Construction of sheaves on the subanalytic site

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Introduction

Let M be a real analytic manifold. The Grothendieck subanalytic topology on M, denoted M_{sa}, and the morphism of sites $\rho_{\mathrm{sa}}\colon M \to M_{\mathrm{sa}}$, were introduced in [KS01]. Recall that the objects of the site M_{sa} are the relatively compact subanalytic open subsets of M and the coverings are, roughly speaking, the finite coverings. In loc. cit. the authors use this topology to construct new sheaves which would have no meaning on the usual topology, such as the sheaf $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$ of $\mathscr{C}^{\infty}$-functions with temperate growth and the sheaf $\mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ of temperate distributions. On a complex manifold X, using the Dolbeault complexes, they constructed the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ (in the derived sense) of holomorphic functions with temperate growth. The last sheaf is implicitly used in the solution of the Riemann-Hilbert problem by Kashiwara [Kas80, Kas84] and is also extremely important in the study of irregular holonomic $\mathscr{D}$-modules (see [KS03, § 7]).

In this paper, we shall modify the preceding construction in order to obtain sheaves of $\mathscr{C}^{\infty}$-functions with a given growth at the boundary. For example, functions whose growth at the boundary is bounded by a given power of the distance (temperate growth of order $s \geq 0$), or by an exponential of a given power of the distance (Gevrey growth of order $s > 1$), as well as their holomorphic counterparts. For that purpose, we have to refine the subanalytic topology and we introduce what we call the linear subanalytic topology, denoted M_{sal}.

Let us describe the contents of this paper with some details.

In **Chapter 1** we construct the linear subanalytic topology on M. Denoting by $\mathrm{Op}_{M_{\mathrm{sa}}}$ the category of open relatively compact subanalytic subsets of M, the presite underlying the site M_{sal} is the same as for M_{sa}, namely $\mathrm{Op}_{M_{\mathrm{sa}}}$, but the coverings are the linear coverings. Roughly speaking, a finite family $\{U_i\}_{i\in I}$ is a linear covering of their union U if there is a constant C such that the distance of any $x \in M$ to $M \setminus U$ is bounded by C-times the maximum of the distance of x to $M \setminus U_i$ ($i \in I$). (See Definition 1.1.1.) In this chapter, we also prove some technical results on linear coverings that we shall need in the course of the paper.

Chapter 2. Let $\mathbf{k}$ be a field. One easily shows that a presheaf F of $\mathbf{k}$-modules on M_{sal} is a sheaf as soon as, for any open sets U_1 and U_2 such that $\{U_1, U_2\}$ is a linear covering of $U_1 \cup U_2$, the Mayer-Vietoris sequence

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2) \tag{0.0.1}$$

is exact. Moreover, if for any such a covering, the sequence

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2) \to 0 \tag{0.0.2}$$

is exact, then the sheaf F is Γ-acyclic, that is, $\mathrm{R}\Gamma(U; F)$ is concentrated in degree 0 for all $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$.

There is a natural morphism of sites $\rho_{\mathrm{sal}} \colon M_{\mathrm{sa}} \to M_{\mathrm{sal}}$ and we shall prove the two results below (see Theorems 2.3.13 and 2.4.17):
(1) the functor $\mathrm{R}\rho_{\mathrm{sal}*} \colon \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}})$ admits a right adjoint $\rho^!_{\mathrm{sal}}$,
(2) if U has a Lipschitz boundary, then the object $\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_U$ is concentrated in degree 0.

Therefore, if a presheaf F on M_{sa} has the property that the Mayer-Vietoris sequences (0.0.2) are exact, it follows that $\mathrm{R}\Gamma(U; \rho^!_{\mathrm{sal}} F)$ is concentrated in degree 0 and is isomorphic to $F(U)$ for any U with Lipschitz boundary. In other words, to a presheaf on M_{sa} satisfying a natural condition, we are able to associate an object of the derived category of sheaves on M_{sa} which has the same sections as F on any Lipschitz open set. This construction is in particular used by Gilles Lebeau [Leb16] who obtains for $s \leq 0$ the "Sobolev sheaves $\mathscr{H}^s_{M_{\mathrm{sa}}}$", objects of $\mathrm{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$ with the property that if $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ has a Lipschitz boundary, then $\mathrm{R}\Gamma(U; \mathscr{H}^s_{M_{\mathrm{sa}}})$ is concentrated in degree 0 and coincides with the classical Sobolev space $H^s(U)$.

The fact that Sobolev sheaves are objects of derived categories and are not concentrated in degree 0 shows that when dealing with spaces of functions or distributions defined on open subsets which are not regular (more precisely, which have not a Lipschitz boundary), it is natural to replace the notion of a space by that of a complex of spaces.

In **Chapter 3**, we briefly study the natural operations on the linear subanalytic sites. The main difficulty is that a morphism $f \colon M \to N$ of real analytic manifolds does not induce a morphism of the linear subanalytic sites. This forces us to treat separately the direct or inverse images of sheaves for closed embeddings and for submersive maps.

In **Chapter 4** we construct some sheaves on M_{sal}. We construct the sheaf $\mathscr{C}^{\infty,s}_{M_{\mathrm{sal}}}$ of $\mathscr{C}^\infty$-functions with growth of order $s \geq 0$ at the boundary and the sheaves $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ of $\mathscr{C}^\infty$-functions with Gevrey growth of type $s > 1$ at the boundary. By using a refined cut-off lemma (which follows from a refined partition of unity due to Hörmander [Hör83]), we prove that these sheaves are Γ-acyclic. Applying the functor $\rho^{!}_{\mathrm{sal}}$, we get new sheaves (in the derived sense) on M_{sa} whose sections on open sets with Lipschitz boundaries are concentrated in degree 0. Then, on a complex manifold X, by considering the Dolbeault complexes of the sheaves of $\mathscr{C}^{\infty}$-functions considered above, we obtain new sheaves of holomorphic functions with various growth.

As already mentioned, Sobolev sheaves are treated in a separate paper by G. Lebeau in [Leb16].

Finally, in **Chapter 5**, we apply these results to endow the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ with a filtration (in the derived sense) that we call the L^{∞}-filtration.

Denote by $\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}$ the sheaf $\mathscr{D}_{M_{\mathrm{sa}}} := \rho_{\mathrm{sa}!}\mathscr{D}_M$ of differential operators on M_{sa}, endowed with its natural filtration and denote by $\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}$ the sheaf $\mathscr{D}_{M_{\mathrm{sal}}} := \rho_{\mathrm{sal}*}\mathscr{D}_{M_{\mathrm{sa}}}$ endowed with its natural filtration. For $\mathscr{T} = M, M_{\mathrm{sa}}, M_{\mathrm{sal}}$, the category $\mathrm{Mod}(\mathrm{F}\mathscr{D}_{\mathscr{T}})$ of filtered $\mathscr{D}$-modules on $\mathscr{T}$ is quasi-abelian in the sense of [Sch99] and its derived category $\mathrm{D}^{+}(\mathrm{F}\mathscr{D}_{\mathscr{T}})$ is well-defined. We shall use here the recent results of [SS16] which give an easy description of these derived categories and we construct a right adjoint $\rho^{!}_{\mathrm{sal}}$ to the derived functor $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}^{+}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{D}^{+}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}})$.

By considering the sheaves $\mathscr{C}^{\infty,s}_{M_{\mathrm{sal}}}$ ($s \geq 0$) we obtain the filtered sheaf $\mathrm{F}_{\infty}\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$. Then, on a complex manifold X, by considering the Dolbeault complex of this filtered sheaf, we obtain the filtration $\mathrm{F}_{\infty}\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ on the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$.

Recall now the Riemann-Hilbert correspondence. Let $\mathscr{M}$ be a regular holonomic $\mathscr{D}_X$-module and let $G := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ be the perverse sheaf of its holomorphic solutions. Kashiwara's theorem of [Kas84] may be formulated by saying that the natural morphism $\mathscr{M} \to \rho^{-1}_{\mathrm{sa}}\mathrm{R}\mathscr{H}om(G, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ is an isomorphism. Replacing the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ with its filtered version $\mathrm{F}_{\infty}\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$, we define the filtered Riemann-Hilbert functors $\mathrm{RHF}_{\infty,\mathrm{sa}}$ and RHF_{∞} by the formulas

$$\begin{aligned} \mathrm{RHF}_{\infty,\mathrm{sa}}\colon \mathrm{D}^{+}_{\mathrm{holreg}}(\mathscr{D}_X) &\to \mathrm{D}^{+}(\mathrm{F}\mathscr{D}_{X_{\mathrm{sa}}}), \\ \mathscr{M} &\mapsto \mathrm{FR}\mathscr{H}om(\mathrm{Sol}(\mathscr{M}), \mathrm{F}_{\infty}\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}), \\ \mathrm{RHF}_{\infty} = \rho^{-1}_{\mathrm{sa}}\mathrm{RHF}_{\infty,\mathrm{sa}}\colon \mathrm{D}^{+}_{\mathrm{holreg}}(\mathscr{D}_X) &\to \mathrm{D}^{+}(\mathrm{F}\mathscr{D}_X) \end{aligned}$$

and we prove that the composition

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X) \xrightarrow{\mathrm{RHF}_{\infty}} \mathrm{D}^{+}(\mathrm{F}\mathscr{D}_X) \xrightarrow{\mathit{for}} \mathrm{D}^{+}(\mathscr{D}_X)$$

is isomorphic to the identity functor. In other words, any regular holonomic $\mathscr{D}_X$-module $\mathscr{M}$ can be *functorially* endowed with a filtration $\mathrm{F}_{\infty}\mathscr{M}$, in the derived sense.

We also briefly introduce an L^2-filtration better suited to apply Hörmander's theory (see [Hör65]) and present some open problems.

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We have also been very much stimulated by the interest of Gilles Lebeau for sheafifying the classical Sobolev spaces and it is a pleasure to thank him here.

Finally Theorem 2.4.11 plays an essential role in the whole paper and we are extremely grateful to Adam Parusinski who has given a proof of this result.

Chapter 1

Subanalytic topologies

1.1 Linear coverings

Notations and conventions

We shall mainly follow the notations of [KS90, KS01] and [KS06].

In this paper, unless otherwise specified, a manifold means a real analytic manifold. We shall freely use the theory of subanalytic sets, due to Gabrielov and Hironaka, after the pioneering work of Lojasiewicz. A short presentation of this theory may be found in [BM88].

For a subset A in a topological space X, $\overline{A}$ denotes its closure, $\operatorname{Int} A$ its interior and ∂A its boundary, $\partial A = \overline{A} \setminus \operatorname{Int} A$.

Recall that given two metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is Lipschitz if there exists a constant $C \geq 0$ such that $d_Y(f(x), f(x')) \leq C \cdot d_X(x, x')$ for all $x, x' \in X$.

(1.1.1) All along this paper, if M is a real analytic manifold, we choose a distance d_M on M such that, for any $x \in M$ and any local chart $(U, \varphi\colon U \hookrightarrow \mathbb{R}^n)$ around x, there exists a neighborhood of x over which d_M is Lipschitz equivalent to the pull-back of the Euclidean distance by φ. If there is no risk of confusion, we write d instead of d_M.

In the following, we will adopt the convention

$$d(x, \emptyset) = D_M + 1, \qquad \text{for all } x \in M, \tag{1.1.2}$$

where $D_M = \sup\{d(y,z);\ y, z \in M\}$. In this way we avoid distinguishing the special case where $M = \bigcup_{i\in I} U_i$ in (1.1.4) below (which can happen if M is compact).

The site M_{sa}

The subanalytic topology was introduced in [KS01].

Let M be a real analytic manifold and denote by $\mathrm{Op}_{M_{\mathrm{sa}}}$ the category of relatively compact subanalytic open subsets of M, the morphisms being the inclusion morphisms. Recall that one endows $\mathrm{Op}_{M_{\mathrm{sa}}}$ with a Grothendieck topology by saying that a family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sa}}}$ is a covering of $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ if $U_i \subset U$ for all $i \in I$ and there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j = U$. It follows from the theory of subanalytic sets that in this situation there exist a constant $C > 0$ and a positive integer N such that

$$d(x, M \setminus U)^N \leq C \cdot (\max_{j\in J} d(x, M \setminus U_j)). \tag{1.1.3}$$

One shall be aware that if U is an open subset of M, we may endow it with the subanalytic topology U_{sa}, but this topology does not coincide in general with the topology induced by M.

We denote by $\rho_{\mathrm{sa}} \colon M \to M_{\mathrm{sa}}$ (or simply ρ) the natural morphism of sites.

The site M_{sal}

Definition 1.1.1. Let $\{U_i\}_{i\in I}$ be a finite family in $\mathrm{Op}_{M_{\mathrm{sa}}}$. We say that this family is 1-regularly situated if there is a constant C such that for any $x \in M$

$$d(x, M \setminus \bigcup_{i\in I} U_i) \leq C \cdot \max_{i\in I} d(x, M \setminus U_i). \tag{1.1.4}$$

Of course, this definition does not depend on the choice of the distance d.

When $M = \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ we have $d(x, M \setminus U) = d(x, \partial U)$, for all $x \in U$. In general we have the following comparison result.

Lemma 1.1.2. *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ be such that ∂U is non empty (that is, U is not a union of connected components of M). Then there exists $C > 0$ such that for all $x \in U$ we have*

$$d(x, M \setminus U) \leq d(x, \partial U) \leq C\, d(x, M \setminus U).$$

Proof. The first inequality is clear and we prove the second one. If it is false, there exist $x_n \in U$, $n \in \mathbb{N}$, such that $d(x_n, \partial U)/d(x_n, M \setminus U) \xrightarrow{n\to\infty} \infty$. Since $\overline{U}$ is compact, up to taking a subsequence we may assume that x_n converges to a point $x \in \overline{U}$. We see easily that $x \in \partial U$. We take a chart around x as in (1.1.1). Since $d_{\mathbb{R}^n}(y, \partial U) = d_{\mathbb{R}^n}(y, M \setminus U)$ for y in the chart near x, we can not have $d(x_n, \partial U)/d(x_n, M \setminus U) \xrightarrow{n\to\infty} \infty$, which proves the result. Q.E.D.

Example 1.1.3. Let $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ be two disjoint open sets. We prove that $\{U_1, U_2\}$ is 1-regularly situated. We set $U = U_1 \cup U_2$. We argue as in the proof of Lemma 1.1.2 and assume by contradiction that there exists a sequence $x_n \in U$, $n \in \mathbb{N}$, such that $d(x_n, M \setminus U)/\max_{i=1,2}\{d(x_n, M \setminus U_i)\}$ converges to ∞. We may as well assume $x_n \in U_1$ for all n. Up to taking a subsequence we may assume that x_n converges to a point $x \in \overline{U_1}$. We see that $x \in \partial U_1$. We take a chart around x as in (1.1.1). Then, for $n \gg 0$, $d_{\mathbb{R}^d}(x_n, M \setminus U_1)$ is realized by a point $y_n \in \partial U_1$. Since $U_2 \cap \partial U_1 = \emptyset$ we have in fact $y_n \in M \setminus U$. Hence $d_{\mathbb{R}^d}(x_n, M \setminus U_1) = d_{\mathbb{R}^d}(x_n, M \setminus U)$. Since d is Lipschitz equivalent to $d_{\mathbb{R}^d}$, the quotient $d(x_n, M \setminus U)/\max_{i=1,2}\{d(x_n, M \setminus U_i)\}$ remains bounded and we have a contradiction.

Example 1.1.4. On $\mathbb{R}^2$ with coordinates (x_1, x_2) consider the open sets:

$$
\begin{aligned}
U_1 &= \{(x_1, x_2);\ x_2 > -x_1^2,\ x_1 > 0\},\\
U_2 &= \{(x_1, x_2);\ x_2 < x_1^2,\ x_1 > 0\},\\
U_3 &= \{(x_1, x_2);\ x_1 > -x_2^2,\ x_2 > 0\}.
\end{aligned}
$$

Then $\{U_1, U_2\}$ is not 1-regularly situated. Indeed, set $W := U_1 \cup U_2 = \{x_1 > 0\}$. Then, if $x = (x_1, 0), x_1 > 0$, $d(x, \mathbb{R}^2 \setminus W) = x_1$ and $d(x, \mathbb{R}^2 \setminus U_i)$ $(i = 1, 2)$ is less that x_1^2.
On the other hand $\{U_1, U_3\}$ is 1-regularly situated. Indeed,

$$d(x, \mathbb{R}^2 \setminus (U_1 \cup U_3)) \leq \sqrt{2} \max(d(x, \mathbb{R}^2 \setminus U_1), d(x, \mathbb{R}^2 \setminus U_3)).$$

Definition 1.1.5. A linear covering of U is a small family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sa}}}$ such that $U_i \subset U$ for all $i \in I$ and

$$\begin{cases} \text{there exists a finite subset } I_0 \subset I \text{ such that the family } \{U_i\}_{i\in I_0} \\ \text{is 1-regularly situated and } \bigcup_{i\in I_0} U_i = U. \end{cases} \tag{1.1.5}$$

Let $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be two families of objects of $\mathrm{Op}_{M_{\mathrm{sa}}}$. Recall that one says that $\{U_i\}_{i\in I}$ is a refinement of $\{V_j\}_{j\in J}$ if for any $i \in I$, there exists $j \in J$ with $U_i \subset V_j$.

Proposition 1.1.6. *The family of linear coverings satisfies the axioms of Grothendieck topologies below* (*see* [KS06, § 16.1]).
COV1 $\{U\}$ *is a covering of* U, *for any* $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$.
COV2 *If a covering* $\{U_i\}_{i\in I}$ *of* U *is a refinement of a family* $\{V_j\}_{j\in J}$ *in* $\mathrm{Op}_{M_{\mathrm{sa}}}$ *with* $V_j \subset U$ *for all* $j \in J$, *then* $\{V_j\}_{j\in J}$ *is a covering of* U.
COV3 *If* $V \subset U$ *are in* $\mathrm{Op}_{M_{\mathrm{sa}}}$ *and* $\{U_i\}_{i\in I}$ *is a covering of* U, *then* $\{V \cap U_i\}_{i\in I}$ *is a covering of* V.
COV4 *If* $\{U_i\}_{i\in I}$ *is a covering of* U *and* $\{V_j\}_{j\in J}$ *is a small family in* $\mathrm{Op}_{M_{\mathrm{sa}}}$ *with* $V_j \subset U$ *such that* $\{U_i \cap V_j\}_{j\in J}$ *is a covering of* U_i *for all* $i \in I$, *then* $\{V_j\}_{j\in J}$ *is a covering of* U.

Proof. We shall use the obvious fact stating that for two subsets $A \subset B$ in M, we have $d(x, M \setminus A) \leq d(x, M \setminus B)$.

COV1 is trivial.
COV2 Let $I_0 \subset I$ be as in (1.1.5). Let $\sigma\colon I \to J$ be such that $U_i \subset V_{\sigma(i)}$, for all $i \in I$. Then, for all $x \in U_i$ we have $d(x, M \setminus U_i) \leq d(x, M \setminus V_{\sigma(i)})$. It follows that $\sigma(I_0)$ satisfies (1.1.5) with respect to $\{V_j\}_{j\in J}$.
COV3 Let $I_0 \subset I$ be as in (1.1.5) and let C be the constant in (1.1.4). Let x be a given point in $V \cap U$. We have $d(x, M \setminus (V \cap U)) \leq d(x, M \setminus U)$. We distinguish two cases.
(a) We assume that $d(x, M \setminus (V \cap U_i)) = d(x, M \setminus U_i)$, for all $i \in I_0$. Then we clearly have $d(x, M \setminus (V \cap U)) \leq C \max_{i\in I_0} d(x, M \setminus (V \cap U_i))$ and I_0 satisfies (1.1.5) with respect to $\{V \cap U_i\}_{i\in I}$.
(b) We assume $d(x, M \setminus (V \cap U_{i_0})) < d(x, M \setminus U_{i_0})$ for some $i_0 \in I_0$. We choose $y \in M \setminus (V \cap U_{i_0})$ such that $d(x, y) = d(x, M \setminus (V \cap U_{i_0}))$. Then we have $d(x, y) < d(x, M \setminus U_{i_0})$. We deduce that $y \in U_{i_0}$ and then that $y \in M \setminus V$. Hence $y \in M \setminus (V \cap U)$ and $d(x, M \setminus (V \cap U)) \leq d(x, y)$. Then

$$\begin{aligned} d(x, M \setminus (V \cap U)) &\leq d(x, M \setminus (V \cap U_{i_0})) \\ &\leq \max_{i\in I_0} d(x, M \setminus (V \cap U_i)). \end{aligned}$$

We obtain (1.1.4) for the family $\{V \cap U_i\}_{i\in I_0}$ with $C = 1$.
COV4 Let $I_0 \subset I$ be as in (1.1.5) and let C be the constant in (1.1.4). For each $i \in I_0$ let $J_i \subset J$ satisfy (1.1.5) with respect to U_i for the family $\{U_i \cap V_j\}_{j\in J}$ and let C_i be the corresponding constant. We set $J_0 = \bigcup_{i\in I_0} J_i$ and $B = \max\{C \cdot C_i;\ i \in I_0\}$. Then we have

$$\begin{aligned} d(x, M \setminus U) &\leq C \max_{i\in I_0} d(x, M \setminus U_i) \\ &\leq C \max_{i\in I_0}(C_i \max_{j\in J_i} d(x, M \setminus (U_i \cap V_j))) \\ &\leq B \max_{i\in I_0} \max_{j\in J_i} d(x, M \setminus V_j) \\ &\leq B \max_{j\in J_0} d(x, M \setminus V_j), \end{aligned}$$

which proves that J_0 satisfies (1.1.5) with respect to $\{V_j\}_{j\in J}$. Q.E.D.

As a particular case of COV4, we get

Corollary 1.1.7. *If $\{U_i\}_{i\in I}$ is a linear covering of $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and $I = \bigsqcup_{\alpha\in A} I_\alpha$ is a partition of I, then setting $U_\alpha := \bigcup_{i\in I_\alpha} U_i$, $\{U_\alpha\}_{\alpha\in A}$ is a linear covering of U.*

The notion of a linear covering is of local nature (in the usual topology). More precisely, we have:

Proposition 1.1.8. *Let $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $\{U_i\}_{i\in I}$ be a finite covering of $\overline{V}$ in M_{sa}. Then $\{V \cap U_i\}_{i\in I}$ is a linear covering of V.*

Proof. Set $U = \bigcup_i U_i$ and let $W \in \mathrm{Op}_{M_{\mathrm{sa}}}$ be a neighborhood of the boundary ∂U such that $V \cap W = \emptyset$. Let us prove that the family $\{W, \{U_i\}_{i\in I}\}$ is a linear covering of $W \cup U$. We set $f(x) = \max\{d(x, M \setminus W), d(x, M \setminus U_i), i \in I\}$ and $Z = \{x \in M;\, d(x, M \setminus (W \cup U)) \geq d(x, U)\}$. Then Z is a compact subset of $W \cup U$. Hence there exists $\varepsilon > 0$ such that $f(x) > \varepsilon$ for all $x \in Z$.

We also see that $\overline{U} \subset Z$. Hence $f(x) = d(x, M \setminus W)$ for $x \notin Z$. Moreover, for a given $x \notin Z$ we have $d(x, M \setminus W) \leq d(x, M \setminus (W \cup U)) < d(x, U)$ by definition of Z. Hence a given $y \in M \setminus W$ realizing $d(x, M \setminus W)$ can not belong to U and we obtain $d(x, M \setminus (W \cup U)) = d(x, M \setminus W)$. Finally $d(x, M \setminus (W \cup U)) = f(x)$ for $x \notin Z$.

Now we deduce that $d(x, M \setminus (W \cup U)) \leq C f(x)$ for some $C > 0$ and for all $x \in M$, that is, $\{W, \{U_i\}_{i\in I}\}$ is a linear covering of $W \cup U$.

Taking the intersection with V we obtain by COV3 that $\{V \cap U_i\}_{i\in I}$ is a linear covering of V. Q.E.D.

Corollary 1.1.9. *Let $\{U_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be two finite families in $\mathrm{Op}_{M_{\mathrm{sa}}}$. We set $U = \bigcup_i U_i$ and we assume that $\overline{U} \subset \bigcup_j B_j$. Then $\{U_i\}_{i\in I}$ is a linear covering of U if and only if $\{U_i \cap B_j\}_{i\in I}$ is a linear covering of $U \cap B_j$ for all $j \in J$.*

Proof. (i) Assume that $\{U_i\}_i$ is a linear covering of U. Applying COV3 to $B_j \cap U \subset U$ we get that the family $\{U_i \cap B_j\}_{i\in I}$ is a linear covering of $U \cap B_j$ for all $j \in J$.

(ii) Assume that the family $\{U_i \cap B_j\}_{i\in I}$ is a linear covering of $U \cap B_j$ for all $j \in J$. By Proposition 1.1.8 the family $\{U \cap B_j\}_{j\in J}$ is a linear covering of U. Hence the result follows from COV4. Q.E.D.

Definition 1.1.10. (a) The linear subanalytic site M_{sal} is the presite M_{sa} endowed with the Grothendieck topology for which the coverings are the linear coverings given by Definition 1.1.5.

(b) We denote by $\rho_{\mathrm{sal}} \colon M_{\mathrm{sa}} \to M_{\mathrm{sal}}$ and by $\rho_{\mathrm{sl}} \colon M \to M_{\mathrm{sal}}$ the natural morphisms of sites.

The morphisms of sites constructed above are summarized by the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho_{\mathrm{sa}}} & M_{\mathrm{sa}} \\ & {\scriptstyle\rho_{\mathrm{sl}}}\searrow & \downarrow{\scriptstyle\rho_{\mathrm{sal}}} \\ & & M_{\mathrm{sal}}. \end{array}$$

Remark 1.1.11. Let M and N be two real analytic manifolds and let $f\colon M \to N$ be a topological isomorphism such that both f and f^{-1} are subanalytic Lipschitz maps. Then $f^{-1}\colon \mathrm{Op}_{M_{\mathrm{sa}}} \to \mathrm{Op}_{N_{\mathrm{sa}}}$ induces an isomorphism of sites $N_{\mathrm{sal}} \xrightarrow{\sim} M_{\mathrm{sal}}$.

1.2 Regular coverings

We shall also use the following:

Definition 1.2.1. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. A regular covering of U is a sequence $\{U_i\}_{i\in[1,N]}$ with $1 \le N \in \mathbb{N}$ such that $U = \bigcup_{i\in[1,N]} U_i$ and, for all $1 \le k \le N$, $\{U_i\}_{i\in[1,k]}$ is a linear covering of $\bigcup_{1\le i\le k} U_i$.

We will use the following recipe to turn an arbitrary covering into a linear covering by a slight enlargement of the open subsets. For an open subset U of M, an arbitrary subset $V \subset U$ and $\varepsilon > 0$ we set

$$V^{\varepsilon,U} = \{x \in M;\ d(x,V) < \varepsilon\, d(x, M \setminus U)\}. \tag{1.2.1}$$

Then $V^{\varepsilon,U}$ is an open subset of U. If the distance d is a subanalytic function on $M \times M$, $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and V is a subanalytic subset, then $V^{\varepsilon,U}$ also belongs to $\mathrm{Op}_{M_{\mathrm{sa}}}$. We see easily that $(U \cap \overline{V}) \subset V^{\varepsilon,U} \subset U$.

Lemma 1.2.2. *We assume that the distance d is a subanalytic function on $M \times M$. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $V \subset U$ be a subanalytic subset. Let $0 < \varepsilon$ and $0 < \delta < 1$. We set $\varepsilon' = \frac{\varepsilon+\delta}{1-\delta}$. Then*

(i) *for any $x \in V^{\varepsilon,U}$ and $y \in M$ such that*

$$d(x,y) < \delta\, d(x, M\setminus U) \text{ or } d(x,y) < \delta\, d(y, M \setminus U),$$

we have $d(y,V) < \varepsilon' d(y, M\setminus U)$, that is, $y \in V^{\varepsilon',U}$,

(ii) *for any $x \in V^{\varepsilon,U}$ we have $d(x, M \setminus V^{\varepsilon',U}) \ge \delta\, d(x, M \setminus U)$,*

(iii) *$\{U \setminus \overline{V}, V^{\varepsilon',U}\}$ is a linear covering of U.*

We remark that any $\varepsilon' > 0$ can be written $\varepsilon' = \frac{\varepsilon+\delta}{1-\delta}$ with ε, δ as in the lemma.

Proof. (i) The triangular inequality $d(x, M\setminus U) \le d(x,y) + d(y, M\setminus U)$ implies

$$\begin{cases} d(x,M\setminus U) < (1-\delta)^{-1} d(y, M\setminus U), & \text{if } d(x,y) < \delta\, d(x, M\setminus U), \\ d(x, M\setminus U) < (1+\delta) d(y, M\setminus U), & \text{if } d(x,y) < \delta\, d(y, M\setminus U). \end{cases}$$

Since $1+\delta < (1-\delta)^{-1}$ we obtain in both cases

$$d(x, M\setminus U) < (1-\delta)^{-1} d(y, M\setminus U). \tag{1.2.2}$$

In particular we have in both cases $d(x,y) < \delta(1-\delta)^{-1}d(y, M\setminus U)$. Now the definition of $V^{\varepsilon,U}$ implies

$$\begin{aligned} d(y,V) &\leq d(x,y) + d(x,V) \\ &< \delta(1-\delta)^{-1}d(y,M\setminus U) + \varepsilon\, d(x, M\setminus U) \\ &< (\varepsilon+\delta)(1-\delta)^{-1}d(y,M\setminus U), \end{aligned}$$

where the last inequality follows from (1.2.2).

(ii) By (i), if a point $y\in M$ does not belong to $V^{\varepsilon',U}$, we have $d(x,y) \geq \delta\, d(x, M\setminus U)$. This gives (ii).

