General Topology

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Introduction

The aim of these Notes is to provide a short and self-contained presentation of the main concepts of general topology.

Of course, we certainly do not claim for any originality here. Indeed, when writing these Notes, we have been deeply influenced by the excellent book [3] of Jacques Dixmier.

We have included a few exercises at the end of the chapters. Here again no originality should be expected.

Besides Dixmier's book, and among a vast literature on the subject, let us only mention the few books below.

For the French students who would learn Mathematical English, we recommand the Notes [5] by Jan Nekovar.

Bibliography

[1] J-Y. Chemin, *Cours LM360, 2009/2010*
http://www.ann.jussieu.fr/ chemin/

[2] G. Choquet, *Cours d'Analyse II, Topologie.* Masson Ed. (1969)

[3] J. Dixmier, *Topologie Générale.* Presses Universitaires de France.

[4] J-P. Marco (sous la direction de) *Mathématiques L3 Analyse.* Pearson Education (2009)

[5] J. Nekovar, *Introduction to Mathematical English.*
http://people.math.jussieu.fr/ nekovar/co/en/

[6] F. Paulin, *Topologie et Calcul Différentiel.*
http://www.fimfa.ens.fr/spip.php?article26

[7] J. Saint Raymond, *Topologie, Calcul Différentiel et variable complexe.*

Calvage et Mounet (2007)

[8] G. Skandalis, *Topologie et analyse.* Mathématiques pour la Licence, Dunod (2001)

Chapter 1

Topological spaces

1.1 A short review of Set Theory

Let us recall a few notations, results and formulas of Set Theory which are of constant use.

- One denotes by $\emptyset$ the empty set and by $\{\mathrm{pt}\}$ a set with a single element.
- For a set X, one denotes by $\mathcal{P}(X)$ the set of all subsets of X.
- For two sets X_1 and X_2, one denotes by $X_1 \times X_2$ their product. There are natural maps $p_i \colon X_1 \times X_2 \to X_i$ $(i = 1, 2)$ called the projections. The set $X_1 \times X_2$ is the set $\{(x_1, x_2); x_i \in X_i,\ i = 1, 2\}$.
- More generally, if I is a set and $\{X_i\}_{i \in I}$ is a family of sets indexed by $i \in I$, one denotes by $\prod_{i \in I} X_i$ their product. An element $x \in \prod_{i \in I} X_i$ is a family $x = \{x_i\}_{i \in I}$ with $x_i \in X_i$ for all $i \in I$. If $X_i \equiv X$ for all i one writes X^I instead of $\prod_{i \in I} X$. For example, $X^{\mathbb{N}}$ is the set of all sequences in X.
- For two sets X_1 and X_2, one denotes by $X_1 \sqcup X_2$ their disjoint union. There are natural maps $\varepsilon_i \colon X_i \to X_1 \sqcup X_2$ $(i = 1, 2)$. If X_1 and X_2 are subsets of another set X, one shall not confuse $X_1 \sqcup X_2$ and $X_1 \cup X_2$.
- If X is a set and $\mathcal{R}$ is an equivalence relation on X, one denotes by $X/\mathcal{R}$ the quotient set which consists in identifying two elements x and x' when $x\mathcal{R}x'$. The map $\gamma \colon X \to X/\mathcal{R}$ is surjective and $x\mathcal{R}x'$ if and only if $\gamma(x) = \gamma(x')$. Conversely, if $f \colon X \to Y$ is a surjective map, it defines an equivalence relation $\mathcal{R}$ on X by $x\mathcal{R}x'$ if and only if $f(x) = f(x')$. The map f factorizes uniquely as $X \xrightarrow{\gamma} X/\mathcal{R} \xrightarrow{\tilde{f}} Y$ and the map $\tilde{f} \colon X/\mathcal{R} \to Y$ is bijective.

- If X and Y are two sets and $f\colon X \to Y$ is a map, its graph Γ_f is the subset of $X \times Y$ given by $\Gamma_f = \{(x, y) \in X \times Y; y = f(x)\}$. In particular, the graph of the identity map $\mathrm{id}_X\colon X \to X$ is the diagonal Δ_X of $X \times X$.

- For two sets X and Y, a map $f\colon X \to Y$ and a subset $A \subset X$, one denotes by $f|_A$ the restriction of f to A, a map $A \to Y$.

- For a set X, there exists a unique map $\emptyset \to X$.

- For a set X, there exists a unique map $X \to \{\mathrm{pt}\}$.

- For a set X and two subsets A, B of X, one denotes by $B \setminus A$ the set of points of X which belong to B and not to A (in particular, $X \setminus A$ is the complement of A in X).

- Given two families of sets $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$, one has $(\bigcup_i A_i) \cap (\bigcup_j B_j) = \bigcup_{i,j}(A_i \cap B_j)$.

Let $f\colon X \to Y$ be a map, let $\{A_i\}_{i\in I}$ be a family of subsets of X and let $\{B_j\}_{j\in J}$ be a family of subsets of Y.

- $f^{-1}(\bigcup_j B_j) = \bigcup_j(f^{-1}(B_j)$,

- $f^{-1}(\bigcap_j B_j) = \bigcap_j f^{-1}(B_j)$,

- $f^{-1}(Y) = X$, $f^{-1}(\emptyset) = \emptyset$,

- for $B \subset Y$, $f^{-1}(Y \setminus B) = f^{-1}(Y) \setminus f^{-1}(B)$,

- $f(\bigcup_i A_i) = \bigcup_i f(A_i)$.

One shall be aware that for subsets A, A_1, A_2 of X, one has

$$\begin{aligned} &f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2), \\ &f(X) \setminus f(A) \subset f(X \setminus A), \end{aligned}$$

but these inclusions are not equalities in general.

1.2 Review: norms and distances

Normed spaces

Let E be a vector space over the field $\mathbf{k} = \mathbb{R}$ or $\mathbf{k} = \mathbb{C}$. A quasi-norm $||\cdot||$ on E is a map $E \to \mathbb{R}_{\geq 0} \sqcup \{+\infty\}$ satisfying

$$(1.1)\begin{cases} \text{N1: } ||x|| = 0 \Leftrightarrow x = 0 \text{ for all } x \in E, \\ \text{N2: } ||\lambda \cdot x|| = |\lambda| \cdot ||x|| \text{ for all } x \in E \text{ and } \lambda \in \mathbf{k}, \\ \text{N3: } ||x+y|| \leq ||x|| + ||y|| \text{ for all } x, y \in E. \end{cases}$$

One calls the inequality in N3, the triangular inequality. It implies $||x|| \leq ||x-y|| + ||y||$, hence, after interchanging x and y:

$$|(||x|| - ||y||)| \leq ||x-y|| \text{ for all } x, y \in X. \tag{1.2}$$

If the quasi-norm $||\cdot||$ takes its values in $\mathbb{R}_{\geq 0}$, then it is called a norm.
A vector space endowed with a norm is called a normed space.
If F is a vector subspace of E, a quasi-norm on E induces a quasi-norm on F, called the induced quasi-norm.

