (\mathrm{id} \otimes \mathrm{id} \otimes \phi)(v_{12}\hat{v}_{13}\hat{v}^*_{13}v^*_{12}) = v(X \otimes 1)v^*,$$

where $X = (\mathrm{id} \otimes \phi)(\hat{v}\hat{v}^*) \in B(L)$. We conclude that X is positive, hence $\phi(xx^*) = \hat{\omega}X\hat{\omega}^* \geq 0$ for any $x \in \mathbb{C}[\mathbb{G}]$. $\qquad\square$

We will be particularly interested in the case when ϕ factors through $C^*_{\mathrm{red}}(\mathbb{G})$, or equivalently, when ϕ is a weak limit of states of the form $\sum_{i=1}^{n} \omega_{\xi_i} \circ \lambda$ with $\xi_i \in H$. In that case we say that f, ϕ are weakly associated to λ, or weakly ℓ^2. We have the following classical Lemma [Fel63, Lemma 1]:

Lemma 3.2. *Let $\pi : C^*(\mathbb{G}) \to B(K)$ be a $*$-representation. Assume that there exists a subset $X \subset K$ such that $\overline{\mathrm{Span}}\ \pi(\mathbb{C}[\mathbb{G}])X = K$ and $\omega_\xi \circ \pi$ is weakly ℓ^2 for all $\xi \in X$. Then π factors through $C^*_{\mathrm{red}}(\mathbb{G})$.*

Proof. As noted in the original paper of Fell, the proof for groups applies in fact to general C^*-algebras, and in particular to discrete quantum groups. More precisely, take $A = C^*(\mathbb{G})$, $T = \pi$ and $\mathcal{S} = \{\lambda\}$ in [Fel63, Rk 1]. $\qquad\square$

On the other hand we say that $\phi \in \mathbb{C}[\mathbb{G}]^*$ is an ℓ^2-form if it is continuous with respect to the ℓ^2-norm on $\mathbb{C}[\mathbb{G}]$, i.e. there exists $K \in \mathbb{R}$ such that $|\phi(x)|^2 \leq Kh(x^*x)$ for all $x \in \mathbb{C}[\mathbb{G}]$. In that case we denote $\|\phi\|_2$ the corresponding norm. Clearly, if ϕ is ℓ^2 then it is weakly ℓ^2.

Although we will not need this in the remainder of this article, the following lemma shows that ℓ^2-forms can also be characterized in terms of the associated "functions" in $C_0(\mathbb{G})$ by mean of the left Haar weight. Note that in the unimodular case we have simply $\varphi(ff^*) = \varphi(f^*f) = \|f\|_2^2$.

Lemma 3.3. *Let $\phi \in \mathbb{C}[\mathbb{G}]^*$ be a linear form and put $f = (\mathrm{id} \otimes \phi)(V)$. Then ϕ is an ℓ^2-form **iff** $\varphi(ff^*) < \infty$.*

Proof. We put $p_0 = (\mathrm{id} \otimes h)(V) \in C_c(\mathbb{G})$, which is also the central support of $\hat{\epsilon}$, and we note the following identity (see the Remark after the proof) in the multiplier Hopf algebra $C_c(\mathbb{G})$:

$$(1) \qquad\qquad \forall c \in C_c(\mathbb{G})\ (\hat{S} \otimes \varphi)(\Delta(p_0)(1 \otimes f)) = f.$$

From this we can deduce that the scalar product in H implements, via the Fourier transform, the natural duality between $C_c(\mathbb{G})$ and $\mathbb{C}[\mathbb{G}]$ which is given, for $x = (\omega \otimes \mathrm{id})(V) \in \mathbb{C}[\mathbb{G}]$ and $f = (\mathrm{id} \otimes \phi)(V) \in C_c(\mathbb{G})$, by $\langle f, x \rangle = (\omega \otimes \phi)(V) = \phi(x) = \omega(f)$. Indeed we have, using the identities $(\hat{S} \otimes \mathrm{id})(V) = V^*$ and $V_{13}V_{23} = (\Delta \otimes \mathrm{id})(V)$:

$$(x\xi_0|\mathcal{F}(f)\xi_0) = h(x^*\mathcal{F}(f)) = (\varphi \otimes h)((1 \otimes x^*)V(f \otimes 1))$$
$$= (\bar{\omega}\hat{S} \otimes \varphi \otimes h)(V_{13}V_{23}(1 \otimes f \otimes 1)) = (\bar{\omega}\hat{S} \otimes \varphi)(\Delta(p_0)(1 \otimes f)) = \bar{\omega}(f).$$

This yields, for $x \in \mathbb{C}[\mathbb{G}]$ and ϕ such that $f = (\mathrm{id} \otimes \phi)(V) \in C_c(\mathbb{G})$, the formula $\phi(x) = \omega(f) = (\mathcal{F}(f^*)\xi_0|x\xi_0)$, which holds in fact for any $x \in C^*_{\mathrm{red}}(\mathbb{G})$ by continuity. We get in particular $\|\phi\|_2 = \|\mathcal{F}(f^*)\|_2 = \|f^*\|_2 = \sqrt{\varphi(ff^*)}$.

Now for a general ϕ and $r \in \mathrm{Irr}\,\mathbb{G}$, denote ϕ_r the restriction of ϕ to $\mathbb{C}[\mathbb{G}]_r$. We have $\|\phi\|_2^2 = \sum_{r \in \mathrm{Irr}\,\mathbb{G}} \|\phi_r\|_2^2$, $(\mathrm{id} \otimes \phi_r)(V) = p_r f$ and $\varphi(ff^*) = \sum_{r \in \mathrm{Irr}\,\mathbb{G}} \varphi(ff^*p_r)$, so that the identity $\|\phi\|_2^2 = \varphi(ff^*)$ still holds in $[0, +\infty]$. $\qquad\square$

Remark 3.4. The identity (1) is equivalent to the fact that the Fourier transform $\mathcal{F}$ is isometric. Indeed a simple computation yields

$$h(\mathcal{F}(a)^*\mathcal{F}(b)) = \varphi[a^*(\hat{S} \otimes \varphi)(\Delta(p_0)(1 \otimes b))].$$

Note that (1) can be easily proved by standard computations in $C_c(\mathbb{G})$, and this is one convenient way of establishing the fact that $\mathcal{F}$ is isometric.

Now we consider the case of the discrete quantum group $\mathbb{F}O_n$, whose irreducible corepresentations $v^k \in B(H_k) \otimes A_o(n)$ are labeled, up to equivalence, by integers k. Recall that we

denote $\mathbb{C}[\mathbb{G}]_k = C^*_{\mathrm{red}}(\mathbb{G})_k$ the corresponding coefficient subspaces, and $(U_k)_k$ the series of dilated Chebyshev polynomials of the second kind. In [Bra12], Brannan shows that the multiplier $T_s : C^*_{\mathrm{red}}(\mathbb{F}O_n) \to C^*_{\mathrm{red}}(\mathbb{F}O_n)$ associated with the functional

$$\tau_s : \mathbb{C}[\mathbb{F}O_n] \to \mathbb{C}, \quad x \in \mathbb{C}[\mathbb{F}O_n]_k \mapsto \frac{U_k(s)}{U_k(n)} \epsilon(x)$$

is a completely positive as above, for every $s \in \,]2, n]$ — see also Section 4.1.

Besides it is known that $\mathbb{F}O_n$ satisfies the Property of Rapid Decay. More precisely, for any $k \in \mathbb{N}$ and any $x \in C^*_{\mathrm{red}}(\mathbb{G})_k$ we have $\|x\| \le C(1+k)\|x\|_2$, where C is constant depending only on n [?]. On the other hand we denote by $l : \mathbb{C}[\mathbb{G}] \to \mathbb{C}$ the length form on $\mathbb{F}O_n$ which coincides with $k\epsilon$ on $\mathbb{C}[\mathbb{G}]_k$, and we recall that the convolution of two linear forms ϕ, $\psi \in \mathbb{C}[\mathbb{G}]^*$ is defined by $\phi * \psi = (\phi \otimes \psi)\Delta$. It is well-known that the convolution exponential $e^\phi = \sum \phi^{*n}/n!$ is well-defined on $\mathbb{C}[\mathbb{G}]$, and in the case of l (or any other central form) we have simply $e^{-\lambda l}(x) = e^{-\lambda k}\epsilon(x)$ if $x \in \mathbb{C}[\mathbb{G}]_k$.