(iii) Since d is subanalytic, the open subset $V^{\varepsilon',U}$ is subanalytic. We also see easily that $U = (U\setminus \overline{V}) \cup V^{\varepsilon',U}$. Now let $x\in M$.
(a) If $x\notin V^{\varepsilon,U}$, then (1.2.1) gives $d(x,V)\geq \varepsilon\, d(x,M\setminus U)$. Since $d(x, M\setminus (U\setminus\overline{V})) = \min\{d(x,M\setminus U), d(x,V)\}$, we deduce $d(x, M\setminus(U\setminus\overline{V})) \geq \min\{\varepsilon, 1\} d(x, M\setminus U)$.
(b) If $x\in V^{\varepsilon,U}$, then (ii) gives $d(x, M\setminus V^{\varepsilon',U}) \geq \delta\, d(x, M\setminus U)$.
We obtain in both cases

$$\max\{d(x, M\setminus(U\setminus\overline{V})), d(x, M\setminus V^{\varepsilon',U})\} \geq C d(x, M\setminus U),$$

where $C = \min\{\delta,\varepsilon\}$. This proves (iii). Q.E.D.

Lemma 1.2.4 below will be used later to obtain subsets satisfying the hypothesis of Lemma 4.3.1. We will prove it by using Lemma 1.2.2 as follows. Let $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $U = U_1\cup U_2$. For $\varepsilon > 0$ we set, using Notation (1.2.1),

$$U_1^\varepsilon = (U_1\setminus U_2)^{\varepsilon,U_1} = \{x\in U_1;\ d(x, U_1\setminus U_2) < \varepsilon\, d(x, M\setminus U_1)\}, \tag{1.2.3}$$
$$U_2^\varepsilon = (U_2\setminus U_1)^{\varepsilon,U_2} = \{x\in U_2;\ d(x, U_2\setminus U_1) < \varepsilon\, d(x, M\setminus U_2)\}. \tag{1.2.4}$$

Lemma 1.2.3. (i) *For $i=1,2$ and for any $\varepsilon>0$, the pair $\{U_i^\varepsilon, U_1\cap U_2\}$ is a linear covering of U_i.*

(ii) *For any $\varepsilon,\varepsilon'>0$ such that $\varepsilon\varepsilon'<1$, we have $\overline{U_1^\varepsilon}\cap\overline{U_2^{\varepsilon'}}\cap U = \emptyset$.*

(iii) *Let $\varepsilon>0$, $0<\delta<1$ and set $\varepsilon' = \frac{\varepsilon+\delta}{1-\delta}$, $\varepsilon'' = \frac{\varepsilon'+\delta}{1-\delta}$. We assume $\varepsilon\varepsilon''<1$. Then, for any $x\in M$,*

$$\begin{cases} d(x, U_2^\varepsilon) \geq \delta\, d(x, M\setminus U_1) & \text{if } x\in U_1^{\varepsilon'}, \\ d(x, U_1^\varepsilon) \geq \delta\, d(x, M\setminus U_1) & \text{if } x\notin U_1^{\varepsilon'}. \end{cases}$$

Proof. (i) By symmetry we can assume $i = 1$. By Lemma 1.2.2, the pair $\{U_1 \setminus \overline{(U_1 \setminus U_2)}, U_1^\varepsilon\}$ is a linear covering of U_1. Since U_2 is open we have $U_1 \setminus \overline{(U_1 \setminus U_2)} = U_1 \cap U_2$ and (i) follows.

(ii) We have

$$\overline{U_1^\varepsilon} \cap U \subset \{x \in U;\ d(x, U_1 \setminus U_2) \leq \varepsilon\, d(x, M \setminus U_1)\},$$
$$\overline{U_2^{\varepsilon'}} \cap U \subset \{x \in U;\ d(x, U_2 \setminus U_1) \leq \varepsilon'\, d(x, M \setminus U_2)\}.$$

We remark that $d(x, M \setminus U_2) \leq d(x, U_1 \setminus U_2)$ and $d(x, M \setminus U_1) \leq d(x, U_2 \setminus U_1)$ for any $x \in M$. Let $x \in \overline{U_1^\varepsilon} \cap \overline{U_2^{\varepsilon'}} \cap U$ and set $d_1 = d(x, U_2 \setminus U_1)$, $d_2 = d(x, U_1 \setminus U_2)$. We deduce $d_i \leq \varepsilon\varepsilon' d_i$, for $i = 1, 2$. Since $\varepsilon\varepsilon' < 1$ we obtain $d_1 = d_2 = 0$. Hence $x \notin U_1$ and $x \notin U_2$. Since $U = U_1 \cup U_2$, this proves (ii).

(iii) By Lemma 1.2.2 (ii), we have $d(x, M \setminus U_1^{\varepsilon''}) \geq \delta\, d(x, M \setminus U_1)$ for any $x \in U_1^{\varepsilon'}$. By (ii) we have $U_2^\varepsilon \subset M \setminus U_1^{\varepsilon''}$ and the first inequality follows.

By Lemma 1.2.2 (i), if $x \notin U_1^{\varepsilon'}$ and $z \in U_1^\varepsilon$, then $d(x, z) \geq \delta\, d(x, M \setminus U_1)$. This gives the second inequality. Q.E.D.

Lemma 1.2.4. *Let $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and set $U = U_1 \cup U_2$. We assume that $\{U_1, U_2\}$ is a linear covering of U. Then there exist $U'_i \subset U_i$, $i = 1, 2$, and $C > 0$ such that*

(i) *$\{U'_i, U_1 \cap U_2\}$ is a linear covering of U_i $(i = 1, 2)$,*

(ii) *$\overline{U'_1} \cap \overline{U'_2} \cap U = \emptyset$,*

(iii) *setting $Z_i = (M \setminus U) \cup \overline{U'_i}$, we have $Z_1 \cap Z_2 = M \setminus U$ and*

$$d(x, Z_1 \cap Z_2) \leq C(d(x, Z_1) + d(x, Z_2)), \quad \textit{for any } x \in M.$$

Proof. We set $\varepsilon = \delta = 1/3$, $\varepsilon' = \frac{\varepsilon+\delta}{1-\delta} = 1$ and $\varepsilon'' = \frac{\varepsilon'+\delta}{1-\delta} = 2$. Using the notations (1.2.3) and (1.2.4) we set $U'_i = U_i^\varepsilon$, $i = 1, 2$.

(i) and (ii) are given by Lemma 1.2.3 (i) and (ii).

(iii) The equality $Z_1 \cap Z_2 = M \setminus U$ follows from (ii). Let C' be the constant in (1.1.4) for the family $\{U_1, U_2\}$. We set $C_1 = \max\{1, \delta^{-1}C'\}$. Let $x \in M$ and let $x_i \in Z_i$ be such that $d(x, x_i) = d(x, Z_i)$. By the definition of Z_1, if $x_1 \notin \overline{U'_1}$, then $x_1 \in M \setminus U$. Hence $d(x, Z_1) = d(x, M \setminus U)$ and the inequality in (iii) is clear.

Hence we can assume $x_1 \in \overline{U'_1}$ and also $x_2 \in \overline{U'_2}$ by symmetry. Then we have $d(x, Z_1) + d(x, Z_2) = d(x, U_1^\varepsilon) + d(x, U_2^\varepsilon)$. Since $\varepsilon\varepsilon'' = 2/3 < 1$, Lemma 1.2.3 (iii) gives $d(x, U_1^\varepsilon) + d(x, U_2^\varepsilon) \geq \delta\, d(x, M \setminus U_1)$. The same holds with $M \setminus U_1$ replaced by $M \setminus U_2$ and (1.1.4) gives

$$d(x, U_1^\varepsilon) + d(x, U_2^\varepsilon) \geq \delta \max_{i=1,2}\{d(x, M \setminus U_i)\} \geq C_1^{-1} d(x, M \setminus U),$$

so that (iii) holds with $C = C_1$. Q.E.D.

Lemma 1.2.5. *We assume that the distance d is a subanalytic function on $M \times M$. Let $\{U_i\}_{i=1}^N$ be a 1-regularly situated family in $\mathrm{Op}_{M_{\mathrm{sa}}}$ and let $C \geq 1$ be a constant satisfying* (1.1.4)*. We choose $D > C$ and $1 > \varepsilon > 0$ such that $\varepsilon D < 1 - \varepsilon$. We define $U_i^0, V_i, U_i' \in \mathrm{Op}_{M_{\mathrm{sa}}}$ inductively on i by $U_1^0 = V_1 = U_1' = U_1$ and*

$$\begin{aligned} U_i^0 &= \{x \in U_i;\ d(x, M \setminus (U_i \cup V_{i-1})) < D\, d(x, M \setminus U_i)\},\\ V_i &= V_{i-1} \cup U_i^0,\\ U_i' &= (U_i^0)^{\varepsilon, V_i} \quad (\textit{using the notation } (1.2.1)). \end{aligned}$$

Then $V_N = \bigcup_{i=1}^N U_i$ and, for all $k = 1, \dots, N$, we have $U_k' \subset U_k$, $V_k = \bigcup_{i=1}^k U_i'$ and $\{U_i'\}_{i=1}^k$ is a 1-regularly situated family in $\mathrm{Op}_{M_{\mathrm{sa}}}$.

Proof. (i) Let us prove that $U_k' \subset U_k$. Let $x \in U_k'$ and let us show that $x \in U_k$. By (1.2.1) we have $x \in V_k$ and there exists $y \in U_k^0$ such that $d(x, y) < \varepsilon\, d(x, M \setminus V_k)$. We deduce $d(x, y) < \varepsilon(d(x, y) + d(y, M \setminus V_k))$ and then

$$d(x, y) < (\varepsilon/(1 - \varepsilon))\, d(y, M \setminus V_k). \tag{1.2.5}$$

On the other hand we have $U_k^0 \subset U_k$, hence $V_k \subset U_k \cup V_{k-1}$. Since $y \in U_k^0$ we deduce

$$d(y, M \setminus V_k) \leq d(y, M \setminus (U_k \cup V_{k-1})) < D\, d(y, M \setminus U_k). \tag{1.2.6}$$

The inequalities (1.2.5), (1.2.6) and the hypothesis on D and ε give $d(x, y) < d(y, M \setminus U_k)$. Hence $x \in U_k$.

(ii) We have $V_i = V_{i-1} \cup U_i^0$. Hence Lemma 1.2.2 implies that $\{V_{i-1}, U_i'\}$ is a covering of V_i in M_{sa}. Let us prove the last part of the lemma by induction on k. We immediately obtain that $V_k = \bigcup_{i=1}^k U_i'$. Moreover, $\{V_{k-1}, U_k'\}$ being a covering of V_k, we get by using COV4 that, for all $k = 1, \dots, N$, $\{U_i'\}_{i=1}^k$ is a 1-regularly situated family in $\mathrm{Op}_{M_{\mathrm{sa}}}$.

(iii) It remains to prove that $V_N = \bigcup_{i=1}^N U_i$. It is clear that $V_k \subset \bigcup_{i=1}^N U_i$, for all $k = 1, \dots, N$. Let $x \in \bigcup_{i=1}^N U_i$. Since $\{U_i\}_{i=1}^N$ is 1-regularly situated, there exists i_0 such that $d(x, M \setminus \bigcup_{i=1}^{N} U_i) \leq C\, d(x, M \setminus U_{i_0})$. In particular $x \in U_{i_0}$ and moreover $d(x, M \setminus (U_{i_0} \cup V_{i_0-1})) \leq C\, d(x, M \setminus U_{i_0}) < D\, d(x, M \setminus U_{i_0})$. Therefore $x \in U^0_{i_0}$. By definition $U^0_{i_0} \subset V_{i_0} \subset V_N$. Hence $x \in V_N$ and we obtain $V_N = \bigcup_{i=1}^{N} U_i$. Q.E.D.

In particular, we have proved:

Proposition 1.2.6. *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Then for any linear covering $\{U_i\}_{i \in I}$ of U there exists a refinement which is a regular covering of U.*

Chapter 2

Sheaves on subanalytic topologies

2.1 Sheaves

Usual notations

We shall mainly follow the notations of [KS90, KS01] and [KS06].

In this paper, we denote by $\mathbf{k}$ a field, although most of the results hold under the hypothesis that $\mathbf{k}$ is a commutative unital Noetherian ring with finite global dimension. Unless otherwise specified, a manifold means a real analytic manifold.

If $\mathscr{C}$ is an additive category, we denote by $\mathrm{C}(\mathscr{C})$ the additive category of complexes in $\mathscr{C}$. For $* = +, -, \mathrm{b}$ we also consider the full additive subcategory $\mathrm{C}^*(\mathscr{C})$ of $\mathrm{C}(\mathscr{C})$ consisting of complexes bounded from below (resp. from above, resp. bounded) and $\mathrm{C}^{\mathrm{ub}}(\mathscr{C})$ means $\mathrm{C}(\mathscr{C})$ ("ub" stands for "unbounded"). If $\mathscr{C}$ is an abelian category, we denote by $\mathrm{D}(\mathscr{C})$ its derived category and similarly with $\mathrm{D}^*(\mathscr{C})$ for $* = +, -, \mathrm{b}, \mathrm{ub}$.

For a site $\mathscr{T}$, we denote by $\mathrm{PSh}(\mathbf{k}_{\mathscr{T}})$ and $\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ the abelian categories of presheaves and sheaves of $\mathbf{k}$-modules on $\mathscr{T}$. We denote by $\iota\colon \mathrm{Mod}(\mathbf{k}_{\mathscr{T}}) \to \mathrm{PSh}(\mathbf{k}_{\mathscr{T}})$ the forgetful functor and by $(\,\bullet\,)^a$ its left adjoint, the functor which associates a sheaf to a presheaf. Note that in practice we shall often not write ι. Recall that $\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ is a Grothendieck category and, in particular, has enough injectives. We write $\mathrm{D}^*(\mathbf{k}_{\mathscr{T}})$ instead of $\mathrm{D}^*(\mathrm{Mod}(\mathbf{k}_{\mathscr{T}}))$ $(* = +, -, \mathrm{b}, \mathrm{ub})$.

For a site $\mathscr{T}$, we will often use the following well-known fact. For any $F \in \mathrm{D}(\mathbf{k}_{\mathscr{T}})$ and any $i \in \mathbb{Z}$, the cohomology sheaf $H^i(F)$ is the sheaf associated with the presheaf $U \mapsto H^i(U; F)$. In particular, if $H^i(U; F) = 0$ for all $U \in \mathscr{T}$, then $H^i(F) \simeq 0$.

For an object U of $\mathscr{T}$, recall that there is a sheaf naturally attached to U (see *e.g.* [KS06, § 17.6]). We shall denote it here by $\mathbf{k}_{U\mathscr{T}}$ or simply $\mathbf{k}_U$ if there is no risk of confusion. This is the sheaf associated with the presheaf (see loc. cit. Lemma 17.6.11):

$$V \mapsto \oplus_{V \to U} \mathbf{k}.$$

The functor "associated sheaf" is exact. If follows that, if $V \to U$ is a monomorphism in $\mathscr{T}$, then the natural morphism $\mathbf{k}_{V\mathscr{T}} \to \mathbf{k}_{U\mathscr{T}}$ also is a monomorphism.

Sheaves on M and M_{sa}

We shall mainly use the subanalytic topology introduced in [KS01]. In loc. cit., sheaves on the subanalytic topology are studied in the more general framework of indsheaves. We refer to [Pre08] for a direct and more elementary treatment of subanalytic sheaves.

Recall that $\rho_{\mathrm{sa}} \colon M \to M_{\mathrm{sa}}$ denotes the natural morphism of sites. The functor $\rho_{\mathrm{sa}*}$ is left exact and its left adjoint ρ_{sa}^{-1} is exact. Hence, we have the pairs of adjoint functors

$$\mathrm{Mod}(\mathbf{k}_M) \underset{\rho_{\mathrm{sa}}^{-1}}{\overset{\rho_{\mathrm{sa}*}}{\rightleftarrows}} \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}), \qquad \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \underset{\rho_{\mathrm{sa}}^{-1}}{\overset{\mathrm{R}\rho_{\mathrm{sa}*}}{\rightleftarrows}} \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sa}}}). \tag{2.1.1}$$

The functor $\rho_{\mathrm{sa}*}$ is fully faithful and $\rho_{\mathrm{sa}}^{-1}\rho_{\mathrm{sa}*} \simeq \mathrm{id}$. Moreover, $\rho_{\mathrm{sa}}^{-1}\mathrm{R}\rho_{\mathrm{sa}*} \simeq \mathrm{id}$ and $\mathrm{R}\rho_{\mathrm{sa}*}$ in (2.1.1) is fully faithful.

The functor ρ_{sa}^{-1} also admits a left adjoint functor $\rho_{\mathrm{sa}!}$. For $F \in \mathrm{Mod}(\mathbf{k}_M)$, $\rho_{\mathrm{sa}!}F$ is the sheaf on M_{sa} associated with the presheaf $U \mapsto F(\overline{U})$. The functor $\rho_{\mathrm{sa}!}$ is exact, fully faithful and commutes with tensor products.

Proposition 2.1.1. *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $F \in \mathrm{Mod}(\mathbf{k}_M)$. Then*

$$\mathrm{R}\Gamma(U; \mathrm{R}\rho_{\mathrm{sa}*}F) \simeq \mathrm{R}\Gamma(U; F).$$

Proof. This follows from $\mathrm{R}\Gamma(U;G) \simeq \mathrm{RHom}\,(\mathbf{k}_U, G)$ for $G \in \mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ ($\mathscr{T} = M$ or $\mathscr{T} = M_{\mathrm{sa}}$) and by adjunction since $\rho_{\mathrm{sa}}^{-1}\mathbf{k}_{UM_{\mathrm{sa}}} \simeq \mathbf{k}_{UM}$. Q.E.D.

Also note that the functor $\rho_{\mathrm{sa}*}$ admitting an exact left adjoint functor, it sends injective objects of $\mathrm{Mod}(\mathbf{k}_M)$ to injective objects of $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$.

One denotes by $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the category of $\mathbb{R}$-constructible sheaves on M. One denotes by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of objects with $\mathbb{R}$-constructible cohomologies.

Recall that $\rho_{\mathrm{sa}*}$ is exact when restricted to the subcategory $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$. Hence we shall consider this last category both as a full subcategory of $\mathrm{Mod}(\mathbf{k}_M)$ and a full subcategory of $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$.

For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ we have the sheaf $\mathbf{k}_{UM_{\mathrm{sa}}} \simeq \rho_{\mathrm{sa}*}\mathbf{k}_{UM}$ on M_{sa} that we simply denote by $\mathbf{k}_U$.

Sheaves on M and M_{sal}

Recall Definition 1.1.10. The functor $\rho_{\mathrm{sal}*}$ is left exact and its left adjoint ρ_{sal}^{-1} is exact since the presites underlying the sites M_{sa} and M_{sal} are the same (see [KS06, Th. 17.5.2]). Hence, we have the pairs of adjoint functors

(2.1.2) $$\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \underset{\rho_{\mathrm{sal}}^{-1}}{\overset{\rho_{\mathrm{sal}*}}{\rightleftarrows}} \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}}), \qquad \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}}) \underset{\rho_{\mathrm{sal}}^{-1}}{\overset{\mathrm{R}\rho_{\mathrm{sal}*}}{\rightleftarrows}} \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}}).$$

Lemma 2.1.2. *The functor $\rho_{\mathrm{sal}*}$ in (2.1.2) is fully faithful and $\rho_{\mathrm{sal}}^{-1}\rho_{\mathrm{sal}*} \simeq \mathrm{id}$. Moreover, $\rho_{\mathrm{sal}}^{-1}\mathrm{R}\rho_{\mathrm{sal}*} \simeq \mathrm{id}$ and $\mathrm{R}\rho_{\mathrm{sal}*}$ in (2.1.2) is fully faithful.*

Proof. (i) By its definition, $\rho_{\mathrm{sal}}^{-1}\rho_{\mathrm{sal}*}F$ is the sheaf associated with the presheaf $U \mapsto (\rho_{\mathrm{sal}*}F)(U) \simeq F(U)$ and this presheaf is already a sheaf.

(ii) Since ρ_{sal}^{-1} is exact, $\rho_{\mathrm{sal}}^{-1}\mathrm{R}\rho_{\mathrm{sal}*}$ is the derived functor of $\rho_{\mathrm{sal}}^{-1}\rho_{\mathrm{sal}*}$. Q.E.D.

In the sequel, if K is a compact subset of M, we set for a sheaf F on M_{sa} or M_{sal}:

$$\Gamma(K;F) := \varinjlim_{K \subset U} \Gamma(U;F), \quad U \in \mathrm{Op}_{M_{\mathrm{sa}}}.$$

Lemma 2.1.3. *Let $F \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}})$. For K compact in M, we have the natural isomorphisms*

$$\Gamma(K;F) \xrightarrow{\sim} \Gamma(K;\rho_{\mathrm{sal}}^{-1}F) \xrightarrow{\sim} \Gamma(K;\rho_{\mathrm{sl}}^{-1}F).$$

Proof. The first isomorphism follows from Proposition 1.1.8. The second one from [KS01, Prop. 6.6.2] since $\rho_{\mathrm{sl}}^{-1} \simeq \rho_{\mathrm{sa}}^{-1}\rho_{\mathrm{sal}}^{-1}$. Q.E.D.

The next result is analogue to [KS01, Prop. 6.6.2].

Proposition 2.1.4. *Let $F \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}})$. For U open in M, we have the natural isomorphism*

$$\Gamma(U;\rho_{\mathrm{sl}}^{-1}F) \simeq \varprojlim_{V \subset\subset U} \Gamma(V;F), \; V \in \mathrm{Op}_{M_{\mathrm{sa}}}.$$

Proof. We have the chain of isomorphisms, the second one following from Lemma 2.1.3:

$$\Gamma(U;\rho_{\mathrm{sl}}^{-1}F)\simeq \varprojlim_{V\subset\subset U}\Gamma(\overline{V};\rho_{\mathrm{sl}}^{-1}F)\simeq \varprojlim_{V\subset\subset U}\Gamma(\overline{V};F)\simeq \varprojlim_{V\subset\subset U}\Gamma(V;F).$$

Q.E.D.

The next result is analogue to [KS01, Prop. 6.6.3, 6.6.4]. Since the proof of loc. cit. extends to our situation with the help of Proposition 2.1.4, we do not repeat it.

Proposition 2.1.5. *The functor ρ_{sl}^{-1} admits a left adjoint that we denote by $\rho_{\mathrm{sl}!}$. For $F \in \mathrm{Mod}(\mathbf{k}_M)$, $\rho_{\mathrm{sl}!}F$ is the sheaf on M_{sal} associated with the presheaf $U \mapsto F(\overline{U})$. The functor $\rho_{\mathrm{sl}!}$ is exact and fully faithful.*

Sheaves on M_{sa} and M_{sal}

Proposition 2.1.6. *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Then we have $\rho_{\mathrm{sal}*}\mathbf{k}_{UM_{\mathrm{sa}}} \simeq \mathbf{k}_{UM_{\mathrm{sal}}}$ and $\rho_{\mathrm{sal}}^{-1}\mathbf{k}_{UM_{\mathrm{sal}}} \simeq \mathbf{k}_{UM_{\mathrm{sa}}}$.*

Proof. The proof of [KS01, Prop. 6.3.1] gives the first isomorphism without any changes other than notational. The second isomorphism follows by Lemma 2.1.2. Q.E.D.

Proposition 2.1.7. *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $F \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$. Then*

$$\mathrm{R}\Gamma(U;\mathrm{R}\rho_{\mathrm{sal}*}F)\simeq \mathrm{R}\Gamma(U;F).$$

The proof goes as for Proposition 2.1.1.

In the sequel we shall simply denote by $\mathbf{k}_U$ the sheaf $\mathbf{k}_{U\mathscr{T}}$ for $\mathscr{T} = M_{\mathrm{sa}}$ or $\mathscr{T} = M_{\mathrm{sal}}$.

Proposition 2.1.8. *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}. Then a presheaf F is a sheaf if and only if it satisfies:*

(i) $F(\emptyset) = 0$,

(ii) *for any $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ such that $\{U_1, U_2\}$ is a covering of $U_1 \cup U_2$, the sequence $0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2)$ is exact.*

Of course, if $\mathscr{T} = M_{\mathrm{sa}}$, $\{U_1, U_2\}$ is always a covering of $U_1 \cup U_2$.

Proof. In the case of the site $M_{\rm sa}$ this is Proposition 6.4.1 of [KS01]. Let F be a presheaf on $M_{\rm sal}$ such that (i) and (ii) are satisfied and let us prove that F is a sheaf. Let $U \in \mathrm{Op}_{M_{\rm sa}}$ and let $\{U_i\}_{i\in I}$ be a linear covering of U. By Proposition 1.2.6 we can find a finite refinement $\{V_j\}_{j\in J}$ of $\{U_i\}_{i\in I}$ which is a regular covering of U. We choose $\sigma\colon J \to I$ such that $V_j \subset U_{\sigma(j)}$ for all $j \in J$ and we consider the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & F(U) & \xrightarrow{u} & \bigoplus_{i\in I} F(U_i) & \xrightarrow{v} & \bigoplus_{i,j\in I} F(U_{ij}) \\ & \Big\| & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ 0 \longrightarrow & F(U) & \longrightarrow & \bigoplus_{k\in J} F(V_k) & \longrightarrow & \bigoplus_{k,l\in J} F(V_{kl}), \end{array} \tag{2.1.3}$$

where a and b are defined as follows. For $s = \{s_i\}_{i\in I} \in \bigoplus_{i\in I} F(U_i)$, we set $a(s) = \{t_k\}_{k\in J} \in \bigoplus_{k\in J} F(V_k)$ where $t_k = s_{\sigma(k)}|_{V_k}$. In the same way we set $b(\{s_{ij}\}_{i,j\in I}) = \{s_{\sigma(k)\sigma(l)}|_{V_{kl}}\}_{k,l\in J}$. The proof of [KS01, Prop. 6.4.1] applies to a regular covering in $M_{\rm sal}$ and we deduce that the bottom row of the diagram (2.1.3) is exact. It follows immediately that $\mathrm{Ker}\, u = 0$. This proves that F is a separated presheaf.

It remains to prove that $\mathrm{Ker}\, v = \mathrm{Im}\, u$. Let $s = \{s_i\}_{i\in I} \in \bigoplus_{i\in I} F(U_i)$ be such that $v(s) = 0$. By the exactness of the bottom row we can find $t \in F(U)$ such that $a(u(t) - s) = 0$. Let us check that $t|_{U_i} = s_i$ for any given $i \in I$. The family $\{U_i \cap V_k\}_{k\in J}$ is a covering of U_i in $M_{\rm sal}$. Since F is separated it is enough to see that $t|_{U_i\cap V_k} = s_i|_{U_i\cap V_k}$ for all $k \in J$. Setting $W = U_i \cap V_k$, we have

$$t|_W = s_{\sigma(k)}|_W = (s_{\sigma(k)}|_{U_i\cap U_{\sigma(k)}})|_W = (s_i|_{U_i\cap U_{\sigma(k)}})|_W = s_i|_W,$$

where the first equality follows from $a(u(t) - s) = 0$ and the third one from $v(s) = 0$. Q.E.D.

Lemma 2.1.9. *Let $\mathscr{T}$ be either the site $M_{\rm sa}$ or the site $M_{\rm sal}$. Let $U \in \mathrm{Op}_{M_{\rm sa}}$ and let $\{F_i\}_{i\in I}$ be an inductive system in $\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ indexed by a small filtrant category I. Then*

$$\varinjlim_i \Gamma(U; F_i) \xrightarrow{\sim} \Gamma(U; \varinjlim_i F_i). \tag{2.1.4}$$

This kind of results is well-known from the specialists (see *e.g.* [KS01,EP]) but for the reader's convenience, we give a proof.

Proof. For a covering $\mathscr{S} = \{U_j\}_j$ of U set

$$\Gamma(\mathscr{S}; F) := \mathrm{Ker}\big(\prod_i F(U_i) \rightrightarrows \prod_{ij} F(U_i \cap U_j)\big).$$

Denote by "$\varinjlim$" the inductive limit in the category of presheaves and recall that $\varinjlim_i F_i$ is the sheaf associated with "$\varinjlim_i$" F_i. The presheaf "$\varinjlim_i$" F_i is separated. Denote by $\mathrm{Cov}(U)$ the family of coverings of U in $\mathscr{T}$ ordered as follows. For $\mathscr{S}_1$ and $\mathscr{S}_2$ in $\mathrm{Cov}(U)$, $\mathscr{S}_1 \preceq \mathscr{S}_2$ if $\mathscr{S}_1$ is a refinement of $\mathscr{S}_2$. Then $\mathrm{Cov}(U)^{\mathrm{op}}$ is filtrant and

$$\begin{aligned}\Gamma(U;\varinjlim_i F_i) &\simeq \varinjlim_{\mathscr{S}\in\mathrm{Cov}(U)} \Gamma(\mathscr{S};\text{“}\varinjlim_i\text{”}\, F_i)\\ &\simeq \varinjlim_{\mathscr{S}}\varinjlim_i \Gamma(\mathscr{S};F_i)\\ &\simeq \varinjlim_i\varinjlim_{\mathscr{S}} \Gamma(\mathscr{S};F_i) \simeq \varinjlim_i \Gamma(U;F_i).\end{aligned}$$

Here, the second isomorphism follows from the fact that we may assume that the covering $\mathscr{S}$ is finite. Q.E.D.