Example 1.2.1. Let $\mathbb{C}^{\mathbb{N}}$ denote the space of all sequences $x = (a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{C}$. One may endow $\mathbb{C}^{\mathbb{N}}$ with the quasi-norms below:

$$||x||_\infty = \sup_n |a_n|, \quad ||x||_2 = \big(\sum_n |a_n|^2\big)^{\frac{1}{2}}, \quad ||x||_1 = \sum_n |a_n|.$$

The fact that $||x||_1$ and $||x||_\infty$ are quasi-norms is obvious as well as the fact that $||x||_2$ satisfies N1 and N2. As for N3, see Exercise 1.1.

Example 1.2.2. Let X be a set. The set $\mathbb{R}^X$ of all real valued functions on X is endowed with the quasi-norm

$$||f||_X = \sup_{x\in X} |f(x)|. \tag{1.3}$$

Definition 1.2.3. One denotes by $l^p(\mathbb{C})$ $(p = 1, 2, \infty)$ the subspace of $\mathbb{C}^{\mathbb{N}}$ consisting of sequences x for which $||x||_p < \infty$. One also denote by $l^p(\mathbb{R})$ the subspace of $l^p(\mathbb{C})$ consisting of real sequences.

Note that the quasi-norm $||\cdot||_p$ becomes a norm when restricted to the space $l^p(\mathbb{C})$. Hence, $(l^p(\mathbb{C}), ||\cdot||_p)$ is a normed space. Also note that for $n \in \mathbb{N}$, $\mathbb{R}^n$ and $\mathbb{C}^n$ are subspaces of $l^p(\mathbb{C})$. In particular, $\mathbb{R}^n$ is endowed with the norms $||\cdot||_p$ $(p = 1, 2, \infty)$. The norm $||\cdot||_2$ is called the Euclidian norm.

Definition 1.2.4. Let E be a $\mathbf{k}$-vector space and let $||\cdot||_1$ and $||\cdot||_2$ be two norms on E. One says that these two norms are equivalent if there exists a constant c with $0 < c \leq 1$ such that

$$c||\cdot||_1 \leq ||\cdot||_2 \leq c^{-1}||\cdot||_1$$

Clearly, this relation is an equivalence relations. We shall see later that all the norms on a finite dimensional vector space are equivalent.

Metric spaces

Let X be a set. A quasi-distance, or generalized distance, d on X is a map $d\colon X \times X \to \mathbb{R}_{\geq 0} \sqcup \{+\infty\}$ satisfying

$$(1.4)\begin{cases} \text{D1: } d(x,y) = d(y,x) \text{ for all } x,y \in X, \\ \text{D2: } d(x,y) = 0 \Leftrightarrow x = y \text{ for all } x,y \in X, \\ \text{D3: } d(x,y) \leq d(x,z) + d(z,y) \text{ for all } x,y,z \in X. \end{cases}$$

One often calls the inequality in D3, the triangular inequality.

If the generalized distance d takes its values in $\mathbb{R}_{\geq 0}$, then it is called a distance. A set X endowed with a distance d is called a metric space and denoted (X, d). In the sequel, unless otherwise specified, we shall consider distances, not quasi-distances.

Using D1 and D3, we get

$$(1.5) \qquad |d(x,z) - d(y,z)| \leq d(x,y) \text{ for all } x,y,z \in X.$$

One also encounters spaces endowed with a distance satisfying a property stronger than D3, namely

$$(1.6) \quad \text{D4: } d(x,y) \leq \max(d(x,z), d(z,y)) \text{ for all } x,y,z \in X.$$

Such spaces are called ultrametric spaces.

If A is a subset of X and d is a distance on X, then d defines a distance on A.

Definition 1.2.5. Let d_1 and d_2 be two distances on the same space X. One says that d_1 and d_2 are equivalent if there exist a positive real numbers $0 < c \leq 1$ such that

$$c \cdot d_1 \leq d_2 \leq c^{-1} d_1.$$

Clearly, this relation is an equivalence relation.

Given a norm $||\cdot||$ on a vector space E, one associates a distance on E by setting

$$d(x,y) = ||x-y||. \tag{1.7}$$

Such a distance is invariant by translation:

$$d(x,y) = d(x+z, y+z).$$

Clearly, two equivalent norms define two equivalent distances.

Definition 1.2.6. Let $f\colon X \to Y$ be a map and assume that X is endowed with a distance d_X and Y with a distance d_Y. One says that f is an isometry if $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.

Note that an isometry is necessarily injective.

1.3 Topologies

Definition 1.3.1. Let X be a set. A topology on X is the data of a subset $\mathcal{T} \subset \mathcal{P}(X)$, called the family of open subsets of X, satisfying the axioms (1.8) below. A topological space is a set endowed with a topology.

$$\begin{cases} \text{T1: } \emptyset \text{ and } X \text{ are open (that is, } \emptyset \in \mathcal{T} \text{ and } X \in \mathcal{T}), \\ \text{T2: for any family } \{U_i\}_{i\in I} \text{ of open subsets (that is, } U_i \in \mathcal{T}), \text{ the} \\ \text{union } \bigcup_i U_i \text{ is open (that is, } \bigcup_i U_i \in \mathcal{T}), \\ \text{T3: for any \textit{finite} family } \{U_j\}_{j\in J} \text{ of open subsets (that is, } U_j \in \mathcal{T}), \\ \text{the intersection } \bigcap_j U_j \text{ is open (that is, } \bigcap_j U_j \in \mathcal{T}). \end{cases} \tag{1.8}$$

The complementary of an open subset is called a closed subset:

$$A \text{ is closed if and only if } X \setminus A \text{ is open.}$$

Hence, the family of closed subsets satisfies:

$$\begin{cases} \text{(i) } \emptyset \text{ and } X \text{ are closed,} \\ \text{(ii) for a family } \{S_i\}_{i\in I} \text{ of closed subsets, the intersection } \bigcap_i S_i \text{ is} \\ \text{closed,} \\ \text{(iii) for a \textit{finite} family } \{S_j\}_{j\in J} \text{ of closed subsets, the union } \bigcup_i S_i \\ \text{is closed.} \end{cases} \tag{1.9}$$

Example 1.3.2. (i) The family $\mathcal{P}(X)$ of all subsets of X defines a topology on X, called the discrete topology on X. Note that for this topology, each point $\{x\}$ is open in X.
(ii) The topology on X for which the only open subsets are X and $\emptyset$ is called the trivial topology.
(iii) On $X = \{\mathrm{pt}\}$ there is a unique topology, the open sets being pt and $\emptyset$.
(iv) Let $X = \mathbb{R}$ and recall that an open interval is an interval $]a, b[$ with $-\infty \leq a \leq b \leq +\infty$. Let us call *open* the subsets which are union of open intervals. This clearly defines a topology on $\mathbb{R}$, called the usual topology. Similarly, an open subset of $\mathbb{R}^n$ for the usual topology is by definition a union of products of open intervals.
(iv) The family of subsets $] - \infty, c[$ with $-\infty \leq c \leq +\infty$ defines a topology on $\mathbb{R}$.
(v) Let $X = \mathbb{C}$ and let us say that a set U if open if there exists a polynomial $P \in \mathbb{C}[z]$ such that $U = \{z \in \mathbb{C}; P(z) \neq 0\}$. In other words, U is open if either U is empty or $\mathbb{C} \setminus U$ is finite. One gets a topology on $\mathbb{C}$ called the Zariski topology.

