Deformation quantization modules

Pierre Schapira
(joint work with M. Kashiwara)

Université Pierre et Marie Curie
Paris, France

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We shall give an overview of recent results on modules over deformation quantization algebroids on complex Poisson manifolds. We study in particular the composition of kernels (finiteness and duality) and the functoriality of Hochschild classes. The classical Riemann-Roch theorem and its non commutative generalizations can now be interpreted as the disjoint union of two distinct results: the functoriality of Hochschild classes and the link between the Hochschild class and the Chern character. Finally, we have a glance to holonomic modules on symplectic manifolds.

References: Kashiwara-S,
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star-products

From now on, $(X, \mathscr{O}_X)$ is a complex manifold. An associative multiplication law $\star$ on $\mathscr{O}_X[[\hbar]]$ is a star-product if it is $\mathbb{C}[[\hbar]]$-bilinear and satisfies

$$f \star g = fg + \sum_{i \geq 1} P_i(f, g)\hbar^i \text{ for } f, g \in \mathscr{O}_X,$$

where the P_i's are bi-differential operators such that $P_i(f, 1) = P_i(1, f) = 0$ for all $f \in \mathscr{O}_X$ and $i > 0$. We call $(\mathscr{O}_X[[\hbar]], \star)$ a star-algebra. (See Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer and also Berezin.)

- A star-product defines a Poisson structure on $(X, \mathscr{O}_X)$:

$$\{f, g\} = \hbar^{-1}(f \star g - g \star f) \bmod \hbar\mathscr{O}_X[[\hbar]],$$

- any morphism of $\mathbb{C}[[\hbar]]$-algebras $(\mathscr{O}_X[[\hbar]], \star) \to (\mathscr{O}_X[[\hbar]], \star')$ is an isomorphism and is given by a sequence of differential operators,
- locally, (globally in the real case), any Poisson manifold $(X, \mathscr{O}_X)$ may be endowed with a star-product to which the Poisson structure is associated. This is the famous theorem of Kontsevich.

Example : the Poisson bracket is symplectic. In this case, the star product is locally isomorphic to the Leibniz product. With symplectic coordinates $(x; u)$

$$f \star g = \sum_{\alpha \in \mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!} (\partial_u^\alpha f)(\partial_x^\alpha g).$$

DQ-algebras

Definition

A DQ-algebra $\mathscr{A}$ on X is a $\mathbb{C}[[\hbar]]$-algebra locally isomorphic to a star-algebra $(\mathscr{O}_X[[\hbar]], \star)$.

Recall that any $\mathbb{C}$-algebra endomorphism of $\mathscr{O}_X$ is the identity. It follows that the $\mathbb{C}$-algebra isomorphism $\mathscr{A}/\hbar\mathscr{A} \xrightarrow{\sim} \mathscr{O}_X$ is unique. Clearly a DQ-algebra $\mathscr{A}$ satisfies the conditions:

$$\begin{cases} \text{(i)} & \hbar\colon \mathscr{A} \to \mathscr{A} \text{ is injective } (\mathscr{A} \text{ has no } \hbar\text{-torsion}), \\ \text{(ii)} & \mathscr{A} \to \varprojlim_n \mathscr{A}/\hbar^n\mathscr{A} \text{ is an isomorphism } (\mathscr{A} \text{ is } \hbar\text{-adic complete}), \\ \text{(iii)} & \mathscr{A}/\hbar\mathscr{A} \text{ is locally isomorphic to } \mathscr{O}_X \text{ as a } \mathbb{C}\text{-algebra.} \end{cases}$$

We denote by σ_0 the composition

$$\sigma_0\colon \mathscr{A} \to \mathscr{A}/\hbar\mathscr{A} \xrightarrow{\sim} \mathscr{O}_X.$$

One proves that $\mathscr{A}$ is right and left Noetherian (in particular, coherent).
In practice, one has to extend the notion of sheaf of algebras to that of an algebroid stack (Kashiwara 96, Kontsevich 2000). Roughly speaking, an algebroid stack is locally a sheaf of algebras but the usual glueing condition (vanishing of a 2-cocycle) does not hold and is replaced by the vanishing of a 3-cocycle.