Example 2.1.10. Let $M=\mathbb{R}^2$ endowed with coordinates $x=(x_1,x_2)$. For $\varepsilon, A>0$ we define the subanalytic open subset

$$U_{A,\varepsilon}=\{x;\ 0<x_1<\varepsilon,\ -Ax_1^2<x_2<Ax_1^2\}. \tag{2.1.5}$$

We define a presheaf F on M_{sal} by setting, for any $V\in\mathrm{Op}_{M_{\mathrm{sa}}}$,

$$F(V)=\begin{cases}\mathbf{k} & \text{if for any } A>0, \text{ there exists } \varepsilon>0 \text{ such that } U_{A,\varepsilon}\subset V,\\ 0 & \text{otherwise.}\end{cases}$$

The restriction map $F(V)\to F(V')$, for $V'\subset V$, is $\mathrm{id}_{\mathbf{k}}$ if $F(V')=\mathbf{k}$. We prove that F is sheaf in (iii) below after the preliminary remarks (i) and (ii).

(i) For given $A,\varepsilon_0>0$ we have $d((\varepsilon,0),M\setminus U_{A,\varepsilon_0})\geq(A/4)\varepsilon^2$, for any $\varepsilon>0$ small enough. In particular, if $F(V)=\mathbf{k}$, then

$$d((\varepsilon,0),M\setminus V)/\varepsilon^2\to+\infty \quad \text{when } \varepsilon\to 0. \tag{2.1.6}$$

(ii) Let us assume that there exist $A>0$ and a sequence $\{\varepsilon_n\}$, $n\in\mathbb{N}$, such that $\varepsilon_n>0$, $\varepsilon_n\to 0$ when $n\to\infty$ and V contains the closed balls $B((\varepsilon_n,0),A\varepsilon_n^2)$ for all $n\in\mathbb{N}$. Then there exists $\varepsilon>0$ such that V contains $\overline{U_{A,\varepsilon}}\setminus\{0\}$.

Before we prove this claim we translate the conclusion in terms of sheaf theory (in the usual site $\mathbb{R}^2$). Let $p\colon\mathbb{R}^2\to\mathbb{R}$ be the projection $(x_1,x_2)\mapsto x_1$. Then, for $x_1>0$, the set $p^{-1}(x_1)\cap V\cap\overline{U_{A,\varepsilon}}$ is a finite disjoint union of intervals, say $I_1,\dots,I_N$. If $p^{-1}(x_1)\cap V$ contains $p^{-1}(x_1)\cap\overline{U_{A,\varepsilon}}$, then $N=1$,

I_1 is closed and $\mathrm{R}\Gamma(\mathbb{R};\mathbf{k}_{I_1}) = \mathbf{k}$. In the other case none of these $I_1,\dots,I_N$ is closed and $H^0(\mathbb{R};\mathbf{k}_{I_j}) = 0$, for all $j = 1,\dots,N$. By the base change formula we deduce that V contains $\overline{U_{A,\varepsilon}} \setminus \{0\}$ if and only if $\mathrm{R}p_*(\mathbf{k}_{V\cap\overline{U_{A,\varepsilon}}})|_{]0,\varepsilon]} \simeq \mathbf{k}_{]0,\varepsilon]}$.

We remark that, for $\varepsilon < 1$, we have $\mathrm{R}p_*(\mathbf{k}_{V\cap\overline{U_{A,\varepsilon}}})|_{]0,\varepsilon]} \simeq \mathrm{R}p_*(\mathbf{k}_{V\cap\overline{U_{A,1}}})|_{]0,\varepsilon]}$. The sheaf $\mathrm{R}p_*(\mathbf{k}_{V\cap\overline{U_{A,1}}})$ is constructible. Hence it is constant on $]0,\varepsilon]$ for $\varepsilon > 0$ small enough. Since $(\mathrm{R}p_*(\mathbf{k}_{V\cap\overline{U_{A,1}}}))_{\varepsilon_n} \simeq \mathbf{k}$ by hypothesis, the conclusion follows.

(iii) Now we check that F is a sheaf on M_{sal} with the criterion of Proposition 2.1.8. Let $U, U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ such that $\{U_1, U_2\}$ is a covering of U.

(iii-a) Let us prove that $F(U) \to F(U_1) \oplus F(U_2)$ is injective. So we assume that $F(U) = \mathbf{k}$ (otherwise this is obvious) and we prove that $F(U_1) = \mathbf{k}$ or $F(U_2) = \mathbf{k}$. Let $A > 0$. By (2.1.6) and (1.1.4) there exists $\varepsilon_0 > 0$ such that

$$\max\{d((\varepsilon,0), M\setminus U_1), d((\varepsilon,0), M\setminus U_2)\} \geq A\varepsilon^2, \quad \text{for all } \varepsilon \in]0,\varepsilon_0[.$$

Hence, for any integer $n \geq 1$, the ball $B((1/n,0), A/n^2)$ is included in U_1 or U_2. One of U_1 or U_2 must contain infinitely many such balls. By (ii) we deduce that it contains U_{A,ε_A}, for some $\varepsilon_A > 0$. When A runs over $\mathbb{N}$ we deduce that one of U_1 or U_2 contains infinitely many sets of the type U_{A,ε_A}, $A \in \mathbb{N}$. Hence $F(U_1) = \mathbf{k}$ or $F(U_2) = \mathbf{k}$.

(iii-b) Now we prove that the kernel of $F(U_1)\oplus F(U_2) \to F(U_{12})$ is $F(U)$. We see easily that the only case where this kernel could be bigger than $F(U)$ is $F(U_1) = F(U_2) = \mathbf{k}$ and $F(U_{12}) = 0$. In this case, for any $A > 0$, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $U_{A,\varepsilon_1} \subset U_1$ and $U_{A,\varepsilon_2} \subset U_2$. This gives $U_{A,\min\{\varepsilon_1,\varepsilon_2\}} \subset U_{12}$ which contradicts $F(U_{12}) = 0$.

(iv) By the definition of F we have a natural morphism $u\colon F \to \rho_{\mathrm{sal}*}\mathbf{k}_{\{0\}}$ which is surjective. We can see that $\rho_{\mathrm{sal}}^{-1}(u)$ is an isomorphism. We define $N \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}})$ by the exact sequence

$$0 \to N \to F \to \rho_{\mathrm{sal}*}\mathbf{k}_{\{0\}} \to 0. \tag{2.1.7}$$

Then $\rho_{\mathrm{sal}}^{-1}N \simeq 0$ but $N \neq 0$. More precisely, for $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we have $N(V) = 0$ if $0 \in V$ and $N(V) \xrightarrow{\sim} F(V)$ if $0 \notin V$.

2.2 Γ-acyclic sheaves

Čech complexes

In this subsection, $\mathscr{T}$ denotes either the site M_{sa} or the site M_{sal}.

For a finite set I and a family of open subsets $\{U_i\}_{i\in I}$ we set for $\emptyset \neq J \subset I$,

$$U_J := \bigcap_{j\in J} U_j.$$

Lemma 2.2.1. *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}. Let $\{U_1, U_2\}$ be a covering of $U_1 \cup U_2$. Then the sequence*

$$0 \to \mathbf{k}_{U_{12}} \to \mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \to \mathbf{k}_{U_1\cup U_2} \to 0 \tag{2.2.1}$$

is exact.

Proof. The result is well-known for the site M_{sa} and the functor $\rho_{\mathrm{sal}*}$ being left exact, it remains to show that $\mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \to \mathbf{k}_{U_1\cup U_2}$ is an epimorphism. This follows from the fact that for any $F \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}})$, the map $\mathrm{Hom}_{\mathbf{k}_{M_{\mathrm{sal}}}}(\mathbf{k}_{U_1\cup U_2}, F) \to \mathrm{Hom}_{\mathbf{k}_{M_{\mathrm{sal}}}}(\mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2}, F)$ is a monomorphism. Q.E.D.

Consider now a finite family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sa}}}$ and let $N := |I|$. We choose a bijection $I = [1, N]$. Then we have the Čech complex in $\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ in which the term corresponding to $|J| = 1$ is in degree 0.

$$\mathbf{k}^\bullet_{\mathscr{U}} := 0 \to \bigoplus_{J\subset I, |J|=N} \mathbf{k}_{U_J} \otimes e_J \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{J\subset I,|J|=1} \mathbf{k}_{U_J} \otimes e_J \to 0. \tag{2.2.2}$$

Recall that $\{e_J\}_{|J|=k}$ is a basis of $\bigwedge^k \mathbb{Z}^N$ and the differential is defined as usual by sending $\mathbf{k}_{U_J} \otimes e_J$ to $\bigoplus_{i\in I} \mathbf{k}_{U_{J\setminus\{i\}}} \otimes e_i \lfloor e_J$ using the natural morphism $\mathbf{k}_{U_J} \to \mathbf{k}_{U_{J\setminus\{i\}}}$.

Proposition 2.2.2. *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $\mathscr{U} := \{U_i\}_i \in I$ be a finite covering of U in $\mathscr{T}$ (a regular covering in case $\mathscr{T} = M_{\mathrm{sal}}$). Then the natural morphism $\mathbf{k}^\bullet_{\mathscr{U}} \to \mathbf{k}_U$ is a quasi-isomorphism.*

Proof. Recall that $N = |I|$. We may assume $I = [1, N]$. For $N = 2$ this is nothing but Lemma 2.2.1. We argue by induction and assume the result is proved for $N - 1$. Denote by $\mathscr{U}'$ the covering of $U' := \bigcup_{1\leq i\leq N-1} U_i$ by the family $\{U_i\}_{i\in[1,\dots,N-1]}$. Consider the subcomplex F_1 of $\mathbf{k}^\bullet_{\mathscr{U}}$ given by

$$F_1 := 0 \to \bigoplus_{N\in J\subset I,|J|=N} \mathbf{k}_{U_J} \otimes e_J \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{N\in J\subset I,|J|=1} \mathbf{k}_{U_J} \otimes e_J \to 0. \tag{2.2.3}$$

Note that F_1 is isomorphic to the complex $\mathbf{k}^\bullet_{\mathscr{U}'\cap U_N} \to \mathbf{k}_{U_N}$ where $\mathbf{k}_{U_N}$ is in degree 0 and we shall represent F_1 by this last complex. By [KS06, Th. 12.4.3], there is a natural morphism of complexes

$$u\colon \mathbf{k}^\bullet_{\mathscr{U}'}[-1] \to \big(\mathbf{k}^\bullet_{\mathscr{U}'\cap U_N} \to \mathbf{k}_{U_N}\big) \tag{2.2.4}$$

such that $\mathbf{k}^\bullet_{\mathscr{U}}$ is isomorphic to the mapping cone of u. Hence, writing the long exact sequence associated with the mapping cone of u, we are reduced, by the induction hypothesis, to prove that the morphism

$$\mathbf{k}_{U'\cap U_N} \to \mathbf{k}_{U'} \oplus \mathbf{k}_{U_N}$$

is a monomorphism and its cokernel is isomorphic to $\mathbf{k}_U$. Since $\{U', U_N\}$ is a covering of U, this follows from Lemma 2.2.1. Q.E.D.

Acyclic sheaves

In this subsection, $\mathscr{T}$ denotes either the site M_{sa} or the site M_{sal}. In the literature, one often encounters sheaves which are $\Gamma(U;\,\bullet\,)$-acyclic for a given $U \in \mathscr{T}$ but the next definition does not seem to be frequently used.

Definition 2.2.3. Let $F \in \mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$. We say that F is Γ-acyclic if we have $H^k(U;F) \simeq 0$ for all $k > 0$ and all $U \in \mathscr{T}$.

We shall give criteria in order that a sheaf F on the site $\mathscr{T}$ be Γ-acyclic.

Let $U \in \mathrm{Op}_{M_{\text{sa}}}$ and let $\mathscr{U} := \{U_i\}_i \in I$ be a finite covering of U in $\mathscr{T}$ (a regular covering in case $\mathscr{T} = M_{\text{sal}}$). We denote by $C^\bullet(\mathscr{U};F)$ the associated Čech complex:

$$C^\bullet(\mathscr{U};F) := \mathrm{Hom}_{\mathbf{k}_{M_{\text{sal}}}}(\mathbf{k}^\bullet_{\mathscr{U}}, F). \tag{2.2.5}$$

One can write more explicitly this complex as the complex:

$$0 \to \bigoplus_{J\subset I, |J|=1} F(U_J)\otimes e_J \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{J\subset I,|J|=N} F(U_J)\otimes e_J \to 0 \tag{2.2.6}$$

where the differential d is obtained by sending $F(U_J)\otimes e_J$ to $\bigoplus_{i\in I} F(U_J \cap U_i)\otimes e_i \wedge e_J$.

Proposition 2.2.4. *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal} and let $F \in \mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$. The conditions below are equivalent.*

(i) *For any $\{U_1, U_2\}$ which is a covering of $U_1 \cup U_2$, the sequence $0 \to F(U_1 \cup U_2) \to F(U_1)\oplus F(U_2) \to F(U_1 \cap U_2) \to 0$ is exact.*

(ii) *The sheaf F is Γ-acyclic.*

(iii) *For any exact sequence in $\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$*

$$G^\bullet := 0 \to \bigoplus_{i_0\in A_0} \mathbf{k}_{U_{i_0}} \to \cdots \to \bigoplus_{i_N\in A_N} \mathbf{k}_{U_{i_N}} \to 0 \tag{2.2.7}$$

the sequence $\mathrm{Hom}_{\mathbf{k}_{\mathscr{T}}}(G^\bullet, F)$ is exact.

(iv) *For any finite covering* $\mathscr{U}$ *of* U *(regular covering in case* $\mathscr{T} = M_{\mathrm{sal}}$*), the morphism* $F(U) \to C^\bullet(\mathscr{U}; F)$ *is a quasi-isomorphism.*

Proof. (i)⇒(ii) (a) Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Let us first show that for any exact sequence of sheaves $0 \to F \xrightarrow{\varphi} F' \xrightarrow{\psi} F'' \to 0$ and any $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, the sequence $0 \to F(U) \to F'(U) \to F''(U) \to 0$ is exact. Let $s'' \in F''(U)$. By the exactness of the sequence of sheaves, there exists a finite covering $U = \bigcup_{i=1}^N U_i$ and $s'_i \in F'(U_i)$ such that $\psi(s'_i) = s''|_{U_i}$. In case $\mathscr{T} = M_{\mathrm{sal}}$, we may assume that the covering is regular by Proposition 1.2.6. For $k = 1, \dots, N$, we set $V_k = \bigcup_{i=1}^k U_i$. Let us prove by induction on k that there exists $t'_k \in F'(V_k)$ such that $\psi(t'_k) = s''|_{V_k}$. Starting with $t'_1 = s'_1$ we assume that we have found t'_k. Since our covering is regular, $\{V_k, U_{k+1}\}$ is a covering of V_{k+1}. We set for short $W = V_k \cap U_{k+1}$. We have $\psi(t'_k|_W) = \psi(s'_{k+1}|_W)$. Hence there exists $s \in F(W)$ such that $\varphi(s) = t'_k|_W - s'_{k+1}|_W$. By hypothesis (i) there exists $s_V \in F(V_k)$ and $s_U \in F(U_{k+1})$ such that $s = s_V|_W - s_U|_W$. Setting $t'_V = t'_k - \varphi(s_V)$ and $s'_U = s'_{k+1} - \varphi(s_U)$ we obtain $t'_U|_W = s'_V|_W$ and we can glue $t'_U|_W$ and $s'_V|_W$ into $t'_{k+1} \in F'(V_{k+1})$. We check easily that $\psi(t'_{k+1}) = s''|_{V_{k+1}}$ and the induction proceeds.

(i)⇒(ii) (b) Denote by $\mathscr{J}$ the full additive subcategory of $\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ consisting of sheaves satisfying the condition (i). We shall show that the category $\mathscr{J}$ is $\Gamma(U; \bullet)$-injective for all $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. The category $\mathscr{J}$ contains the injective sheaves. By the first part of the proof, it thus remains to show that, for any short exact sequence of sheaves $F^\bullet := 0 \to F' \to F \to F'' \to 0$, if both F' and F belong to $\mathscr{J}$, then F'' belongs to $\mathscr{J}$.

Let U_1, U_2 as in (i) and denote by $\mathbf{k}^\bullet_{\mathscr{U}}$ the exact sequence $0 \to \mathbf{k}_{U_1 \cap U_2} \to \mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \to \mathbf{k}_{U_1 \cup U_2} \to 0$. Consider the double complex $\mathrm{Hom}_{\mathbf{k}_{\mathscr{T}}}(\mathbf{k}^\bullet_{\mathscr{U}}, F^\bullet)$. By the preceding result all rows and columns except at most one (either one row or one column depending how one writes the double complex) are exact. It follows that the double complex is exact.

(ii)⇒(iii) Consider an injective resolution $I^\bullet$ of F, that is, a complex $I^\bullet$ of injective sheaves such that the sequence $I^{\bullet,+} := 0 \to F \to I^\bullet$ is exact. The hypothesis implies that $\Gamma(W; I^{\bullet,+})$ remains exact for all $W \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Then the argument goes as in the proof of (i)⇒(ii) (b). Recall that $G^\bullet$ denotes the complex of (2.2.7) and consider the double complex $\mathrm{Hom}_{\mathbf{k}_{\mathscr{T}}}(G^\bullet, I^{\bullet,+})$. Then all its rows and columns except one (either one row or one column depending how one writes the double complex) will be exact. It follows that all rows and columns are exact.

(iii)⇒(iv) follows from Proposition 2.2.2.

(iv)⇒(i) is obvious. Q.E.D.

Corollary 2.2.5. *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}. A small filtrant inductive limit of Γ-acyclic sheaves is Γ-acyclic.*

Proof. Since small filtrant inductive limits are exact in $\mathrm{Mod}(\mathbf{k})$, the family of sheaves satisfying condition (i) of Proposition 2.2.4 is stable by such limits by Lemma 2.1.9. Q.E.D.

Definition 2.2.6. Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}. One says that $F \in \mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ is flabby if for any U and V in $\mathrm{Op}_{M_{\mathrm{sa}}}$ with $V \subset U$, the natural morphism $F(U) \to F(V)$ is surjective.

Lemma 2.2.7. *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}.*

(i) *Injective sheaves are flabby.*

(ii) *Flabby sheaves are Γ-acyclic.*

(iii) *The category of flabby sheaves is stable by small filtrant inductive limits.*

Proof. (i) Let F be an injective sheaf and let U and V in $\mathrm{Op}_{M_{\mathrm{sa}}}$ with $V \subset U$. Recall that the sequence $0 \to \mathbf{k}_V \to \mathbf{k}_U$ is exact. Applying the functor $\mathrm{Hom}_{\mathbf{k}_{\mathscr{T}}}(\,\bullet\,, F)$ we get the result.

(ii) If $F \in \mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ is flabby then it satisfies condition (i) of Proposition 2.2.4.

(iii) The proof of Corollary 2.2.5 also works in this case. Q.E.D.

2.3 The functor $\rho^!_{\mathrm{sal}}$

In this section we make an essential use of the Brown representability theorem (see for example [KS06, Th 14.3.1]).

Direct sums in derived categories

In this subsection, we state and prove some elementary results that we shall need, some of them being well-known from the specialists.

Lemma 2.3.1. *Let $\mathscr{C}$ be a Grothendieck category and let $d \in \mathbb{Z}$. Then the cohomology functor H^d and the truncation functors $\tau^{\leq d}$ and $\tau^{\geq d}$ commute* *with small direct sums in* $\mathrm{D}(\mathscr{C})$*. In other words, if* $\{F_i\}_{i\in I}$ *is a small family of objects of* $\mathrm{D}(\mathscr{C})$*, then*

$$\bigoplus_i \tau^{\leq d} F_i \xrightarrow{\sim} \tau^{\leq d}(\bigoplus_i F_i) \tag{2.3.1}$$

and similarly with $\tau^{\geq d}$ *and* H^d*.*

Proof. (i) The case of H^d follows from [KS06, Prop. 10.2.8, Prop. 14.1.1].
(ii) The morphism in (2.3.1) is well-defined and it is enough to check that it induces an isomorphism on the cohomology. This follows from (i) since for any object $Y \in \mathrm{D}(\mathscr{C})$, $H^j(\tau^{\leq d}Y)$ is either 0 or $H^j(Y)$. Q.E.D.

Lemma 2.3.2. *Let* $\mathscr{C}$ *and* $\mathscr{C}'$ *be two Grothendieck categories and let* $\rho\colon \mathscr{C} \to \mathscr{C}'$ *be a left exact functor. Let* I *be a small category. Assume*

(i) I *is either filtrant or discrete,*

(ii) ρ *commutes with inductive limits indexed by* I*,*

(iii) *inductive limits indexed by* I *of injective objects in* $\mathscr{C}$ *are acyclic for the functor* ρ*.*

Then for all $j \in \mathbb{Z}$*, the functor* $R^j\rho\colon \mathscr{C} \to \mathscr{C}'$ *commutes with inductive limits indexed by* I*.*

Proof. Let $\alpha\colon I \to \mathscr{C}$ be a functor. Denote by $\mathscr{I}$ the full additive subcategory of $\mathscr{C}$ consisting of injective objects. It follows for example from [KS06, Cor. 9.6.6] that there exists a functor $\psi\colon I \to \mathscr{I}$ and a morphism of functors $\alpha \to \psi$ such that for each $i \in I$, $\alpha(i) \to \psi(i)$ is a monomorphism. Therefore one can construct a functor $\Psi\colon I \to \mathrm{C}^+(\mathscr{I})$ and a morphism of functor $\alpha \to \Psi$ such that for each $i \in I$, $\alpha(i) \to \Psi(i)$ is a quasi-isomorphism. Set $X_i = \alpha(i)$ and $G_i^\bullet = \Psi(i)$. We get a qis $X_i \to G_i^\bullet$, hence a qis

$$\varinjlim_i X_i \to \varinjlim_i G_i^\bullet .$$

On the other hand, we have

$$\begin{aligned} \varinjlim_i R^j\rho(X_i) &\simeq \varinjlim_i H^j(\rho(G_i^\bullet)) \\ &\simeq H^j\rho(\varinjlim_i G_i^\bullet) \end{aligned}$$

where the second isomorphism follows from the fact that H^j commutes with direct sums and with filtrant inductive limits. Then the result follows from hypothesis (iii). Q.E.D.

Lemma 2.3.3. *We make the same hypothesis as in Lemma 2.3.2. Let* $-\infty < a \leq b < \infty$*, let* I *be a small set and let* $X_i \in \mathrm{D}^{[a,b]}(\mathscr{C})$*. Then*

$$\bigoplus_i R\rho(X_i) \xrightarrow{\sim} R\rho(\bigoplus_i X_i). \tag{2.3.2}$$

Proof. The morphism in (2.3.2) is well-defined and we have to prove it is an isomorphism. If $b = a$, the result follows from Lemma 2.3.2. The general case is deduced by induction on $b - a$ by considering the distinguished triangles

$$H^a(X_i)\,[-a] \to X_i \to \tau^{\geq a+1} X_i \xrightarrow{+1} .$$

Q.E.D.

Proposition 2.3.4. *Let* $\mathscr{C}$ *and* $\mathscr{C}'$ *be two Grothendieck categories and let* $\rho\colon \mathscr{C} \to \mathscr{C}'$ *be a left exact functor. Assume that*

(a) *ρ has finite cohomological dimension,*

(b) *ρ commutes with small direct sums,*

(c) *small direct sums of injective objects in $\mathscr{C}$ are acyclic for the functor ρ.*

Then

(i) *the functor* $R\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ *commutes with small direct sums,*

(ii) *the functor* $R\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ *admits a right adjoint* $\rho^!\colon \mathrm{D}(\mathscr{C}') \to \mathrm{D}(\mathscr{C})$*,*

(iii) *the functor* $\rho^!$ *induces a functor* $\rho^!\colon \mathrm{D}^+(\mathscr{C}') \to \mathrm{D}^+(\mathscr{C})$*.*

Proof. (i) Let $\{X_i\}_{i\in I}$ be a family of objects of $\mathrm{D}(\mathscr{C})$. It is enough to check that the natural morphism in $\mathrm{D}(\mathscr{C}')$

$$\bigoplus_{i\in I} R\rho(X_i) \to R\rho(\bigoplus_{i\in I} X_i) \tag{2.3.3}$$

induces an isomorphism on the cohomology groups. Assume that ρ has cohomological dimension $\leq d$. For $X \in \mathrm{D}(\mathscr{C})$ and for $j \in \mathbb{Z}$, we have

$$\tau^{\geq j} R\rho(X) \simeq \tau^{\geq j} R\rho(\tau^{\geq j-d-1} X).$$

The functor ρ being left exact we get for $k \geq j$:

$$H^k R\rho(X) \simeq H^k R\rho(\tau^{\leq k}\tau^{\geq j-d-1} X). \tag{2.3.4}$$

We have the sequence of isomorphisms:

$$\begin{aligned} H^kR\rho(\bigoplus_i X_i) &\simeq H^kR\rho(\tau^{\leq k}\tau^{\geq j-d-1}\bigoplus_i X_i) \\ &\simeq H^kR\rho(\bigoplus_i \tau^{\leq k}\tau^{\geq j-d-1}X_i) \\ &\simeq \bigoplus_i H^kR\rho(\tau^{\leq k}\tau^{\geq j-d-1}X_i) \\ &\simeq \bigoplus_i H^kR\rho(X_i). \end{aligned}$$

The first and last isomorphisms follow from (2.3.4).
The second isomorphism follows from Lemma 2.3.1.
The third isomorphism follows from Lemma 2.3.3.
(ii) follows from (i) and the Brown representability theorem (see for example [KS06, Th 14.3.1]).

(iii) This follows from hypothesis (a) and (the well-known) Lemma 2.3.5 below. Q.E.D.

Lemma 2.3.5. *Let $\rho\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor between two Grothendieck categories. Assume that $\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ admits a right adjoint $\rho^!\colon \mathrm{D}(\mathscr{C}') \to \mathrm{D}(\mathscr{C})$ and assume moreover that ρ has finite cohomological dimension. Then the functor $\rho^!$ sends $\mathrm{D}^+(\mathscr{C}')$ to $\mathrm{D}^+(\mathscr{C})$.*

Proof. By the hypothesis, we have for $X \in \mathrm{D}(\mathscr{C})$ and $Y \in \mathrm{D}(\mathscr{C}')$

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{C}')}(\rho(X), Y) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, \rho^!(Y)).$$

Assume that the cohomological dimension of the functor ρ is $\leq r$. Let $Y \in \mathrm{D}^{\geq 0}(\mathscr{C}')$. Then $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, \rho^!(Y)) \simeq 0$ for all $X \in \mathrm{D}^{<-r}(\mathscr{C})$. This means that $\rho^!(Y)$ belongs to the right orthogonal to $\mathrm{D}^{<-r}(\mathscr{C})$ and this implies that $\rho^!(Y) \in \mathrm{D}^{\geq -r}(\mathscr{C}')$. Q.E.D.

The functor $\mathrm{R}\Gamma(U; \bullet)$

Lemma 2.3.6. *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal} and let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Let I be a small filtrant category and $\alpha\colon I \to \mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ a functor. Set for short $F_i = \alpha(i)$. Then for any $j \in \mathbb{Z}$*

$$\varinjlim_i H^j\mathrm{R}\Gamma(U; F_i) \xrightarrow{\sim} H^j\mathrm{R}\Gamma(U; \varinjlim_i F_i). \tag{2.3.5}$$

Proof. By Lemma 2.1.9, the functor $\Gamma(U;\,\bullet\,)$ commutes with small filtrant inductive limits and such limits of injective objects are $\Gamma(U;\,\bullet\,)$-acyclic by Lemma 2.2.7. Hence, we may apply Lemma 2.3.2. Q.E.D.

Proposition 2.3.7. *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. The functor $\Gamma(U;\,\bullet\,)\colon \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathbf{k})$ has cohomological dimension $\leq \dim M$.*

Proof. We know that if $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$, then $H^j\mathrm{R}\Gamma(U;F) \simeq 0$ for $j > \dim M$. Since any $F \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ is a small filtrant inductive limit of constructible sheaves, the result follows from Lemma 2.3.6. Q.E.D.

Corollary 2.3.8. *Let $\mathscr{J}$ be the subcategory of $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ consisting of sheaves which are Γ-acyclic. For any $F \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$, there exists an exact sequence $0 \to F \to F^0 \to \cdots \to F^n \to 0$ where $n = \dim M$ and the F^j's belong to $\mathscr{J}$.*

Proof. Consider a resolution $0 \to F \to I^0 \xrightarrow{d^0} I^1 \to \cdots$ with the I^j's injective and define $F^j = I^j$ for $j \leq n-1$, $F^j = 0$ for $j > n$ and $F^n = \mathrm{Ker}\, d^n$. It follows from Proposition 2.3.7 that F^n is Γ-acyclic. Q.E.D.

Proposition 2.3.9. *Let I be a small set and let $F_i \in \mathrm{D}(\mathbf{k}_{M_{\mathrm{sa}}})$ $(i \in I)$. For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we have the natural isomorphism*

$$\bigoplus_{i\in I} \mathrm{R}\Gamma(U;F_i) \xrightarrow{\sim} \mathrm{R}\Gamma(U;\bigoplus_{i\in I} F_i) \text{ in } \mathrm{D}(\mathbf{k}). \tag{2.3.6}$$

Proof. The functor $\Gamma(U;\,\bullet\,)$ has finite cohomological dimension by Proposition 2.3.7, it commutes with small direct sums by Lemma 2.1.9 and inductive limits of injective objects are $\Gamma(U;\,\bullet\,)$-acyclic by Lemma 2.2.7. Hence, we may apply Proposition 2.3.4. Q.E.D.