Topology associated with a distance

Let (X, d) denote a metric space. For $a \in X$ and $\varepsilon \geq 0$, one sets

$$\begin{aligned} B(a, \varepsilon) &= \{x \in X; d(x, a) < \varepsilon\}, \\ \overline{B}(a, \varepsilon) &= \{x \in X; d(x, a) \leq \varepsilon\}. \end{aligned}$$

One calls $B(a, \varepsilon)$ (resp. $\overline{B}(a, \varepsilon)$) the open (resp. closed) ball with center a and radius ε.

Example 1.3.3. Draw the picture of the closed ball of center 0 and radius 1 for $X = \mathbb{R}^2$ endowed with one of the norms $|| \cdot ||_p$, $p = 1, 2, \infty$.

Definition 1.3.4. A subset U of X is called an open subset if it is a union of open balls.

This definition will be justified by Proposition 1.3.6 below.

Lemma 1.3.5. *Let U be an open subset and let $x \in U$. Then there exists $\varepsilon_x > 0$ such that the open ball $B(x, \varepsilon_x)$ is contained in U.*

Proof. Since U is a union of open balls, there exist $a \in X$ and $\eta > 0$ such that $x \in B(a, \eta)$. Hence, $d(x, a) < \eta$. Choose $0 < \varepsilon_x < \eta - d(a, x)$. Then $B(x, \varepsilon_x) \subset B(a, \eta) \subset U$. q.e.d.

In particular, we get $U = \bigcup_x B(x; \varepsilon_x)$.

Proposition 1.3.6. *The family of open subsets of X satisfies the axioms* T1-T2-T3*, hence defines a topology on X.*

Proof. The axioms T1 and T2 are clearly satisfied. Let us prove T3. Let U_1 and U_2 be two open subsets and let $x \in U_1 \cap U_2$. By Lemma 1.3.5, there exist ε_i such that $B(x, \varepsilon_i) \subset U_i$ $(i = 1, 2)$. Setting $\varepsilon = \inf(\varepsilon_1, \varepsilon_2)$ we get that $B(x, \varepsilon) \subset U_1 \cap U_2$. Hence, $U_1 \cap U_2$ is a union of open balls. q.e.d.

From now on, we shall consider a metric space as a topological space for the topology defined by this proposition.

Proposition 1.3.7. (i) *An open ball is open in X.*

(ii) *A closed ball is closed in X.*

Proof. (i) is obvious.
(ii) Let $\overline{B}(a, \varepsilon)$ be a closed ball and let $x \notin \overline{B}(a, \varepsilon)$. Then $d(x, a) > \varepsilon$. Choose $0 < \eta < d(x, a) - \varepsilon$. Then $B(x, \eta) \subset X \setminus \overline{B}(a, \varepsilon)$. This shows that the complementary set of $\overline{B}(a, \varepsilon)$ is a union of open balls, hence is open. q.e.d.

Proposition 1.3.8. *Two equivalent distances on X define the same topology on X.*

The proof is obvious and left as an exercise.

However, one shall be aware that two non equivalent distances may induce the same topology on X. See Example 1.4.7 below.

Example 1.3.9. Let X be a space and define a distance d on X by setting $d(x, x) = 0$ and $d(x, y) = 1$ for $x \neq y$. Then the associated topology is the discrete topology on X.

Neighborhoods

Definition 1.3.10. Let $V \subset X$ and let $x \in X$. One says that V is a neighborhood of x if V contains an open set U which contains x, that is, there exists U open with $x \in U \subset V$.

Note that a set V is open if and only if it is a neighborhood of each of its points. In fact, if V is open and $x \in V$, then V is a neighborhood of x, and conversely, if for each $x \in V$ there exists an open set U_x with $x \in U_x \subset V$, then $V = \bigcup_{x \in V} U_x$ is a union of open subset, hence is open.

Also note that the family of neighborhoods of $x \in X$ is stable by finite intersection.

Definition 1.3.11. Let $x \in X$. A family $\{V_i\}_{i \in I}$ of subsets of X is a fundamental system of neighborhoods of x, or simply, a neighborhoods system of x if any V_i is a neighborhood of x and for any neighborhood V of x, there exists $i \in I$ with $V_i \subset V$.

For example, the family of open subsets of X which contain x is a fundamental system of neighborhoods of x.

Proposition 1.3.12. *Let (X, d) be a metric space and let $a \in X$. The family $\{B(a, \frac{1}{n})\}_{n \in \mathbb{N}_{>0}}$ is a fundamental system of neighborhoods of a in X.*

Proof. This follows from Lemma 1.3.5. q.e.d.

Note that the family $\{\overline{B}(a, \frac{1}{n})\}_{n \in \mathbb{N}_{>0}}$ is also a fundamental system of neighborhoods of a.

Interior and closure

Definition 1.3.13. Let $A \subset X$.

(i) The closure of A, ("la fermeture" or "l'adhérence" in French) denoted $\overline{A}$ is the smallest closed subset of X which contains A, that is, $\overline{A} = \bigcap S$ where S ranges through the family of all closed subsets of X which contain A.

(ii) The interior of A, denoted $\mathrm{Int(A)}$ or IntA or also $\mathring{A}$ is the biggest open subset contained in A, that is, $\mathrm{IntA} = \bigcup \mathrm{U}$ where U ranges through the family of open subsets of X contained in A.

(iii) The boundary of A, denoted ∂A is the set $\overline{A} \setminus \mathrm{IntA}$. Hence ∂A is the closed subset $X \setminus (\mathrm{IntA} \cup (\mathrm{Int(X \setminus A)})$.