When $X = T^*M$ is the cotangent bundle to a complex manifold M, there is a canonical DQ-algebra $\widehat{W}_X(0)$ on X constructed by using the sheaves of formal microdifferential operators of Sato-Kashiwara-Kawai. It is constructed as follows. Denote by t the coordinate on $\mathbb{C}$ and by $(t;\tau)$ the associated coordinates on $T^*\mathbb{C}$. Consider the map

$$\rho\colon T^*M \times T^*_{\tau\neq 0}\mathbb{C} \to T^*M$$
$$((x;\xi),(t,\tau)) \mapsto (x;\xi/\tau).$$

Denote by $\widehat{E}_{X\times T^*\mathbb{C}}(0)$ the sheaf of formal microdifferential operators of order ≤ 0 and by $\widehat{E}_{X\times T^*\mathbb{C},\hat{t}}(0)$ the subsheaf consisting of operators commuting with ∂_t. Setting $\hbar = \partial_t^{-1}$, we get the DQ-algebra

$$\widehat{W}_X(0) := \rho_* \widehat{E}_{X\times T^*\mathbb{C},\hat{t}}(0).$$

Properties of DQ-algebroids

- We set $\mathscr{A}_{X^a} = \mathscr{A}_X^{\mathrm{op}}$.
- Let X and Y be complex manifolds endowed with DQ-algebroids $\mathscr{A}_X$ and $\mathscr{A}_Y$ respectively. There is a canonical DQ-algebroid $\mathscr{A}_{X\times Y} := \mathscr{A}_X \underline{\boxtimes} \mathscr{A}_Y$ on $X \times Y$ locally equivalent to the stack associated with the exterior product of the DQ-algebras. For an $\mathscr{A}_X$-module $\mathscr{M}$ and an $\mathscr{A}_Y$-module $\mathscr{N}$, one defines their exterior product

$$\mathscr{M} \underline{\boxtimes} \mathscr{N} := \mathscr{A}_{X\times Y} \otimes_{\mathscr{A}_X \boxtimes \mathscr{A}_Y} \mathscr{M} \boxtimes \mathscr{N}.$$

- Let $\mathscr{A}_X$ be a DQ-algebroid on X and let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$. Its dual $\mathrm{D}'_{\mathscr{A}_X}\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X^a})$ is given by

$$\mathrm{D}'_{\mathscr{A}_X}\mathscr{M} := \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M}, \mathscr{A}_X).$$

- Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$. Then, locally, $\mathscr{M}$ admits a resolution by free modules of finite rank of length $\leq d_X + 1$.
- We denote by $\delta\colon X \hookrightarrow X \times X^a$ the diagonal embedding. Then $\delta_* \mathscr{A}_X$ is a coherent $\mathscr{A}_{X\times X^a}$-module. The dualizing complex $\omega_X^{\mathscr{A}} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X\times X^a})$ is defined by:

$$\omega_X^{\mathscr{A}} = \mathrm{D}'_{\mathscr{A}_X}(\mathrm{D}'_{\mathscr{A}_{X\times X^a}}(\delta_* \mathscr{A}_X)).$$

See also Dolgushev for a related construction and Van den Bergh and Yekutieli for abstract studies of the dualizing complex in algebraic geometry.

$\mathrm{gr}_\hbar(\mathscr{A}_X)$ and $\mathscr{A}_X^{\mathrm{loc}}$

We set

$$\begin{aligned} &\mathrm{gr}_\hbar(\mathscr{A}_X) := \mathscr{A}_X/\hbar\mathscr{A}_X \text{ the associated graded ring} \\ &\mathscr{A}_X^{\mathrm{loc}} := \mathbb{C}_X((\hbar)) \otimes_{\mathbb{C}_X[[\hbar]]} \mathscr{A}_X \text{ the localization .} \end{aligned}$$