The functor $\mathrm{R}\rho_{\mathrm{sal}*}$

Lemma 2.3.10. *Let $\mathscr{J}$ be the subcategory of $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ consisting of sheaves which are Γ-acyclic. The category $\mathscr{J}$ is $\rho_{\mathrm{sal}*}$-injective (see* [KS06, Cor. 13.3.8]*).*

Proof. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$.
(i) We see easily that if both F' and F belong to $\mathscr{J}$, then F'' belongs to $\mathscr{J}$.
(ii) It remains to prove that if $F' \in \mathscr{J}$, then the sequence $0 \to \rho_{\mathrm{sal}*}F' \to \rho_{\mathrm{sal}*}F \to \rho_{\mathrm{sal}*}F'' \to 0$ is exact. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. By Proposition 2.1.7 and the hypothesis, the sequence $0 \to \rho_{\mathrm{sal}*}F'(U) \to \rho_{\mathrm{sal}*}F(U) \to \rho_{\mathrm{sal}*}F''(U) \to 0$ is exact. Q.E.D.

Applying Corollary 2.3.8, we get:

Proposition 2.3.11. *The functor* $\rho_{\mathrm{sal}*}$ *has cohomological dimension* $\leq \dim M$*.*

Proposition 2.3.12. *Let* I *be a small set and let* $F_i \in \mathrm{D}(\mathbf{k}_{M_{\mathrm{sa}}})$ $(i \in I)$*. We have the natural isomorphism*

$$\bigoplus_{i\in I} \mathrm{R}\rho_{\mathrm{sal}*}F_i \xrightarrow{\sim} \mathrm{R}\rho_{\mathrm{sal}*}(\bigoplus_{i\in I} F_i) \text{ in } \mathrm{D}(\mathbf{k}_{M_{\mathrm{sal}}}). \tag{2.3.7}$$

Proof. By Proposition 2.3.11, the functor $\rho_{\mathrm{sal}*}$ has finite cohomological dimension and by Lemma 2.1.9 it commutes with small direct sums. Moreover, inductive limits of injective objects are $\rho_{\mathrm{sal}*}$-acyclic by Lemmas 2.3.10 and 2.2.7. Hence, we may apply Proposition 2.3.4 (i). Q.E.D.

Theorem 2.3.13. (i) *The functor* $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}(\mathbf{k}_{M_{\mathrm{sa}}}) \to \mathrm{D}(\mathbf{k}_{M_{\mathrm{sal}}})$ *admits a right adjoint* $\rho_{\mathrm{sal}}^{!}\colon \mathrm{D}(\mathbf{k}_{M_{\mathrm{sal}}}) \to \mathrm{D}(\mathbf{k}_{M_{\mathrm{sa}}})$*.*

(ii) *The functor* $\rho_{\mathrm{sal}}^{!}$ *induces a functor* $\rho_{\mathrm{sal}}^{!}\colon \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}}) \to \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}})$*.*

Proof. These results follow from Propositions 2.3.12 and 2.3.11, as in Proposition 2.3.4. Q.E.D.

Corollary 2.3.14. *One has an isomorphism of functors on* $\mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}})$*:*

$$\mathrm{id} \xrightarrow{\sim} \rho_{\mathrm{sal}}^{!}\mathrm{R}\rho_{\mathrm{sal}*}. \tag{2.3.8}$$

Proof. This follows from the fact that $(\mathrm{R}\rho_{\mathrm{sal}*}, \rho_{\mathrm{sal}}^{!})$ is a pair of adjoint functors and that $\mathrm{R}\rho_{\mathrm{sal}*}$ is fully faithful by Lemma 2.1.2. Q.E.D.

Remark 2.3.15. (i) We don't know if the category M_{sal} has finite flabby dimension. We don't even know if for any $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sal}}})$ and any $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we have $\mathrm{R}\Gamma(U;F) \in \mathsf{D}^{\mathrm{b}}(\mathbf{k})$.
(ii) We don't know if the functor $\rho_{\mathrm{sal}}^{!}\colon \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}}) \to \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}})$ constructed in Theorem 2.3.13 induces a functor $\rho_{\mathrm{sal}}^{!}\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sal}}}) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sa}}})$.

2.4 Open sets with Lipschitz boundaries

Normal cones and Lipschitz boundaries

In this paragraph $\mathbb{R}^n$ is equipped with coordinates (x', x_n), $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$.

Definition 2.4.1. We say that $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ has Lipschitz boundary or simply that U is Lipschitz if, for any $x \in \partial U$, there exist an open neighborhood V of x and a bi-Lipschitz subanalytic homeomorphism $\psi\colon V \xrightarrow{\sim} W$ with W an open subset of $\mathbb{R}^n$ such that $\psi(V \cap U) = W \cap \{x_n > 0\}$.

Remark 2.4.2. (i) The property of being Lipschitz is local and thus the preceding definition extends to subanalytic but not necessarily relatively compact open subsets of M.
(ii) If U_i is Lipschitz in M_i $(i = 1, 2)$ then $U_1 \times U_2$ is Lipschitz in $M_1 \times M_2$.
(iii) If U is Lipschitz and $x \in \partial U$, there exist a constant $C > 0$ and a sequence $\{y_n\}_{n\in\mathbb{N}}$, $y_n \in U$, such that $d(y_n, x) \to 0$ and $d(y_n, x) \le Cd(y_n, \partial U)$, for all $n \in \mathbb{N}$ (in the notations of the definition, assume $\psi(x) = (x', 0)$ and set $y_n = \psi^{-1}(x', 1/n)$).

Example 2.4.3. (i) Lemma 2.4.5 below will provide many examples of Lipschitz open sets.
(ii) Let (x, y) denotes the coordinates on $\mathbb{R}^2$. Using (iii) of Remark 2.4.2 we see that the open set $U = \{(x, y); 0 < y < x^2\}$ is not Lipschitz.

Lemma 2.4.4. *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. We assume that, for any $x \in \partial U$, there exist an open neighborhood V of x and a bi-analytic isomorphism $\psi\colon V \xrightarrow{\sim} W$ with W an open subset of $\mathbb{R}^n$ such that $\psi(V \cap U) = W \cap \{(x', x_n);\ x_n > \varphi(x')\}$ for a Lipschitz subanalytic function φ. Then U is Lipschitz.*

Proof. We define $\psi_1\colon \mathbb{R}^n \to \mathbb{R}^n$, $(x', x_n) \mapsto (x', x_n - \varphi(x'))$. Then ψ_1 is a bi-Lipschitz subanalytic homeomorphism and we have $(\psi_1 \circ \psi)(V \cap U) = \psi_1(W) \cap \{x_n > 0\}$. Hence U is Lipschitz. Q.E.D.

Lemma 2.4.5. *Let $\mathbb{V}$ be a vector space and let γ be a proper closed convex cone with non empty interior. Let $U \in \mathrm{Op}_{\mathbb{V}_{\mathrm{sa}}}$. Then the open set $U + \gamma$ has Lipschitz boundary.*

Proof. Let $p \in \partial(U + \gamma)$. We identify $\mathbb{V}$ with $\mathbb{R}^n$ so that p is the origin and γ contains the cone $\gamma_0 = \{(x', x_n);\ x_n > \|x'\|\}$. We have in particular

$$\gamma_0 \subset (U + \gamma) \subset (\mathbb{R}^n \setminus (-\gamma_0)). \tag{2.4.1}$$

For $x' \in \mathbb{R}^{n-1}$ we set $l_{x'} = (U + \gamma) \cap (\{x'\} \times \mathbb{R})$. Then $l_{x'} = l_{x'} + [0, +\infty[$. By (2.4.1) we also have $l_{x'} \neq \emptyset$ and $l_{x'} \neq \mathbb{R}$. Hence we can write $l_{x'} =]\varphi(x'), +\infty[$, for a well-defined function $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$.

Let us prove that φ is Lipschitz. Let $x' \in \mathbb{R}^{n-1}$ and let us set $q = (x', \varphi(x')) \in \partial(U + \gamma)$. We have the similar inclusion as (2.4.1), $(q + \gamma_0) \subset (U + \gamma) \subset (\mathbb{R}^n \setminus (q - \gamma_0))$. Hence $\partial(U + \gamma) \subset (\mathbb{R}^n \setminus ((q + \gamma_0) \cup (q - \gamma_0)))$. For any $y' \in \mathbb{R}^{n-1}$ we have $(y', \varphi(y')) \in \partial(U + \gamma)$ and the last inclusion translates into $|\varphi(y') - \varphi(x')| \le \|y' - x'\|$. Hence φ is Lipschitz and $U + \gamma$ is Lipschitz by Lemma 2.4.4. Q.E.D.

We refer to [KS90, Def 4.1.1] for the definition of the normal cone $C(A, B)$ associated with two subsets A and B of M.

Definition 2.4.6. (See [KS90, § 5.3].) Let S be a subset of M. The strict normal cone $N_x(S)$ and the conormal cone $N_x^*(S)$ of S at $x \in M$ as well as the strict normal cone $N(S)$ and the conormal cone $N^*(S)$ of S are given by

$$\begin{aligned} &N_x(S) = T_xM \setminus C(M \setminus S, S), \text{ an open cone in } T_xM, \\ &N_x^*(S) = N_x(S)^\circ \text{ (where } ^\circ \text{ denotes the polar cone)}, \\ &N(S) = \bigcup_{x \in M} N_x(S), \text{ an open convex cone in } TM, \\ &N^*(S) = \bigcup_{x \in M} N_x^*(S). \end{aligned}$$

By loc. cit. Prop. 5.3.7, we have:

Lemma 2.4.7. *Let U be an open subset of M and let $x \in \partial U$. Then the conditions below are equivalent:*

(i) *$N_x(U)$ is non empty,*

(ii) *$N_y(U)$ is non empty for all y in a neighborhood of x,*

(iii) *$N_x^*(U)$ is contained in a closed convex proper cone with non empty interior in T_x^*M,*

(iv) *there exists a local chart in a neighborhood of x such that identifying M with an open subset of $\mathbb{V}$, there exists a closed convex proper cone with non empty interior γ in $\mathbb{V}$ such that U is γ-open in an open neighborhood W of x, that is,*

$$W \cap ((U \cap W) + \gamma) \subset U.$$

Definition 2.4.8. We shall say that an open subset U of M satisfies a cone condition if for any $x \in \partial U$, $N_x(U)$ is non empty.

By Lemmas 2.4.5 and 2.4.7 we have:

Proposition 2.4.9. *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. If U satisfies a cone condition, then U is Lipschitz.*

Remark 2.4.10. One shall be aware that our definition of being Lipschitz differs from that of Lebeau in [Leb16]. By Lemma 2.4.4, if U is Lipschitz in Lebeau's sense, then it is Lipschitz in our sense.

A vanishing theorem

The next theorem is a key result for this paper and its proof is due to A. Parusinski [Par16].

Theorem 2.4.11. (A. Parusinski) *Let* $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$*. Then there exists a finite covering* $V = \bigcup_{j\in J} V_j$ *with* $V_j \in \mathrm{Op}_{M_{\mathrm{sa}}}$ *such that the family* $\{V_j\}_{j\in J}$ *is a covering of* V *in* M_{sal} *and moreover* $H^k(V_j;\mathbf{k}_M) \simeq 0$ *for all* $k > 0$ *and all* $j \in J$*.*

Recall that one denotes by $\rho_{\mathrm{sal}}\colon M_{\mathrm{sa}} \to M_{\mathrm{sal}}$ the natural morphism of sites.

Lemma 2.4.12. *We have* $\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_{M_{\mathrm{sa}}} \simeq \mathbf{k}_{M_{\mathrm{sal}}}$*.*

Proof. The sheaf $H^k(\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_{M_{\mathrm{sa}}})$ is the sheaf associated with the presheaf $U \mapsto H^k(U;\mathbf{k}_{M_{\mathrm{sa}}})$. This sheaf if zero for $k > 0$ by Theorem 2.4.11. Q.E.D.

Lemma 2.4.13. *Let* $M = \mathbb{R}^n$ *and set* $U =]0,+\infty[\times\mathbb{R}^{n-1}$*. Then we have* $\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_U \simeq \mathbf{k}_U$*.*

Proof. (i) The sheaf $H^k(\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_U)$ is the sheaf associated with the presheaf $V \mapsto H^k(V;\mathbf{k}_U)$. Hence it is enough to show that any $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$ admits a finite covering $V = \bigcup_{j\in J} V_j$ in M_{sal} such that $H^k(V_j;\mathbf{k}_U) \simeq 0$ for all $k > 0$. We assume that the distance d is a subanalytic function. Let us set $V_{<0} = V \cap (]-\infty,0[\times\mathbb{R}^{n-1})$ and $V' = V_{<0}^{1,V}$, where we use the notation (1.2.1) with $\varepsilon = 1$. In our case we can write (1.2.1) as follows

$$V' = \{x \in V;\ d(x, V\setminus U) < d(x, M\setminus V)\}.$$

This is a subanalytic open subset of V. By Lemma 1.2.2 we have

(2.4.2) $\{V', V\cap U\}$ is a covering of V in M_{sal}.

(ii) Let us prove that $\mathrm{R}\Gamma(V';\mathbf{k}_U) \simeq 0$. We denote by (x_1, x') the coordinates on $M = \mathbb{R}^n$. For $x = (x_1, x')$ with $x_1 \geq 0$, we have $d(x, V\setminus U) \geq d(x, M\setminus U) = x_1$. If $(x_1, x') \in V'$ we obtain $d(x, M\setminus V) > x_1$, hence $\overline{B(x,x_1)} \subset V$, where $B(x,x_1)$ is the ball with center x and radius x_1. This proves that $V'\cap\overline{U}$ is contained in the right hand side of the following equality

(2.4.3) $$V'\cap\overline{U} = \{x = (x_1,x') \in V;\ x_1 \geq 0 \text{ and } \overline{B(x,x_1)} \subset V\}$$

and the reverse inclusion is easily checked. It follows that, if $(x_1,x') \in V'\cap\overline{U}$, then $(y_1,x') \in V'\cap\overline{U}$, for all $y_1 \in [0,x_1]$. Let $q\colon \mathbb{R}^n \to \{0\}\times\mathbb{R}^{n-1}$ be the projection. We deduce:

(a) q maps $V' \cap \overline{U}$ onto $V \cap \partial U$,

(b) $q^{-1}(x) \cap V' \cap U$ is an open interval, for any $x = (0, x') \in V \cap \partial U$.

For any $c < 0 < d$ we have $\mathrm{R}\Gamma(]c, d[; \mathbf{k}_{]0,d[}) \simeq 0$. Hence (a) and (b) give $\mathrm{R}q_* \mathrm{R}\Gamma_{V'} \mathbf{k}_U \simeq 0$, by the base change formula, and we obtain $\mathrm{R}\Gamma(V'; \mathbf{k}_U) \simeq \mathrm{R}\Gamma(\mathbb{R}^{n-1}; \mathrm{R}q_* \mathrm{R}\Gamma_{V'} \mathbf{k}_U) \simeq 0$.

(iii) By Theorem 2.4.11 we can choose a finite covering of $V \cap U$ in M_{sal}, say $\{W_j\}_{j \in J}$, such that $H^k(W_j; \mathbf{k}_U) \simeq 0$ for all $k > 0$. By (2.4.2) the family $\{V', \{W_j\}_{j \in J}\}$ is a covering of V in M_{sal}. By (ii) this covering satisfies the required condition in (i), which proves the result. Q.E.D.

We need to extend Definition 2.4.1.

Definition 2.4.14. We say that $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ is weakly Lipschitz if for each $x \in M$ there exists a neighborhood $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$ of x, a finite set I and $U_i \in \mathrm{Op}_{M_{\mathrm{sa}}}$, $i \in I$, such that $U \cap V = \bigcup_{i \in I} U_i$ and

$$\begin{cases} \text{for all } \emptyset \neq J \subset I, \text{ the set } U_J = \bigcap_{j \in J} U_j \text{ is a disjoint union of} \\ \text{Lipschitz open sets.} \end{cases} \tag{2.4.4}$$

By its definition, the property of being weakly Lipschitz is local on M.

Example 2.4.15. The open subset $U = \mathbb{R}^2 \setminus \{0\}$ of $\mathbb{R}^2$ is not Lipschitz but it is weakly Lipschitz: setting $U_\pm = \{(x, y) \in \mathbb{R}^2; \pm y > -|x|\}$ we have $U = U_+ \cup U_-$ and U_+, U_-, $U_+ \cap U_-$ are disjoint unions of Lipschitz open subsets.

Proposition 2.4.16. *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and consider a finite family of smooth submanifolds $\{Z_i\}_{i \in I}$, closed in a neighborhood of $\overline{U}$. Set $Z = \bigcup_{i \in I} Z_i$. Assume that*

(a) *U is Lipschitz,*

(b) *$Z_i \cap Z_j \cap \partial U = \emptyset$ for $i \neq j$, ∂U is smooth in a neighborhood of $Z \cap \partial U$ and the intersection is transversal,*

(c) *in a neighborhood of each point of $Z \cap U$ there exist a local coordinate system $(x_1, \dots, x_n)$ and for each $i \in I$, a subset I_i of $\{1, \dots, n\}$ such that $Z_i = \bigcap_{j \in I_i} \{x; x_j = 0\}$.*

Then $U \setminus Z$ is weakly Lipschitz.

Proof. Since the property of being weakly Lipschitz is local on M, it is enough to prove the result in a neighborhood of each point $p \in \overline{U}$.
(i) Assume $p \in \partial U$. We choose a local coordinate system $(x_1, \dots, x_n)$ centered at p such that $U = \{x; x_n > 0\}$ and $Z = \{x; x_1 = \dots = x_r = 0\}$ (with $r < n$). For $1 \le i \le r$, define $U_i = \{x; x_n > 0, x_i \neq 0\}$. Then the family $\{U_i\}_{i=1,\dots,r}$ satisfies (2.4.4).
(ii) Assume $p \in U$. We choose a local coordinate system $(x_1, \dots, x_n)$ such that $Z_i = \bigcap_{j_i \in I_i} \{x; x_{j_i} = 0\}$. For each $j_i \in I_i$ and $\varepsilon_i = \pm 1$, define $U_i^{\varepsilon_i, j_i} = \{x;\ \varepsilon_i x_{j_i} > 0\}$. Set $A = \prod_{i \in I}(\{\pm 1\} \times I_i)$ and for $\alpha \in A$ set $U_\alpha = \bigcap_{i \in I} U_i^{\alpha_i}$. Then the family $\{U_\alpha\}_{\alpha \in A}$ satisfies (2.4.4). Q.E.D.

Theorem 2.4.17. *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and assume that U is weakly Lipschitz. Then*

(i) $\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_{UM_{\mathrm{sa}}} \simeq \rho_{\mathrm{sal}*}\mathbf{k}_{UM_{\mathrm{sa}}} \simeq \mathbf{k}_{UM_{\mathrm{sal}}}$ *is concentrated in degree zero.*

(ii) *For $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sal}}})$, one has* $\mathrm{R}\Gamma(U; \rho_{\mathrm{sal}}^! F) \simeq \mathrm{R}\Gamma(U; F)$.

(iii) *Let $F \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}})$ and assume that F is Γ-acyclic. Then $\mathrm{R}\Gamma(U; \rho_{\mathrm{sal}}^! F)$ is concentrated in degree 0 and is isomorphic to $F(U)$.*

Note that the result in (i) is local and it is not necessary to assume here that U is relatively compact.

Proof. (i)–(a) First we assume that U is Lipschitz. The first isomorphism is a local problem. Hence, by Remark 1.1.11 and by the definition of "Lipschitz boundary" the first isomorphism follows from Lemma 2.4.13. The second isomorphism is given in Proposition 2.1.6.
(i)–(b) The first isomorphism is a local problem and we may assume that U has a covering by open sets U_i as in Definition 2.4.14. By using the Čech resolution associated with this covering, we find an exact sequence of sheaves in $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$:

$$0 \to L_r \to \cdots \to L_0 \to \mathbf{k}_U \to 0$$

where each L_i is a finite sum of sheaves isomorphic to $\mathbf{k}_V$ for some $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$ with V Lipschitz. Therefore, $\mathrm{R}\rho_{\mathrm{sal}*}L_i$ is concentrated in degree 0 by (i)–(a) and the result follows.

(ii) follows from (i) and the adjunction between $\mathrm{R}\rho_{\mathrm{sal}*}$ and $\rho_{\mathrm{sal}}^!$.

(iii) follows from (ii). Q.E.D.

Example 2.4.18. Let $M = \mathbb{R}^2$ endowed with coordinates $x = (x_1, x_2)$. Let $R > 0$ and denote by B_R the open Euclidian ball with center 0 and radius R. Consider the subanalytic sets:

$$U_1 = \{x \in B_R; x_1 > 0, x_2 < x_1^2\}, \quad U_2 = \{x \in B_R; x_1 > 0, x_2 > -x_1^2\},$$
$$U_{12} = U_1 \cap U_2, \quad U = U_1 \cup U_2 = \{x \in B_R; x_1 > 0\}.$$

Note that $\{U_1, U_2\}$ is a covering of U in M_{sa} but not in M_{sal}. Denote for short by $\rho\colon M_{\mathrm{sa}} \to M_{\mathrm{sal}}$ the morphism ρ_{sal}. We have the distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sal}}})$:

$$\text{(2.4.5)} \qquad \mathrm{R}\rho_*\mathbf{k}_{U_{12}} \to \mathrm{R}\rho_*\mathbf{k}_{U_1} \oplus \mathrm{R}\rho_*\mathbf{k}_{U_2} \to \mathrm{R}\rho_*\mathbf{k}_U \xrightarrow{+1}.$$

Since U_1, U_2 and U are Lipschitz, $\mathrm{R}\rho_*\mathbf{k}_V$ is concentrated in degree 0 for $V = U_1, U_2, U$. It follows that $\mathrm{R}\rho_*\mathbf{k}_{U_{12}}$ is concentrated in degrees 0 and 1. Hence, we have the distinguished triangle

$$\text{(2.4.6)} \qquad \rho_*\mathbf{k}_{U_{12}} \to \mathrm{R}\rho_*\mathbf{k}_{U_{12}} \to R^1\rho_*\mathbf{k}_{U_{12}}[-1] \xrightarrow{+1}.$$

Let us prove that $R^1\rho_*\mathbf{k}_{U_{12}}$ is isomorphic to the sheaf N introduced in (2.1.7). We easily see that there exists a natural morphism $\mathbf{k}_U \to N$ which is surjective. Hence we have to prove that the sequence

$$\mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \to \mathbf{k}_U \to N$$

is exact. This reduces to the following assertion: if $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$ satisfies $V \subset U$ and $N(V) = 0$, then $\{V \cap U_1, V \cap U_2\}$ is a linear covering of V. We prove this claim now.

Let $V \subset U$ be such that $N(V) = 0$. By the definition of N, there exists $A > 0$ such that $U_{A,\varepsilon} \not\subset V$ for all $\varepsilon > 0$, where $U_{A,\varepsilon}$ is defined in (2.1.5). Hence there exists a sequence $\{(x_{1,n}, x_{2,n})\}_{n\in\mathbb{N}}$ such that $x_{1,n} > 0$, $x_{1,n} \to 0$ when $n \to \infty$, $|x_{2,n}| < Ax_{1,n}^2$ and $(x_{1,n}, x_{2,n}) \notin V$, for all $n \in \mathbb{N}$. We define $f(x) = d((x,0), M \setminus V)$, for $x \in \mathbb{R}$. Then f is a continuous subanalytic function and $f(x_{1,n}) < Ax_{1,n}^2$, for all $n \in \mathbb{N}$. The set $\{x \in]0,1[; f(x) < Ax^2\}$ is subanalytic and relatively compact, hence it is a finite disjoint union of points (but it is open) and intervals. Since it contains a sequence converging to 0, it must contain some interval $]0, x_0[$. We have then $f(x) \le Ax^2$ for all $x \in]0, x_0[$. We deduce, for any $(x_1, x_2) \in \mathbb{R}^2$ with $x_1 \in]0, x_0[$,

$$\text{(2.4.7)} \qquad d((x_1,x_2), M \setminus V) \le |x_2| + d((x_1,0), M \setminus V) \le |x_2| + Ax_1^2.$$

On the other hand we can find $B > 0$ such that, for any $(x_1, x_2) \in U$,

$$\text{(2.4.8)} \qquad \max\{d((x_1,x_2), M \setminus U_1), d((x_1,x_2), M \setminus U_2)\} \ge |x_2| + Bx_1^2.$$

We deduce easily from (2.4.7) and (2.4.8) that $\{V \cap U_1, V \cap U_2\}$ is a linear covering of V.

Chapter 3

Operations on sheaves

All along this chapter, we follow Convention (1.1.1).

In this chapter we study the natural operations on sheaves for the linear subanalytic topology. In particular, given a morphism of real analytic manifolds, our aim is to define inverse and direct images for sheaves on the linear subanalytic topology. We are not able to do it in general (see Remark 3.3.7) and we shall distinguish the case of a closed embedding and the case of a submersion.

3.1 Tensor product and internal hom

Since M_{sal} is a site, the category $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}})$ admits a tensor product, denoted $\bullet \otimes \bullet$ and an internal hom, denoted $\mathscr{H}om$. The functor $\otimes$ is exact and the functor $\mathscr{H}om$ admits a right derived functor. For more details, we refer to [KS06, § 18.2].

3.2 Operations for closed embeddings

f-regular open sets

In this section, $f\colon M \hookrightarrow N$ will be a closed embedding. We identify M with a subset of N. We assume for simplicity that d_M is the restriction of d_N to M and we write d for d_M or d_N. We also keep the convention (1.1.2) for $d(x, \emptyset)$.

Definition 3.2.1. Let $V \in \mathrm{Op}_{N_{\mathrm{sa}}}$. We say that V is f-regular if there exists $C > 0$ such that

$$d(x, M \setminus M \cap V) \leq C\, d(x, N \setminus V) \quad \text{for all } x \in M. \tag{3.2.1}$$

- The property of being f-regular is local on M. More precisely, if $M = \bigcup_{i\in I} U_i$ is an open covering and $V \in \mathrm{Op}_{N_{\mathrm{sa}}}$ is $f|_{U_i}$-regular for each $i \in I$, then V is f-regular.
- If V and W belong to $\mathrm{Op}_{N_{\mathrm{sa}}}$ with $f^{-1}(V) = f^{-1}(W)$, $V \subset W$ and V is f-regular, then W is f regular.

Lemma 3.2.2. *Let $f\colon M \hookrightarrow N$ be a closed embedding. The family $\{V \in \mathrm{Op}_{N_{\mathrm{sa}}};\ V$ is f-regular$\}$ is stable by finite intersections.*

Proof. We shall use the obvious fact which asserts that for two closed sets F_1 and F_2 in a metric space,

$$d(x, F_1 \cup F_2) = \inf(d(x, F_1), d(x, F_2)).$$

Let V_1 and V_2 be two f-regular objects of $\mathrm{Op}_{N_{\mathrm{sa}}}$ and let C_1 and C_2 be the corresponding constants as in (3.2.1). Let $x \in M$. We have

$$\begin{aligned} d(x, M \setminus (M \cap V_1 \cap V_2)) &= \inf_i d(x, M \setminus (M \cap V_i)) \\ &\leq \inf_i (C_i \cdot d(x, N \setminus V_i)) \\ &\leq (\max_i C_i) \cdot (\inf_i d(x, N \setminus V_i)) \\ &= (\max_i C_i) \cdot d(x, N \setminus (V_1 \cap V_2)). \end{aligned}$$

Q.E.D.

Lemma 3.2.3. *Let $f\colon M \hookrightarrow N$ be a closed embedding and let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Then there exists $V \in \mathrm{Op}_{N_{\mathrm{sa}}}$ such that V is f-regular and $M \cap V = U$.*

Proof. We choose $V_0 \in \mathrm{Op}_{N_{\mathrm{sa}}}$ such that $\overline{U} \subset V_0$. We set

$$\delta = \inf\{d(x, N \setminus V_0);\ x \in U\}$$

and $V = (V_0 \setminus (V_0 \cap M)) \cup U$. We have $\delta > 0$. Let $x \in M$ and $y \in N$ be such that $d(x, N \setminus V) = d(x, y)$. If $y \in M$, then $d(x, N \setminus V) = d(x, M \setminus U)$. If $y \notin M$, then $d(x, N \setminus V) = d(x, N \setminus V_0) \geq \delta$. In any case we have $d(x, N \setminus V) \geq \min\{d(x, M \setminus U), \delta\}$. Hence (3.2.1) is satisfied with $C = \max\{1, D/\delta\}$, where $D = \max\{d(x, M \setminus U);\ x \in M\} < \infty$. Q.E.D.

Lemma 3.2.4. *Let $f\colon M \hookrightarrow N$ be a closed embedding. Let $V \in \mathrm{Op}_{N_{\mathrm{sal}}}$ be an f-regular open set and let $\{V_i\}_{i\in I}$ be a linear covering of V, that is, a covering in $\mathrm{Op}_{N_{\mathrm{sal}}}$. Then there exists a refinement $\{W_j\}_{j\in J}$ of $\{V_i\}_{i\in I}$ such that $\{W_j\}_{j\in J}$ is a linear covering of V and W_j is f-regular for all $j \in J$. We can even choose $J = I$ and $W_i \subset V_i$, for all $i \in I$.*

Proof. Let C be a constant as in (3.2.1). Let $I_0 \subset I$ be a finite subset and let $C' > 0$ be such that

$$d(x, N \setminus V) \leq C' \cdot \max_{i \in I_0} d(x, N \setminus V_i) \quad \text{for all } x \in N. \tag{3.2.2}$$

Then, for any $x \in M$ we have

$$\begin{aligned} d(x, M \setminus (M \cap V)) &\leq C \cdot d(x, N \setminus V) \\ &\leq CC' \cdot \max_{i \in I_0} d(x, N \setminus V_i). \end{aligned} \tag{3.2.3}$$

We set $D = 2CC'$. For $i \in I_0$ we define $W_i \in \mathrm{Op}_{N_{\mathrm{sal}}}$ by

$$W_i = (V_i \setminus M) \cup \{x \in M \cap V_i;\ d(x, M \setminus (M \cap V)) < D\, d(x, N \setminus V_i)\}$$

and for $i \in I \setminus I_0$ we set $W_i = \emptyset$.