Note that, for two subsets A and B of X:

$$\begin{cases} \mathrm{IntA \subset A \subset \overline{A}}, \\ A \text{ is open} \Leftrightarrow A = \mathrm{IntA}, \\ A \text{ is closed} \Leftrightarrow A = \overline{A}, \\ \mathrm{IntA = X \setminus \overline{(X \setminus A)}}, \\ \overline{A} = X \setminus \mathrm{Int(X \setminus A)}, \\ x \in \overline{A} \text{ if and only if any neighborhood of } x \text{ intersects } A, \\ \overline{A \cup B} = \overline{A} \cup \overline{B}, \\ \mathrm{Int(A \cap B) = IntA \cap IntB}. \end{cases}$$

Definition 1.3.14. (a) A subset A is dense in X if $\overline{A} = X$.

(b) A subset A has no interior points if Int(A) $= \emptyset$ or equivalently, $X \setminus A$ is dense in X.

For example, $\mathbb{Q}$ is dense in $\mathbb{R}$ and has no interior points.

Note that $x \in \overline{A}$ if and only if any neighborhood of x intersects A.

One shall be aware that in a metric space (X, d) the closed ball $\overline{B}(a, R)$ is not necessarily the closure of the open ball $B(a, R)$.

Example 1.3.15. On $\mathbb{Z}$ endowed with the distance induced by that of $\mathbb{R}$, consider the open ball $B(0; 1)$ of center 0 and radius 1. Then $B(0; 1) = \{0\}$ and this ball is a closed set. Hence $\overline{B(0; 1)} \neq \overline{B}(0; 1)$.

Definition 1.3.16. Let $A \subset X$ and let $x \in X$.

(a) One says that x is an accumulation point of A if any neighborhood of x intersects $A \setminus \{x\}$.

(b) One says that x is isolated in A if $x \in A$ and x is not an accumulation point of A, that is, there exists an open set U in X with $x \in U$ and $(A \setminus \{x\}) \cap U = \emptyset$.

Hausdorff spaces

Definition 1.3.17. A topological space X is Hausdorff ("séparé" in French) if for any $x, y \in X$ with $x \neq y$, there exist a neighborhood V_x of x and a neighborhood V_y of y with $V_x \cap V_y = \emptyset$.

Example 1.3.18. (i) The space $\mathbb{R}^n$ endowed with its natural topology is Hausdorff.
(ii) If a topological space X has more than one element, then the trivial topology on X is not Hausdorff.

Proposition 1.3.19. *Let (X, d) be a metric space. Then X is Hausdorff.*

Proof. Let $x, y \in X$ with $x \neq y$. Then $d(x, y) > 0$. Choose $\varepsilon < \frac{d(x,y)}{2}$. Then $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$. q.e.d.

Induced topology

Let X be a topological space and let $A \subset X$. The *induced topology* on A is defined as follows: a set $V \subset A$ is open in A if and only if there exists U open in X such that $V = U \cap A$. It follows that a set $S \subset A$ is closed in A if and only if there exists Z closed in X such that $S = Z \cap A$.

Moreover, if A is open (resp. closed) in X, then the open (resp. closed) sets in A are exactly the open (resp. closed) sets in X contained in A.

Note that if X is a Hausdorff space and A is a subset of X endowed with the induced topology, then A is Hausdorff.

Definition 1.3.20. A subset A of X is discrete if the induced topology by X on A is the discrete topology, or equivalently, if any point of A is isolated in A.

Example 1.3.21. (i) We endow $\mathbb{Z}$ or $\mathbb{N}$ with the topology induced by that of $\mathbb{R}$. Hence, $\mathbb{Z}$ and a fortiori $\mathbb{N}$ are discrete.
(ii) Let $A = \{\frac{1}{n}\}_{n\in\mathbb{N},n>0}$. Then A is discrete for the induced topology by $\mathbb{R}$. Note that $A \cup \{0\}$ is no longer discrete.

Example 1.3.22. Set $\overline{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$. We endow $\overline{\mathbb{N}}$ with the following topology. The open sets of $\overline{\mathbb{N}}$ are the union of the subsets of $\mathbb{N}$ and the sets $[n, +\infty]$ $(n \in \mathbb{N})$. Then $\overline{\mathbb{N}}$ induces on $\mathbb{N}$ its discrete topology but $\overline{\mathbb{N}}$ is not discerete.

1.4 Continuous maps

Definition 1.4.1. Let X and Y be two topological spaces and $f\colon X \to Y$ a map. One says that f is continuous if for any open subset $V \subset Y$, the set $f^{-1}(V)$ is open in X.

Note that f is continuous if and only if for any closed subset $Z \subset Y$, the set $f^{-1}(Z)$ is closed in X. This is also equivalent to the following property:

$$\text{for any } x \in X \text{ and any neighborhood } W \text{ of } f(x) \text{ in } Y,\ f^{-1}(W) \text{ is a neighborhood of } x \text{ in } X. \tag{1.10}$$

It follows immediately from the definition that if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, then the composition $g \circ f\colon X \to Z$ is continuous.

Example 1.4.2. (i) If X is endowed with the discrete topology and Y is a topological space, any map $f\colon X \to Y$ is continuous.
(ii) If Y is endowed with the trivial topology and X is a topological space, any map $f\colon X \to Y$ is continuous.

Definition 1.4.3. Let X and Y be two topological spaces and $f\colon X \to Y$ a map. One says that f is continuous at $a \in X$ if for any neighborhood W of $b = f(a)$ in Y, $f^{-1}(W)$ is a neighborhood of a in X.

Using (1.10), one gets that f is continuous if and only if f is continuous at each $x \in X$.

One can obtain Definition 1.4.3 as a particular case of Definition 1.4.1 as follows. Denote by $\widetilde{X}$ the space X endowed with the new topology for which the open subsets are $\emptyset$ and the open sets which contain a. Defines similarly $\widetilde{Y}$, replacing a with $f(a)$, and denote by $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ the map f. Then f is continuous at a if and only if $\widetilde{f}$ is continuous. In particular, we get that if $f\colon X \to Y$ is continuous at $a \in X$ and $g\colon Y \to Z$ is continuous at $f(a)$, then the composition $g \circ f\colon X \to Z$ is continuous at a.

One also introduces the notion of an open (resp. closed) map.

Definition 1.4.4. A map $f\colon X \to Y$ is open (resp. closed) if the image by f of any open (resp. closed) subset of X is open (resp. closed) in Y.

Example 1.4.5. (i) Let $a, b \in \mathbb{R}$ and let us endow the interval $]a, b]$ of $\mathbb{R}$ with the induced topology. The embedding $j\colon]a, b] \hookrightarrow \mathbb{R}$ is continuous, but it is neither open nor closed.
(ii) If X is an open (resp. a closed) subset of a topological space endowed with the induced topology, then the embedding $X \hookrightarrow Y$ is open (resp. closed).
(iii) Let $n \geq 1$. The projections $\mathbb{R}^n \to \mathbb{R}$ are open.
(iv) The projection $\mathbb{R}^2 \to \mathbb{R}$ is not closed. Indeed, consider the closed subset of $\mathbb{R}^2$:

$$Z = \{(x, y) \in \mathbb{R}^2; xy = 1, y > 0\}.$$

The image of Z by the projection $p_1\colon (x, y) \mapsto x$ is the set $]0, +\infty[$ which is not closed in $\mathbb{R}$.