Note that $\mathrm{gr}_\hbar(\mathscr{A}_X)$ is locally isomorphic to $\mathscr{O}$ (globally in the algebraic case). We introduce the functor:

$$\begin{aligned} &\mathrm{gr}_\hbar\colon \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{gr}_\hbar(\mathscr{A}_X)), \quad \mathscr{M} \mapsto \mathbb{C}_X \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X[[\hbar]]} \mathscr{M} \\ &\mathrm{loc}\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}}), \quad \mathscr{M} \mapsto \mathbb{C}_X((\hbar)) \otimes_{\mathbb{C}_X[[\hbar]]} \mathscr{M}. \end{aligned}$$

Note that

- the functor $\mathrm{gr}_\hbar$ is conservative on coherent modules,
- the support of a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module is co-isotropic (Gabber's theorem),
- for $X = T^*M$ a cotangent bundle, $\pi\colon X \to M$ the projection, there is a canonical DQ-algebra $\mathscr{A}_X$ and $\pi^{-1}\mathscr{D}_M$ is naturally a sub-algebra of $\mathscr{A}_X^{\mathrm{loc}}$.

Hence, in some sense, the theory of $\mathscr{A}$-modules contains that of $\mathscr{O}$-modules and the theory of $\mathscr{A}^{\mathrm{loc}}$-modules contains that of $\mathscr{D}$-modules.

Kernels

Consider three complex manifolds X_i $(i = 1, 2, 3)$ endowed with DQ-algebroids $\mathscr{A}_{X_i}$. We denote by p_i the i-th projection and by p_{ij} the (i,j)-th projection. We set $X_{ij} = X_i \times X_j$, $X_{ij^a} = X_i \times X_j^a$, $\mathscr{A}_{ij} = \mathscr{A}_{X_{ij}}$, etc.

$$\begin{array}{ccccc} & & X_{123} & & \\ & \swarrow^{p_{12}} & \downarrow^{p_{13}} & \searrow^{p_{23}} & \\ X_{12} & & X_{13} & & X_{23} \end{array}$$

Let $\Lambda_{ij} \subset X_{ij}$ be closed subsets and let $\mathscr{K}_{ij} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda_{ij}}(\mathscr{A}_{ij^a})$ $(ij = 12$ or $ij = 23)$. We set $\Lambda_{12} \circ \Lambda_{23} = p_{13}(p_{12}^{-1}\Lambda_{12} \cap p_{23}^{-1}\Lambda_{23})$ and

$$\mathscr{K}_{12} \underset{X_2}{\circ} \mathscr{K}_{23} \quad = \quad \mathrm{R}p_{13!}(p_{12}^{-1}\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{p_{12}^{-1}\mathscr{A}_{1^a2}} \mathscr{A}_{123} \overset{\mathrm{L}}{\otimes}_{p_{23}^{-1}\mathscr{A}_{23^a}} p_{23}^{-1}\mathscr{K}_2) \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{13^a}).$$

Theorem

Assume that the projection p_{13} is proper on $p_{12}^{-1}\Lambda_{12} \cap p_{23}^{-1}\Lambda_{23}$. Set $\Lambda_{13} = \Lambda_{12} \circ \Lambda_{23}$. Then

(a) *the object $\mathscr{K}_{12} \circ \mathscr{K}_{23}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda_{13}}(\mathscr{A}_{13^a})$,*

(b) *we have a natural isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{1^a3})$:*

$$\mathrm{D}'_{\mathscr{A}}(\mathscr{K}_{12}) \underset{X_2^a}{\circ} \omega^{\mathscr{A}}_{X_2^a} \underset{X_2^a}{\circ} \mathrm{D}'_{\mathscr{A}}(\mathscr{K}_{23}) \xrightarrow{\sim} \mathrm{D}'_{\mathscr{A}}(\mathscr{K}_{12} \underset{X_2}{\circ} \mathscr{K}_{23}).$$

Sketch of proof

The proof of the coherency of the direct image uses Grauert's theorem (Grothendieck in the algebraic setting) and a general tool

Definition

One says that an object $\mathscr{M}$ of $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$ is cohomologically complete if $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{A}_X^{\mathrm{loc}}, \mathscr{M}) \simeq \mathrm{R}\mathscr{H}om_{\mathbb{Z}[\hbar]}(\mathbb{Z}[\hbar, \hbar^{-1}], \mathscr{M}) \simeq 0$,

Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A})$. Then $\mathscr{M}$ is cohomologically complete.