Lemma 3.5. *A state $\phi \in C^*(\mathbb{F}O_n)^*$ factors through $C^*_{\mathrm{red}}(\mathbb{F}O_n)$ iff $e^{-\lambda l} * \phi$ is ℓ^2 for all $\lambda > 0$.*

Proof. We denote by ϕ_k the restriction of ϕ to the f.-d. subspace of $\mathbb{C}[\mathbb{G}]_k$. Since these subspaces are pairwise orthogonal, we have $\|\phi\|_2^2 = \sum \|\phi_k\|^2$. Now for every $s < n$ there is a $\lambda > 0$ such that $U_k(s)/U_k(n) \le e^{-\lambda k}$ for all k. We can then write

$$\|\tau_s * \phi\|_2^2 = \sum \|(\tau_s * \phi)_k\|^2 = \sum \frac{U_k(s)^2}{U_k(n)^2}\|\phi_k\|^2 \le \sum e^{-2\lambda k}\|\phi_k\|^2 = \|e^{-\lambda l} * \phi\|_2^2.$$

Now if $e^{-\lambda l} * \phi$ is ℓ^2 for all $\lambda > 0$, we conclude that this is also the case of $\tau_s * \phi$ for all $s \in \,]2, n]$. In particular $\tau_s * \phi$ is weakly ℓ^2 for all s. We have $(\tau_s * \phi)(x) \to \phi(x)$ as $s \to n$ for all $x \in \mathbb{C}[\mathbb{G}]$, and $\tau_s * \phi$ is a state for all s, hence $\tau_s * \phi \to \phi$ weakly and it follows that ϕ is weakly ℓ^2.

Conversely, if ϕ factors through $C^*_{\mathrm{red}}(\mathbb{G})$, using Property RD we can write for any $x \in C^*_{\mathrm{red}}(\mathbb{G})_k$:

$$|\phi(x)| \le \|x\| \le C(1+k)\|x\|_2,$$

hence $\|\phi_k\|_2 \le C(1+k)$. This clearly implies that $\|e^{-\lambda l} * \phi\|_2$ is finite for every $\lambda > 0$. $\qquad\square$

3.2. **The adjoint representation.** Recall that we denote by $\lambda : C^*(\mathbb{G}) \to B(H)$ the GNS representation of the Haar state h, and consider the corresponding right regular representation $\rho : C^*(\mathbb{G}) \to B(H)$, $x \mapsto U\lambda(x)U$ given by the unitary $U(\Lambda_h(x)) = \Lambda_h(f_1 * S(x)) = \Lambda_h(S(x * f_{-1}))$. We have in particular $\rho(v_{ij})\Lambda_h(x) = \Lambda_h(x(f_1 * v_{ji}^*))$, if $v_{ij} = v_{e_i,e_j}$ are the coefficients of a unitary corepresentation in an ONB. Recall that $[UC^*_{\mathrm{red}}(\mathbb{G})U, C^*_{\mathrm{red}}(\mathbb{G})] = 0$.

The adjoint representation of $\mathbb{G}$ is $\mathrm{ad} : C^*(\mathbb{G}) \to B(H)$, $x \mapsto \sum \lambda(x_{(1)})\rho(x_{(2)})$. Here $\Delta : x \mapsto \sum x_{(1)} \otimes x_{(2)}$ is the coproduct from $C^*(\mathbb{G})$ to $C^*_{\mathrm{red}}(\mathbb{G}) \otimes_{\max} C^*_{\mathrm{red}}(\mathbb{G})$, which is well-defined because a product of two commuting corepresentations of V is again a corepresentation. We have $(\mathrm{id} \otimes \mathrm{ad})(v) = (\mathrm{id} \otimes \lambda)(v)(\mathrm{id} \otimes \rho)(v)$ if v is a corepresentation of $\mathbb{G}$. Here are two explicit formulae, for $x \in \mathbb{C}[\mathbb{G}]$ and v_{ij} coefficient of a unitary corepresentation in an ONB:

$$\mathrm{ad}(x)\Lambda_h(y) = \sum \Lambda_h(x_{(1)}y(f_1 * S(x_{(2)}))),$$
$$\mathrm{ad}(v_{ij})\Lambda_h(y) = \sum \Lambda_h(v_{ik}y(f_1 * v_{jk}^*)).$$

The corepresentation of V associated to ρ is $W = (1 \otimes U)V(1 \otimes U)$, and the one associated to ad is $A = VW$. Note that we have, in the notation of [BS93], $W = \Sigma\tilde{V}\Sigma$. In particular [BS93, Prop. 6.8] shows that $W(1 \otimes f)W^* = \sigma\Delta(f)$ for $f \in C_0(\mathbb{G})$. This means that the multiplicative unitary W^* is associated to the discrete quantum group $\mathbb{G}^{\mathrm{co-op}}$.

Lemma 3.6. *The canonical line $\mathbb{C}\xi_0 \subset H$ is invariant for the adjoint representation $\mathrm{ad} : C^*(\mathbb{G}) \to B(H)$ iff $\mathbb{G}$ is unimodular. In that case, ξ_0 is a fixed vector for ad.*

Proof. For $x \in \mathbb{C}[\mathbb{G}]$ one can compute $\mathrm{ad}(x)\xi_0 = \sum x_{(1)}(f_1 * S(x_{(2)}))\xi_0$, which equals in the unimodular case $\sum x_{(1)}S(x_{(2)})\xi_0 = \epsilon(x)\xi_0$. So in that case ξ_0 is fixed.

Furthermore, for $v \in \operatorname{Irr}\mathbb{G}$ the previous computation together with Woronowicz' orthogonality relations lead to

$$(\xi_0| \operatorname{ad}(v_{ij})\xi_0) = \sum_k h(v_{ik}(f_1 * v_{jk}^*)) = \sum_k h((v_{jk}^* * f_{-1})v_{ik})$$

$$= \sum_k \frac{\delta_{ji}\delta_{kk}}{\operatorname{qdim} v} = \frac{\dim v}{\operatorname{qdim} v}\epsilon(v_{ij}).$$

In particular, denoting $\phi : x \mapsto (\xi_0| \operatorname{ad}(x)\xi_0)$, we see that $(\operatorname{id} \otimes \phi)(v) = (\dim v / \operatorname{qdim} v)\operatorname{id}$. Now if the line $\mathbb{C}\xi_0$ is invariant, ϕ is a character and since v is unitary we must have $\dim v = \operatorname{qdim} v$ for all $v \in \operatorname{Irr}\mathbb{G}$. This happens exactly when $\mathbb{G}$ is unimodular. $\qquad\square$

Definition 3.7. If $\mathbb{G}$ is a unimodular discrete quantum group, we denote ad° the restriction of the adjoint representation of $C^*(\mathbb{G})$ to the subspace $H^\circ = \xi_0^\perp \subset H$.

Our aim in the remainder of the section is to show that ad° factors through λ in the case of $\mathbb{G} = \mathbb{F}O_n$, $n \geq 3$ — or more generally for unimodular orthogonal free quantum groups. We will need some estimates involving coefficients of irreducible corepresentations of $\mathbb{G}$.

In order to carry on the computations, we will need to be more precise about the irreducible corepresentations of $\mathbb{F}O(Q)$. We denote by $v = v^1 \in B(\mathbb{C}^n) \otimes A_o(Q)$ the fundamental corepresentation. Then we introduce recursively the irreducible corepresentation $v^k \in B(H_k) \otimes A_o(Q)$ as the unique subcorepresentation of $v^{\otimes k}$ not equivalent to any v^l, $l < k$. In this way we have $H_k \subset H_1^{\otimes k}$, with $H_0 = \mathbb{C}$ and $H_1 = \mathbb{C}^n$. We denote by $P_k \in B(H_1^{\otimes k})$ the orthogonal projection onto H_k. Recall that each irreducible corepresentation of $A_o(Q)$ is equivalent to exactly one v^k, and that we have the equivalence of corepresentations

$$H_k \otimes H_l \simeq H_{|k-l|} \oplus H_{|k-l|+2} \oplus \cdots \oplus H_{k+l-2} \oplus H_{k+l}.$$

It is known that $\dim H_k = [k+1]_\rho = (\rho^{k+1} - \rho^{-k-1})/(\rho - \rho^{-1})$. Note in particular that we have $D_1\rho^k \leq \dim H_k \leq D_2\rho^k$ with constants $0 < D_1 < D_2$ depending only on n.

Let us also denote by $Q_r^k \in L(H_1^{\otimes k})$ the orthogonal projection onto the sum of all subspaces equivalent to H_r, so that $P_k = Q_k^k$. If p_r is the minimal central projection of $C_0(\mathbb{F}O_n)$ associated with v^r, then Q_r^k also corresponds to $\Delta^{k-1}(p_r)$ via the natural action of $C_0(\mathbb{F}O_n)$ on H_1.