Definition 1.4.6. Let X and Y be two topological spaces. A topological isomorphism $f\colon X \to Y$ is a continuous map such that there exists a continuous map $g\colon Y \to X$ satisfying $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

In the literature, topological isomorphisms are often called "homeomorphisms".

Example 1.4.7. (i) Let $X =]-\frac{\pi}{2}, +\frac{\pi}{2}[$. Then the map $\tan\colon]-\dfrac{\pi}{2}, \dfrac{\pi}{2}[\to \mathbb{R}$ is a topological isomorphism.
(ii) Let d_1 denote the distance on X induced by that of $\mathbb{R}$, that is, $d_1(x, y) = |x - y|$. Let d_2 be the distance $d_2(x, y) = |\tan(x) - \tan(y)|$. Since d_1 is bounded and d_2 is not, these distances are not equivalent. Since the map $\tan\colon (X, d_2) \to (\mathbb{R}, |\cdot|)$ is an isometry, the topology on X induced by d_2 is the inverse image topology of $\mathbb{R}$ by the map tan, and we have seen in (i) that this is the usual topology of X. Hence, d_1 and d_2 induce the same topology

on $X =]-\frac{\pi}{2}, +\frac{\pi}{2}[$. Hence, two distances may be not equivalent although they define the same topology.

If two spaces X and Y are topologically isomorphic, then, as far as one is interested in their topological properties, they can be identified, similarly as two real vector spaces of the same dimension can be identified as far as one is only interested in their linear properties. However, if the spaces $]-\dfrac{\pi}{2}, \dfrac{\pi}{2}[$ and $\mathbb{R}$ can be identified as topological spaces, they cannot be identified as metric spaces (see below).

Note that a topological isomorphism is bijective. However, a continuous map can be bijective without being a topological isomorphism. For example denote by Y the space $\mathbb{R}$ endowed with its natural topology and denote by X the space $\mathbb{R}$ endowed with the discrete topology. Then the identity map $X \to Y$ is bijective and continuous without being a topological isomorphism.

Consider a continuous and bijective map $f\colon X \to Y$. Then f is a topological isomorphism if and only if f is open (resp. closed).

Limits and sequences

Let X and Y be two topological spaces, $A \subset X$, $a \in \overline{A}$ (in general, $a \notin A$), $b \in Y$ and let $f\colon A \to Y$ be a map.

Definition 1.4.8. One says that $f(x)$ converges (or goes) to b when x goes to a with $x \in A$, if for any neighborhood V of b in Y, $f^{-1}(V) = U \cap A$ for some neighborhood U of a in X. In this case, one writes: $f(x) \to b$ when $A \ni x \to a$ or, for short, $f(x) \xrightarrow{x \to a} b$.

Consider the space $\widetilde{X} = X \cup \{a\}$, endowed with the induced topology and define the map $\widetilde{f}\colon \widetilde{X} \to Y$ by setting $\widetilde{f}(x) = f(x)$ for $x \in A$ and $\widetilde{f}(a) = b$. Then $f(x)$ converges to b when x goes to a if and only if the map $\widetilde{f}$ is continuous at a.

Now consider a sequence $(y_n)_{n\in\mathbb{N}}$ in a topological space Y. This is nothing but a map $\chi\colon \mathbb{N} \to Y$ where $\chi(n) = y_n$. Let $b \in Y$. Now embed $\mathbb{N}$ in $\overline{\mathbb{N}}$ where this last space is endowed with the topology of Example 1.3.22, and extends χ to $\overline{\mathbb{N}}$ by setting $\chi(\infty) = b$:

$$\chi\colon \overline{\mathbb{N}} \to X, \quad \chi(n) = y_n, \chi(\infty) = b. \tag{1.11}$$

The sequence $(y_n)_{n\in\mathbb{N}}$ in Y converges to b when n goes to infinity (one writes writes $y_n \xrightarrow{n} b$) if and only if $\chi(n)$ goes to b when n goes to infinity. In other words

(1.12) the sequence $(y_n)_{n\in\mathbb{N}}$ converges to b when n goes to infinity if and only if for any neighborhood V of b there exists $N \in \mathbb{N}$ such that $y_n \in V$ for all $n \geq N$.

1.5 Inverse image topology

Let X and Y be two sets and let $f\colon X \to Y$ be a map. Assume that Y is endowed with a topology. One then defines a topology on X, called *the inverse image topology*, by deciding that the open subsets of X are the inverse image by f of the open subsets of Y, that is,

$U \subset X$ is open in X if and only if $U = f^{-1}(V)$ for an open subset V of Y

One checks easily that the axioms (1.8) are satisfied and moreover, the map f is continuous.

Example 1.5.1. (i) The induced topology is a particular case of the inverse image topology. Denote by $j\colon A \hookrightarrow X$ the injection of a subset A of X. The induced topology on A is the inverse image topology by j.
(ii) Set $X = Y = \mathbb{R}$ and consider the map $f\colon X \to Y$, $x \mapsto x^2$. Endow Y with its natural topology and X with the inverse image topology. Then a set A is open in X if and only if it is open for the usual topology and moreover $A = -A$, that is, $x \in A$ if and only if $-x \in A$.

Proposition 1.5.2. *Let $f\colon X \to Y$ be a continuous map and assume that the topology on X is the inverse image topology. Let S be a topological space and let $g\colon S \to X$ be a map. Then g is continuous if and only if $h := f \circ g\colon S \to Y$ is continuous.*

$$\begin{tikzcd} S \arrow[d, "g"'] \arrow[dr, "h"] & \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Proof. The map g is continuous if and only if for any U open in X, $g^{-1}(U)$ is open in S. Since any U open in X is of the type $f^{-1}(V)$ for V open in Y, this is equivalent to saying that for any V open in Y, $g^{-1}(f^{-1}(V)) = h^{-1}(V)$ is open in S. q.e.d.

1.6 Product topology

Let X_1 and X_2 be two topological spaces [1] and let $X = X_1 \times X_2$ be their product. Hence, X is the set of points (x_1, x_2) with $x_i \in X_i$ $(i = 1, 2)$. One denotes by

$$p_i\colon X_1 \times X_2 \to X_i \quad (i = 1, 2) \tag{1.13}$$

[1] The reader will extend the definitions and results of this section to the case of n topological spaces $X_1, X_2, \dots, X_n$.

the projections.