Theorem

Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$ and assume:

(a) $\mathscr{M}$ *is cohomologically complete,*

(b) $\mathrm{gr}_\hbar(\mathscr{M}) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}_\hbar(\mathscr{A}_X))$.

Then $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$.

Hochschild classes

We define the Hochschild homology of $\mathscr{A}_X$ by setting:

$$\mathcal{HH}(\mathscr{A}_X) := \delta_*\mathscr{A}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} \delta_*\mathscr{A}_X, \text{ an object of } \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X[[\hbar]]).$$

Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$. We have the chain of morphisms

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M},\mathscr{M}) &\xleftarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M},\mathscr{A}_X)\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X}\mathscr{M} \\ &\simeq \delta_*\mathscr{A}_{X^a}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}(\mathscr{M}\underline{\boxtimes}\mathrm{D}'_{\mathscr{A}}\mathscr{M}) \\ &\to \delta_*\mathscr{A}_{X^a}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}\delta_*\mathscr{A}_X = \mathcal{HH}(\mathscr{A}_X). \end{aligned}$$

We get a map

$$\mathrm{Hom}_{\mathscr{A}_X}(\mathscr{M},\mathscr{M}) \to H^0_{\mathrm{Supp}(\mathscr{M})}(X;\mathcal{HH}(\mathscr{A}_X)).$$

The image of $\mathrm{id}_{\mathscr{M}}$ defines the Hochschild class of $\mathscr{M}$

$$\mathrm{hh}_X(\mathscr{M}) \in H^0_{\mathrm{Supp}(\mathscr{M})}(X;\mathcal{HH}(\mathscr{A}_X)).$$

Operations on Hochschild classes

There is a natural morphism

$$\circ\colon \mathrm{R}p_{13!}(p_{12}^{-1}\mathcal{HH}(\mathscr{A}_{X_1\times X_2^a})\overset{\mathrm{L}}{\otimes}p_{23}^{-1}\mathcal{HH}(\mathscr{A}_{X_2\times X_3^a}))\to\mathcal{HH}(\mathscr{A}_{X_1\times X_3^a}).$$

Let Λ be a closed subset of X. We set

$$\mathrm{HH}_\Lambda(\mathscr{A}_X):=H^0\mathrm{R}\Gamma_\Lambda(X;\mathcal{HH}(\mathscr{A}_X)).$$

Let $\Lambda_{ij}\subset X_{ij}$ ($ij=12$ or $ij=23$) and assume that

$$p_{12}^{-1}\Lambda_{12}\cap p_{23}^{-1}\Lambda_{23}\text{ is proper over }X_{13}. \tag{1}$$

Set $\Lambda_{13}=\Lambda_{12}\circ\Lambda_{23}$. We get a map

$$\circ\colon \mathrm{HH}_{\Lambda_{12}}(\mathscr{A}_{X_{12^a}})\otimes\mathrm{HH}_{\Lambda_{23}}(\mathscr{A}_{X_{23^a}})\longrightarrow\mathrm{HH}_{\Lambda_{13}}(\mathscr{A}_{X_{13^a}}).$$