(i) Since $D \geq CC'$, the inequality (3.2.3) gives $V = \bigcup_{i \in I_0} W_i$. Let us prove that $\{W_i\}_{i \in I_0}$ is a linear covering of V. We first prove the following claim, for given $\varepsilon > 0$, $i \in I_0$ and $x \in N$:

$$\begin{aligned} &\text{if } d(x, N \setminus W_i) \leq \varepsilon d(x, N \setminus V), \\ &\text{then } d(x, N \setminus V_i) \leq (\varepsilon(1 + \tfrac{C}{D}) + \tfrac{C}{D}) d(x, N \setminus V). \end{aligned} \tag{3.2.4}$$

If $d(x, N \setminus W_i) = d(x, N \setminus V_i)$, the claim is obvious. In the other case we choose $y \in N$ such that $d(x, N \setminus W_i) = d(x, y)$. Then we have $y \in V_i \setminus W_i$. Hence $y \in M$ and the definition of W_i gives $d(y, N \setminus V_i) \leq D^{-1} d(y, M \setminus (M \cap V))$. We deduce

$$\begin{aligned} d(x, N \setminus V_i) &\leq d(x, y) + d(y, N \setminus V_i) \\ &\leq d(x, y) + D^{-1} d(y, M \setminus (M \cap V)) \\ &\leq d(x, y) + CD^{-1} d(y, N \setminus V) \\ &\leq (1 + CD^{-1}) d(x, y) + CD^{-1} d(x, N \setminus V) \\ &\leq (\varepsilon(1 + CD^{-1}) + CD^{-1}) d(x, N \setminus V), \end{aligned}$$

which proves (3.2.4).

Now we prove that $\{W_i\}_{i \in I_0}$ is a linear covering of V. We choose ε small enough so that $(\varepsilon(1 + \frac{C}{D}) + \frac{C}{D}) < \frac{1}{C'}$ (recall that $D = 2CC'$) and we prove, for all $x \in N$,

$$d(x, N \setminus V) \leq \varepsilon^{-1} \cdot \max_{i \in I_0} d(x, N \setminus W_i). \tag{3.2.5}$$

Indeed, if (3.2.5) is false, then (3.2.4) implies $d(x, N \setminus V_i) < \frac{1}{C'} d(x, N \setminus V)$ for some $x \in V$ and all $i \in I_0$. But this contradicts (3.2.2).

(ii) Let us prove that W_i is f-regular, for any $i \in I_0$. We remark that $W_i \setminus M = V_i \setminus M$. Hence $d(x, N \setminus W_i) = d(x, N \setminus V_i)$ or $d(x, N \setminus W_i) = d(x, M \setminus (M \cap W_i))$, for all $x \in M$. In the first case we have, assuming $x \in M \cap W_i$ (the case $x \notin M \cap W_i$ being trivial),

$$\begin{aligned} d(x, M \setminus (M \cap W_i)) &\leq d(x, M \setminus (M \cap V)) \\ &\leq D\, d(x, N \setminus V_i) = D\, d(x, N \setminus W_i). \end{aligned}$$

In the second case we have obviously $d(x, M \setminus (M \cap W_i)) \leq d(x, N \setminus W_i)$. Hence (3.2.1) holds for W_i with the constant $\max\{D, 1\}$. Q.E.D.

Thanks to Lemma 3.2.2, to f we can associate a new site.

Definition 3.2.5. Let $f\colon M \to N$ be a closed embedding.

(i) The presite N^f is given by $\mathrm{Op}_{N^f} = \{V \in N_{\mathrm{sa}}; V \text{ is } f\text{-regular}\}$.

(ii) The site N^f_{sal} is the presite N^f endowed with the topology such that a family $\{V_i\}_{i\in I}$ of objects Op_{N^f} is a covering of V in N^f if it is a covering in N_{sal}.

One denotes by $\rho_f\colon N_{\mathrm{sal}} \to N^f_{\mathrm{sal}}$ the natural morphism of sites.

Proposition 3.2.6. *The functor* $f^t\colon \mathrm{Op}_{N^f_{\mathrm{sal}}} \to \mathrm{Op}_{M_{\mathrm{sa}}}$, $V \mapsto f^{-1}(V)$, *induces a morphism of sites* $\widetilde{f}\colon M_{\mathrm{sal}} \to N^f_{\mathrm{sal}}$. *Moreover, this functor of sites is left exact in the sense of* [KS06, Def. 17.2.4].

Proof. (i) Let C be a constant as in (3.2.1). Let $\{V_i\}_{i\in I}$ be a covering of V in N_{sal} and let $I_0 \subset I$ be a finite subset and $C' > 0$ be such that $d(y, N \setminus V) \leq C' \cdot \max_{i\in I_0} d(y, N \setminus V_i)$ for all $y \in N$. We deduce, for $x \in M$,

$$\begin{aligned} d(x, M \setminus M \cap V) &\leq C \cdot d(x, N \setminus V) \\ &\leq CC' \cdot \max_{i\in I_0} d(x, N \setminus V_i) \\ &\leq CC' \cdot \max_{i\in I_0} d(x, M \setminus M \cap V_i). \end{aligned}$$

(ii) We have to prove that the functor $f^t\colon \mathrm{Op}_{N^f_{\mathrm{sal}}} \to \mathrm{Op}_{M_{\mathrm{sa}}}$ is left exact in the sense of [KS06, Def. 3.3.1], that is, for each $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, the category whose objects are the inclusions $U \to f^{-1}(V)$ ($V \in \mathrm{Op}_{N^f_{\mathrm{sal}}}$) is cofiltrant.

This category is non empty by Lemma 3.2.3 and then it is cofiltrant by Lemma 3.2.2. Q.E.D.

Hence, we have the morphisms of sites

$$
\begin{array}{ccc}
 & & N_{\mathrm{sal}} \\
 & & \Big\downarrow{\scriptstyle \rho_f} \\
M_{\mathrm{sal}} & \xrightarrow[\widetilde{f}]{} & N^f_{\mathrm{sal}}.
\end{array}
\tag{3.2.6}
$$

Now we consider two closed embeddings $f\colon M \to N$ and $g\colon N \to L$ of real analytic manifolds and we set $h := g \circ f$. We get the diagram of presites:

$$
\begin{array}{ccccc}
N_{\mathrm{sa}} & \xrightarrow{\widetilde{g}} & L^g & \xleftarrow{\rho_g} & L_{\mathrm{sa}} \\
\Big\downarrow{\scriptstyle \rho_f} & & & \searrow{\scriptstyle \lambda_h} & \Big\downarrow{\scriptstyle \rho_h} \\
N^f & & \xrightarrow{\overline{g}} & & \\
\Big\uparrow{\scriptstyle \widetilde{f}} & & & & \\
M_{\mathrm{sa}} & & \xrightarrow{\widetilde{h}} & & L^g \cap L^h,
\end{array}
\tag{3.2.7}
$$

where the objects of $L^g \cap L^h$ are the open sets $U \in \mathrm{Op}_{L_{\mathrm{sa}}}$ which are both g-regular and h-regular, $\overline{g}$ is induced by $\widetilde{g}$ and λ_h is the obvious inclusion. We will use the following lemma to prove that the direct images defined in the next section are compatible with the composition.

Lemma 3.2.7. (i) *Let $W \in \mathrm{Op}_{L^h}$. Then $W \cap N \in \mathrm{Op}_{N^f}$.*

(ii) *Let $W \in \mathrm{Op}_{L^g}$ be such that $N \cap W \in \mathrm{Op}_{N^f}$. Then $W \in \mathrm{Op}_{L^h}$.*

(iii) *Let $W \in \mathrm{Op}_{L^g}$ and $V \in \mathrm{Op}_{N^f}$ be such that $V \subset N \cap W$. Then there exists $U \in \mathrm{Op}_{L^g} \cap \mathrm{Op}_{L^h}$ such that $U \subset W$ and $V \subset N \cap U$.*

Proof. (i) By hypothesis there exists $C > 0$ such that $d(x, M \setminus M \cap W) \leq C\, d(x, L \setminus W)$, for any $x \in M$. Since $d(x, L \setminus W) \leq d(x, N \setminus N \cap W)$ we deduce (i).

(ii) By hypothesis there exist $C_1, C_2 > 0$ such that, for any $x \in M$,

$$d(x, M \setminus M \cap W) \leq C_1 d(x, N \setminus N \cap W) \leq C_1 C_2 d(x, L \setminus W),$$

which proves the result.

(iii) By Lemma 3.2.3 there exists $U_0 \in \mathrm{Op}_{L^g}$ such that $N \cap U_0 = V$. Then $U = U_0 \cap W$ is g-regular by Lemma 3.2.2 and $N \cap U = V$. Hence U is also h-regular by (ii). Q.E.D.

Inverse and direct images by closed embeddings

Let us first recall the inverse and direct images of presheaves.

Notation 3.2.8. (i) For a morphism $f\colon \mathscr{T}_1 \to \mathscr{T}_2$ of presites, we denote by f_* and $f^\dagger$ the direct and inverse image functors for presheaves.
(ii) We recall that the direct image functor f_* has a right adjoint $f^\ddagger\colon \operatorname{PSh}(\mathbf{k}_{\mathscr{T}_2}) \to \operatorname{PSh}(\mathbf{k}_{\mathscr{T}_1})$ defined as follows (see [KS06, (17.1.4)]). For $P \in \operatorname{PSh}(\mathbf{k}_{\mathscr{T}_2})$ and $U \in \operatorname{Op}_{\mathscr{T}_1}$ we have $(f^\ddagger P)(U) = \varprojlim_{f^t(V)\to U} P(V)$.

Lemma 3.2.9. *Let $f\colon M \to N$ be a closed embedding and let $G \in \operatorname{Mod}(\mathbf{k}_{N^f_{\mathrm{sal}}})$. Then, using the notations of (3.2.6), we have $\rho_f^\ddagger G \in \operatorname{Mod}(\mathbf{k}_{N_{\mathrm{sal}}})$.*

Proof. We have to prove that, for any $V \in \operatorname{Op}_{N_{\mathrm{sa}}}$ and any covering of V in N_{sal}, say $\{V_i\}_{i\in I}$, the following sequence is exact

$$0 \to \varprojlim_{W\subset V} G(W) \to \prod_{i\in I} \varprojlim_{W_i\subset V_i} G(W_i) \to \prod_{i,j\in I} \varprojlim_{W_{ij}\subset V_i\cap V_j} G(W_{ij}), \tag{3.2.8}$$

where W, W_i, W_{ij} run respectively over the f-regular open subsets of V, V_i, $V_i \cap V_j$. The limit in the second term of (3.2.8) can be replaced by the limit over the pairs (W, W_i) of f-regular open subsets with $W \subset V$, $W_i \subset W \cap V_i$. Then the family $\{W \cap V_i\}_{i\in I}$ is a covering of W in N_{sal}. By Lemma 3.2.4 it admits a refinement $\{W_i'\}_{i\in I}$ where the W_i''s are f-regular and $W_i' \subset V_i$. We may as well assume that W_i contains W_i', for any $i \in I$. Then $\{W_i\}_{i\in I}$ is a covering of W in N^f_{sal}. Hence the second term of (3.2.8) can be replaced by

$$\varprojlim_{W\subset V} \varprojlim_{\{W_i\}_{i\in I}} \prod_{i\in I} G(W_i),$$

where W runs over the f-regular open subsets of V and the family $\{W_i\}_{i\in I}$ runs over the coverings of W in N^f_{sal} such that $W_i \subset W \cap V_i$.

Now in the third term of (3.2.8) we may assume that W_{ij} contains $W_i \cap W_j$ and the exactness of the sequence follows from the hypothesis that $G \in \operatorname{Mod}(\mathbf{k}_{N^f_{\mathrm{sal}}})$. Q.E.D.

Definition 3.2.10. Let $f\colon M \to N$ be a closed embedding. We use the notations of (3.2.6).

(i) We denote by $f_{\mathrm{sal}*}\colon \operatorname{Mod}(M_{\mathrm{sal}}) \to \operatorname{Mod}(N_{\mathrm{sal}})$ the functor $\rho_f^\ddagger \circ \widetilde{f}_*$ and we call $f_{\mathrm{sal}*}$ the direct image functor.

(ii) We denote by $f^{-1}_{\mathrm{sal}}\colon \operatorname{Mod}(N_{\mathrm{sal}}) \to \operatorname{Mod}(M_{\mathrm{sal}})$ the functor $\widetilde{f}^{-1} \circ \rho_{f*}$ and we call f^{-1}_{sal} the inverse image functor.

For $F \in \mathrm{Mod}(M_{\mathrm{sal}})$, $G \in \mathrm{Mod}(N_{\mathrm{sal}})$, $U \in \mathrm{Op}_{M_{\mathrm{sal}}}$ and $V \in \mathrm{Op}_{N_{\mathrm{sal}}}$, we obtain

$$\Gamma(V; f_{\mathrm{sal}*}F) \simeq \varprojlim_{W \in \mathrm{Op}_{N^f}, W \subset V} F(M \cap W), \tag{3.2.9}$$

$$\Gamma(U; f_{\mathrm{sal}}^{-1}G) \simeq \varinjlim_{W \in \mathrm{Op}_{N^f}, W \cap M = U} G(W). \tag{3.2.10}$$

Lemma 3.2.11. *Let $f\colon M \to N$ and $g\colon N \to L$ be closed embeddings and let $h = g \circ f$. We use the notations of the diagram* (3.2.7)*. There is a natural isomorphism of functors*

$$\widetilde{g}_* \circ \rho_f^{\ddagger} \xrightarrow{\sim} \lambda_h^{\ddagger} \circ \overline{g}_*. \tag{3.2.11}$$

Proof. The morphisms of functors $\lambda_{h*} \circ \widetilde{g}_* \circ \rho_f^{\ddagger} \simeq \overline{g}_* \circ \rho_{f*} \circ \rho_f^{\ddagger} \to \overline{g}_*$ gives by adjunction the morphism in (3.2.11). To prove that this morphism is an isomorphism, let us choose $G \in \mathrm{PSh}(\mathbf{k}_{N^f})$ and $W \in \mathrm{Op}_{L^g}$. We get the morphism

$$\Gamma(W; (\widetilde{g}_* \circ \rho_f^{\ddagger})(G)) \to \Gamma(W; (\lambda_h^{\ddagger} \circ \overline{g}_*)(G)), \tag{3.2.12}$$

where $\Gamma(W; (\widetilde{g}_* \circ \rho_f^{\ddagger})(G)) \simeq \varprojlim_{V \in \mathrm{Op}_{N^f}, V \subset N \cap W} G(V)$ and $\Gamma(W; (\lambda_h^{\ddagger} \circ \overline{g}_*)(G)) \simeq \varprojlim_{U \in \mathrm{Op}_{L^h}, U \subset W} G(N \cap U)$. Then the result follows from Lemma 3.2.7. Q.E.D.

Proposition 3.2.12. *Let $f\colon M \to N$ and $g\colon N \to L$ be closed embeddings and let $h = g \circ f$. There is a natural isomorphism of functors $g_{\mathrm{sal}*} \circ f_{\mathrm{sal}*} \xrightarrow{\sim} h_{\mathrm{sal}*}$.*

Proof. Applying Lemma 3.2.11, we define the isomorphism as the composition $\rho_g^{\ddagger} \circ \widetilde{g}_* \circ \rho_f^{\ddagger} \circ \widetilde{f}_* \xrightarrow{\sim} \rho_g^{\ddagger} \circ \lambda_h^{\ddagger} \circ \overline{g}_* \circ \widetilde{f}_* \simeq \rho_h^{\ddagger} \circ \widetilde{h}_*$. Q.E.D.

Theorem 3.2.13. *Let $f\colon M \to N$ be a closed embedding.*

(i) *The functor $f_{\mathrm{sal}*}$ is right adjoint to the functor f_{sal}^{-1}.*

(ii) *The functor $f_{\mathrm{sal}*}$ is left exact and the functor f_{sal}^{-1} is exact.*

(iii) *If $g\colon N \to L$ is another closed embedding, we have $(g \circ f)_{\mathrm{sal}*} \simeq g_{\mathrm{sal}*} \circ f_{\mathrm{sal}*}$ and $(g \circ f)_{\mathrm{sal}}^{-1} \simeq f_{\mathrm{sal}}^{-1} \circ g_{\mathrm{sal}}^{-1}$.*

Proof. (i) We have $f_{\mathrm{sal}*} = \rho_f^{\ddagger} \circ \widetilde{f}_*$ and $f_{\mathrm{sal}}^{-1} = \widetilde{f}^{-1} \circ \rho_{f*}$. Since $(\widetilde{f}^{-1}, \widetilde{f}_*)$ and $(\rho_{f*}, \rho_f^{\ddagger})$ are pairs of adjoint functors between categories of presheaves and since the category of sheaves is a fully faithful subcategory of the category of presheaves, the result follows.
(ii) By the adjunction property, it remains to show that functor f_{sal}^{-1} is left exact, hence that the functor $\widetilde{f}^{-1}$ is exact. By Proposition 3.2.6, the morphism of sites $\widetilde{f}\colon M_{\mathrm{sal}} \to N_{\mathrm{sal}}^f$ is left exact in the sense of [KS06, Def. 17.2.4]. Then the result follows from [KS06, Th. 17.5.2].
(iii) The functoriality of direct images follows from Proposition 3.2.12 and that of inverse images results by adjunction. Q.E.D.

3.3 Operations for submersions

Let $f\colon M \to N$ denote a morphism of real analytic manifolds. In this section we assume that f is a submersion. If f is proper, it induces a morphism of sites $M_{\mathrm{sal}} \to N_{\mathrm{sal}}$, but otherwise, it does not even give a morphism of presites. Following [KS01] we shall introduce other sites M_{sb} (denoted M_{sa} in loc. cit.), similar to M_{sa} but containing all open subanalytic subsets of M, and M_{sbl}, similar to M_{sal}. Then M_{sbl} has the same category of sheaves as M_{sal} and any submersion $f\colon M \to N$ induces a morphism of sites $f_{\mathrm{sbl}}\colon M_{\mathrm{sbl}} \to N_{\mathrm{sbl}}$.

Another subanalytic topology

One denotes by $\mathrm{Op}_{M_{\mathrm{sb}}}$ the category of open subanalytic subsets of M and says that a family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sb}}}$ is a covering of $U \in \mathrm{Op}_{M_{\mathrm{sb}}}$ if $U_i \subset U$ for all $i \in I$ and, for each compact subset K of M, there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j \cap K = U \cap K$. We denote by M_{sb} the site so-defined. The next result is obvious (and already mentioned in [KS01]).

Proposition 3.3.1. *The morphism of sites $M_{\mathrm{sb}} \to M_{\mathrm{sa}}$ induces an equivalence of categories* $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sb}}}) \simeq \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$.

Similarly, we introduce another linear subanalytic topology M_{sbl} as follows. The objects of the presite M_{sbl} are those of M_{sb}, namely the open subanalytic subsets of M. In order to define the topology, we have to generalize Definitions 1.1.1 and 1.1.5.

Definition 3.3.2. Let $\{U_i\}_{i\in I}$ be a finite family in $\mathrm{Op}_{M_{\mathrm{sb}}}$. We say that this family is 1-regularly situated if for any compact subset $K \subset M$, there is a constant C such that for any $x \in K$

$$d(x, M \setminus \bigcup_{i\in I} U_i) \leq C \cdot \max_{i\in I} d(x, M \setminus U_i). \tag{3.3.1}$$

Definition 3.3.3. A linear covering of $U \in \mathrm{Op}_{M_{\mathrm{sb}}}$ is a small family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sb}}}$ such that $U_i \subset U$ for all $i \in I$ and

$$(3.3.2)\begin{cases}\text{for each relatively compact subanalytic open subset } W \subset M \text{ there}\\ \text{exists a finite subset } I_0 \subset I \text{ such that the family } \{W \cap U_i\}_{i\in I_0} \text{ is}\\ \text{1-regularly situated in } W \text{ and } \bigcup_{i\in I_0}(U_i \cap W) = U \cap W.\end{cases}$$

Proposition 3.3.4. (i) *The family of linear coverings satisfies the axioms of Grothendieck topologies.*

(ii) *The functor ρ_* associated with the morphism of sites $\rho\colon M_{\mathrm{sbl}} \to M_{\mathrm{sal}}$ defines an equivalence of categories $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sbl}}}) \simeq \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}})$.*

The verification is left to the reader.

Inverse and direct images

Proposition 3.3.5. *Let $f\colon M \to N$ be a morphism of real analytic manifolds. We assume that f is a submersion. Then f induces a morphism of sites $f_{\mathrm{sbl}}\colon M_{\mathrm{sbl}} \to N_{\mathrm{sbl}}$.*

Proof. Let $V \in \mathrm{Op}_{N_{\mathrm{sb}}}$ and let $\{V_i\}_{i\in I}$ be a linear covering of V. We have to prove that $\{f^{-1}V_i\}_{i\in I}$ is a linear covering of $f^{-1}V$. As in the case of M_{sa}, the definition of the linear coverings is local (see Corollary 1.1.9). Hence we can assume that $M = N \times L$. We can also assume that $d_M((x,y),(x',y')) = \max\{d_N(x,x'), d_L(y,y')\}$, for $x, x' \in N$ and $y, y' \in L$. Then for any $(x,y) \in M$ we have $d_M((x,y), N \setminus f^{-1}V) = d_N(x, N \setminus V)$ and the result follows easily. Q.E.D.

By Propositions 3.3.4 and 3.3.5 any submersion $f\colon M \to N$ between real analytic manifolds induces a pair of adjoint functors $(f_{\mathrm{sal}}^{-1}, f_{\mathrm{sal}*})$ between $\mathrm{Mod}(M_{\mathrm{sal}})$ and $\mathrm{Mod}(N_{\mathrm{sal}})$ and we get the analogue of Theorem 3.2.13:

Theorem 3.3.6. *Let $f\colon M \to N$ be a submersion.*

(i) *The functor $f_{\mathrm{sal}*}$ is right adjoint to the functor f_{sal}^{-1}.*

(ii) *The functor $f_{\mathrm{sal}*}$ is left exact and the functor f_{sal}^{-1} is exact.*

(iii) *If $g\colon N \to L$ is another submersion, we have $(g \circ f)_{\mathrm{sal}*} \simeq g_{\mathrm{sal}*} \circ f_{\mathrm{sal}*}$ and $(g \circ f)_{\mathrm{sal}}^{-1} \simeq f_{\mathrm{sal}}^{-1} \circ g_{\mathrm{sal}}^{-1}$.*

Remark 3.3.7. Our two definitions of $f_{\mathrm{sal}*}$ for closed embeddings and submersions do not give a definition for a general f by composition. For example let us consider the following commutative diagram

$$\xymatrix{M = \mathbb{R}^2 \ar[r]^-{i} \ar[d]_{p} \ar[rd]^{f} & \mathbb{R}^3 \ar[d]^{q} \\ \mathbb{R} \ar[r]^-{j} & N = \mathbb{R}^2,}$$

where $i(x, y) = (x, y, 0)$, $p(x, y) = x$, $q(x, y, z) = (x, z)$ and $j(x) = (x, 0)$. For $V \in \mathrm{Op}_{N_{\mathrm{sb}}}$ we define two families of open subsets of $f^{-1}(V)$:

$$I_1 = \{M \cap W;\ W \in \mathrm{Op}_{\mathbb{R}^3_{\mathrm{sb}}},\ W \subset q^{-1}V,\ W \text{ is } i\text{-regular}\},$$
$$I_2 = \{p^{-1}(\mathbb{R} \cap V');\ V' \in \mathrm{Op}_{N_{\mathrm{sb}}},\ V' \subset V,\ V' \text{ is } j\text{-regular}\}.$$

Then, for any $F \in \mathrm{Mod}(M_{\mathrm{sbl}})$ we have

$$\Gamma(V; q_{\mathrm{sal}*} i_{\mathrm{sal}*} F) \simeq \Gamma(q^{-1}V; i_{\mathrm{sal}*} F) \simeq \varprojlim_{U \in I_1} F(U), \tag{3.3.3}$$

$$\Gamma(V; j_{\mathrm{sal}*} p_{\mathrm{sal}*} F) \simeq \varprojlim_{V' \subset V,\, V' \ j\text{-regular}} \Gamma(\mathbb{R} \cap V'; p_{\mathrm{sal}*} F) \simeq \varprojlim_{U \in I_2} F(U). \tag{3.3.4}$$

Let us take for V the open set $V = \{(x, z);\ x^3 > z^2\}$. Then the two families I_1 and I_2 of open subsets of $f^{-1}(V) = \{(x, y);\ x > 0\}$ are not cofinal. Indeed the set $W_0 \subset \mathbb{R}^3$ given by $W_0 = \{(x, y, z);\ x^3 > y^2 + z^2\}$ is i-regular. Hence $M \cap W_0 = \{(x, y);\ x^3 > y^2\}$ belongs to I_1. On the other hand we see easily that, if V' is j-regular and $V' \subset V$, then $\mathbb{R} \cap V' \subset]\varepsilon, +\infty[$, for some $\varepsilon > 0$. Hence $M \cap W_0$ is not contained in any set of the family I_2.

Let us define $F = \varinjlim_{\varepsilon > 0} \mathbf{k}_{[0,\varepsilon] \times \{0\}} \in \mathrm{Mod}(M_{\mathrm{sbl}})$. Taking $U = M \cap W_0$ in (3.3.3) we can see that $\Gamma(V; q_{\mathrm{sal}*} i_{\mathrm{sal}*} F) \simeq \mathbf{k}$. On the other hand (3.3.4) implies $\Gamma(V; j_{\mathrm{sal}*} p_{\mathrm{sal}*} F) \simeq 0$. Hence $q_{\mathrm{sal}*} i_{\mathrm{sal}*} \not\simeq j_{\mathrm{sal}*} p_{\mathrm{sal}*}$.

Chapter 4

Construction of sheaves

On the site M_{sa}, the sheaves $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$ and $\mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ below have been constructed in [KS96, KS01]. By using the linear topology we shall construct sheaves on M_{sal} associated with more precise growth conditions.

All along this chapter, we follow Convention 1.1.1.

4.1 Sheaves on the subanalytic site

Temperate growth

For the reader's convenience, let us recall first some definitions of [KS96, KS01]. As usual, we denote by $\mathscr{C}^\infty_M$ (resp. $\mathscr{A}_M$) the sheaf of complex valued functions of class $\mathscr{C}^\infty$ (resp. real analytic), by $\mathcal{D}b_M$ (resp. $\mathscr{B}_M$) the sheaf of Schwartz's distributions (resp. Sato's hyperfunctions) and by $\mathscr{D}_M$ the sheaf of finite-order differential operators with coefficients in $\mathscr{A}_M$.

Definition 4.1.1. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $f \in \mathscr{C}^\infty_M(U)$. One says that f has *polynomial growth* at $p \in M \setminus U$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} \big(d(x, K \setminus U)\big)^N |f(x)| < \infty. \tag{4.1.1}$$

We say that f is temperate at p if all its derivatives have polynomial growth at p. We say that f is temperate if it is temperate at any point $p \in M \setminus U$.

For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we shall denote by $\mathscr{C}^{\infty,\mathrm{tp}}_M(U)$ the subspace of $\mathscr{C}^\infty_M(U)$ consisting of temperate functions.

For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we shall denote by $\mathcal{D}b^{\mathrm{tp}}_M(U)$ the space of temperate distributions on U, defined by the exact sequence

$$0 \to \Gamma_{M\setminus U}(M;\mathcal{D}b_M) \to \Gamma(M;\mathcal{D}b_M) \to \mathcal{D}b^{\mathrm{tp}}_M(U) \to 0.$$

It follows from (1.1.3) that $U \mapsto \mathscr{C}^{\infty,\mathrm{tp}}_M(U)$ is a sheaf and it follows from the work of Lojasiewicz [Loj59] that $U \mapsto \mathcal{D}b^{\mathrm{tp}}_M(U)$ is also a sheaf. We denote by $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$ and $\mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ these sheaves on M_{sa}. The first one is called the sheaf of $\mathscr{C}^\infty$-functions with temperate growth and the second the sheaf of temperate distributions. Note that both sheaves are Γ-acyclic (see [KS01, Lem 7.2.4] or Proposition 4.1.4 below) and the sheaf $\mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ is flabby (see Definition 2.2.6).

We also introduce the sheaf $\mathscr{C}^\infty_{M_{\mathrm{sa}}}$ of $\mathscr{C}^\infty$-functions on M_{sa} as

$$\mathscr{C}^\infty_{M_{\mathrm{sa}}} := \rho_{\mathrm{sa}*}\mathscr{C}^\infty_M.$$

We denote as usual by $\mathscr{D}_M$ the sheaf of rings of finite order differential operators on the real analytic manifold M. If $\iota_M\colon M \hookrightarrow X$ is a complexification of M, then $\mathscr{D}_M \simeq \iota_M^{-1}\mathscr{D}_X$. We set, following [KS01]:

$$\mathscr{D}_{M_{\mathrm{sa}}} := \rho_{\mathrm{sa}!}\mathscr{D}_M. \tag{4.1.2}$$

The sheaves $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$, $\mathscr{C}^\infty_{M_{\mathrm{sa}}}$ and $\mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ are $\mathscr{D}_{M_{\mathrm{sa}}}$-modules.

Remark 4.1.2. The sheaves $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$ and $\mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ are respectively denoted by $\mathscr{C}^{\infty,t}_M$ and $\mathcal{D}b^t_M$ in [KS01].

A cutoff lemma on M_{sa}

Lemma 4.1.3 below is an immediate corollary of a result of Hörmander [Hör83, Cor.1.4.11] and was already used in [KS96, Prop. 10.2].