Note that the subspace $\operatorname{Im} Q_r^{a+b+c}(P_{a+b} \otimes P_c)$ (resp. $\operatorname{Im} Q_r^{a+b+c}(P_a \otimes P_{b+c})$) corresponds to the unique subcorepresentation of $v^{a+b} \otimes v^c$ (resp. $v^a \otimes v^{b+c}$) equivalent to v^r, when it is non-zero. When $r = a+b+c$ both spaces coincide with H_{a+b+c}. On the other hand one can show that these subspaces of $H_1^{\otimes a+b+c}$ are pairwise "far from each other" when $r < a+b+c$ and b is big. More precisely, Lemma A.4 of [VV07] shows that

$$\|(P_{a+b} \otimes P_c)Q_r^{a+b+c}(P_a \otimes P_{b+c})\| \leq C_1 q^b$$

for some constant $C_1 > 0$ depending only on q. Indeed when r varies up to $a+b+c-2$ the maps on the left-hand side live in pairwise orthogonal subspaces of $H_1^{\otimes a+b+c}$ and sum up to $(P_{a+b} \otimes P_c)(P_a \otimes P_{b+c}) - P_{a+b+c}$.

Since we are interested in ad°, we assume in the remainder of this section that $\mathbb{F}O(Q)$ is unimodular : equivalently, Q is a scalar multiple of a unitary matrix, or $q\rho = 1$.

In the following lemma we give an upper estimate for $|(\omega_\zeta \otimes \operatorname{Tr}_k)((P_l \otimes P_k)Q_r^{k+l}\Sigma_{lk})|$, where ζ is any vector in H_l, Tr_k is the trace on $B(H_k)$, and $\Sigma_{lk} : H_l \otimes H_k \to H_k \otimes H_l$ is the flip map. Note that $\|(P_l \otimes P_k)Q_r^{k+l}\Sigma_{lk}\| \leq 1$ so that $(\dim H_k)\|\zeta\|^2$ is a trivial upper bound, which grows exponentially with k. In the lemma we derive an upper bound which is linear in k (and even constant for $r < k+l$).

Note that it is quite natural to consider maps like $(P_l \otimes P_k)Q_r^{k+l}\Sigma_{lk}$ when studying the adjoint representation of $A_o(n)$. A non-trivial upper bound for the norm $\|(P_1 \otimes P_k)Q_{k-1}^{k+1}\Sigma_{1k}\|$ was given in [VV07, Lemma 7.11], but it does not imply our tracial estimate below.

Lemma 3.8. *Let $(e_p^k)_p$ be an ONB of H_k. There exists a constant $C \geq 1$, depending only on n, such that we have, for any $0 < l \leq k$, $\zeta \in H_l$ and $r = k-l, k-l+2, \ldots, k+l-2$:*

$$\left| \sum_p (\zeta \otimes e_p^k | Q_r^{k+l}(e_p^k \otimes \zeta)) \right| \leq Cq^{-l}\|\zeta\|^2 \ \text{and} \ \left| \sum_p (\zeta \otimes e_p^k | P_{k+l}(e_p^k \otimes \zeta)) \right| \leq Ckq^{-l}\|\zeta\|^2.$$

Proof. In this proof C denotes a generic constant depending only on n. Let us denote by $S_r^{k,l}$, $S_+^{k,l}$ the sums in the statement as well as $S^{k,l} = S_+^{k,l} + \sum_r S_r^{k,l}$. We first prove the estimate for $S_r^{k,l}$: using the Lemma A.4 from [VV07] recalled above, with $a = c = l$ and $b = k - l$, we can write

$$|S_r^{k,l}| = \left| \sum_p (\zeta \otimes e_p^k | Q_r^{k+l}(e_p^k \otimes \zeta)) \right| \leq \sum_p |(\zeta \otimes e_p^k | (P_l \otimes P_k) Q_r^{k+l}(P_k \otimes P_l)(e_p^k \otimes \zeta))|$$

$$\leq \|(P_l \otimes P_k) Q_r^{k+l}(P_k \otimes P_l)\| \sum_p \|\zeta\|^2 \|e_p^k\|^2 \leq C_1 q^{k-l}(\dim H_k)\|\zeta\|^2 \leq C_1 D_2 q^{-l}\|\zeta\|^2.$$

On the other hand, we can compute $S^{k,l}$ using scalar products in $H_l \otimes H_{k-l} \otimes H_l$ to obtain the following induction relation:

$$S^{k,l} = \sum_{p,q,i,j} (\zeta \otimes e_p^k | e_i^l \otimes e_q^{k-l} \otimes e_j^l) \times (e_i^l \otimes e_q^{k-l} \otimes e_j^l | e_p^k \otimes \zeta)$$

$$= \sum_{p,q} (e_p^k | e_q^{k-l} \otimes \zeta) \times (\zeta \otimes e_q^{k-l} | e_p^k) = \sum_{p,q} (\zeta \otimes e_q^{k-l} | e_p^k) \times (e_p^k | e_q^{k-l} \otimes \zeta)$$

$$= \sum_q (\zeta \otimes e_q^{k-l} | P_k(e_q^{k-l} \otimes \zeta)) = S_+^{k-l,l}.$$

As a result we have $|S_+^{k,l}| = |S^{k,l} - \sum_r S_r^{k,l}| \leq |S_+^{k-l,l}| + Clq^{-l}\|\zeta\|^2$. For $k = 0$ we have $S_+^{k,l} = \|\zeta\|^2$. For $k = 1, \ldots, l-1$ we use the trivial estimate $|S_+^{k,l}| \leq (\dim H_k)\|\zeta\|^2 \leq Cq^{-l}\|\zeta\|^2$, and an easy induction on k for fixed l finally yields the result. □

From Lemma 3.8 one can deduce a similar estimate concerning coefficients of corepresentations. Given ONB's $(e_i^k)_i$, $(e_a^l)_a$ of H_k, H_l, we denote by v_{ij}^k, v_{ab}^l the associated coefficients of v^k, v^l.

Lemma 3.9. *There exists numbers C_l, depending only on n and l, such that for all $k \geq l > 0$ we have*

$$\sum_{ij} \left| \sum_p (\Lambda_h(v_{jp}^k v_{ab}^l) | \Lambda_h(v_{ab}^l v_{ip}^k)) \right|^2 \leq C_l k^2 q^k.$$

Proof. Let us recall some facts from the Woronowicz-Peter-Weyl theory in the Kac case. Since $v^k \otimes v^l \simeq v^{|k-l|} \oplus \cdots \oplus v^{k+l-2} \oplus v^{k+l}$, the product of coefficients $v_{jp}^k v_{ab}^l$ decomposes as a sum of coefficients of v^r, $r = |k-l|, \ldots, k+l-2, k+l$. More precisely the decomposition corresponds to projecting $e_j^k \otimes e_a^l$ and $e_p^k \otimes e_b^l$ onto the subspace equivalent to H_r. Besides, the scalar product of two coefficients $(\omega_{x,y} \otimes \text{id})(w)$, $(\omega_{x',y'} \otimes \text{id})(w)$ of an irreducible corepresentation w corresponds to the scalar product $(x' \otimes y | x \otimes y')$ up to a factor $(\text{qdim } w)^{-1}$.

Denote by C_{ij} the sums over p on the left-hand side of the statement. We compute now, using Lemma 3.8:

$$|C_{ij}| = \left| \sum_r \frac{1}{\text{qdim } v^r} \sum_p (e_a^l \otimes e_i^k | Q_r^{k+l}(e_j^k \otimes e_a^l))(e_p^k \otimes e_b^l | Q_r^{k+l}(e_b^l \otimes e_p^k)) \right|$$

$$\leq \sum_r \frac{1}{\text{qdim } v^r} |(e_a^l \otimes e_i^k | Q_r^{k+l}(e_j^k \otimes e_a^l))| \times \left| \sum_p (e_b^l \otimes e_p^k | Q_r^{k+l}(e_p^k \otimes e_b^l)) \right|$$

$$\leq \sum_r \frac{Ckq^{-l}}{\text{qdim } v^r} |(e_a^l \otimes e_i^k | Q_r^{k+l} \Sigma_{lk}(e_a^l \otimes e_j^k))|,$$

where $\Sigma_{lk} : H_l \otimes H_k \to H_k \otimes H_l$ is the flip map. Note that the sum over r contains at most $(l+1)$ terms. Using Cauchy-Schwarz' inequality and summing $|C_{ij}|^2$ over i, j we recognize the squares of the Hilbert-Schmidt norms of $A_r = (\omega_{e_a^l} \otimes \text{id})(Q_r^{k+l} \Sigma_{lk})$, which are dominated by $\dim H_k$ because the operator norm of A_r is less than 1:

$$\sum_{ij} |C_{ij}|^2 \leq (l+1) \sum_r \frac{(Ckq^{-l})^2}{(\text{qdim } v^r)^2} \sum_{ij} |(e_i^k | A_r(e_j^k))|^2 \leq (l+1) \sum_r \frac{(Ckq^{-l})^2}{(\text{qdim } v^r)^2} \dim H_k.$$