Theorem

The composition of kernels commutes to the Hochschild class. More precisely, assuming (1)*, the diagram below commutes:*

$$\begin{array}{ccc}
\mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda_{12}}(\mathscr{A}_{X_{12^a}})\times\mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda_{23}}(\mathscr{A}_{X_{23^a}}) & \xrightarrow{\ \underset{X_2}{\circ}\ } & \mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda_{13}}(\mathscr{A}_{X_{13^a}}) \\
\Big\downarrow{\scriptstyle \mathrm{hh}\times\mathrm{hh}} & & \Big\downarrow{\scriptstyle \mathrm{hh}} \\
\mathrm{HH}_{\Lambda_{12}}(\mathscr{A}_{X_{12^a}})\otimes\mathrm{HH}_{\Lambda_{23}}(\mathscr{A}_{X_{23^a}}) & \xrightarrow{\ \circ\ } & \mathrm{HH}_{\Lambda_{13}}(\mathscr{A}_{X_{13^a}}).
\end{array}$$

Absolute case

Let us particularize to the absolute case ($X_1 = X_3 = X$ and $X_2 = \mathrm{pt}$). Denote by $(\bullet)^\star = \mathrm{RHom}_{\mathbb{C}[[\hbar]]}(\bullet, \mathbb{C}[[\hbar]])$ the duality functor on $\mathrm{D}^{\mathrm{b}}_{\mathit{finite}}(\mathbb{C}[[\hbar]])$. Assuming $\Lambda_1 \cap \Lambda_2$ is compact, we have a natural map

$$\circ\colon \mathrm{HH}_{\Lambda_1}(\mathscr{A}_X) \times \mathrm{HH}_{\Lambda_2}(\mathscr{A}_X) \longrightarrow \mathbb{C}[[\hbar]].$$

Corollary

Let $\mathscr{M}$ and $\mathscr{N}$ be two objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ and assume that $\mathrm{supp}(\mathscr{M}) \cap \mathrm{supp}(\mathscr{N})$ *is compact. Then*

(a) *the object* $\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{N})$ *belongs to* $\mathrm{D}^{\mathrm{b}}_{\mathit{finite}}(\mathbb{C}[[\hbar]])$,

(b) $\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{N}, \omega_X^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{M}) \simeq (\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{N}))^\star$,

(c) $\chi(\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{N}^{\mathrm{loc}})) = \mathrm{hh}_X(\mathscr{M}) \circ \mathrm{hh}_X(\mathscr{N})$.

Let us say that an $\mathscr{A}_X^{\mathrm{loc}}$-module is good if it is generated by a coherent $\mathscr{A}_X$-module and denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the corresponding triangulated category. Then if X is compact, the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ is Ext-finite and $\mathscr{M} \mapsto \omega_X^{\mathscr{A}} \otimes_{\mathscr{A}_X} \mathscr{M}$ is a Serre functor.
If X is compact and the Poisson structure associated with $\mathscr{A}_X$ is symplectic, one proves the isomorphism $\omega_X^{\mathscr{A}} \simeq \mathscr{A}_X[d_X]$. Hence, $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ is a Calabi-Yau triangulated category over $\mathbb{C}((\hbar))$ of dimension d_X.

Commutative case

When a star product $(\mathscr{O}_X[[\hbar]], \star)$ is commutative, it is isomorphic to the usual product and we may forget the formal parameter $\hbar$. Then $\mathcal{HH}(\mathscr{O}_X)$ is nothing but $\delta_X^* \delta_{X*}\mathscr{O}$ and we recover the fact that the Hochschild class of coherent $\mathscr{O}$-modules commute to external product, inverse image and proper direct image (see e.g. Caldararu and Shklyarov.). Denote by ω_X the dualizing complex for $\mathscr{O}$-modules. There are natural isomorphisms α_X and β_X in $\mathrm{D^b}(\mathscr{O})$ (Hochschild-Kostant-Rosenberg):

$$\begin{array}{ccc} \delta_X^*\delta_{X*}\mathscr{O} & \xrightarrow[\mathrm{td}]{\sim} & \delta_X^!\delta_{X!}\omega_X \\ {\scriptstyle \alpha_X}\downarrow{\scriptstyle \wr} & & {\scriptstyle \wr}\uparrow{\scriptstyle \beta_X} \\ \bigoplus_{i=0}^{d_X}\Omega_X^i\,[i] & \xrightarrow{\tau_X} & \bigoplus_{i=0}^{d_X}\Omega_X^i\,[i]. \end{array}$$