Lemma 4.1.3. *Let Z_1 and Z_2 be two closed subanalytic subsets of M. Then there exists $\psi \in \mathscr{C}^{\infty,\mathrm{tp}}_M(M \setminus (Z_1 \cap Z_2))$ such that $\psi = 0$ on a neighborhood of $Z_1 \setminus Z_2$ and $\psi = 1$ on a neighborhood of $Z_2 \setminus Z_1$.*

Proposition 4.1.4. *Let $\mathscr{F}$ be a sheaf of $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$-modules on M_{sa}. Then $\mathscr{F}$ is Γ-acyclic.*

Proof. By Proposition 2.2.4, it is enough to prove that for U_1, U_2 in $\mathrm{Op}_{M_{\mathrm{sa}}}$, the sequence $0 \to \mathscr{F}(U_1 \cup U_2) \to \mathscr{F}(U_1) \oplus \mathscr{F}(U_2) \to \mathscr{F}(U_1 \cap U_2) \to 0$ is exact. This follows from Lemma 4.1.3 (see [KS96, Prop. 10.2] or Proposition 4.3.4 below). Q.E.D.

Gevrey growth

The definition below of the sheaf $\mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sa}}}$ is inspired by the definition of the sheaves of $\mathscr{C}^{\infty}$-functions of Gevrey classes, but is completely different from the classical one. Here we are interested in the growth of functions at the boundary contrarily to the classical setting where one is interested in the Taylor expansion of the function. As usual, there are two kinds of regularity which can be interesting: regularity at the interior or at the boundary. Since we shall soon consider the Dolbeault complexes of our new sheaves, the interior regularity is irrelevant and we are only interested in the growth at the boundary.

We refer to [Kom73b] for an exposition on classical Gevrey functions or distributions and their link with Sato's theory of boundary values of holomorphic functions. Note that there is also a recent study by [HM11] of these sheaves using the tools of subanalytic geometry.

In § 4.2 we shall define more refined sheaves by using the linear subanalytic topology.

Definition 4.1.5. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $f \in \mathscr{C}^{\infty}_{M}(U)$. We say that f has 0-*Gevrey growth* at $p \in M \setminus U$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p, $h > 0$ and $s > 1$ such that

$$\sup_{x \in K \cap U} \big(\exp(-h \cdot d(x, K \setminus U)^{1-s})\big)|f(x)| < \infty. \tag{4.1.3}$$

We say that f has Gevrey growth at p if all its derivatives have 0-Gevrey growth at p. We say that f has Gevrey growth if it has such a growth at any point $p \in M \setminus U$.

We denote by $G_M(U)$ the subspace of $\mathscr{C}^{\infty}_{M}(U)$ consisting of functions with Gevrey growth and by $\mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sa}}}$ the presheaf $U \mapsto G_M(U)$ on M_{sa}.

The next result is clear in view of (1.1.3) and Proposition 4.1.4.

Proposition 4.1.6. (a) *The presheaf* $\mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sa}}}$ *is a sheaf on* M_{sa}*,*

(b) *the sheaf* $\mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sa}}}$ *is a* $\mathscr{D}_{M_{\mathrm{sa}}}$*-module,*

(c) *the sheaf* $\mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sa}}}$ *is a* $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$*-module,*

(d) *the sheaf* $\mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sa}}}$ *is* Γ*-acyclic.*

4.2 Sheaves on the linear subanalytic site

By Lemma 2.3.10, if a sheaf $\mathscr{F}$ on M_{sa} is Γ-acyclic, then $\mathrm{R}\rho_{\mathrm{sal}*}\mathscr{F}$ is concentrated in degree 0. This applies in particular to the sheaves $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$, $\mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sa}}}$.

In the sequel, we shall use the following notations. We set

$$\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}} := \rho_{\mathrm{sal}*}\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}, \quad \mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sal}}} := \rho_{\mathrm{sal}*}\mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}, \quad \mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sal}}} := \rho_{\mathrm{sal}*}\mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sa}}}.$$

Temperate growth of a given order

Definition 4.2.1. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, let $f \in \mathscr{C}^\infty_M(U)$ and let $t \in \mathbb{R}_{\geq 0}$. We say that f has *polynomial growth of order* $\leq t$ at $p \in M \setminus U$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exists a sufficiently small compact neighborhood K of p such that

$$\sup_{x \in K \cap U} \big(d(x, K \setminus U)\big)^t |f(x)| < \infty. \tag{4.2.1}$$

We say that f is temperate of order t at p if, for each $m \in \mathbb{N}$, all its derivatives of order $\leq m$ have polynomial growth of order $\leq t + m$ at p. We say that f is temperate of order t if it is temperate of order t at any point $p \in M \setminus U$.

For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we denote by $\mathscr{C}^{\infty,t}_M(U)$ the subspace of $\mathscr{C}^\infty_M(U)$ consisting of functions temperate of order t and we denote by $\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}}$ the presheaf on M_{sal} so obtained.

The next result is clear by Proposition 2.1.8.

Proposition 4.2.2. (i) *The presheaves* $\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}}$ $(t \geq 0)$ *are sheaves on* M_{sal}*,*

(ii) *the sheaf* $\mathscr{C}^{\infty,0}_{M_{\mathrm{sal}}}$ *is a sheaf of rings,*

(iii) *for* $t \geq 0$*,* $\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}}$ *is a* $\mathscr{C}^{\infty,0}_{M_{\mathrm{sal}}}$*-module and there are natural morphisms* $\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}} \otimes_{\mathscr{C}^{\infty,0}_{M_{\mathrm{sal}}}} \mathscr{C}^{\infty,t'}_{M_{\mathrm{sal}}} \to \mathscr{C}^{\infty,t+t'}_{M_{\mathrm{sal}}}$.

We also introduce the sheaf

$$\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}} := \varinjlim_t \mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}}.$$

(Of course, the limit is taken in the category of sheaves on M_{sal}.) Then, for $0 \leq t \leq t'$, there are natural monomorphisms of sheaves on M_{sal} :

$$\mathscr{C}^{\infty,0}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,t'}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}. \tag{4.2.2}$$

Note that the inclusion $\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$ is strict since there exists a function f (say on an open subset U of $\mathbb{R}$) with polynomial growth of order $\leq t$ and such that its derivative does not have polynomial growth of order $\leq t + 1$.

Gevrey growth of a given order

Definition 4.2.3. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, let $s \in]1,+\infty[$ and let $f \in \mathscr{C}^\infty_M(U)$. We say that f has 0-*Gevrey growth of type* (s) at $p \in M \setminus U$ if it satisfies the following condition. For a local coordinate system $(x_1,\dots,x_n)$ around p, there exists a sufficiently small compact neighborhood K of p such that

$$\sup_{x \in K\cap U} \big(\exp(-h\cdot d(x,K\setminus U)^{1-s})\big)|f(x)| < \infty \tag{4.2.3}$$

for all $h \in]0,+\infty[$. We say that f has Gevrey growth of type (s) if all its derivatives have 0-Gevrey growth of type (s) at p. We say that f has Gevrey growth of type (s) if it has such a growth at any point $p \in M \setminus U$.

Similarly, one defines f of Gevrey growth of type $\{s\}$ when replacing (4.2.3) for all $h \in]0,+\infty[$ with the same condition for some $h \in]0,+\infty[$.

Definition 4.2.4. For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and $s \in]1,+\infty[$, we denote by $G^{(s)}_M(U)$ and $G^{\{s\}}_M(U)$ the spaces of functions of Gevrey growth of type (s) and $\{s\}$, respectively.

We denote by $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ the presheaves on M_{sal} so obtained.

Clearly, the presheaves $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ do not depend on the choice of the distance.

Proposition 4.2.5. (i) *The presheaves* $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ *and* $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ *are sheaves on* M_{sal}*,*

(ii) *the sheaves* $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ *and* $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ *are* $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$*-modules,*

(iii) *the presheaves* $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ *and* $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ *are* Γ*-acyclic,*

(iv) *we have natural monomorphisms of sheaves on* M_{sal} *for* $1 < s < s'$

$$\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev}(s')}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev}\{s'\}}_{M_{\mathrm{sal}}}.$$

Proof. (i), (ii) and (iv) are obvious and (iii) will follow from (ii) and Proposition 4.3.4 below (see Corollary 4.3.5). Q.E.D.

We set

$$\mathscr{C}^{\infty,\mathrm{gev\,st}}_{M_{\mathrm{sal}}} := \varinjlim_{s>1} \mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}.$$

Hence, we have monomorphisms of sheaves on M_{sal} for $0 \le t$ and $1 < s$

$$\begin{aligned}\mathscr{C}^{\infty,0}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}} \\ \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev\,st}}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty}_{M_{\mathrm{sal}}}.\end{aligned}$$

Definition 4.2.6. If $\mathscr{F}_{M_{\rm sal}}$ is one of the sheaves $\mathscr{C}^{\infty,t}_{M_{\rm sal}}$, $\mathscr{C}^{\infty,{\rm tp}\,st}_{M_{\rm sal}}$, $\mathscr{C}^{\infty,{\rm gev}(s)}_{M_{\rm sal}}$, $\mathscr{C}^{\infty,{\rm gev}\{s\}}_{M_{\rm sal}}$ or $\mathscr{C}^{\infty,{\rm gev\,st}}_{M_{\rm sal}}$, we set $\mathscr{F}_{M_{\rm sa}} := \rho_{\rm sal}^! \mathscr{F}$.

Let us apply Theorem 2.4.17 and Corollary 4.3.5. We get that if $U \in {\rm Op}_{M_{\rm sa}}$ is weakly Lipschitz and if $\mathscr{F}_{M_{\rm sal}}$ denotes one of the sheaves above, then

$$\mathrm{R}\Gamma(U; \mathscr{F}_{M_{\rm sa}}) \simeq \Gamma(U; \mathscr{F}_{M_{\rm sal}}).$$

We call $\mathscr{C}^{\infty,t}_{M_{\rm sa}}$, $\mathscr{C}^{\infty,{\rm tp}\,st}_{M_{\rm sa}}$, $\mathscr{C}^{\infty,{\rm gev}(s)}_{M_{\rm sa}}$, $\mathscr{C}^{\infty,{\rm gev}\{s\}}_{M_{\rm sa}}$ and $\mathscr{C}^{\infty,{\rm gev\,st}}_{M_{\rm sa}}$ the sheaves on $M_{\rm sa}$ of $\mathscr{C}^{\infty}$-functions of growth t, strictly temperate growth, Gevrey growth of type (s) and $\{s\}$ and strictly Gevrey growth, respectively. Recall that on $M_{\rm sa}$, we also have the sheaf $\mathscr{C}^{\infty,{\rm tp}}_{M_{\rm sa}}$ of $\mathscr{C}^{\infty}$-functions of temperate growth, the sheaf $\mathcal{D}b^{\rm tp}_{M_{\rm sa}}$ of temperate distributions and the sheaf $\mathscr{C}^{\infty,{\rm gev}}_{M_{\rm sa}}$ of $\mathscr{C}^{\infty}$-functions of Gevrey growth.

Rings of differential operators

Let M be a real analytic manifold. Recall that $\mathscr{D}_M$ denotes the sheaf of finite order analytic differential operators on M and that we have set in (4.1.2)

$$\mathscr{D}_{M_{\rm sa}} := \rho_{{\rm sa}!}\mathscr{D}_M. \tag{4.2.4}$$

Now we set

$$\mathscr{D}_{M_{\rm sal}} := \rho_{{\rm sal}*}\mathscr{D}_{M_{\rm sa}}. \tag{4.2.5}$$

Hence, $\mathscr{D}_{M_{\rm sa}}$ is the sheaf on $M_{\rm sa}$ associated with the presheaf $U \mapsto \mathscr{D}_M(\overline{U})$ and $M_{\rm sal}$ is its direct image on $M_{\rm sal}$. We define similarly the sheaves $\mathscr{D}_{\mathscr{T}}(m)$ of differential operators of order $\le m$ on the site $\mathscr{T} = M, M_{\rm sa}, M_{\rm sal}$.

Lemma 4.2.7. *There are natural morphisms* $\mathscr{D}_{M_{\rm sal}}(m) \otimes \mathscr{C}^{\infty,t}_{M_{\rm sal}} \to \mathscr{C}^{\infty,t+m}_{M_{\rm sal}}$ *making* $\mathscr{C}^{\infty,{\rm tp}\,st}_{M_{\rm sal}}$ *and* $\mathscr{C}^{\infty,{\rm tp}}_{M_{\rm sal}}$ *left* $\mathscr{D}_{M_{\rm sal}}$*-modules.*
The sheaves $\mathscr{C}^{\infty,{\rm gev}(s)}_{M_{\rm sal}}$ *and* $\mathscr{C}^{\infty,{\rm gev}\{s\}}_{M_{\rm sal}}$ *are naturally left* $\mathscr{D}_{M_{\rm sal}}$*-modules.*

Proof. This follows immediately from Definitions 4.2.1 and 4.2.3. Q.E.D.

By using the functor $\rho_{\rm sal}^!$, we will construct new sheaves (in the derived sense) on $M_{\rm sa}$ associated with the sheaves previously constructed on $M_{\rm sal}$.

Theorem 4.2.8. (i) *The functor* $\rho_{{\rm sal}*}\colon {\rm Mod}(\mathscr{D}_{M_{\rm sa}}) \to {\rm Mod}(\mathscr{D}_{M_{\rm sal}})$ *has finite cohomological dimension.*

(ii) *The functor* $\mathrm{R}\rho_{{\rm sal}*}\colon {\rm D}(\mathscr{D}_{M_{\rm sa}}) \to {\rm D}(\mathscr{D}_{M_{\rm sal}})$ *commutes with small direct sums.*

(iii) *The functor* $\mathrm{R}\rho_{\mathrm{sal}*}$ *in* (ii) *admits a right adjoint* $\rho_{\mathrm{sal}}^{!}\colon \mathrm{D}(\mathscr{D}_{M_{\mathrm{sal}}}) \to \mathrm{D}(\mathscr{D}_{M_{\mathrm{sa}}})$.

(iv) *The functor* $\rho_{\mathrm{sal}}^{!}$ *induces a functor* $\rho_{\mathrm{sal}}^{!}\colon \mathrm{D}^{+}(\mathscr{D}_{M_{\mathrm{sal}}}) \to \mathrm{D}^{+}(\mathscr{D}_{M_{\mathrm{sa}}})$.

Proof. Consider the quasi-commutative diagram of categories

$$\begin{array}{ccc} \mathrm{Mod}(\mathscr{D}_{M_{\mathrm{sa}}}) & \xrightarrow{\rho_{\mathrm{sal}*}} & \mathrm{Mod}(\mathscr{D}_{M_{\mathrm{sal}}}) \\ {\scriptstyle for}\downarrow & & \downarrow{\scriptstyle for} \\ \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}}) & \xrightarrow{\rho_{\mathrm{sal}*}} & \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sal}}}). \end{array}$$

The functor *for*: $\mathrm{Mod}(\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}})$ is exact and sends injective objects to injective objects, and similarly with M_{sal} instead of M_{sa}. It follows that the diagram below commutes:

$$\begin{array}{ccc} \mathrm{D}(\mathscr{D}_{M_{\mathrm{sa}}}) & \xrightarrow{\mathrm{R}\rho_{\mathrm{sal}*}} & \mathrm{D}(\mathscr{D}_{M_{\mathrm{sal}}}) \\ {\scriptstyle for}\downarrow & & \downarrow{\scriptstyle for} \\ \mathrm{D}(\mathbb{C}_{M_{\mathrm{sa}}}) & \xrightarrow{\mathrm{R}\rho_{\mathrm{sal}*}} & \mathrm{D}(\mathbb{C}_{M_{\mathrm{sal}}}). \end{array}$$

Moreover, the two functors *for* in the last diagram above are conservative. Then
(i) and (ii) follow from the corresponding result for $\mathbb{C}_{M_{\mathrm{sa}}}$-modules.
(iii) and (iv) follow from the Brown representability theorem, (see Proposition 2.3.4). Q.E.D.

4.3 A refined cutoff lemma

Lemma 4.3.1 below will play an important role in this paper and is an immediate corollary of a result of Hörmander [Hör83, Cor.1.4.11]. Note that Hörmander's result was already used in [KS96, Prop. 10.2] (see Lemma 4.1.3 above).

Hörmander's result is stated for $M = \mathbb{R}^n$ but we check in Lemma 4.3.2 that it can be extended to an arbitrary manifold.

Lemma 4.3.1. *Let* Z_1 *and* Z_2 *be two closed subsets of* $M := \mathbb{R}^n$. *Assume that there exists* $C > 0$ *such that*

$$d(x, Z_1 \cap Z_2) \le C(d(x, Z_1) + d(x, Z_2)) \text{ for any } x \in M. \tag{4.3.1}$$

Then there exists $\psi \in \mathscr{C}_M^{\infty,0}(M \setminus (Z_1 \cap Z_2))$ *such that* $\psi = 0$ *on a neighborhood of* $Z_1 \setminus Z_2$ *and* $\psi = 1$ *on a neighborhood of* $Z_2 \setminus Z_1$.

Lemma 4.3.2. *Let M be a manifold. Let Z_1 and Z_2 be two closed subsets of M such that $M \setminus (Z_1 \cap Z_2)$ is relatively compact and such that* (4.3.1) *holds for some $C > 0$. Then the conclusion of Lemma 4.3.1 holds true.*

Proof. We consider an embedding of M in some $\mathbb{R}^N$ and we denote by d_M, $d_{\mathbb{R}^N}$ the distance on M or $\mathbb{R}^N$. We have a constant $D \geq 1$ such that $D^{-1} d_{\mathbb{R}^N}(x,y) \leq d_M(x,y) \leq D\, d_{\mathbb{R}^N}(x,y)$, for all $x, y \in M \setminus (Z_1 \cap Z_2)$.

Let $x \in \mathbb{R}^N$ and let $x' \in M$ such that $d_{\mathbb{R}^N}(x,x') = d_{\mathbb{R}^N}(x,M)$. In particular $d_{\mathbb{R}^N}(x,x') \leq d_{\mathbb{R}^N}(x,Z_1)$. Then we have, assuming $x' \notin Z_1 \cap Z_2$,

$$\begin{aligned} d_{\mathbb{R}^N}(x, Z_1 \cap Z_2) &\leq d_{\mathbb{R}^N}(x,x') + D\, d_M(x', Z_1 \cap Z_2) \\ &\leq d_{\mathbb{R}^N}(x,x') + DC(d_M(x',Z_1) + d_M(x',Z_2)) \\ &\leq d_{\mathbb{R}^N}(x,x') + D^2C(d_{\mathbb{R}^N}(x',Z_1) + d_{\mathbb{R}^N}(x',Z_2)) \\ &\leq (1+2D^2C) d_{\mathbb{R}^N}(x,x') + D^2C(d_{\mathbb{R}^N}(x,Z_1) + d_{\mathbb{R}^N}(x,Z_2)) \\ &\leq (1+3D^2C)(d_{\mathbb{R}^N}(x,Z_1) + d_{\mathbb{R}^N}(x,Z_2)). \end{aligned}$$

If $x' \in Z_1 \cap Z_2$, then $d_{\mathbb{R}^N}(x, Z_1 \cap Z_2) = d_{\mathbb{R}^N}(x,M) \leq d_{\mathbb{R}^N}(x,Z_1)$ and the same inequality holds trivially. Hence we can apply Lemma 4.3.1 to $Z_1, Z_2 \subset \mathbb{R}^N$ and obtain a function $\psi \in \mathscr{C}^{\infty,0}_{\mathbb{R}^N}(\mathbb{R}^N \setminus (Z_1 \cap Z_2))$. Then $\psi|_{M \setminus (Z_1 \cap Z_2)}$ belongs to $\mathscr{C}^{\infty,0}_M(M \setminus (Z_1 \cap Z_2))$ and satisfies the required properties. Q.E.D.

Lemma 4.3.3. *Let $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and set $U = U_1 \cup U_2$. We assume that $\{U_1, U_2\}$ is a linear covering of U. Then there exist $U_i' \subset U_i$, $i = 1, 2$, and $\psi \in \mathscr{C}^{\infty,0}_M(U)$ such that*

(i) *$\{U_i', U_1 \cap U_2\}$ is a linear covering of U_i,*

(ii) *$\psi|_{U_1'} = 0$ and $\psi|_{U_2'} = 1$.*

Proof. We choose $U_i' \subset U_i$, $i = 1, 2$, as in Lemma 1.2.4 and we set $Z_i = (M \setminus U) \cup \overline{U_i'}$. Then the result follows from Lemmas 1.2.4 and 4.3.2. Q.E.D.

Proposition 4.3.4. *Let $\mathscr{F}$ be a sheaf of $\mathscr{C}^{\infty,0}_{M_{\mathrm{sal}}}$-modules on M_{sal}. Then $\mathscr{F}$ is Γ-acyclic.*

Proof. By Proposition 2.2.4, it is enough to prove that for any $\{U_1, U_2\}$ which is a covering of $U_1 \cup U_2$, the sequence $0 \to \mathscr{F}(U_1 \cup U_2) \to \mathscr{F}(U_1) \oplus \mathscr{F}(U_2) \to \mathscr{F}(U_1 \cap U_2) \to 0$ is exact. This follows from Lemma 4.3.3, similarly as in the proof of [KS96, Prop. 10.2]. The only non trivial fact is the surjectivity at the last term, which we check now.

We choose $U_i' \subset U_i$, $i = 1, 2$, and $\psi \in \mathscr{C}^{\infty,0}_M(U)$ as in Lemma 4.3.3. Let $s \in \Gamma(U_1 \cap U_2; \mathscr{F})$. Since $\{U_i', U_1 \cap U_2\}$ is a linear covering of U_i, $i = 1, 2$, we can define $s_1 \in \Gamma(U_1; \mathscr{F})$ and $s_2 \in \Gamma(U_2; \mathscr{F})$ by

$$s_1|_{U_1 \cap U_2} = \psi \cdot s,\ s_1|_{U_1'} = 0 \quad \text{and} \quad s_2|_{U_1 \cap U_2} = (1-\psi) \cdot s,\ s_2|_{U_2'} = 0.$$

Then $s_1|_{U_1\cap U_2} + s_2|_{U_1\cap U_2} = s$, as required. Q.E.D.

Corollary 4.3.5. *The sheaves* $\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}}$, $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$, $\mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sal}}}$, $\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}}$ $(t \in \mathbb{R}_{\geq 0})$, $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ *and* $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ $(s > 1)$, $\mathscr{C}^{\infty,\mathrm{gev\,st}}_{M_{\mathrm{sal}}}$ *and* $\mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sal}}}$ *are* Γ*-acyclic.*

Let $\mathscr{F}_{M_{\mathrm{sal}}}$ denote one of the sheaves appearing in Corollary 4.3.5 and let $\mathscr{F}_{M_{\mathrm{sa}}} := \rho^!_{\mathrm{sal}}\mathscr{F}_{M_{\mathrm{sal}}} \in \mathrm{D}^+(\mathscr{D}_{M_{\mathrm{sa}}})$. Then, if U is weakly Lipschitz, $\mathrm{R}\Gamma(U;\mathscr{F}_{M_{\mathrm{sa}}})$ is concentrated in degree 0 and coincides with $\mathscr{F}_{M_{\mathrm{sal}}}(U)$.

4.4 A comparison result

In the next lemma, we set $M := \mathbb{R}^n$ and we denote by dx the Lebesgue measure. As usual, for $\alpha \in \mathbb{N}^n$ we denote by D^α_x the differential operator $(\partial/\partial_{x_1})^{\alpha_1}\dots(\partial/\partial_{x_n})^{\alpha_n}$ and we denote by $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ the Laplace operator on M.

In all this section, we consider an open set $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. We set for short

$$d(x) = d(x, M \setminus U).$$

For a locally integrable function φ on U and $s \in \mathbb{R}_{\geq 0}$, we set

$$\|\varphi\|_\infty = \sup_{x\in U}|\varphi(x)|, \quad \|\varphi\|^s_\infty = \|d(x)^s\varphi(x)\|_\infty. \tag{4.4.1}$$

Proposition 4.4.1. *There exists a constant* C_α *such that for any locally integrable function* φ *on* U*, one has the estimate for* $s \geq 0$*:*

$$\|D^\alpha_x\varphi\|^{s+|\alpha|}_\infty \leq C_\alpha\big(\|\varphi\|^s_\infty + \|\Delta D^\alpha_x\varphi\|^{s+|\alpha|+2}_\infty\big). \tag{4.4.2}$$

Proof. We shall adapt the proof of [KS96, Prop. 10.1].
(i) Let us take a distribution $K(x)$ and a $\mathscr{C}^\infty$ function $R(x)$ such that

$$\delta(x) = \Delta K(x) + R(x)$$

(where $\delta(x)$ is the Dirac distribution at the origin) and the support of $K(x)$ and the support of $R(x)$ are contained in $\{x \in M; |x| \leq 1\}$. Then $K(x)$ is integrable. For $c > 0$ and for a function ψ set:

$$\psi_c(x) = \psi(c^{-1}x),\ \widetilde{K}_c = c^{2-n}K_c \text{ and } \widetilde{R}_c = c^{-n}R_c.$$

Then we have again

$$\delta(x) = \Delta\widetilde{K}_c(x) + \widetilde{R}_c(x)\,.$$

Hence we have for any distribution ψ

$$\psi(x) = \int \widetilde{K}_c(x-y)(\Delta\psi)(y)dy + \int \widetilde{R}_c(x-y)\psi(y)dy\,. \tag{4.4.3}$$

Now for $x \in U$, set $c(x) = d(x)/2$. We set

$$A_\alpha(x) = |\int \widetilde{K}_{c(x)}(x-y)(\Delta D_y^\alpha \varphi)(y)dy|,$$
$$B_\alpha(x) = |\int \widetilde{R}_{c(x)}(x-y) D_y^\alpha \varphi(y)dy|.$$

Since $\int |\widetilde{K}_{c(x)}(x-y)|dy = c(x)^2 \int |K(\frac{x}{c(x)} - y)|dy$, we get

$$\int |\widetilde{K}_{c(x)}(x-y)|dy \le C_1 d(x)^2$$

for some constant C_1.
(ii) We have

$$\begin{aligned} A_\alpha(x) &\le \Big(\sup_{|x-y|\le c(x)} |(D_y^\alpha \Delta\varphi)(y)|\Big) \int |\widetilde{K}_{c(x)}(x-y)|dy \\ &\le C_1 \Big(\sup_{|x-y|\le c(x)} |(D_y^\alpha \Delta\varphi)(y)|\Big) \cdot d(x)^2. \end{aligned}$$

Hence,

$$\begin{aligned} d(x)^{s+|\alpha|} A_\alpha(x) &\le C_1 \Big(\sup_{|x-y|\le c(x)} |(D_y^\alpha \Delta\varphi)(y)|\Big) \cdot d(x)^{s+|\alpha|+2} \\ &\le 2^{s+|\alpha|+2} C_1 \Big(\sup_{|x-y|\le c(x)} |d(y)^{s+|\alpha|+2} (D_y^\alpha \Delta\varphi)(y)|\Big) \\ &\le 2^{s+|\alpha|+2} C_1 \|\Delta D_x^\alpha \varphi\|_\infty^{s+|\alpha|+2}. \end{aligned} \tag{4.4.4}$$

Here we have used the fact that on the ball centered at x and radius $c(x)$, we have $d(x) \le 2d(y)$.
(iii) Since $\widetilde{R}_c(x-y)$ is supported by the ball of center x and radius $c(x)$, we have

$$\begin{aligned} B_\alpha(x) &= |\int_{B(x,c(x))} D_y^\alpha \widetilde{R}_{c(x)}(x-y)\varphi(y)dy| \\ &= c(x)^{-|\alpha|} |\int_{B(x,c(x))} c(x)^{-n} (D_y^\alpha R)_{c(x)}(x-y)\varphi(y)dy| \\ &\le c(x)^{-|\alpha|} \sup_{|x-y|\le c(x)} |\varphi(y)| \cdot \int |D_y^\alpha R(y)|dy. \end{aligned}$$

Here we have used the fact that $D^\alpha_y R_{c(x)}(y) = c(x)^{-|\alpha|}(D^\alpha_y R)_{c(x)}(y)$.

As in (ii), we deduce that

$$
\begin{aligned}
d(x)^{s+|\alpha|}B_\alpha(x) &\leq C_2 \sup_{|x-y|\leq c(x)} |d(y)^s\varphi(y)| \\
&\leq C_2\|\varphi\|^s_\infty. \qquad (4.4.5)
\end{aligned}
$$

for some constant C_2.

(iv) By choosing $\psi = D^\alpha_x\varphi$ in (4.4.3) the estimate (4.4.2) follows from (4.4.4) and (4.4.5). Q.E.D.

4.5 Sheaves on complex manifolds

Let X be a complex manifold of complex dimension d_X and denote by $X_\mathbb{R}$ the real analytic underlying manifold. Denote by $\overline{X}$ the complex manifold conjugate to X. (The holomorphic functions on $\overline{X}$ are the anti-holomorphic functions on X.) Then $X \times \overline{X}$ is a complexification of $X_\mathbb{R}$ and $\mathscr{O}_{\overline{X}}$ is a $\mathscr{D}_{X\times\overline{X}}$-module which plays the role of the Dolbeault complex. In the sequel, when there is no risk of confusion, we write for short X instead of $X_\mathbb{R}$.