In our case $\dim = \operatorname{qdim}$ and $\rho = q^{-1}$. Since $r \geq k - l$, we obtain

$$\sum_{ij} |C_{ij}^k|^2 \leq (l+1)^2 \frac{(Ckq^{-l})^2}{(\dim H_{k-l})^2} \dim(H_k) \leq C^2 D_1^2 D_2 (l+1)^2 k^2 q^{-2l} q^{k-2l},$$

where C, D_1, D_2 are constants depending only on n. $\square$

Theorem 3.10. *Consider an orthogonal free quantum group $\mathbb{F}O(Q)$ which is unimodular. Then the representation ad° of $C^*(\mathbb{F}O(Q))$ factors through λ.*

Proof. By Lemma 3.2 it suffices to prove that the states $\phi = \omega_\xi \circ \mathrm{ad} : x \mapsto (\xi | \mathrm{ad}(x)\xi)$ on $A_o(n)$ are weakly associated to the regular representation, for a set of vectors ξ spanning a dense subspace of H°. In particular, we can assume that ξ is a coefficient of some irreducible corepresentation, and we will take in fact $\xi = \Lambda_h(v_{ab}^{l*})$.

We already saw that $\|\phi\|_2^2 = \sum \|\phi_k\|_2^2$, where ϕ_k is the restriction of ϕ to the subspace $\mathbb{C}[\mathbb{G}]_k = \operatorname{Span}\{v_{ij}^k\}$. Moreover in the unimodular case $(\sqrt{\dim H_k} v_{ij}^k)_{ij}$ is an ONB of $\mathbb{C}[\mathbb{G}]_k$ with respect to the ℓ^2 norm, and hence $\|\phi_k\|_2^2 = \dim H_k \sum |\phi(v_{ij}^k)|^2$. Using the fact that h is a trace, we have

$$\phi(v_{ij}^k) = \sum_p (\Lambda_h(v_{ab}^{l*}) | \Lambda_h(v_{ip}^k v_{ab}^{l*} v_{jp}^{k*})) = \sum_p (\Lambda_h(v_{ip}^{k*} v_{ab}^{l*}) | \Lambda_h(v_{ab}^{l*} v_{jp}^{k*}))$$

$$= \sum_p (\Lambda_h(v_{jp}^k v_{ab}^l) | \Lambda_h(v_{ab}^l v_{ip}^k)).$$

Now we can use Lemma 3.9 and we obtain $\|\phi_k\|_2^2 \leq \dim(H_k) C_l k^2 q^k \leq D_2 C_l k^2$. In particular it is clear now that $\|e^{-\lambda l} * \phi\|_2^2 \leq D_2 C_l \sum e^{-2\lambda k} k^2 < \infty$ for all $\lambda > 0$, and so ϕ is weakly associated to λ by Lemma 3.5. $\square$

3.3. Inner amenability.

The notion of inner amenability for locally compact quantum groups has been defined in [GNI13]. We will only consider this notion for discrete quantum groups. Recall that, when $\mathbb{G}$ is a discrete quantum group, we denote by $p_0 \in L^\infty(\mathbb{G})$ the minimal central projection corresponding to the trivial corepresentation. Following Effros [Eff75] we define inner amenability as follows.

Definition 3.11. A discrete quantum group $\mathbb{G}$ is called *inner amenable* if there exists a state $m \in L^\infty(\mathbb{G})^*$ such that $m(p_0) = 0$ and,

$$m((\mathrm{id} \otimes \omega)\Delta(f)) = m((\omega \otimes \mathrm{id})\Delta(f)) \quad \text{for all} \quad \omega \in L^\infty(\mathbb{G})_*, \ f \in L^\infty(\mathbb{G}).$$

Remark 3.12. Our terminology is different from [GNI13] where they call *strictly inner amenable* a discrete quantum group satisfying Definition 3.11. Note however that, according to [GNI13, Remark 3.1(c)], all discrete quantum groups are inner amenable in the sense of [GNI13, Def. 3.1].

Theorem 3.13. *Let $\mathbb{G}$ be a unimodular discrete quantum group. The following are equivalent.*

   (1) $\mathbb{G}$ *is inner amenable.*
   (2) *The trivial representation $\epsilon : C^*(\mathbb{G}) \to \mathbb{C}$ is weakly contained in ad°.*

Moreover, if $\mathbb{G}$ is countable and $\mathcal{L}(\mathbb{G})$ has property Gamma then $\mathbb{G}$ is inner amenable.

Proof. $1 \Rightarrow 2$. The proof of this implication is similar to the one of the implication $1 \Rightarrow 2$ in [Tom16, Theorem 3.8], and moreover we are in the unimodular case. Let us give a sketch of the proof. Denoting $A = (\mathrm{id} \otimes \mathrm{ad})(\mathbb{V}) = VW$, it is known that ϵ is weakly contained in ad° iff there exists a net of unit vectors $\xi_n \in H^\circ = \xi_0^\perp$ such that $\|A(\eta \otimes \xi_n) - \eta \otimes \xi_n\| \to 0$ for all $\eta \in H$.

Let J_φ and J_h be the modular conjugations of φ and h respectively, with respect to the GNS constructions as in Section 2. It is known that $(J_h \otimes J_\varphi)V(J_h \otimes J_\varphi) = V^*$, and since $U = J_\varphi J_h = J_h J_\varphi$ in the discrete case we also have $(J_h \otimes J_\varphi)W(J_h \otimes J_\varphi) = W^*$. Moreover $W = (1 \otimes U)V(1 \otimes U) = (J_h \otimes J_h)V^*(J_h \otimes J_h)$, and [?, Prop. 2.15] yields the second formula below:

$$\begin{aligned} V^*(\Lambda_\varphi(g) \otimes \Lambda_\varphi(f)) &= (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(f)(g \otimes 1)) \text{ and} \\ W(\Lambda_\varphi(g) \otimes \Lambda_\varphi(f)) &= (\Lambda_\varphi \otimes \Lambda_\varphi)(\sigma\Delta(f)(g \otimes 1)), \end{aligned} \tag{2}$$

for all $f, g \in \mathcal{N}_\varphi$, and σ the flip map on $L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$. Note that for the second formula to hold in the non-unimodular case one has to replace Λ_φ by a GNS construction for the right Haar weight ψ.

We use also the identification $C_0(\mathbb{G}) = \bigoplus_\alpha B(H_\alpha)$, where α runs over $\mathrm{Irr}\,\mathbb{G}$, and we recall that $\varphi = \sum_\alpha \dim(H_\alpha)\,\mathrm{Tr}_\alpha$ in this identification. We denote $(e_{ij}^\alpha)_{ij}$ the matrix units of $B(H_\alpha)$ associated to a chosen orthonormal basis $(e_i^\alpha)_i$ of H_α. Recall finally the notation $\omega_{\zeta,\xi} \in B(H)_*$ for $\zeta, \xi \in H$. In what follows we restrict these linear forms to $L^\infty(\mathbb{G})$ and we convolve them according to Δ. Now we have:

Claim 1. *Let $f = f^* \in \mathcal{N}_\varphi \subset L^\infty(\mathbb{G})$ and put $X = \sigma\Delta(f^2) - \Delta(f^2) = \sum_{\alpha,i,j} e_{ij}^\alpha \otimes X(\alpha)_{ij}$. For all $\alpha \in \mathrm{Irr}(\mathbb{G})$ we have:*

$$\|\omega_{\Lambda_\varphi(f)} * \omega_{J_h\Lambda_\varphi(e_{i1}^\alpha),J_h\Lambda_\varphi(e_{j1}^\alpha)} - \omega_{J_h\Lambda_\varphi(e_{i1}^\alpha),J_h\Lambda_\varphi(e_{j1}^\alpha)} * \omega_{\Lambda_\varphi(f)}\| \geq \varphi(e_{11}^\alpha)\varphi(|X(\alpha)_{ij}|).$$