Definition 1.6.1. An elementary open subset of X is a product $U_1 \times U_2$ with U_i open in X_i $(i = 1, 2)$.

An open subset of X is a union of elementary open subsets.

Proposition 1.6.2. *The family of open subsets of X defines a topology on X.*

The topology defined by the above proposition is called the product topology.

Proof. Let us check the axioms of (1.8).
T1: $\emptyset = \emptyset \times X_2$ and $X = X_1 \times X_2$,
T2: is obvious,
T3: Let U and V be two open subsets of X. Then $U = \bigcup_i U_i$ and $V = \bigcup_j V_j$ where the U_i's and the V_j's are elementary open subsets. Hence $U_i = U_i^1 \times U_i^2$ and $V_j = V_j^1 \times V_j^2$. Then $U \cap V = \bigcup_{i,j}(U_i \cap V_j)$ and each $U_i \cap V_j$ is an elementary open set since

$$U_i \cap V_j = (U_i^1 \cap V_j^1) \times (U_i^2 \cap V_j^2).$$

q.e.d.

Proposition 1.6.3. *The projections p_i (see (1.13)) are continuous. Moreover, given a topological space Z, a map $f\colon Z \to X$ is continuous if and only if the two compositions $p_i \circ f$ $(i = 1, 2)$ are continuous.*

(1.14)

$$\begin{tikzcd}
 & & X_1 \\
Z \arrow[urr, "p_1\circ f"] \arrow[r, "f"] \arrow[drr, "p_2\circ f"'] & X_1 \times X_2 \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & \\
 & & X_2
\end{tikzcd}$$

Proof. (i) Let U be an open subset of X_1. Then $p_1^{-1}(U) = U \times X_2$ is an elementary open subset of X.
(ii) If $f\colon Z \to X$ is continuous, then the compositions $p_i \circ f$ $(i = 1, 2)$ are continuous by (i). Conversely, assume that the compositions $p_i \circ f$ $(i = 1, 2)$ are continuous. Let U be an open subset of X. Then U is a union of elementary open subsets and since f^{-1} commutes with unions, it is enough

to check that the inverse image of an elementary open subset $U_1 \times U_2$ is open in Z. This follows from

$$\begin{aligned} f^{-1}(U_1 \times U_2) &= f^{-1}(U_1 \times X_2) \cap f^{-1}(X_1 \times U_2) \\ &= (p_1 \circ f)^{-1}(U_1) \cap (p_2 \circ f)^{-1}(U_2). \end{aligned}$$

q.e.d.

Proposition 1.6.4. *The space X is Hausdorff if and only if each space X_i $(i = 1, 2)$ is Hausdorff.*

Proof. (i) Assume X_1 and X_2 are Hausdorff. Let $x, y \in X$ with $x \neq y$. Setting $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we have $x_1 \neq y_1$ or $x_2 \neq y_2$ (or both). Assume $x_1 \neq y_1$ and choose two open subsets U_1 and V_1 in X_1 with $x_1 \in U_1$, $y_1 \in V_1$ and $U_1 \cap V_1 = \emptyset$. Then $x \in U_1 \times X_2$, $y \in V_1 \times X_2$ and $(U_1 \times X_2) \cap (V_1 \times X_2) = \emptyset$.
(ii) The proof of the converse statement is similar and left to the reader. q.e.d.

For a set X, recall that one denotes by Δ_X the diagonal of $X \times X$:

$$\Delta_X = \{(x, y) \in X \times X; x = y\}.$$

If X is a topological space, one endows Δ_X with the topology induced by $X \times X$. Then one checks easily that the projections p_i $(i = 1, 2)$ induce topological isomorphisms $p_i \colon \Delta_X \xrightarrow{\sim} X$.

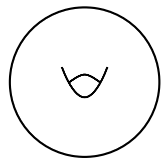
Proposition 1.6.5. *The space X is Hausdorff if and only if the diagonal Δ_X is closed in $X \times X$.*

Proof. (i) Assume X is Hausdorff. Let us prove that the complementary set of Δ_X is a union of elementary open sets, hence is open. Let $(x, y) \in X \times X$ with $(x, y) \notin \Delta_X$, that is, $x \neq y$. Let $U \in x$, $V \ni y$ be two open sets with $U \cap V = \emptyset$. Then $(x, y) \in U \times V$ and $\Delta_X \cap (U \times V) = \emptyset$.
(ii) The proof of the converse statement is similar and left to the reader. q.e.d.

Corollary 1.6.6. *Let $f, g \colon X \to Y$ be two continuous maps and let $A \subset X$ be a dense subset. Assume that $f = g$ on A and Y is Hausdorff. Then $f = g$.*

Proof. The map $(f, g) \colon X \to Y \times Y$ is continuous. Hence the inverse image of the diagonal Δ_Y by this map is closed in X. Since its contains A, it is equal to X. q.e.d.

Example 1.6.7. Consider the Euclidian circle $\mathbb{S}^1 \subset \mathbb{R}^2$. One sets $\mathbb{T}{:=}\mathbb{S}^1 \times \mathbb{S}^1$ and calls it the torus. It is naturally a subset of $\mathbb{R}^4$ but it can be drawn in $\mathbb{R}^3$, visualized by the picture below.



(1.15)

1.7 Direct image topology

Let X and Y be two sets and let $f\colon X \to Y$ be a map. Assume that X is endowed with a topology. One then defines a topology on Y, called *the direct image topology*, by deciding that a subset V of Y is open if and only if $f^{-1}(V)$ is open in X.

$$V \subset Y \text{ is open in } Y \text{ if and only if } f^{-1}(V) \text{ is open in } X.$$

One checks easily that the axioms (1.8) are satisfied.

Proposition 1.7.1. *Let $f\colon X \to Y$ be a map and assume that the topology on Y is the direct image topology. Let Z be a topological space and let $g\colon Y \to Z$ be a map. Then g is continuous if and only if $h{:=}g \circ f\colon X \to Z$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z. \end{tikzcd}$$

Proof. The map g is continuous if and only if for any W open in Z, $g^{-1}(W)$ is open in Y. By the definition of the image topology on Y, this is equivalent to saying that $f^{-1}g^{-1}(W) = h^{-1}(W)$ is open in X. q.e.d.