Notation 4.5.1. In the sequel, we will often have to consider the composition $\mathrm{R}\rho_{\mathrm{sal}*} \circ \rho_{\mathrm{sa}!}$. For convenience, we introduce a notation. We set

$$
\rho_{\mathrm{sl}*!} := \rho_{\mathrm{sal}*} \circ \rho_{\mathrm{sa}!}. \qquad (4.5.1)
$$

Sheaves on complex manifolds

By applying the Dolbeault functor $\mathrm{R}\mathscr{H}om_{\mathscr{D}_{\overline{X}_{\mathrm{sal}}}}(\rho_{\mathrm{sl}*!}\mathscr{O}_{\overline{X}}, \bullet)$ to one of the sheaves

$$
\mathscr{C}^{\infty,\mathrm{tp}\,st}_{X_{\mathrm{sal}}},\ \mathscr{C}^{\infty,\mathrm{tp}}_{X_{\mathrm{sal}}},\ \mathscr{C}^{\infty,\mathrm{gev}(s)}_{X_{\mathrm{sal}}},\ \mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{X_{\mathrm{sal}}},\ \mathscr{C}^{\infty,\mathrm{gev\,st}}_{X_{\mathrm{sal}}},\ \mathscr{C}^{\infty,\mathrm{gev}}_{X_{\mathrm{sal}}},\ \mathscr{C}^{\infty}_{X_{\mathrm{sal}}},
$$

we obtain respectively the sheaves

$$
\mathscr{O}^{\mathrm{tp}\,st}_{X_{\mathrm{sal}}},\quad \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}},\quad \mathscr{O}^{\mathrm{gev}(s)}_{X_{\mathrm{sal}}},\quad \mathscr{O}^{\mathrm{gev}\{s\}}_{X_{\mathrm{sal}}},\quad \mathscr{O}^{\mathrm{gev\,st}}_{X_{\mathrm{sal}}},\quad \mathscr{O}^{\mathrm{gev}}_{X_{\mathrm{sal}}},\quad \mathscr{O}_{X_{\mathrm{sal}}}.
$$

All these objects belong to $\mathrm{D}^+(\mathscr{D}_{X_{\mathrm{sal}}})$. Then we can apply the functor $\rho^!_{\mathrm{sal}}$ and we obtain the sheaves

$$
\mathscr{O}^{\mathrm{tp}\,st}_{X_{\mathrm{sa}}},\quad \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}},\quad \mathscr{O}^{\mathrm{gev}(s)}_{X_{\mathrm{sa}}},\quad \mathscr{O}^{\mathrm{gev}\{s\}}_{X_{\mathrm{sa}}},\quad \mathscr{O}^{\mathrm{gev\,st}}_{X_{\mathrm{sa}}},\quad \mathscr{O}^{\mathrm{gev}}_{X_{\mathrm{sa}}},\quad \mathscr{O}_{X_{\mathrm{sa}}}.
$$

Note that the functor $\rho^!_{\mathrm{sal}}$ commutes with the Dolbeault functor. More precisely:

Lemma 4.5.2. *Let* $\mathscr{C}$ *be an object of* $\mathrm{D}^+(\mathscr{D}_{X^{\mathbb{R}}_{\mathrm{sal}}})$. *There is a natural isomorphism*

$$\rho^!_{\mathrm{sal}}\mathrm{R}\mathscr{H}om_{\mathscr{D}_{\overline{X}_{\mathrm{sal}}}}(\rho_{\mathrm{sl}*!}\mathscr{O}_{\overline{X}}, \mathscr{C}_{X_{\mathrm{sal}}}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_{\overline{X}_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{O}_{\overline{X}}, \rho^!_{\mathrm{sal}}\mathscr{C}_{X_{\mathrm{sal}}}). \tag{4.5.2}$$

Proof. This follows from the fact that the $\mathscr{D}_X$-module $\mathscr{O}_{\overline{X}}$ admits a global locally finite free resolution. Q.E.D.

Recall the natural isomorphism [KS96, Th. 10.5]

$$\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_{\overline{X}_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{O}_{\overline{X}}, \mathcal{D}b^{\mathrm{tp}}_{X_{\mathrm{sa}}}).$$

Proposition 4.5.3. *The natural morphism*

$$\mathscr{O}^{\mathrm{tp}\,st}_{X_{\mathrm{sal}}} \to \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}$$

is an isomorphism in $\mathrm{D}^+(\mathscr{D}_{X_{\mathrm{sal}}})$.

Proof. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Consider the diagram (in which $M = \mathbb{R}^{2n}$)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(U;\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}}) & \xrightarrow{\Delta} & \Gamma(U;\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(U;\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}) & \xrightarrow{\Delta} & \Gamma(U;\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}) & \longrightarrow & 0. \end{array}$$

As in the proof of [KS96, Th. 10.5], we are reduced to prove that the vertical arrows induce a qis from the top line to the bottom line. We shall apply Proposition 4.4.1.
(i) Let $\varphi \in \Gamma(U;\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}})$ with $\Delta\varphi = 0$. There exists some $s \geq 0$ such that $\|d(x)^s\varphi\|_\infty < \infty$. Then $\|d(x)^{s+|\alpha|}D^\alpha_x\varphi\|_\infty < \infty$ by (4.4.2).
(ii) It follows from [KS96, Prop.10.1] that the arrow in the bottom is surjective. Now let $\psi \in \Gamma(U;\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}})$. There exists $\varphi \in \Gamma(U;\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}})$ with $\Delta\varphi = \psi$. Then it follows from (4.4.2) that $\varphi \in \Gamma(U;\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}})$. Q.E.D.

Remark 4.5.4. It is natural to expect that the morphism

$$\mathscr{O}^{\mathrm{gev\,st}}_{X_{\mathrm{sal}}} \to \mathscr{O}^{\mathrm{gev}}_{X_{\mathrm{sal}}}$$

is an isomorphism in $\mathrm{D}^+(\mathscr{D}_{X_{\mathrm{sal}}})$. The proof of Proposition 4.5.3 can be adapted with the exception that one does not know if the map $\Delta\colon \mathscr{C}^{\infty,\mathrm{gev}}_{X_{\mathrm{sa}}}(U) \to \mathscr{C}^{\infty,\mathrm{gev}}_{X_{\mathrm{sa}}}(U)$ is surjective.

Solutions of holonomic $\mathscr{D}$-modules

The next result is a reformulation of a theorem of Kashiwara [Kas84].

Theorem 4.5.5. *Let $\mathscr{M}$ be a regular holonomic $\mathscr{D}_X$-module. Then the natural morphism*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}_{X_{\mathrm{sa}}})$$

is an isomorphism.

The next result was a conjecture of [KS03] and has recently been proved by Morando [Mor13] (see also [KS15] for a rather different proof) by using the deep results of Mochizuki [Moc09] (completed by those of Kedlaya [Ked10, Ked11] for the analytic case).

Theorem 4.5.6. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then for any $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X)$,*

$$\rho_{\mathrm{sa}}^{-1}\mathrm{R}\mathscr{H}om(G, \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})) \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X).$$

It is natural to conjecture that this theorem still holds when replacing the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ with one of the sheaves $\mathscr{O}^{\mathrm{gev}(s)}_{X_{\mathrm{sa}}}$ or $\mathscr{O}^{\mathrm{gev}\{s\}}_{X_{\mathrm{sa}}}$.

In [KS03], the object $\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ is explicitly calculated when $X = \mathbb{C}$ and, denoting by t a holomorphic coordinate on X, $\mathscr{M}$ is associated with the operator $t^2\partial_t + 1$, that is, $\mathscr{M} = \mathscr{D}_X \exp(1/t)$.

It is well-known, after [Ram78] (see also [Kom73]), that the holomorphic solutions of an ordinary linear differential equation singular at the origin have Gevrey growth, the growth being related to the slopes of the Newton polygon.

Conjecture 4.5.7. Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then the natural morphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}^{\mathrm{gev}}_{X_{\mathrm{sa}}}) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}_{X_{\mathrm{sa}}})$$

is an isomorphism, or, equivalently,

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}^{\mathrm{gev}}_{X_{\mathrm{sa}}}) \xrightarrow{\sim} \mathrm{R}\rho_{\mathrm{sa}*}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X).$$

Moreover, there exists a discrete set $Z \subset \mathbb{R}_{>1}$ such that the morphisms $\mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\mathscr{M}, \mathscr{O}^{\mathrm{gev}(s)}_{X_{\mathrm{sa}}}) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\mathscr{M}, \mathscr{O}^{\mathrm{gev}(t)}_{X_{\mathrm{sa}}})$ are isomorphisms for $s \leq t$ in the same components of $\mathbb{R}_{>1} \setminus Z$.

Chapter 5

Filtrations

5.1 Derived categories of filtered objects

In this section, we shall recall results of [Sch99] completed in [SS16].

Complements on abelian categories

In this subsection we state and prove some elementary results (some of them being well-known) on abelian and derived categories that we shall need.

Let $\mathscr{C}$ be an abelian category and let Λ be a small category. As usual, one denotes by $\mathrm{Fct}(\Lambda, \mathscr{C})$ the abelian category of functors from Λ to $\mathscr{C}$. Recall that the kernel of a morphism $u\colon X \to Y$ is the functor $\lambda \mapsto \operatorname{Ker} u(\lambda)$ and similarly with the cokernel or more generally with limits and colimits.

Lemma 5.1.1. *Assume that $\mathscr{C}$ is a Grothendieck category. Then*

(a) *the category $\mathrm{Fct}(\Lambda, \mathscr{C})$ is a Grothendieck category,*

(b) *if $F \in \mathrm{Fct}(\Lambda, \mathscr{C})$ is injective, then for $\lambda \in \Lambda$, $F(\lambda)$ is injective in $\mathscr{C}$.*

Proof. The category $\mathrm{Fct}(\Lambda, \mathscr{C})$ is equivalent to the category $\mathrm{PSh}(\Lambda^{\mathrm{op}}, \mathscr{C})$ of preshaves on Λ^{op} with values in $\mathscr{C}$. It follows that, for any given $\lambda \in \Lambda$, the functor $\mathrm{Fct}(\Lambda, \mathscr{C}) \to \mathscr{C}$, $F \mapsto F(\lambda)$ has a left adjoint. We can define it as follows (see *e.g.* [KS06, Not. 17.6.13]). For $G \in \mathscr{C}$ we define $G_\lambda \in \mathrm{Fct}(\Lambda, \mathscr{C})$ by

$$G_\lambda(\mu) = \bigoplus_{\mathrm{Hom}_{\Lambda}(\lambda,\mu)} G.$$

Then we can check directly that

(5.1.1) the functor $\mathscr{C} \ni G \mapsto G_\lambda \in \mathrm{Fct}(\Lambda, \mathscr{C})$ is exact,

(5.1.2) $\mathrm{Hom}_{\mathrm{Fct}(\Lambda,\mathscr{C})}(G_\lambda, F) \simeq \mathrm{Hom}_{\mathscr{C}}(G, F(\lambda))$ for any $F \in \mathrm{Fct}(\Lambda, \mathscr{C})$.

(a) Applying *e.g.* Th. 17.4.9 of loc. cit., it remains to show that $\mathrm{Fct}(\Lambda,\mathscr{C})$ admits a small system of generators. Let G be a generator of $\mathscr{C}$. It follows from (5.1.2) that the family $\{G_\lambda\}_{\lambda\in\Lambda}$ is a small system of generators in $\mathrm{Fct}(\Lambda,\mathscr{C})$.
(b) Follows from (5.1.2) and (5.1.1). Q.E.D.

We consider two abelian categories $\mathscr{C}$ and $\mathscr{C}'$ and a left exact functor $\rho\colon \mathscr{C}\to\mathscr{C}'$. The functor ρ induces a functor

$$\widetilde{\rho}\colon\ \mathrm{Fct}(\Lambda,\mathscr{C})\to\mathrm{Fct}(\Lambda,\mathscr{C}'). \tag{5.1.3}$$

Lemma 5.1.2. *Assume that $\mathscr{C}$ is a Grothendieck category.*

(a) *The functor $\widetilde{\rho}$ is left exact.*

(b) *Let I be a small category and assume that ρ commutes with colimits indexed by I. Then the functor $\widetilde{\rho}$ in* (5.1.3) *commutes with colimits indexed by I.*

(c) *Assume that ρ has cohomological dimension $\leq d$, that is, $R^j\rho=0$ for $j>d$. Then $\widetilde{\rho}$ has cohomological dimension $\leq d$.*

(d) *Assume that ρ commutes with small direct sums and that small direct sums of injective objects in $\mathscr{C}$ are acyclic for the functor ρ. Then small direct sums of injective objects in* $\mathrm{Fct}(\Lambda,\mathscr{C})$ *are acyclic for the functor $\widetilde{\rho}$.*

Proof. (a) is obvious.
(b) follows from the equivalence $\mathrm{Fct}(I,\mathrm{Fct}(\Lambda,\mathscr{C}))\simeq\mathrm{Fct}(\Lambda,\mathrm{Fct}(I,\mathscr{C}))$ and similarly with $\mathscr{C}'$.
(c) By Lemma 5.1.1 (a), the category $\mathrm{Fct}(\Lambda,\mathscr{C})$ admits enough injectives. Let $F\in\mathrm{Fct}(\Lambda,\mathscr{C})$ and let $F\to F^\bullet$ be an injective resolution of F, that is, $F^\bullet$ is a complex in degrees ≥ 0 of injective objects and $F\to F^\bullet$ is a qis. By Lemma 5.1.1 (b), for $\lambda\in\Lambda$, $F^\bullet(\lambda)$ is an injective resolution of $F(\lambda)$ and by the hypothesis, $H^j(\rho(F^\bullet(\lambda)))\simeq 0$ for $j>d$ and $\lambda\in\Lambda$. This implies that $R^j\rho(F)\simeq H^j(\rho(F^\bullet))$ is 0 for $j>d$.
(d) For a given $\lambda\in\Lambda$ we denote by $i^{\mathscr{C}}_\lambda$ the functor $\mathrm{Fct}(\Lambda,\mathscr{C})\to\mathscr{C}$, $F\mapsto F(\lambda)$. Then $i^{\mathscr{C}}_\lambda$ is exact and, by Lemma 5.1.1 (b), we have $i^{\mathscr{C}'}_\lambda\circ R\widetilde{\rho}\simeq R\rho\circ i^{\mathscr{C}}_\lambda$. Let $F\in\mathrm{Fct}(\Lambda,\mathscr{C})$ be a small direct sum of injective objects. Since $i^{\mathscr{C}}_\lambda$ commutes with direct sums, it follows from Lemma 5.1.1 (b) again that $i^{\mathscr{C}}_\lambda(F)$ is a small direct sum of injective objects in $\mathscr{C}$. By the hypothesis we obtain $R^j\rho\circ i^{\mathscr{C}}_\lambda(F)\simeq 0$, for all $j>0$. Hence $i^{\mathscr{C}'}_\lambda\circ R^j\widetilde{\rho}(F)\simeq 0$, for all $j>0$. Since this holds for all $\lambda\in\Lambda$ we deduce $R^j\widetilde{\rho}(F)\simeq 0$, for all $j>0$, as required. Q.E.D.

Abelian tensor categories

Recall (see *e.g.* [KS06, Ch. 5]) that a tensor Grothendieck category $\mathscr{C}$ is a Grothendieck category endowed with a biadditive functor $\otimes\colon \mathscr{C}\times\mathscr{C}\to\mathscr{C}$ satisfying functorial associativity isomorphisms. We do not recall here what is a tensor category with unit, a ring object A in $\mathscr{C}$, a ring object with unit and an A-module M. In the sequel, all tensor categories will be with unit and a ring object means a ring object with unit.

We shall consider

$$\begin{cases}\text{a Grothendieck tensor category } \mathscr{C} \text{ (with unit) in which small}\\ \text{inductive limits commute with } \otimes.\end{cases} \tag{5.1.4}$$

Lemma 5.1.3. *Let $\mathscr{C}$ be as in* (5.1.4) *and let A be a ring object* (*with unit*) *in $\mathscr{C}$. Then*

(a) *The category* $\mathrm{Mod}(A)$ *is a Grothendieck category,*

(b) *the forgetful functor* $\mathit{for}\colon \mathrm{Mod}(A)\to\mathscr{C}$ *is exact and conservative,*

(c) *the natural functor* $\widetilde{\mathit{for}}\colon \mathrm{D}(A)\to\mathrm{D}(\mathscr{C})$ *is conservative.*

Proof. (a) and (b) are proved in [SS16, Prop. 4.4].
(c) Since $\mathrm{D}(A)$ and $\mathrm{D}(\mathscr{C})$ are triangulated, it is enough to check that if $X\in\mathrm{D}(A)$ verifies $\widetilde{\mathit{for}}(X)\simeq 0$, then $X\simeq 0$. Let X be such an object and let $j\in\mathbb{Z}$. Since *for* is exact, $\mathit{for} H^j(X)\simeq H^j(\widetilde{\mathit{for}}(X))\simeq 0$. Since *for* is conservative, we get $H^j(X)\simeq 0$. Q.E.D.

Derived categories of filtered objects

We shall consider

$$\begin{cases}\text{a filtrant preordered additive monoid } \Lambda \text{ (viewed as a tensor}\\ \text{category with unit),}\\ \text{a category } \mathscr{C} \text{ as in (5.1.4).}\end{cases} \tag{5.1.5}$$

Denote by $\mathrm{Fct}(\Lambda,\mathscr{C})$ the abelian category of functors from Λ to $\mathscr{C}$. It is naturally endowed with a structure of a tensor category with unit by setting for $M_1, M_2\in\mathrm{Fct}(\Lambda,\mathscr{C})$,

$$(M_1\otimes M_2)(\lambda) = \varinjlim_{\lambda_1+\lambda_2\leq\lambda} M_1(\lambda_1)\otimes M_2(\lambda_2).$$

A Λ-ring A of $\mathscr{C}$ is a ring with unit of the tensor category $\mathrm{Fct}(\Lambda,\mathscr{C})$ and we denote by $\mathrm{Mod}(A)$ the abelian category of A-modules.

We denote by $\mathrm{F}_\Lambda\mathscr{C}$ the full subcategory of $\mathrm{Fct}(\Lambda,\mathscr{C})$ consisting of functors M such that for each morphism $\lambda \to \lambda'$ in Λ, the morphism $M(\lambda) \to M(\lambda')$ is a monomorphism. This is a quasi-abelian category. Let

$$\iota\colon \mathrm{F}_\Lambda\mathscr{C} \to \mathrm{Fct}(\Lambda,\mathscr{C})$$

denote the inclusion functor. This functor admits a left adjoint κ and the category $\mathrm{F}_\Lambda\mathscr{C}$ is again a tensor category by setting

$$M_1 \otimes_F M_2 = \kappa(\iota(M_1) \otimes \iota(M_2)).$$

A ring object in the tensor category $\mathrm{F}_\Lambda\mathscr{C}$ will be called a Λ-filtered ring in $\mathscr{C}$ and usually denoted FA. An FA-module FM is then simply a module over FA in $\mathrm{F}_\Lambda\mathscr{C}$ and we denote by $\mathrm{Mod}(FA)$ the quasi-abelian category of FA-modules.

It follows from Lemmas 5.1.1 and 5.1.3 that $\mathrm{Mod}(\iota FA)$ is a Grothendieck category.

Notation 5.1.4. In the sequel, for a ring object B in a tensor category, we shall write $\mathrm{D}^*(B)$ instead of $\mathrm{D}^*(\mathrm{Mod}(B))$, $* = +, -, \mathrm{b}, \mathrm{ub}$.

The next theorem is due to [SS16] and generalizes previous results of [Sch99].

Theorem 5.1.5. *Assume* (5.1.5). *Let FA be a Λ-filtered ring in $\mathscr{C}$. Then the category* $\mathrm{Mod}(FA)$ *is quasi-abelian, the functor* $\iota\colon \mathrm{Mod}(FA) \to \mathrm{Mod}(\iota FA)$ *is strictly exact and induces an equivalence of categories for* $* = \mathrm{ub}, +, -, \mathrm{b}$*:*

$$\iota\colon \mathrm{D}^*(FA) \xrightarrow{\sim} \mathrm{D}^*(\iota FA). \tag{5.1.6}$$

Notation 5.1.6. Let Λ and $\mathscr{C}$ be as in (5.1.5). The functor $\varinjlim\colon \mathrm{Fct}(\Lambda,\mathscr{C}) \to \mathscr{C}$ is exact. Let FA be a Λ-filtered ring in $F_\Lambda\mathscr{C}$) and set

$$A := \varinjlim_\lambda A(\lambda). \tag{5.1.7}$$

(For short, we write $A(\lambda)$ instead of $FA(\lambda)$.) The functor $\varinjlim$ induces an exact functor

$$\varinjlim\colon \mathrm{Mod}(FA) \to \mathrm{Mod}(A), \tag{5.1.8}$$

thus, using Theorem 5.1.5, for $* = \mathrm{ub}, +, -, \mathrm{b}$, a functor

$$\varinjlim\colon \mathrm{D}^*(FA) \to \mathrm{D}^*(A). \tag{5.1.9}$$

Since one often considers FA as a filtration on the ring A, we shall denote by *for* (forgetful) the functor $\varinjlim$:

$$\mathit{for}\colon \mathrm{D}^*(FA) \to \mathrm{D}^*(A), \quad \mathit{for} := \varinjlim. \tag{5.1.10}$$

Complements on filtered objects

Lemma 5.1.7. *Let Λ and $\mathscr{C}$ be as in* (5.1.5) *and let $\mathscr{C}'$ be another Grothendieck tensor category satisfying the same hypotheses as $\mathscr{C}$. Let FB be a Λ-filtered ring in $\mathscr{C}'$.*

(a) *Let $\sigma\colon \mathscr{C}' \to \mathscr{C}$ be an exact functor of tensor categories* (*see Definition 4.2.2 in* [KS06]). *Denote by $\widetilde{\sigma}\colon \mathrm{Fct}(\Lambda, \mathscr{C}') \to \mathrm{Fct}(\Lambda, \mathscr{C})$ the natural functor associated with σ. Then*

  (i) *$FA{:=}\widetilde{\sigma}(FB)$ has a natural structure of a Λ-filtered ring with values in $\mathscr{C}$,*

  (ii) *the functor $\widetilde{\sigma}$ induces an exact functor $\widetilde{\sigma}_\Lambda\colon \mathrm{Mod}(\iota FB) \to \mathrm{Mod}(\iota FA)$ hence a functor $\sigma_\Lambda\colon \mathrm{Mod}(FB) \to \mathrm{Mod}(FA)$.*

(b) *Assume moreover that the functor σ has a right adjoint ρ which is fully faithful* (*hence ρ is left exact and $\sigma\rho \simeq \mathrm{id}_{\mathscr{C}}$*). *Denote by $\widetilde{\rho}\colon \mathrm{Fct}(\Lambda, \mathscr{C}) \to \mathrm{Fct}(\Lambda, \mathscr{C}')$ the natural functor associated with ρ. Then*

  (i) *$\widetilde{\rho}$ is fully faithful and right adjoint to $\widetilde{\sigma}$,*

  (ii) *$\widetilde{\rho}$ induces a left exact fully faithful functor $\widetilde{\rho}_\Lambda\colon \mathrm{Mod}(\iota FA) \to \mathrm{Mod}(\iota FB)$ right adjoint to $\widetilde{\sigma}_\Lambda$ and a fully faithful functor $\rho_\Lambda\colon \mathrm{Mod}(FA) \to \mathrm{Mod}(FB)$ right adjoint to σ_Λ.*

(c) *The diagram below, in which the horizontal arrows are the forgetful functors, is commutative when composing horizontal and down vertical arrows, or when composing horizontal and up vertical arrows*

$$\begin{array}{ccccc}
\mathrm{Mod}(FA) & \longrightarrow & \mathrm{Mod}(\iota FA) & \longrightarrow & \mathrm{Fct}(\Lambda, \mathscr{C}) \\
{\scriptstyle\sigma_\Lambda}\uparrow\downarrow{\scriptstyle\rho_\Lambda} & & {\scriptstyle\widetilde{\sigma}_\Lambda}\uparrow\downarrow{\scriptstyle\widetilde{\rho}_\Lambda} & & {\scriptstyle\widetilde{\sigma}}\uparrow\downarrow{\scriptstyle\widetilde{\rho}} \\
\mathrm{Mod}(FB) & \longrightarrow & \mathrm{Mod}(\iota FB) & \longrightarrow & \mathrm{Fct}(\Lambda, \mathscr{C}').
\end{array}$$

Proof. (a) We first recall that a Λ-ring A of a tensor category $\mathscr{C}$ is the data of $A(\lambda) \in \mathscr{C}$, for each $\lambda \in \Lambda$, morphisms $\mu_A^{\lambda,\lambda'}\colon A(\lambda) \otimes A(\lambda') \to A(\lambda+\lambda')$, for all $\lambda, \lambda' \in \Lambda$, and $\varepsilon_A\colon \mathbf{1}_{\mathscr{C}} \to A(0)$, where $\mathbf{1}_{\mathscr{C}}$ is the unit of $\mathscr{C}$ and 0 the unit of Λ. These morphisms satisfy three commutative diagrams (which we do not recall here) expressing the associativity of μ_A and the fact that ε_A is a unit. Similarly a module M over A is the data of $M(\lambda) \in \mathscr{C}$, for each $\lambda \in \Lambda$, and morphisms $\mu_M^{\lambda,\lambda'}\colon A(\lambda) \otimes M(\lambda') \to M(\lambda+\lambda')$, for all $\lambda, \lambda' \in \Lambda$, satisfying two commutative diagrams left to the reader.

Let us go back to the situation of the lemma. For a Λ-filtered ring FB of $\mathscr{C}'$ and an FB-module N, setting $FA = \widetilde{\sigma}(FB)$, the morphisms $\mu_N^{\lambda,\lambda'}$ induce

$$\mu_{\widetilde{\sigma}(N)}^{\lambda,\lambda'} \colon A(\lambda) \otimes \sigma(N(\lambda')) \simeq \sigma(B(\lambda) \otimes N(\lambda')) \to \sigma(N(\lambda + \lambda')).$$

For $N = \iota FB$ we obtain $\mu_A^{\lambda,\lambda'}$. We define $\varepsilon_A = \sigma(\varepsilon_B)$. We leave to the reader the verification that ε_A, $\mu_A^{\cdot,\cdot}$ and $\mu_{\widetilde{\sigma}(N)}^{\cdot,\cdot}$ satisfy the required commutative diagrams. This defines the functor $\widetilde{\sigma}_\Lambda$. We see easily that $\widetilde{\sigma}_\Lambda$ is exact. Since FB is Λ-filtered, the exactness of σ implies that FA is Λ-filtered and that $\widetilde{\sigma}_\Lambda$ induces the functor σ_Λ of the lemma.
(b) The statement (i) is straightforward. Let us define $\widetilde{\rho}_\Lambda$. For a ιFA-module M the data of

$$\mu_M^{\lambda,\lambda'} \colon \sigma(B(\lambda) \otimes \rho(M(\lambda'))) \simeq A(\lambda) \otimes M(\lambda') \to M(\lambda + \lambda')$$

give by adjunction $\mu_{\widetilde{\rho}(M)}^{\lambda,\lambda'} \colon B(\lambda) \otimes \rho(M(\lambda')) \to \rho(M(\lambda + \lambda'))$ and define a structure of ιFB-module on $\rho(M)$. Since ρ is left exact $\widetilde{\rho}_\Lambda$ induces ρ_Λ. The adjunction properties are clear, as well as $\widetilde{\sigma}_\Lambda \widetilde{\rho}_\Lambda \simeq \mathrm{id}$ and $\sigma_\Lambda \rho_\Lambda \simeq \mathrm{id}$. Hence $\widetilde{\rho}_\Lambda$ and ρ_Λ are fully faithful.
(c) is clear. Q.E.D.

Theorem 5.1.8. (1) *We make the assumptions of* Lemma 5.1.7 (a)-(b) *and assume moreover that*

(i) *ρ has cohomological dimension $\leq d$,*

(ii) *for any $M \in \mathrm{Mod}(\iota FA)$, there exists a monomorphism $M \to I$ in $\mathrm{Mod}(\iota FA)$ such that $I(\lambda)$ is ρ-acyclic, for all $\lambda \in \Lambda$.*

Then the derived functor $R\rho_\Lambda \colon \mathrm{D}^*(FA) \to \mathrm{D}^*(FB)$ $(* = \mathrm{ub}, +)$ exists. It is fully faithful and admits a left adjoint $\rho_\Lambda^{-1} \colon \mathrm{D}^*(FB) \to \mathrm{D}^*(FA)$ $(* = \mathrm{ub}, +)$.
(2) *Assume moreover that*

(iii) *ρ commutes with small direct sums,*

(iv) *small direct sums of injective objects in $\mathscr{C}$ are acyclic for the functor ρ.*

Then the derived functor $R\rho_\Lambda \colon \mathrm{D}(FA) \to \mathrm{D}(FB)$ commutes with small direct sums and admits a right adjoint $\rho_\Lambda^! \colon \mathrm{D}(FB) \to \mathrm{D}(FA)$. Moreover, $\rho_\Lambda^!$ induces a functor $\mathrm{D}^+(FB) \to \mathrm{D}^+(FA)$.
(3) *We make the assumptions of* Lemma 5.1.7 (a) *and assume moreover that σ is fully faithful and has a right adjoint ρ which is exact. Then the derived functor $\sigma_\Lambda \colon \mathrm{D}^*(FA) \to \mathrm{D}^*(FB)$ $(* = \mathrm{ub}, +, \mathrm{b})$ is well defined, is fully faithful and admits a right adjoint $\rho_\Lambda \colon \mathrm{D}^*(FB) \to \mathrm{D}^*(FA)$ $(* = \mathrm{ub}, +, \mathrm{b})$.*

Proof. By Theorem 5.1.5, it is enough to prove the statements when replacing FA and FB with ιFA and ιFB, respectively and ρ_Λ with $\widetilde{\rho}_\Lambda$.

(1) Let us first prove that $\widetilde{\rho}_\Lambda\colon \mathrm{Mod}(\iota FA) \to \mathrm{Mod}(\iota FB)$ admits a derived functor and has cohomological dimension $\leq d$.