Proof of Claim 1. Let z be the adjoint of the phase of $X(\alpha)_{ij}$ and write

$$\omega_{\Lambda_\varphi(f)} * \omega_{J_h\Lambda_\varphi(e_{i1}^\alpha),J_h\Lambda_\varphi(e_{j1}^\alpha)}(z) - \omega_{J_h\Lambda_\varphi(e_{i1}^\alpha),J_h\Lambda_\varphi(e_{j1}^\alpha)} * \omega_{\Lambda_\varphi(f)}(z) =$$
$$= \left(\omega_{J_h\Lambda_\varphi(e_{i1}^\alpha),J_h\Lambda_\varphi(e_{j1}^\alpha)} \otimes \omega_{\Lambda_\varphi(f)}\right)(\sigma\Delta(z) - \Delta(z)).$$

Using the formulas $\sigma\Delta(z) = W(1 \otimes z)W^*$, $\Delta(z) = V^*(1 \otimes z)V$, $(J_h \otimes J_\varphi)V(J_h \otimes J_\varphi) = V^*$, $(J_h \otimes J_\varphi)W(J_h \otimes J_\varphi) = W^*$ and Equations (2), one obtains, as in the proof of [Tom16, Lemma 3.14], the formula:

$$\omega_{\Lambda_\varphi(f)} * \omega_{J_h\Lambda_\varphi(e_{i1}^\alpha),J_h\Lambda_\varphi(e_{j1}^\alpha)}(z) - \omega_{J_h\Lambda_\varphi(e_{i1}^\alpha),J_h\Lambda_\varphi(e_{j1}^\alpha)} * \omega_{\Lambda_\varphi(f)}(z) = \varphi(e_{11}^\alpha)\varphi(|X(\alpha)_{ij}|).$$
$$\square$$

Claim 2. *Let $f \in \mathcal{N}_\varphi \cap L^\infty(\mathbb{G})_+$. Then, for all $\alpha \in \mathrm{Irr}(\mathbb{G})$, one has:*

$$\|A(\Lambda_\varphi(e_{ij}^\alpha) \otimes \Lambda_\varphi(f)) - \Lambda_\varphi(e_{ij}^\alpha) \otimes \Lambda_\varphi(f)\|^2 \leq 2\varphi(f^2)^{\frac{1}{2}}(\varphi \otimes \varphi)(|X(\alpha)|)^{\frac{1}{2}}.$$

Proof of Claim 2. The proof is the same as the proof of [Tom16, Lemma 3.16] without modular factors, by using Equation (2), the Cauchy-Schwarz inequality and the Powers-Størmer inequality.
$$\square$$

We can now finish the proof of $1 \Rightarrow 2$. Let $m \in L^\infty(\mathbb{G})^*$ be a state such that $m(p_0) = 0$ and $m((\mathrm{id} \otimes \omega)\Delta(f)) = m((\omega \otimes \mathrm{id})\Delta(f))$ for all $\omega \in L^\infty(\mathbb{G})_*$, $f \in L^\infty(\mathbb{G})$. By the weak* density of the normal states $\omega \in L^\infty(\mathbb{G})_*$ such that $\omega(p_0) = 0$ in the set of states $\mu \in L^\infty(\mathbb{G})^*$ such that $\mu(p_0) = 0$, there exists a net of normal states $(\omega_n)_n$ such that $\omega_n(p_0) = 0$ for all n and $\omega_n * \omega - \omega * \omega_n \to 0$ weak* for all $\omega \in L^\infty(\mathbb{G})_*$.

By the standard convexity argument, we may and will assume that $\|\omega_n * \omega - \omega * \omega_n\| \to 0$. Now, since $L^\infty(\mathbb{G})$ is standardly represented on H, and by a straightforward cut-off argument, we can assume that $\omega_n = \omega_{\Lambda_\varphi(f_n)}$ with $f_n \in L^\infty(\mathbb{G})_+ \cap C_c(\mathbb{G})$, $p_0 f_n = 0$ and $\|f_n\|_2 = 1$. Applying Claim 1 (instead of [Tom16, Lemma 3.14]), [Tom16, Lemma 3.15] and Claim 2 (instead of [Tom16, Lemma 3.16]), we obtain

$$\|A(\Lambda_\varphi(e_{ij}^\alpha) \otimes \Lambda_\varphi(f_n)) - \Lambda_\varphi(e_{ij}^\alpha) \otimes \Lambda_\varphi(f_n)\| \to 0$$

for all α, i, j. Putting $\xi_n = \Lambda_\varphi(f_n) \in H^\circ$, this easily implies that $\|A(\eta \otimes \xi_n) - \eta \otimes \xi_n\| \to 0$ for all $\eta \in H$, hence ϵ factors through ad°.

$2 \Rightarrow 1$. Take a net of unit vectors $\xi_n \in H^\circ = \xi_0^\perp$ such that $\|A(\eta \otimes \xi_n) - \eta \otimes \xi_n\| \to 0$ for all $\eta \in H$, and put $\widetilde{\xi}_n = J_\varphi \xi_n$. Then $\widetilde{\xi}_n$ has norm one, it is orthogonal to ξ_0 for all n and we have for all $\eta \in H$:

$$\|W^*(\eta \otimes \widetilde{\xi}_n) - V(\eta \otimes \widetilde{\xi}_n)\| = \|W(J_h\eta \otimes \xi_n) - V^*(J_h\eta \otimes \xi_n)\|$$
$$= \|A(J_h\eta \otimes \xi_n) - J_h\eta \otimes \xi_n\| \to 0.$$

Let $m \in L^\infty(\mathbb{G})^*$ be a weak* accumulation point of the net of states $(\omega_{\widetilde{\xi}_n})_n$. One has:

$$m((\omega_\eta \otimes \mathrm{id})\Delta(f)) = \lim\langle(1 \otimes f)V(\eta \otimes \widetilde{\xi}_n), V(\eta \otimes \widetilde{\xi}_n)\rangle = \lim\langle(1 \otimes f)W^*(\eta \otimes \widetilde{\xi}_n), W^*(\eta \otimes \widetilde{\xi}_n)\rangle$$
$$= m((\omega_\eta \otimes \mathrm{id})\sigma\Delta(f)) = m((\mathrm{id} \otimes \omega_\eta)\Delta(f)) \quad \text{for all} \quad \eta \in H, \; f \in L^\infty(\mathbb{G}).$$

Moreover, $m(p_0) = \lim\langle p_0\widetilde{\xi}_n, \widetilde{\xi}_n\rangle = 0$.

Finally, suppose that $\mathbb{G}$ is a countable unimodular discrete quantum group such that $\mathcal{L}(\mathbb{G})$ has property Gamma. To show that the co-unit is weakly contained in ad° we follow the proof of Effros

[Eff75]. Write $\mathrm{Irr}(\mathbb{G}) = \{u^k \mid k \in \mathbb{N}\}$ where $u^k \in B(H_k) \otimes C^*(\mathbb{G})$. Given $n \in \mathbb{N}$, choose a unitary $u_n \in \mathcal{L}(\mathbb{G})$ such that $h(u_n) = 0$ and

$$||u_n\lambda(u_{ij}^k) - \lambda(u_{ij}^k)u_n||_2 < \frac{1}{n\max\{\dim(u^k) \mid k \leq n\}}$$

for all $k = 1,\ldots,n$ and $1 \leq i,j \leq \dim u^k$. From this inequality, it is easy to check that, for all $k \leq n$ and all $\eta \in H_k$, one has

$$||(\mathrm{id} \otimes \lambda)(u^k)(1 \otimes u_n)(\eta \otimes \xi_0) - (1 \otimes u_n)(\mathrm{id} \otimes \lambda)(u^k)(\eta \otimes \xi_0)|| < \frac{1}{n}||\eta||.$$

Recall that ξ_0 is a fixed vector for ad. Hence, for all $\eta \in H_k$, one has $(\mathrm{id} \otimes \mathrm{ad})(u^k)(\eta \otimes \xi_0) = \eta \otimes \xi_0$. Moreover, since the representations λ and ρ commute we find, with $\xi_n = u_n\xi_0$, $k \leq n$ and $\eta \in H_k$,

$$||(\mathrm{id} \otimes \mathrm{ad})(u^k)(\eta \otimes \xi_n) - (\eta \otimes \xi_n)|| =$$
$$= ||(\mathrm{id} \otimes \mathrm{ad})(u^k)(1 \otimes u_n)(\mathrm{id} \otimes \mathrm{ad})(u^k)^*(\eta \otimes \xi_0) - (\eta \otimes \xi_n)||$$
$$= ||(\mathrm{id} \otimes \lambda)(u^k)(1 \otimes u_n)(\mathrm{id} \otimes \lambda)(u^k)^*(\eta \otimes \xi_0) - (1 \otimes u_n)(\eta \otimes \xi_0)||$$
$$< \frac{1}{n}||\eta||.$$