The arrow α_X commutes to external product and inverse image, the arrow β_X commutes to external product and proper direct image.
One defines the Chern and Euler classes of $\mathscr{F}$:

$$\mathrm{ch}(\mathscr{F}) = \alpha_X \circ \mathrm{hh}_X(\mathscr{F}), \quad \mathrm{eu}(\mathscr{F}) = \beta_X^{-1} \circ \mathrm{td} \circ \mathrm{hh}_X(\mathscr{F}).$$

It was conjectured in 1991 by Kashiwara that the map τ_X is the cup product by the Todd class of X, that is $\mathrm{eu}(\mathscr{F}) = \mathrm{ch}(\mathscr{F}) \cup Td_X$.
This conjecture has been recently proved by Ramadoss (after work of Markarian) in the algebraic case and by Grivaux in the analytic case.

Symplectic case

Recall that $\mathscr{A}_X^{\mathrm{loc}} = \mathbb{C}((\hbar)) \otimes_{\mathbb{C}[[\hbar]]} \mathscr{A}_X$ and assume $\mathrm{gr}_\hbar \mathscr{A}_X \simeq \mathscr{O}_X$ for short.

Theorem

Assume X is symplectic. Then there is a natural isomorphism $\mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}) \xrightarrow[\tau_X]{\sim} \mathbb{C}_X((\hbar))\,[d_X]$ which commutes to the composition of kernels.

It follows that there is a canonical non-zero section in $H^{-d_X}(X; \mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}))$. This section is called the "trace density density map" See Feigin-Tsygan, Bressler-Nest-Tsygan, Engeli-Felder etc. We define the Euler class of a coherent $\mathscr{A}_X$-module as the composition

$$\begin{array}{ccccc} & & \mathcal{HH}(\mathscr{A}_X) & & \\ & \swarrow^{\mathrm{gr}_\hbar} & \searrow^{\mathrm{loc}} & \searrow^{\mathrm{eu}} & \\ \mathcal{HH}(\mathscr{O}_X) & & \mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}) & \xrightarrow[\sim]{\tau_X} & \mathbb{C}_X((\hbar))\,[d_X]. \end{array}$$

For two coherent $\mathscr{A}_X$-modules $\mathscr{M}$ and $\mathscr{N}$ whose intersection of supports is compact, we get:

$$\chi(\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{N})) = \int_X \mathrm{eu}(\mathscr{M}) \cup \mathrm{eu}(\mathscr{N}).$$

Holonomic modules

Here, we assume X is symplectic. Recall that the support of a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is a complex subvariety of X and is co-isotropic. One says that $\mathscr{M}$ is holonomic if its support is Lagrangian.

Theorem

Let X be a complex symplectic manifold and let $\mathscr{M}$ and $\mathscr{L}$ be two holonomic $\mathscr{A}_X^{\mathrm{loc}}$-modules. Then

(i) *the object $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L})$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}c}(\mathbb{C}((\hbar))_X)$,*

(ii) *There is a canonical isomorphism:*

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L}) \xrightarrow{\sim} \left(\mathrm{D}'_X\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L},\mathscr{M})\right)[d_X].$$

(iii) *the object $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L})[d_X/2]$ is perverse.*

In general, one cannot estimate the microsupport $\mathrm{SS}(F)$ of the complex $F := \mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{L})$.
Let Λ be a smooth Lagrangian submanifold of X and let $\mathscr{L}$ be a coherent $\mathscr{A}_X$-module supported by Λ. One says that $\mathscr{L}$ is simple along Λ if $\mathrm{gr}_\hbar(\mathscr{L})$ is an invertible $\mathscr{O}_\Lambda \otimes_{\mathscr{O}_X} \mathrm{gr}_\hbar(\mathscr{A}_X)$-module. For example, if $X = T^*M$ and $\Lambda = M$, the module $\mathscr{O}_M[[\hbar]]$ is simple along M.
For two subsets Λ_0 and Λ_1 of X one defines their Whitney normal cone $\mathrm{C}(\Lambda_0, \Lambda_1)$, a closed subset of T^*X, after identifying T^*X and TX by the Hamiltonian isomorphism.