Example 1.7.2. Set $\mathbb{T}{:=}\mathbb{R}/2\pi\mathbb{Z}$ endowed with the quotient topology. Here, $\mathbb{R}/2\pi\mathbb{Z}$ means the quotient of $\mathbb{R}$ by the equivalence relation $x \sim y \Leftrightarrow (x{-}y) \in 2\pi\mathbb{Z}$. As a topological space, one also have $\mathbb{T} = [0, 2\pi]/\sim$ where $\sim$ is the equivalence relation which identifies the two points 0 and 2π, that is, $0 \sim 2\pi$ and otherwise $x \sim y$ if and only if $x = y$. The space $\mathbb{T}$ endowed with the quotient topology is Hausdorff. Indeed, consider $x \neq y$ in $\mathbb{T}$. If x and y are both different from the image of 0, we choose two open neighborhoods U and V of x and y in $]0, 2\pi[$ with $U \cap V = \emptyset$. If x is the image of 0 (hence, y is not) we choose two open neighborhoods U and V of $0 \cup 2\pi$ and y in $[0, 2\pi]$ with $U \cap V = \emptyset$.

Example 1.7.3. Denote by $\mathbb{S}^n$ the Euclidian n-sphere, that is, the unit sphere of the space $\mathbb{R}^{n+1}$ endowed with its Euclidian norm. On the other hand, consider the space $(\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^+$, endowed with the quotient topology and denote by g the quotient map:

$$g\colon \mathbb{R}^{n+1} \setminus \{0\} \to (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^+.$$

Here, $(\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^+$ means the quotient of $\mathbb{R}^n \setminus \{0\}$ by the equivalence relation $x \sim y \Leftrightarrow x = \lambda y$ for some $\lambda \in \mathbb{R}^+$. We call this quotient space, the topological n-sphere.

Clearly, the quotient map g induces an isomorphism from the Euclidian n-sphere to the topological n-sphere and this last one is in particular Hausdorff. In the sequel, we shall often identify the Euclidian and the topological n-spheres, and denote them by $\mathbb{S}^n$.

Example 1.7.4. Consider the map

$$f\colon \mathbb{T} \to \mathbb{S}^1, \quad \theta \mapsto \exp(i\theta).$$

This map is bijective, and one checks that it is a topological isomorphism. (See also Example 3.1.10.)

Glueing topological spaces

Assume we have two topological spaces X_1 and X_2. One endows the disjoint union $X_1 \sqcup X_2$ of a topology by taking as open subsets the union of the open subsets of X_1 and of X_2. We have natural maps $\varepsilon_i\colon X_i \to X_1 \sqcup X_2$.

The next result is obvious.

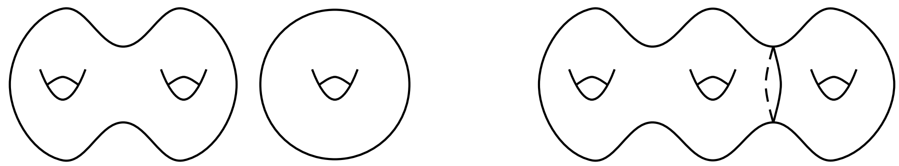
Proposition 1.7.5. *The maps ε_i are continuous. Moreover, given a topological space Z, a map $f\colon X_1 \sqcup X_2 \to Z$ is continuous if and only if the two compositions $f \circ \varepsilon_i$ $(i = 1, 2)$ are continuous.*

$$\begin{tikzcd}
X_1 \arrow[dr, "\varepsilon_1"'] \arrow[drr, "f\circ\varepsilon_1"] & & \\
& X_1 \sqcup X_2 \arrow[r, "f"] & Z \\
X_2 \arrow[ur, "\varepsilon_2"] \arrow[urr, "f\circ\varepsilon_2"'] & &
\end{tikzcd}$$

Now suppose that we also have two closed subsets $S_1 \subset X_1$ and $S_2 \subset X_2$ and a topological isomorphism $H\colon S_1 \xrightarrow{\sim} S_2$. Then one can consider the space

$$X_1 \sqcup_S X_2 := (X_1 \sqcup X_2)/\mathcal{R}$$

where $\mathcal{R}$ is the equivalence relation which identifies S_1 and S_2. We do not give more details, referring to the picture below as an example.

(1.16) 

Exercises to Chapter 1

Exercise 1.1. (i) For $(a_j)_{j\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ in $\mathbb{C}^{\mathbb{N}}$, prove the Cauchy-Schwarz's inequality:

$$\sum_{j=1}^{N}|a_jb_j| \le \big(\sum_{j=1}^{N}|a_j|^2\big)^{\frac{1}{2}}\big(\sum_{j=1}^{N}|b_j|^2\big)^{\frac{1}{2}}, \tag{1.17}$$

(ii) Deduce the inequality

$$\big(\sum_{j=1}^{N}|a_j+b_j|^2\big)^{\frac{1}{2}} \le \big(\sum_{j=1}^{N}|a_j|^2\big)^{\frac{1}{2}} + \big(\sum_{j=1}^{N}|b_j|^2\big)^{\frac{1}{2}}.$$

(iii) Prove that $||\cdot||_2$ is a quasi-norm on $\mathbb{C}^{\mathbb{N}}$.
(Hint: for (i), use the fact that for any real λ, $\sum_{j=1}^{N}(a_j+\lambda b_j)^2 \ge 0$ and deduce that the discrimant of the polynomial $\sum_{j=1}^{N}a_j^2 + 2\lambda\sum_{j=1}^{N}a_jb_j + \lambda^2\sum_{j=1}^{N}b_j^2$ is negative.)

Exercise 1.2. Let X be a topological space and let $A \subset X$. One sets $\alpha(A) = \mathrm{Int}(\overline{\mathrm{A}})$ and $\beta(A) = \overline{\mathrm{Int}(\mathrm{A})}$. Prove that if A is open, then $A \subset \alpha(A)$ and if A is closed, then $\beta(A) \subset A$. Deduce that for any $A \subset X$, $\alpha(\alpha(A)) = \alpha(A)$ and $\beta(\beta(A)) = \beta(A)$.

Exercise 1.3. let X and Y be two topological spaces and let $f\colon X \to Y$ be a map. One denotes by $\Gamma_f = \{(x,y) \in X \times Y; y = f(x)\}$ the graph of f, endowed with the induced topology by $X \times Y$.
(i) Prove that f is continuous if and only if the map $X \to \Gamma_f$ given by $x \mapsto (x, f(x))$ is a topological isomorphism.
(ii) Prove that if f is continuous and Y is Hausdorff, then Γ_f is closed in $X \times Y$.
(iii) Give an example in which X and Y are Hausdorff, Γ_f is closed but f is not continuous.

Exercise 1.4. Let us denote by $\overline{\mathbb{R}}$ the space $\{-\infty\}\cup\mathbb{R}\cup\{+\infty\}$. One endows $\overline{\mathbb{R}}$ with a structure of a topological space by taking for open subsets those which are a union of open subsets of $\mathbb{R}$ and the sets $[-\infty, a[$ and $]b, +\infty]$. Prove that the topological isomorphism $\tan\colon]-\frac{\pi}{2}, +\frac{\pi}{2}[\xrightarrow{\sim} \mathbb{R}$ extends as a topological isomorphism

$$\tan\colon [-\frac{\pi}{2}, +\frac{\pi}{2}] \xrightarrow{\sim} \overline{\mathbb{R}}. \tag{1.18}$$

Exercise 1.5. Let X be a topological space and let $A \subset X$, $x \in \overline{A}$. Let $f, g\colon X \to \overline{\mathbb{R}}$ be two continuous maps and assume that $f|_A \leq g|_A$. Prove that $f(x) \leq g(x)$.