Since $\xi_n \in H^\circ = \xi_0^\perp$, it follows that the co-unit is weakly contained in ad°. $\qquad\square$

Corollary 3.14 (cf [VV07]). *For $n \geq 3$ the discrete quantum group $\mathbb{F}O_n$ is not inner amenable, and in particular the von Neumann algebra $\mathcal{L}(\mathbb{F}O_n)$ is a full factor.*

Proof. For $n \geq 3$ it is known that $\mathbb{F}O_n$ is not amenable, hence λ does not weakly contain ϵ. On the other hand by Theorem 3.10 the representation ad° is weakly contained in λ. Consequently ϵ is not weakly contained in ad°, hence $\mathbb{F}O_n$ is not inner amenable and $\mathcal{L}(\mathbb{F}O_n)$ is full by Theorem 3.13. $\quad\square$

4. PROPERTY (HH) FOR $\mathcal{L}(\mathbb{F}O_n)$

4.1. **A Deformation.** Recall that $\Delta : A_o(n) \to A_o(n) \otimes A_o(n)$ factors to $\Delta' : C_{\mathrm{red}}^*(\mathbb{F}O_n) \to C_{\mathrm{red}}^*(\mathbb{F}O_n) \otimes A_o(n)$ by Fell's absorption principle. On the other hand, for any element $g \in O_n$ we have a character $\omega_g : A_o(n) \to \mathbb{C}$ defined by putting $\omega_g(v_{ij}) = g_{ij}$ and using the universal property of $A_o(n)$. It is easy to check that $(\omega_g \otimes \omega_h)\Delta = \omega_{gh}$ and that $\omega_e = \epsilon$, where e is the unit of O_n and ϵ is the co-unit of $A_o(n)$.

Combining these objects we get $*$-homomorphisms $\alpha_g = (\mathrm{id} \otimes \omega_g) \circ \Delta' : C_{\mathrm{red}}^*(\mathbb{F}O_n) \to C_{\mathrm{red}}^*(\mathbb{F}O_n)$ such that $\alpha_g \circ \alpha_h = \alpha_{gh}$, $\alpha_e = \mathrm{id}$ and $h \circ \alpha_g = h$. In particular each α_g is an automorphism of $C_{\mathrm{red}}^*(\mathbb{F}O_n)$ and we have got an action of O_n on $C_{\mathrm{red}}^*(\mathbb{F}O_n)$ by trace preserving automorphisms. Note that there is also an action α' of O_n^{op} on $C_{\mathrm{red}}^*(\mathbb{F}O_n)$ given by $\alpha_g' = (\omega_g \otimes \mathrm{id}) \circ \Delta''$, where $\Delta'' : C_{\mathrm{red}}^*(\mathbb{F}O_n) \to A_o(n) \otimes C_{\mathrm{red}}^*(\mathbb{F}O_n)$ is the homomorphism analogous to Δ'.

Let $n \geq 2$. Denote $M = \mathcal{L}(\mathbb{F}O_n)$ the von Neumann algebra of $\mathbb{F}O_n$ and $\tilde{M} = M \otimes M$. We identify M inside $\tilde{M}$ via the unital normal faithful trace preserving $*$-homomorphism $\iota := \Delta$. Denote by E the canonical conditional expectation from $\tilde{M}$ to $\iota(M)$. We let O_n act on $\tilde{M}$ by putting $A_g = (\alpha_g \otimes \mathrm{id}) : \tilde{M} \to \tilde{M}$.

Proposition 4.1 (cf [Bra12]). *Denote $(U_k)_k$ the dilated Chebyshev polynomials of the second kind and consider, for each $s \in \mathbb{R}$, the densely defined map $T_s : C_{\mathrm{red}}^*(\mathbb{F}O_n) \to C_{\mathrm{red}}^*(\mathbb{F}O_n)$ with domain $\mathbb{C}[\mathbb{F}O_n]$ such that $T_s = \frac{U_k(s)}{U_k(n)}\mathrm{id}$ on $\mathbb{C}[\mathbb{F}O_n]_k$ for all $k \in \mathbb{N}$.*

Then for each $g \in O_n$ we have $E \circ A_g \circ \iota = T_s$ where $s = \mathrm{Tr}(g)$. In particular, for such s the map T_s extends to a trace preserving completely positive map on $C_{\mathrm{red}}^*(\mathbb{F}O_n)$.

Proof. For $r \in \mathbb{N}$, denote v_{ij}^r the coefficients of the rth irreducible corepresentation v^r of $\mathbb{F}O_n$, with respect to a given ONB of the corresponding space. It is easy to check that $E(v_{ij}^r \otimes v_{kl}^s) = \delta_{rs}\delta_{jk}v_{il}^r/U_r(n)$, where $U_r(n) = \dim v^r$.

On the other hand, denoting u^r the image of v^r as a representation of O_n, we have by definition $\omega_g(v_{ij}^r) = u_{ij}^r(g)$. The character $\chi_r = \mathrm{Tr} \circ u^r$ of u^r is given by $\chi_r(g) = U_r(\mathrm{Tr}\,g)$, since $u^0 = 1$, $u^1 = \mathrm{id}$ and $u^1 \otimes u^r \simeq u^{r-1} \oplus u^{r+1}$. Now it suffices to compute as follows:

$$E \circ A_g \circ \Delta(v_{ij}^r) = \sum_{k,l} E(v_{ik}^r u_{kl}^r(g) \otimes v_{lj}^r) = \sum_k v_{ij}^r \frac{u_{kk}^r(g)}{U_r(n)} = v_{ij}^r \frac{U_r(\mathrm{Tr}\,g)}{U_r(n)}.$$

$\square$

Example 4.2. Define $g_t = \mathrm{id}_{n-2} \oplus R_t \in O_n$ where

$$R_t = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Denoting $A_t = A_{g_t}$, we get in this way a 1-parameter group of automorphisms of $\tilde{M}$ such that $E \circ A_t \circ \iota = T_s$ with $s = n - 2 + 2\cos t \in [n-4, n]$.

Fix $0 < t_0 < \frac{\pi}{3}$. For all $0 < t < t_0$ one has $2 < 1 + 2\cos(t_0) < \mathrm{Tr}(g_t) < n$. By [Bra12, Proposition 4.4.1], there exists a constant $C > 0$ such that

$$\frac{U_k(\mathrm{Tr}(g_t))}{U_k(n)} \le C \left(\frac{\mathrm{Tr}(g_t)}{n} \right)^k$$

for all $0 < t < t_0$ and all $k \in \mathbb{N}$. Hence, for $0 < t < t_0$, the map $T_s = E \circ A_t \circ \iota$ is L^2-compact.

Finally, consider $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $h = \mathrm{id}_{n-2} \oplus S \in O_n$ and $B = \alpha_h \otimes \alpha'_h \in \mathrm{Aut}(\tilde{M})$. We clearly have $B^2 = \mathrm{id}$, $BA_t = A_{-t}B$, and one can check on the generators v_{ij} that $B \circ \Delta = \Delta$ so that B restricts to the identity on $\iota(M)$.

Remark 4.3. The constructions and results of this Section 4 remain valid for the other unimodular orthogonal free quantum groups $\mathbb{F}O(Q)$. Up to isomorphism, the only missing cases are $\mathbb{F}O(Q_{2n})$ with $Q_{2n} = \mathrm{diag}(Q_2, \ldots, Q_2)$ and $Q_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. But these discrete quantum groups all admit the dual of $SU(2)$ as a commutative quotient, and one can build a deformation using the same matrices R_t as in Example 4.2.

On the other hand, in the non unimodular case there is no trace preserving conditional expectation $E : M \otimes M \to \Delta(M)$, so that the arguments of Proposition 4.1 do not apply anymore. Recall however that Haagerup's Property still holds in that case, as shown in [?].

Let us give some applications of the preceding construction. It is known that Haagerup's property is equivalent to the existence of a "proper conditionally negative type function" and/or of a proper cocycle in some representation. Denoting τ_s the linear form on $\mathbb{C}[\mathbb{F}O_n]$ given by $\tau_s(v_{ij}^r) = \delta_{ij} \frac{U_r(s)}{U_r(n)}$, it follows from the proposition above and Lemma 3.1 that τ_s extends to a state of $C^*(\mathbb{F}O_n)$. Differentiating at $s = n$ we obtain a conditionally negative form $\psi : \mathbb{C}[\mathbb{F}O_n] \to \mathbb{C}$ given by $\psi(v_{ij}^r) = \delta_{ij} \frac{U'_r(n)}{U_r(n)}$ — this was also observed in [CFK12, Crl. 10.3], where all ad-invariant conditionally negative forms on $\mathbb{C}[\mathbb{F}O_n]$ are classified. It is clear that $\psi(x^*x) \le 0$ for all $x \in \mathrm{Ker}\,\epsilon$.