Theorem

Let $\mathscr{L}_i$ be a simple holonomic $\mathscr{A}_X^{\mathrm{loc}}$-module along a smooth Lagrangian manifold Λ_i $(i = 0, 1)$. Then

$$\mathrm{SS}\big(\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L}_1, \mathscr{L}_0)\big) \subset \mathrm{C}(\Lambda_0, \Lambda_1).$$

Applications to $\mathscr{D}$-modules

Let M be a complex manifold, $\pi\colon T^*M \to M$ its cotangent bundle and set $X = T^*M$. As already mentionned, there is a canonical DQ-algebra on X denoted $\widehat{W}_X(0)$. Setting $\widehat{W}_X = (\widehat{W}_X(0))^{\mathrm{loc}}$, there is a flat morphism of algebras $\pi^{-1}\mathscr{D}_M \hookrightarrow \widehat{W}_X$. For a coherent $\mathscr{D}$-module $\mathscr{M}$, set

$$\widetilde{\mathscr{M}} = \widehat{W}_X \otimes_{\pi^{-1}\mathscr{D}_M} \pi^{-1}\mathscr{M}$$

In 1994, with J-P. Schneiders, we define the Euler class of $\mathscr{D}$-modules and proved its functorial properties. Since $\mathrm{char}(\mathscr{M}) = \mathrm{supp}(\widetilde{\mathscr{M}})$ and the Euler class of $\widetilde{\mathscr{M}}$ is the image of the Euler class of $\mathscr{M}$ by the natural map $H^{d_X}_{\mathrm{supp}\,\mathscr{M}}(X;\mathbb{C}_X) \to H^{d_X}_{\mathrm{supp}\,\mathscr{M}}(X;\mathbb{C}_X((\hbar)))$, these properties may be deduced from those for $\mathscr{A}_X$-modules.

Main Problem: How to calculate the Euler class?

Assume $\mathscr{M}$ is endowed with a good filtration and set:

$$\mu\mathrm{ch}(\mathscr{M}) \quad = \quad ch(\mathrm{gr}_\hbar(\widetilde{\mathscr{M}})) \cup \pi^* td_X(T^*X).$$

Then Laumon proved (using the RR th) that $\mu\mathrm{ch}(\bullet)$ commutes to direct image and that's why with Schneiders we made the conjecture that $\mathrm{eu}(\mathscr{M}) = [\mu\mathrm{ch}(\mathscr{M})]^{d_X}$. This conjecture has been proved by Bressler-Nest-Tsygan (2002). For that purpose, they have to replace Hochschild homology with periodic cyclic homology.

Open problems

A natural problem would be to replace Hochschild homology with cyclic homology in the preceding construction, and:
(i) to prove its compatibility with the composition of kernels,
(ii) to make a link with De Rham cohomology, that is to obtain a commutative diagram:

$$\begin{array}{ccccc} & & \mathcal{HC}^-(\mathscr{A}_X) & & \\ & \swarrow^{\mathrm{gr}_\hbar} & & \searrow^{\mathrm{loc}} & \\ \mathcal{HC}^-(\mathscr{O}_X) & & & & \mathcal{HC}^-(\mathscr{A}_X^{\mathrm{loc}}) \\ \downarrow & & & & \downarrow \\ \bigoplus_i \mathbb{C}_X[i] & & \xrightarrow{\textit{unknown}} & & \bigoplus_i \mathbb{C}_X((\hbar))[i]. \end{array}$$

The Chern character in this framework has recently been constructed by Bressler-Gorokhovsky-Nest-Tsygan. However, to my opinion, there is still not a good understanding of cyclic homology in the framework of derived or DG categories. Such an understanding, that is, giving an intrinsic meaning to the Connes operator without calculating explicitely the Hochschild homology, would need the tools of derived algebraic geometry of Lurie and Toën.