We let $\mathscr{I}$ be the subcategory of $\mathrm{Mod}(\iota FA)$ which consists of the $I \in \mathrm{Mod}(\iota FA)$ such that $I(\lambda)$ is ρ-acyclic, for all $\lambda \in \Lambda$. Using the hypothesis (iv) and the relation $\mathit{for} \circ \widetilde{\rho}_\Lambda \simeq \widetilde{\rho} \circ \mathit{for}$ we see that the subcategory $\mathscr{I}$ is $\widetilde{\rho}_\Lambda$-injective. Hence $R\widetilde{\rho}_\Lambda$ exists. We also see that $\mathit{for}(\mathscr{I})$ is a $\widetilde{\rho}$-injective family. Hence $\mathit{for} \circ R\widetilde{\rho}_\Lambda \simeq R\widetilde{\rho} \circ \mathit{for}$. Now the assertion on the cohomological dimension follows from Lemma 5.1.2-(c).

By Lemma 5.1.7, the functor $\widetilde{\rho}_\Lambda$ is right adjoint to $\widetilde{\sigma}_\Lambda$. This functor $\widetilde{\sigma}_\Lambda$ induces $\widetilde{\rho}_\Lambda^{-1}$ on the derived category which is left adjoint to $R\widetilde{\rho}_\Lambda$. The relation $\widetilde{\sigma}_\Lambda \widetilde{\rho}_\Lambda \simeq \mathrm{id}$ gives $\widetilde{\rho}_\Lambda^{-1} R\widetilde{\rho}_\Lambda \simeq \mathrm{id}$. Hence $R\widetilde{\rho}_\Lambda$ is fully faithful.

(2) By the Brown representability theorem, it is enough to prove that

$$R\widetilde{\rho}_\Lambda \text{ commutes with small direct sums.} \tag{5.1.11}$$

We consider the functor $\widetilde{\rho}\colon \mathrm{Fct}(\Lambda, \mathscr{C}) \to \mathrm{Fct}(\Lambda, \mathscr{C}')$. The hypotheses of Proposition 2.3.4 are satisfied by Lemma 5.1.2. Therefore the functor $\widetilde{\rho}$ has cohomological dimension $\leq d$ and the functor $R\widetilde{\rho}\colon \mathrm{D}(\mathrm{Fct}(\Lambda, \mathscr{C})) \to \mathrm{D}(\mathrm{Fct}(\Lambda, \mathscr{C}'))$ commutes with small direct sums.

Now we prove (5.1.11). Let $\{X_i\}_{i\in I}$ be a family of objects of $\mathrm{D}(\iota FA)$. There is a natural morphism $\bigoplus_{i\in I} R\widetilde{\rho}_\Lambda(X_i) \to R\widetilde{\rho}_\Lambda(\bigoplus_{i\in I} X_i)$ in $\mathrm{D}(\iota FB)$ and it follows from Lemma 5.1.3 that this morphism is an isomorphism.

(3) is obvious. Q.E.D.

5.2 Filtrations on $\mathscr{O}_{X_{\mathrm{sal}}}$

In the sequel, if FM is a filtered object in $\mathscr{C}$ over the ordered additive monoid $\mathbb{R}$, we shall write F^sM instead of $(FM)(s)$ to denote the image of the functor FM at $s \in \mathbb{R}$. This induces a functor $\mathrm{D}(\mathrm{F}_{\mathbb{R}}\mathscr{C}) \to \mathrm{D}(\mathscr{C})$ denoted in the same way $FM \mapsto F^sM$.

The filtered ring of differential operators

Recall that the sheaf $\mathscr{D}_M$ of finite order differential operators on M has a natural $\mathbb{N}$-filtration given by the order.

Recall that the rings $\mathscr{D}_{M_{\mathrm{sa}}}$ and $\mathscr{D}_{M_{\mathrm{sal}}}$ as well as the sheaves $\mathscr{D}_{M_{\mathrm{sa}}}(m)$ and $\mathscr{D}_{M_{\mathrm{sal}}}(m)$ are defined in (4.2.4) and (4.2.5). We remark that $\rho_{\mathrm{sa}}^{-1}(\mathscr{D}_{M_{\mathrm{sa}}}(m)) \simeq \mathscr{D}_M(m)$ and $\rho_{\mathrm{sal}}^{-1}(\mathscr{D}_{M_{\mathrm{sal}}}(m)) \simeq \mathscr{D}_{M_{\mathrm{sa}}}(m)$.

Definition 5.2.1. Let $\mathscr{T}$ be the site M or M_{sa} or M_{sal}. We define the filtered sheaf $\mathrm{F}\mathscr{D}_{\mathscr{T}}$ over $\Lambda = \mathbb{R}$ by setting:

$$\mathrm{F}^s\mathscr{D}_{\mathscr{T}} = \mathscr{D}_{\mathscr{T}}([s])$$

where $[s]$ is the integral part of s and $\mathscr{D}_{\mathscr{T}}([s])$ is the sheaf of differential operators of order $\leq [s]$. In particular, $\mathrm{F}^s\mathscr{D}_{\mathscr{T}} = 0$ for $s < 0$. We denote by $\mathrm{Mod}(\mathrm{F}\mathscr{D}_{\mathscr{T}})$ the category of filtered modules over $\mathscr{D}_{\mathscr{T}}$.

Let $M_{\mathscr{T}}$ be either M, M_{sa} or M_{sal}. In the sequel, we look at $\mathrm{Mod}(\mathbb{C}_{M_{\mathscr{T}}})$ as an abelian Grothendieck tensor category with unit and at $\mathrm{F}\mathscr{D}_{M_{\mathscr{T}}}$ as a Λ-ring object in $F_\Lambda\mathscr{C}$ (with $\Lambda = \mathbb{R}$) and $\mathscr{C} = \mathrm{Mod}(\mathbb{C}_{M_{\mathscr{T}}})$. Note that Definition 5.2.1 is in accordance with Lemma 5.1.7 (a) (i).

Since $\rho_{\mathrm{sa}}^{-1}(\mathscr{D}_{M_{\mathrm{sa}}}(m)) \simeq \mathscr{D}_M(m)$ and $\rho_{\mathrm{sal}}^{-1}(\mathscr{D}_{M_{\mathrm{sal}}}(m)) \simeq \mathscr{D}_{M_{\mathrm{sa}}}(m)$ we can apply Lemma 5.1.7 (a) with the exact functors $\sigma = \rho_{\mathrm{sa}}^{-1}$ or $\sigma = \rho_{\mathrm{sal}}^{-1}$. We obtain the functors

$$\begin{aligned} &\rho_{\mathrm{sa}}^{-1}\colon \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathrm{F}\mathscr{D}_M), \\ &\rho_{\mathrm{sal}}^{-1}\colon \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}) \to \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}). \end{aligned} \tag{5.2.1}$$

We will also use the fully faithful right adjoint of ρ_{sal}^{-1} given by Lemma 5.1.7 (b)

$$\rho_{\mathrm{sal}*}\colon \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}). \tag{5.2.2}$$

Theorem 5.2.2. (i) *The functor* $\rho_{\mathrm{sal}*}$ *in* (5.2.2) *admits a right derived functor* $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}^*(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{D}^*(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}})$ $(* = \mathrm{ub}, +)$ *which is fully faithful and admits a left adjoint functor* $\rho_{\mathrm{sal}}^{-1}\colon \mathrm{D}^*(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}) \to \mathrm{D}^*(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$ $(* = \mathrm{ub}, +)$.

(ii) *The functor* $\mathrm{R}\rho_{\mathrm{sal}*}$ $(* = \mathrm{ub}, +)$ *commutes with small direct sums and admits a right adjoint* $\rho_{\mathrm{sal}}^{!}\colon \mathrm{D}^*(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}) \to \mathrm{D}^*(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$ $(* = \mathrm{ub}, +)$.

(iii) *The functor* $\rho_{\mathrm{sa}}^{-1}\colon \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathrm{F}\mathscr{D}_M)$ *has a fully faithful right adjoint* $\mathrm{R}\rho_{\mathrm{sa}!}\colon \mathrm{D}^+(\mathrm{F}\mathscr{D}_M) \to \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$.

Proof. (i)–(ii) We shall apply Theorem 5.1.8 (1)-(2) with $\mathscr{C} = \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}})$, $\mathscr{C}' = \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sal}}})$, $\rho = \rho_{\mathrm{sal}*}$, $\sigma = \rho_{\mathrm{sal}}^{-1}$, $\Lambda = \mathbb{R}$, $FA = \mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}$, $FB = \mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}$. Let us check hypotheses (i)–(iv) of Theorem 5.1.8. Hypothesis (i) follows from Proposition 2.3.11. The hypotheses (iii) and (iv) follow from Lemma 2.1.9. By Lemma 5.1.3 we know that $\mathrm{Mod}(\iota\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$ has enough injectives. Hence to check the hypothesis (ii) it is enough to prove that if $I \in \mathrm{Mod}(\iota\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$ is injective, then $I(\lambda)$ is $\rho_{\mathrm{sal}*}$-acyclic for any $\lambda \in \Lambda$.

By Lemmas 2.3.10 and 2.2.7 it is enough to prove that $I(\lambda)$ is flabby. For any $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ we have

$$\Gamma(U; I(\lambda)) \simeq \mathrm{Hom}_{\mathrm{Mod}(\iota\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})}((\mathscr{D}_{M_{\mathrm{sa}}}^{[-\lambda]})_U, I), \tag{5.2.3}$$

where $\mathscr{D}_{M_{\mathrm{sa}}}^{[-\lambda]}$ denotes the object $\iota\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}$ with the filtration shifted by λ, that is, $F^s\mathscr{D}_{M_{\mathrm{sa}}}^{[-\lambda]} = F^{s-\lambda}\mathscr{D}_{M_{\mathrm{sa}}}$; this isomorphism sends a section s of $I(\lambda)$ to the morphism $1 \mapsto s$ (which is filtered because $1 \in F^\lambda\mathscr{D}_{M_{\mathrm{sa}}}^{[-\lambda]}$). Hence the flabbiness of $I(\lambda)$ follows from the injectivity of I and the exact sequence $0 \to (\mathscr{D}_{M_{\mathrm{sa}}}^{[\lambda]})_U \to (\mathscr{D}_{M_{\mathrm{sa}}}^{[\lambda]})_V$, for any inclusion $U \subset V$. This completes the proof of (i)–(ii).
(iii) We apply Theorem 5.1.8 (3) with $\rho = \rho_{\mathrm{sa}}^{-1}$, $\sigma = \rho_{\mathrm{sa}!}$. Q.E.D.

We define a functor

$$\mathrm{F}\mathscr{H}om : \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \times \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$$

by setting for $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ and $\mathrm{F}\mathscr{M} \in \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$

$$\mathscr{H}om\,(G, \mathrm{F}\mathscr{M})(\lambda) = \mathscr{H}om\,(G, \mathscr{M}(\lambda)).$$

Using Theorem 5.1.5, this functor admits a derived functor

$$\mathrm{FR}\mathscr{H}om : \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \times \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}).$$

Recall the functor *for* in (5.1.10).

Lemma 5.2.3. *Let $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_M)$ and let $\mathrm{F}\mathscr{M} \in \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$. Then*

$$\mathrm{F}^\lambda\mathrm{R}\mathscr{H}om\,(G, \mathrm{F}\mathscr{M}) \simeq \mathrm{R}\mathscr{H}om\,(G, \mathrm{F}^\lambda\mathscr{M}),$$
$$\mathit{for}\mathrm{FR}\mathscr{H}om\,(G, \mathrm{F}\mathscr{M}) \simeq \mathrm{R}\mathscr{H}om\,(G, \mathit{for}\mathrm{F}\mathscr{M}).$$

Proof. The first isomorphism follows directly from Lemma 5.1.1 (b) and we only prove the second one.
(i) Since the problem is local on M, we may assume that G has compact support.
(ii) By standard arguments, we may then reduce to the case where $G = \mathbb{C}_U$, $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$.
(iii) Using Theorem 5.1.5, we may replace $\mathrm{F}\mathscr{M} \in \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$ with an object $\widetilde{\mathscr{M}} \in \mathrm{D}^+(\mathrm{Fct}(\mathbb{R}, \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}})))$. Let us represent $\widetilde{\mathscr{M}}$ by a complex of injective

objects $I^\bullet \in \mathrm{C}^+(\mathrm{Fct}(\mathbb{R}, \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}})))$. Then,

$$\begin{aligned} \mathit{for}\mathrm{F}\mathrm{R}\mathscr{H}\!om\,(\mathbb{C}_U, \mathrm{F}\mathscr{M}) &\simeq \varinjlim \mathrm{R}\Gamma(\mathbb{C}_U, \widetilde{\mathscr{M}}) \\ &\simeq \varinjlim \Gamma(U; I^\bullet) \\ &\underset{(a)}{\simeq} \Gamma(U; \varinjlim I^\bullet) \underset{(b)}{\simeq} \mathrm{R}\Gamma(U; \varinjlim I^\bullet) \\ &\simeq \mathrm{R}\Gamma(U; \varinjlim \widetilde{\mathscr{M}}) \simeq \mathrm{R}\Gamma(U; \mathit{for}\mathrm{F}\mathscr{M}). \end{aligned}$$

Isomorphism (a) follows from Lemma 2.1.9 and isomorphism (b) follows from Lemma 5.1.1 (b) and Corollary 2.2.5. Q.E.D.

On a complex manifold X, we endow the $\mathscr{D}_X$-module $\mathscr{O}_X$ with the filtration $\mathrm{F}\mathscr{O}_X$ given by

$$\mathrm{F}^s\mathscr{O}_X = \begin{cases} 0 & \text{if } s < 0, \\ \mathscr{O}_X & \text{if } s \geq 0. \end{cases} \tag{5.2.4}$$

By applying the functors $\rho_{\mathrm{sa}!}$ and $\rho_{\mathrm{sal}*}$, we get the objects $\rho_{\mathrm{sa}!}\mathscr{O}_X$ and $\rho_{\mathrm{sl}*!}\mathscr{O}_X$ of $\mathrm{Mod}(F\mathscr{D}_{X_{\mathrm{sa}}})$ and $\mathrm{Mod}(F\mathscr{D}_{X_{\mathrm{sal}}})$, respectively. One shall be aware that these objects are in degree 0 contrarily to the sheaf $\mathscr{O}_{X_{\mathrm{sa}}}$ (when $d_X > 1$).

The L^∞-filtration on $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$

Recall that on the site M_{sal}, the sheaf $\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}}$ is endowed with a filtration, given by the sheaves $\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}}$ ($t \in \mathbb{R}_{\geq 0}$). We also set

$$\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}} = 0 \text{ for } t < 0.$$

Using Lemma 4.2.7 and Theorem 5.2.2, we set:

Definition 5.2.4. (a) We denote by $\mathrm{F}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$ the object of $\mathrm{Mod}(F\mathscr{D}_{M_{\mathrm{sal}}})$ given by the sheaves $\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}}$ ($t \in \mathbb{R}$).

(b) We set $\mathrm{F}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}} := \rho^!_{\mathrm{sal}}\,\mathrm{F}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$, an object of $\mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$.

We call these filtrations the L^∞-filtration on $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$, respectively.

Hence,

- $\mathrm{F}^s_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}} = \mathscr{C}^{\infty,s}_{M_{\mathrm{sal}}}$ for $s \in \mathbb{R}$,
- we have morphisms $\mathrm{F}^r\mathscr{D}_{M_{\mathrm{sal}}} \otimes \mathrm{F}^s_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}} \to \mathrm{F}^{s+r}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$,

- using Notation 5.1.6, $for \mathrm{F}_\infty \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}} \simeq \mathscr{C}^{\infty,\mathrm{tp}\, st}_{M_{\mathrm{sal}}}$ and similarly with M_{sa} instead of M_{sal}.

If $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ is weakly Lipschitz, we thus have for $s \geq 0$:

$$\mathrm{R}\Gamma(U; \mathrm{F}^s_\infty \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}) \simeq \mathscr{C}^{\infty,s}_M(U). \tag{5.2.5}$$

Remark 5.2.5. One could have also endowed $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$ with the L^2-filtration constructed similarly as the L^∞-filtration, when replacing the norm in (4.4.1) with the L^2-norm:

$$\|\varphi\|_2 = (\int_U |\varphi(x)|^2 dx)^{1/2}, \quad \|\varphi\|_2^s = \|d(x)^s \varphi(x)\|_2. \tag{5.2.6}$$

One gets the filtered sheaves $\mathrm{F}_2 \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$ and $\mathrm{F}_2 \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$.

The L^∞-filtration on $\mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}$

On a complex manifold X, we set:

$$\mathrm{F}_\infty \mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}} := \mathrm{R}\mathscr{H}om_{\mathrm{F}\mathscr{D}_{\overline{X}_{\mathrm{sal}}}}(\rho_{\mathrm{sl}*!}\mathcal{O}_{\overline{X}}, \mathrm{F}_\infty \mathscr{C}^{\infty,\mathrm{tp}}_{X_{\mathrm{sal}}}) \in \mathrm{D}^+(\mathrm{F}\mathscr{D}_{X_{\mathrm{sal}}}), \tag{5.2.7}$$

$$\begin{aligned} \mathrm{F}_\infty \mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} &:= \mathrm{R}\mathscr{H}om_{\mathrm{F}\mathscr{D}_{\overline{X}_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathcal{O}_{\overline{X}}, \mathrm{F}_\infty \mathscr{C}^{\infty,\mathrm{tp}}_{X_{\mathrm{sa}}}) \\ &\simeq \rho^!_{\mathrm{sal}} \mathrm{F}_\infty \mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}} \in \mathrm{D}^+(\mathrm{F}\mathscr{D}_{X_{\mathrm{sa}}}). \end{aligned} \tag{5.2.8}$$

Proposition 5.2.6. *The object $\mathrm{F}^s_\infty \mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}$ is represented by the complex of sheaves on $X^{\mathbb{R}}_{\mathrm{sal}}$:*

$$0 \to \mathrm{F}^s_\infty \mathscr{C}^{\infty,(0,0)}_{X_{\mathrm{sal}}} \xrightarrow{\overline{\partial}} \mathrm{F}^{s+1}_\infty \mathscr{C}^{\infty,(0,1)}_{X_{\mathrm{sal}}} \to \cdots \to \mathrm{F}^{s+d_X}_\infty \mathscr{C}^{\infty,(0,d_X)}_{X_{\mathrm{sal}}} \to 0. \tag{5.2.9}$$

Proof. Recall that the Spencer complex $\mathrm{SP_X}(\mathscr{D}_\mathrm{X})$ is the complex of left $\mathscr{D}_X$-modules

$$\mathrm{SP_X}(\mathscr{D}_\mathrm{X}) := \quad 0 \to \mathscr{D}_\mathrm{X} \otimes_{\mathcal{O}} \bigwedge^{\mathrm{d_x}} \Theta_\mathrm{X} \xrightarrow{\mathrm{d}} \cdots \to \mathscr{D}_\mathrm{X} \otimes_{\mathcal{O}} \Theta_\mathrm{X} \to \mathscr{D}_\mathrm{X} \to 0. \tag{5.2.10}$$

Moreover, there is an isomorphism of complexes, in any local chart,

$$\mathrm{SP_X}(\mathscr{D}_\mathrm{X}) \simeq \mathrm{K}_\bullet(\mathscr{D}_\mathrm{X}; \cdot\partial_1, \ldots, \cdot\partial_{\mathrm{d_X}}) \tag{5.2.11}$$

where the right hand side is the co-Koszul complex of the sequence $\cdot\partial_1, \ldots, \cdot\partial_{d_X}$ acting on the right on $\mathscr{D}_X$. This implies that the left $\mathscr{D}$-linear morphism $\mathscr{D}_X \to \mathcal{O}_X$ induces an isomorphism $\mathrm{SP_X}(\mathscr{D}_\mathrm{X}) \xrightarrow{\sim} \mathcal{O}_\mathrm{X}$ in $\mathrm{D^b}(\mathscr{D}_X)$.

If we endow $\mathscr{D}_X \otimes_{\mathcal{O}} \bigwedge^k \Theta_X$, $k = 0, \ldots, d_X$, with the filtration $\mathrm{F}^s(\mathscr{D}_X \otimes_{\mathcal{O}} \bigwedge^i \Theta_X) = \mathrm{F}^{s-k}(\mathscr{D}_X) \otimes_{\mathcal{O}} \bigwedge^i \Theta_X$, then $\mathrm{SP_X}(\mathscr{D}_\mathrm{X})$ gives a complex in $\mathrm{Mod}(\mathrm{F}\mathscr{D}_{\mathscr{T}})$ and we obtain $\mathrm{SP_X}(\mathscr{D}_\mathrm{X}) \xrightarrow{\sim} \mathcal{O}_\mathrm{X}$ in $\mathrm{D^b}(\mathrm{F}\mathscr{D}_X)$. Applying this to $\overline{X}$ and using the definition (5.2.8) we obtain the result. Q.E.D.

Corollary 5.2.7. *Let $U \subset X$ be an open relatively compact subanalytic subset. Assume that U is weakly Lipschitz. Then the object* $\mathrm{R}\Gamma(U; \mathrm{F}^s_\infty \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ *is represented by the complex*

(5.2.12)
$$0 \to \mathscr{C}_X^{\infty,s,(0,0)}(U) \xrightarrow{\overline{\partial}} \mathscr{C}_X^{\infty,s+1,(0,1)}(U) \to \cdots \to \mathscr{C}_X^{\infty,s+d_X,(0,d_X)}(U) \to 0.$$

Applying the functor ρ_{sa}^{-1}, one recovers the filtration introduced in (5.2.4):

$$\rho_{\mathrm{sa}}^{-1} \mathrm{F}_\infty \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} \simeq \mathrm{F}\mathscr{O}_X. \tag{5.2.13}$$

5.3 A functorial filtration on regular holonomic modules

Good filtrations on holonomic modules already exist in the literature, in the regular case (see [KK81, BK86, Sai88, Sai90]) and also in the irregular case (see [Mal96]). But these filtrations are constructed on each holonomic module and are by no means functorial. Here we directly construct objects of $\mathrm{D}^+(\mathrm{F}\mathscr{D}_X)$, the derived category of filtered $\mathscr{D}$-modules.

Denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ consisting of objects with regular holonomic cohomology. To $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X)$, one associates

$$\mathrm{Sol}(\mathscr{M}) := \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O}_X).$$

We know by [Kas75] that $\mathrm{Sol}(\mathscr{M})$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$, that is, $\mathrm{Sol}(\mathscr{M})$ has $\mathbb{C}$-constructible cohomology. Moreover, one can recover $\mathscr{M}$ from $\mathrm{Sol}(\mathscr{M})$ by the formula:

$$\mathscr{M} \simeq \rho_{\mathrm{sa}}^{-1} \mathrm{R}\mathscr{H}om(\mathrm{Sol}(\mathscr{M}), \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}). \tag{5.3.1}$$

This is the Riemann-Hilbert correspondence obtained by Kashiwara in [Kas80, Kas84].

Using the filtration $\mathrm{F}_\infty \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ on $\mathscr{O}_{X_{\mathrm{sa}}}$ we can set:

Definition 5.3.1. Let $\mathscr{M}$ be a regular holonomic module. We define the filtered Riemann-Hilbert functors $\mathrm{RHF}_{\infty,\mathrm{sa}}$ and RHF_∞ by the formulas

$$\begin{aligned} \mathrm{RHF}_{\infty,\mathrm{sa}} \colon \mathrm{D}^+_{\mathrm{holreg}}(\mathscr{D}_X) &\to \mathrm{D}^+(\mathrm{F}\mathscr{D}_{X_{\mathrm{sa}}}), \\ \mathscr{M} &\mapsto \mathrm{F}\mathrm{R}\mathscr{H}om(\mathrm{Sol}(\mathscr{M}), \mathrm{F}_\infty \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}), \\ \mathrm{RHF}_\infty = \rho_{\mathrm{sa}}^{-1} \mathrm{RHF}_{\infty,\mathrm{sa}} \colon \mathrm{D}^+_{\mathrm{holreg}}(\mathscr{D}_X) &\to \mathrm{D}^+(\mathrm{F}\mathscr{D}_X). \end{aligned}$$

Note that $\mathrm{RHF}_{\infty,\mathrm{sa}}$ and RHF_∞ are triangulated functors.
Recall Notation 5.1.6 and the functor *for*.

Proposition 5.3.2. *In the diagram below*

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X) \xrightarrow{\mathrm{RHF}_\infty} \mathrm{D}^+(\mathrm{F}\mathscr{D}_X) \xrightarrow{for} RD^+(\mathscr{D}_X)$$

the composition is isomorphic to the identity functor.

Proof. Since ρ_{sa}^{-1} commutes with inductive limits, the diagram below commutes:

$$\begin{array}{ccccc}
\mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X) & \xrightarrow{\mathrm{RHF}_{\infty,\mathrm{sa}}} & \mathrm{D}^+(\mathrm{F}\mathscr{D}_{X_{\mathrm{sa}}}) & \xrightarrow{for} & \mathrm{D}^+(\mathscr{D}_{X_{\mathrm{sa}}}) \\
 & & \downarrow{\scriptstyle \rho_{\mathrm{sa}}^{-1}} & & \downarrow{\scriptstyle \rho_{\mathrm{sa}}^{-1}} \\
 & & \mathrm{D}^+(\mathrm{F}\mathscr{D}_X) & \xrightarrow{for} & \mathrm{D}^+(\mathscr{D}_X).
\end{array}$$

Now let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X)$ and set for short $G = \mathrm{Sol}_X(\mathscr{M})$. By using Lemma 5.2.3 we get

$$\begin{aligned}
for\mathrm{FR}\mathscr{H}om\,(G, \mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}) &\simeq \mathrm{R}\mathscr{H}om\,(G, for\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}) \\
&\simeq \mathrm{R}\mathscr{H}om\,(G, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})
\end{aligned}$$

and we conclude with (5.3.1). Q.E.D.

Notation 5.3.3. The module $\mathscr{M}$ endowed with the filtration obtained by applying the functor $\mathrm{RHF}_{\infty,\mathrm{sa}}$ or RHF_∞, will simply be denoted by $\mathrm{F}_{\infty,\mathrm{sa}}\mathscr{M}$ or $\mathrm{F}_\infty\mathscr{M}$, respectively.

Example 5.3.4. Let D be a normal crossing divisor in X and let $\mathscr{M}$ be a regular holonomic module such that $\mathrm{Sol}(\mathscr{M}) \simeq \mathbb{C}_{X\setminus D}$. Let $W \in \mathrm{Op}_{X_{\mathrm{sa}}}$ with smooth boundary transversal to the strata of D so that $W \setminus D$ is weakly Lipschitz. Set $U := W \setminus D$. Then, by Lemma 5.2.3, $\mathrm{R}\Gamma(W; \mathrm{F}^s_{\infty,\mathrm{sa}}\mathscr{M}) \simeq \mathrm{R}\Gamma(U; \mathrm{F}^s_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ and therefore the object $\mathrm{R}\Gamma(W; \mathrm{F}^s_\infty\mathscr{M})$ is represented by the complex (5.2.12).

Remark 5.3.5. By using the filtration F_2 on $\mathscr{C}^{\infty,\mathrm{tp}}_{X_{\mathrm{sal}}}$ (see Remark 5.2.5), one can also endow $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}$ with an L^2-filtration and define similarly $\mathrm{F}_2\mathscr{O}^{\mathrm{tp}}_{\mathrm{sal}}$. Unfortunately, Hörmander's theory does not apply immediately to this situation. More precisely, for U open in $\mathbb{R}^n$, denote by $L^2(U; loc)$ the space of functions φ which are locally in L^2 for the Lebesgue measure and define

$$L^{2,s}(U) = \{\varphi \in L^2(U; loc); \|\varphi\|^s_2 < \infty, \}, \tag{5.3.2}$$

where $\|\varphi\|_2^s$ is defined in (5.3.2).

For U relatively compact and open in $\mathbb{C}^n$, denote by $W^{2,s,(p,q)}(U)$ the space of (p,q)-forms with coefficients in $L^{2,s}(U)$ and set

$$W_0^{2,s,(p,q)}(U) = \{\varphi \in W^{2,s,(p,q)}(U); \overline{\partial}\varphi \in W^{2,s,(p,q+1)}(U)\}.$$

Now we define $\widetilde{\mathrm{F}}_2\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}$ as the Dolbeault complex

$$\widetilde{\mathrm{F}}_2^s\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}(U) := 0 \to W_0^{2,s,(0,0)}(U) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} W_0^{2,s,(0,n)}(U) \to 0.$$

Then [Hör65, Th 2.2.3] asserts that if U is pseudoconvex, $\widetilde{\mathrm{F}}_2^s\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}(U)$ is concentrated in degree 0. However $\mathrm{F}^m\mathscr{D}_{X_{\mathrm{sal}}}$ does not send $W_0^{2,s,\cdot}$ in $W_0^{2,s+m,\cdot}$ and $\widetilde{\mathrm{F}}_2\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}$ is not defined as an object of $\mathrm{D}(\mathrm{F}\mathscr{D}_{X_{\mathrm{sal}}})$.

Given a regular holonomic $\mathscr{D}_X$-module $\mathscr{M}$, natural questions arise.

(i) Does there exist an integer r such that $H^j(\mathrm{F}^s_\infty\mathscr{L}) \to H^j(\mathrm{F}^{s+r}_\infty\mathscr{L})$ is the zero morphism for $s \gg 0$ and $j \neq 0$.

(ii) Is the filtration $H^0(\mathrm{F}_\infty\mathscr{M})$ a good filtration?

(iii) Does there exist a discrete set $Z \subset \mathbb{R}_{\geq 0}$ such that the morphisms $\mathrm{F}^s_\infty\mathscr{M} \to \mathrm{F}^t_\infty\mathscr{M}$ $(s \leq t)$ are isomorphisms for $[s,t]$ contained in a connected component of $\mathbb{R}_{\geq 0} \setminus Z$?

Note that it may be convenient to use better the L^2-filtration (see Remark 5.3.5).

One can also ask the question of comparing these filtrations with other filtrations already existing in the literature.

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