Moreover the following lemma shows that ψ is indeed proper, and more precisely that the associated function $\Psi = (\mathrm{id} \otimes \psi)(V)$ behaves like the "naive" length function $L = \sum r p_r$ at infinity:

Lemma 4.4. We have $\frac{U'_r(n)}{U_r(n)} = \frac{r}{\sqrt{n^2-4}} + O(1)$ as $r \to \infty$.

Proof. We have the well-known formula $U_r(n) = (q^{r+1} - q^{-r-1})/(q - q^{-1})$, where $q = \frac{1}{2}(n - \sqrt{n^2 - 4})$. Differentiating first with respect to q, we get

$$\frac{U'_r(n)}{U_r(n)} = \frac{r(1 - q^{-2}) - 2q^{-2} + (r+2)q^{-2r-2} - rq^{-2r-4}}{(q - q^{-1})(1 - q^{-2r-2})} \times \frac{\mathrm{d}q}{\mathrm{d}n}.$$

Noticing that $\mathrm{d}q/\mathrm{d}n = q/\sqrt{n^2 - 4}$, we obtain finally

$$\frac{U'_r(n)}{U_r(n)} = \frac{1}{\sqrt{n^2 - 4}} \left(r - \frac{2}{q^2 - 1} + o(1) \right).$$

$\square$

4.2. **Property (HH).** It is then natural to ask also for the explicit construction of a proper cocycle establishing Haagerup's property. Notice that by [Ver12] such a cocycle cannot live in the regular representation or a finite multiple of it. Now an explicit cocycle can easily be obtained by differentiating the 1-parameter group above. More precisely, for any $X \in \mathfrak{o}_n$ we define $\delta_X = d_X A_g \circ \iota : \mathbb{C}[\mathbb{F}O_n] \to \tilde{M}$. Denoting $u^r \in B(H_r) \otimes C(O_n)$ the image of v^r, we have explicitly

$$\delta_X(v_{ij}^r) = \sum_{k,l} v_{ik}^r d_X u_{kl}^r(g) \otimes v_{lj}^r.$$

Here $d_X u_{kl}^r$ is the differential of $u_{kl}^r : O_n \to \mathbb{C}$, evaluated on the tangent vector X at e. Since $A_g \circ \iota$ is a $*$-homomorphism, we get indeed derivations with respect to the $\mathbb{C}[\mathbb{F}O_n]$-bimodule structure coming from the embedding $\iota = \Delta$, and in fact we get a Lie algebra map $(X \mapsto \delta_X)$ from $\mathfrak{o}_n$ to the space of derivations $\mathrm{Der}(\mathbb{C}[\mathbb{F}O_n], \tilde{M})$.

We moreover denote $f_X = (\mathrm{id} \otimes \delta_X)(V)$, which is an unbounded multiplier of the Hilbert $C_0(\mathbb{F}O_n)$-module $C_0(\mathbb{F}O_n) \otimes \tilde{M}$. Then $f_X^* f_X$ is an unbounded multiplier of $C_0(\mathbb{F}O_n)$, the "conditionally negative" type function associated to δ_X. The following Lemma shows in particular that $f_X^* f_X$ corresponds to the cond. negative form ψ obtained by differentiating the deformation $T_s = (\mathrm{id} \otimes \tau_s)\Delta$.

Lemma 4.5. *The derivation δ_X takes its values in the orthogonal complement $\tilde{M}^\circ$ of $\iota(M)$ with respect to $h \otimes h$. Moreover we have*

$$f_X^* f_X = 2 \,\mathrm{Tr}(X^* X) \times (\mathrm{id} \otimes \psi)(V).$$

In particular for any $X \in \mathfrak{o}_n$, $X \neq 0$, the derivation δ_X is proper in the sense that $p_r f_X^* f_X \geq c_r p_r$ for any r, with $c_r \to \infty$.

Proof. The beginning of the proof is quite general and probably well-known to experts.

We choose a 1-parameter subgroup $(g_t)_t$ of O_n such that $g_0' = X$, we put $A_t = A_{g_t}$, $s(t) = \mathrm{Tr}(g_t)$ and we differentiate the identity $E \circ A_t \circ \iota = T_{s(t)}$ between linear maps on $\mathbb{C}[\mathbb{F}O_n]$ from Proposition 4.1. We obtain $\mathrm{Tr}(g_t') \times T'_{s(t)} = E \circ A_t' \circ \iota$, where T_s' is the derivative with respect to s, and the other derivatives are relative to t. In particular for $t = 0$ this yields $E \circ \delta_X = E \circ d_X A_g \circ \iota = \mathrm{Tr}(X) \times T_n'$. Since $\mathrm{Tr}\, X = 0$ for $X \in \mathfrak{o}_n$ we obtain $E \circ \delta_X = 0$, hence $\delta_X(\mathbb{C}[\mathbb{F}O_n]) \subset \tilde{M}^\circ$.

Now we differentiate once more at $t = 0$, obtaining $\mathrm{Tr}(g_0'') \times T_n' = E \circ A_0'' \circ \iota$ as linear maps on $\mathbb{C}[\mathbb{F}O_n]$. Since $(g_t)_t$ is a 1-parameter group and $X^* + X = 0$, we have $g_0'' = g_0'^2 = -X^* X$ in $M_n(\mathbb{R})$. Similarly, since $(A_t)_t$ is a 1-parameter group of trace-preserving automorphisms we have $A_0'' = A_0'^2 = -A_0'^* A_0'$, where $A_0'^*$ denotes the adjoint of A_0' with respect to the hilbertian structure of $\tilde{M}$. Moreover we have $E = \iota^*$ at the hilbertian level, so that $\delta_X^* \delta_X = (A_0' \iota)^* (A_0' \iota) = -E \circ A_0'' \circ \iota = \mathrm{Tr}(X^* X) \times T_n'$.

The last identity can also be written $(\delta_X(a)|\delta_X(b)) = \mathrm{Tr}(X^* X) h(a^* T_n'(b))$ for all a, $b \in \mathbb{C}[\mathbb{F}O_n]$. As a result we obtain

$$f_X^* f_X = (\mathrm{id} \otimes \delta_X)(V)^*(\mathrm{id} \otimes \delta_X)(V) = \mathrm{Tr}(X^* X) \times (\mathrm{id} \otimes h)(V^*(\mathrm{id} \otimes T_n')(V))$$

as unbounded multipliers — in other words, the identity above makes sense in the f.-d. algebra $p_r C_0(\mathbb{F}O_n) \simeq B(H_r)$ for any r. But by definition of ψ we have $T_n' = (\mathrm{id} \otimes \psi) \circ \Delta$, and using the identity $(\mathrm{id} \otimes \Delta)(V) = V_{12} V_{13}$ we can write

$$f_X^* f_X = \mathrm{Tr}(X^* X) \times (\mathrm{id} \otimes h \otimes \psi)(V_{12}^* V_{12} V_{13}) = \mathrm{Tr}(X^* X) \times (\mathrm{id} \otimes \psi)(V).$$

Finally we have $(p_r \otimes \psi)(V) = c_r p_r$ with $c_r = U_r'(n)/U_r(n)$ by the computation of ψ before Lemma 4.4, and the properness results from that lemma. $\qquad\square$

Recall the construction of the bimodule K_π associated to a $*$-representation $\pi : C^*(\mathbb{G}) \to B(H_\pi)$. We put $K_\pi = H \otimes H_\pi$ where H is the GNS space of the Haar state h. The space K_π is endowed with two representations, $\tilde{\pi} = (\lambda \otimes \pi)\Delta' : C^*_{\mathrm{red}}(\mathbb{F}O_n) \to B(K_\pi)$ and $\tilde{\rho} = \lambda^{\mathrm{op}} \otimes 1 : C^*_{\mathrm{red}}(\mathbb{F}O_n)^{\mathrm{op}} \to B(K_\pi)$, where we put $\lambda^{\mathrm{op}}(x)\Lambda_h(y) = \Lambda_h(yx)$.

Lemma 4.6. *The space $L^2(\tilde{M})$, viewed as an M, M−bimodule via the left and right actions of $\iota(M) = \Delta(M)$, is isomorphic to the M, M−bimodule K_π naturally associated with the adjoint representation $\pi = \mathrm{ad}$ of $\mathbb{G}$. The trivial part $\mathbb{C}\xi_0$ of the adjoint representation corresponds to the trivial sub-bimodule $\iota(M) \subset \tilde{M}$.*

Proof. We consider the unitary $V(1 \otimes U) : L^2(\tilde{M}) = H \otimes H \to H \otimes H$. Recall that $V(\Lambda_h \otimes \Lambda_h)(x \otimes y) = (\Lambda_h \otimes \Lambda_h)(\Delta(x)(1 \otimes y))$ and $U\Lambda_h(y) = \Lambda_h(f_1 * S(y))$ for x, $y \in \mathbb{C}[\mathbb{G}]$, so that

$$V(1 \otimes U)(\Lambda_h \otimes \Lambda_h)(x \otimes y) = (\Lambda_h \otimes \Lambda_h)(\Delta(x)(1 \otimes f_1 * S(y))).$$

Recall that $\mathrm{ad}(z)\Lambda_h(y) = \Lambda_h(z_{(1)} y (f_1 * S(z_{(2)})))$. Then we have:

$$V(1 \otimes U)(\Lambda_h \otimes \Lambda_h)(\Delta(z)(x \otimes y)) = (\Lambda_h \otimes \Lambda_h)((z_{(1)} \otimes z_{(2)})\Delta(x)(1 \otimes f_1 * (S(y)S(z_{(3)}))))$$
$$= (\lambda \otimes \mathrm{ad})\Delta(z)V(1 \otimes U)(\Lambda_h \otimes \Lambda_h)(x \otimes y).$$

Moreover in the Kac case we can write:

$$V(1 \otimes U)(\Lambda_h \otimes \Lambda_h)((x \otimes y)\Delta(z)) = (\Lambda_h \otimes \Lambda_h)(\Delta(x)(z_{(1)} \otimes z_{(2)})(1 \otimes S(z_{(3)})S(y)))$$
$$= (\Lambda_h \otimes \Lambda_h)(\Delta(x)(z \otimes S(y)))$$
$$= (\lambda^{op}(z) \otimes \mathrm{id})V(1 \otimes U)(\Lambda_h \otimes \Lambda_h)(x \otimes y).$$

This shows that $V(1 \otimes U)$ yields an isomorphism $L^2(\tilde{M}) \simeq K_{\mathrm{ad}}$ as $M, M-$bimodules. Moreover, putting $x = y = 1$ we see that $V(1 \otimes U)(\Lambda_h \otimes \Lambda_h)\iota(M) = \Lambda_h(M) \otimes \xi_0$. $\qquad \square$

Recall now from [OP10b] that a discrete group G has Property strong (HH) if it admits a proper cocycle with values in a representation weakly contained in the regular representation. These notions make sense in the quantum case, and combining the lemma above with Theorem 3.10 and Lemma 4.5 we obtain:

Corollary 4.7. *The discrete quantum groups $\mathbb{F}O_n$ satisfy the Property strong (HH).*

Remark 4.8. Lemma 4.6 works also at the level of the derivations δ_X and yields a formula for the associated cocycles with values in the adjoint representation $\pi = \mathrm{ad}$. More precisely, a simple computation shows that $V(1 \otimes U)\delta_X$ is of the form $(\Lambda_h \otimes c_X)\Delta$, with $c_X : \mathbb{C}[\mathbb{F}O_n] \to H$ given by $c_X = \Lambda_h m(d_X \alpha_g \otimes S)\Delta$, or in other terms:

$$c_X(v_{ij}^r) = \sum (d_X u_{kl}^r) \times \Lambda_h(v_{ik}^r v_{jl}^{r*}).$$

The fact that $V(1 \otimes U)\delta_X$ is a derivation implies that c_X is a cocycle, i.e. $c_X(xy) = \pi(x)c_X(y) + c_X(x)\epsilon(y)$ for $x, y \in \mathbb{C}[\mathbb{F}O_n]$. Denoting $g_X = (\mathrm{id} \otimes c_X)(V)$ the unbounded multiplier of the Hilbert module $C_0(\mathbb{F}O_n) \otimes H$ associated with c_X, we have $f_X^* f_X = g_X^* g_X$. Hence Lemma 4.5 also shows that the cocycle c_X is proper.

For the generators v_{ij} of $\mathbb{C}[\mathbb{F}O_n]$ we have $c_X(v_{ij}) = \sum X_{kl}\Lambda_h(v_{ik}v_{jl})$. For a particular choice of X one gets e.g. $c(v_{ij}) = \Lambda_h(v_{i1}v_{j2} - v_{i2}v_{j1})$, and the other values of c can be deduced recursively using the cocycle relation.

4.3. **Strong solidity.** We first recall the following result due to Ozawa and Popa [OP10a].

Theorem 4.9. *Let M be a tracial von Neumann algebra which is weakly amenable and admits the following deformation property: there exists a tracial von Neumann algebra $\tilde{M}$, a trace preserving inclusion $M \subset \tilde{M}$ and a one-parameter group $(\alpha_t)_{t \in \mathbb{R}}$ of trace-preserving automorphism of $\tilde{M}$ such that*

- $\lim_{t \to 0} \|\alpha_t(x) - x\|_2 = 0$ *for all $x \in M$.*
- $_M\left(L^2(\tilde{M}) \ominus L^2(M)\right)_M \prec \, _M\left(L^2(M) \otimes L^2(M)\right)_M$.
- $E_M \circ \alpha_t$ *is compact on $L^2(M)$ for all t small enough.*

Then for any diffuse amenable von Neumann subalgebra $P \subset M$ we have that $\mathcal{N}_M(P)''$ is amenable — in other words M is strongly solid.

Proof. If $K = L^2(\tilde{M}) \ominus L^2(M)$ is (strongly) contained in an amplification of the coarse bimodule, this is a particular case of [OP10a, Thm. 4.9], with $k = 1$, $Q = Q_1 = \mathbb{C}$ and $\mathcal{G} = \mathcal{N}_M(P)$. Indeed in that case "compactness over Q" for $E_M \circ \alpha_t$ means that its extension to $L^2(M)$ is compact, $L^2\langle M, e_{Q_1}\rangle$ is the coarse bimodule $L^2(M) \otimes L^2(M)$, and $P \npreceq_M Q$ means that P is diffuse. Moreover if M is weakly amenable and P is amenable, then the action of $\mathcal{G} = \mathcal{N}_M(P)$ on P is weakly compact by [Oza12, Thm. B]. Hence the hypotheses of [OP10a, Thm. 4.9] are satisfied, and we can conclude that $N = \mathcal{N}_M(P)''$ is amenable relative to Q inside M, which just means that N is amenable in our case.

Note that the proof of [OP10a, Thm. 4.9] shows the existence, for all non-zero central projection $p \in M$, all $F \subset \mathcal{N}_M(P)$ finite and all $\epsilon > 0$ the existence of a vector $\zeta \in K \otimes L^2(M)$ such that $\|p\zeta\|_2 \geq \|p\|_2/8$, $\|[u \otimes \bar{u}, \zeta]\|_2 < \epsilon/2$ for all $u \in F$ and $\|x\zeta\|_2 \leq \|x\|_2$ for all $x \in M$. In particular one can then show as in [Sin11, Thm. 3.1] that K is left amenable over $\mathcal{N}_M(P)'' \subset M$. Now, if K is only weakly contained in the coarse bimodule, [Sin11, Thm. 3.2] still allows to conclude that $\mathcal{N}_M(P)''$ is amenable. $\qquad \square$

Now by Proposition 4.1, Lemma 4.6 and Theorem 3.10 one can apply the preceding Theorem to the deformation of $\mathcal{L}(\mathbb{F}O_n)$ presented in Section 4.1. Moreover the weak amenability assumption is satisfied by [Fre13, Thm. 6.1]. As a result we obtain:

Theorem 4.10. *For all $n \geq 3$, the II_1 factor $\mathcal{L}(\mathbb{F}O_n)$ is strongly solid.*

Note that this result was already proved in [Iso12] using the more recent approach from [PV12] to strong solidity, and the Akemann-Ostrand Property, established in [Ver05, Thm. 8.3] for free quantum groups. Finally, it seems likely that strong solidity can also be deduced from Property strong (HH) as in [OP10b, Crl. B], by adapting Peterson's techniques to the proper derivation $\delta_X : \mathbb{C}[\mathbb{F}O_n] \to \tilde{M}$ in the quantum setting. Notice in particular that $\Delta_X = \delta_X^* \delta_X$ is a "central multiplier" of $\mathbb{C}[\mathbb{F}O_n]$ by the proof of Lemma 4.5, so that the passage from the classical to the quantum setting is probably straightforward.

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