Exercise 1.6. Let X be a set with two elements $X = \{a, b\}$. Describe all possible topologies on X.

Exercise 1.7. Let $f\colon X \to Y$ be a continuous map and let $Z \subset X$. Prove the inclusion $f(\overline{Z}) \subset \overline{f(Z)}$. (Hint: otherwise, there exists $x \in \overline{Z}$ and an open neighborhood V of $f(x)$ such that $V \cap f(Z) = \emptyset$.)

Exercise 1.8. Let E be a finite dimensional real vector space and let γ be a closed convex cone in E with $0 \in E$. (Recall that a subset A of E is convex if $x, y \in A$ implies $tx + (1-t)y \in A$ for all $t \in [0, 1]$, that is, the segment $[x, y]$ is containbed in A.)

Let us say that a subset $U \subset E$ is γ-open if $U = U + \gamma$.
(i) Prove that the family of γ-open subsets defines a topology on E.
(i) Assume that γ contains no line. (One says that γ is a proper cone.) Prove that for $x \neq y$, there exists a γ-open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.
(iii) Prove that the γ-topology is Hausdorff if and only if $\gamma = \{0\}$.

Chapter 2

Metric spaces

2.1 Basic properties of metric spaces

We have defined the notions of a distance and a metric space in Section 1.2.

Let (X, d) be a metric space. Consider a sequence $(x_n)_{n \in \mathbb{N}}$ in X and let $a \in X$. As a particular case of (1.12), we have that $x_n \xrightarrow{n} a$ if and only if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(a, x_n) \leq \varepsilon$ for all $n \geq N$. Hence

Proposition 2.1.1. *Let $(x_n)_n$ be a sequence in the metric space (X, d) and let $a \in X$. Then $x_n \xrightarrow{n} a$ if and only if $d(a, x_n)$ goes to 0 when n goes to infinity.*

Definition 2.1.2. Consider a sequence $(x_n)_n$ in the metric space (X, d). One says that $a \in X$ is a limit point (valeur d'adhérence, in French) of the sequence if for any neighborhood U of a in X and any $N \in \mathbb{N}$, there exists $n \geq N$ such that $x_n \in U$.

Proposition 2.1.3. *Consider a sequence $(x_n)_n$ and set $F_p := \overline{\bigcup_{n \geq p} \{x_n\}}$. Then the set F of limit points of $(x_n)_n$ is the set $F = \bigcap_p F_p$.*

Proof. By the definition, $a \in X$ is a limit point if and only if a belongs to the closure of all F_p's. q.e.d.

Consider a sequence $(x_n)_{n \in \mathbb{N}}$ in X. Recall that an extracted sequence is a sequence $(x_{p(n)})_{n \in \mathbb{N}}$ where the map $n \to p(n)$ is strictly increasing (hence, $p(n) \geq n$).

Corollary 2.1.4. *Consider a sequence $(x_n)_n$ and let $a \in X$. Then a is a limit point of the sequence if and only if there exists an extracted sequence which converges to a.*

Proof. (i) If there exists an extracted sequence $(x_{p(n)})_n$ of the sequence $(x_n)_n$ which converges to a then, for each neighborhood U of a there exists $N \in \mathbb{N}$ such that $x_{p(n)} \in U$ for all $n \geq N$. Since $p(n) \geq n$, this implies that a is a limit point of the sequence.
(ii) Conversely, let a be a limit point. By Proposition 2.1.3, $a \in \bigcup_p F_p$. Define a sequence by choosing $x_n \in F_n$. then the sequence converges to a. q.e.d.

Proposition 2.1.5. *Let $A \subset X$. Then $x \in \overline{A}$ if and only if there exists a sequence $(x_n)_n$ in A with $x_n \xrightarrow{n} x$.*

In particular, a is a limit point of a sequence $(x_n)_n$ if and only if there exists an extracted sequence $(x_{n(p)})_p$ which goes to a when p goes to infinity.

Proof. (i) Assume that $x \in \overline{A}$. Then any ball $B(x, \frac{1}{n})$ intersects A. Hence there exists $x_n \in A$ with $d(x, x_n) < \frac{1}{n}$.
(ii) Assume $x_n \xrightarrow{n} x$ with $x_n \in A$ for all n. Let U be a neighborhood of x. There exists some $x_n \in U$. Hence, $U \cap A \neq \emptyset$. q.e.d.

Proposition 2.1.6. *Let X and Y be two metric spaces and let $f\colon X \to Y$ be a map. Then f is continuous at $a \in X$ if and only if, for any sequence $(x_n)_n$ in X with $x_n \xrightarrow{n} a$, the sequence $(f(x_n))_n$ converges to $f(a)$.*

Proof. First, note that f is continuous at a if and only if, for any $\varepsilon > 0$, $f^{-1}(B(f(a), \varepsilon))$ is a neighborhood of a, that is, contains a ball $B(a, \eta)$ for some $\eta > 0$. In other words, f is continuous at a if and only if for any $\varepsilon > 0$ there exists $\eta > 0$ such that

$$d(a, x) < \eta \Rightarrow d(f(a), f(x)) < \varepsilon. \tag{2.1}$$

(i) Assume f is continuous at a and let $(x_n)_n$ be a sequence in X with $x_n \xrightarrow{n} a$. Let $\varepsilon > 0$ and let us choose $\eta > 0$ such that (2.1) is satisfied. Let $N \in \mathbb{N}$ such that $d(a, x_n) < \eta$ for $n \geq N$. Then $d(f(a), f(x_n)) < \varepsilon$ which proves that $f(x_n) \xrightarrow{n} f(a)$.
(ii) Conversely, assume that for any sequence $(x_n)_n$ in X with $x_n \xrightarrow{n} a$, the sequence $f(x_n)$ converges to $f(a)$. Assume that f is not continuous at a. Then there exists $\varepsilon > 0$ such that for any $\eta > 0$ there exists x_η with $d(a, x_\eta) < \eta$ and $d(f(a), f(x_\eta)) \geq \varepsilon$. Choosing $\eta = \frac{1}{n}$ and setting $x'_n = x_{\frac{1}{n}}$, we get a sequence $(x'_n)_n$ with $x'_n \xrightarrow{n} a$ and $d(f(x'_n), f(a)) \geq \varepsilon$. This is a contradiction. q.e.d.

Proposition 2.1.7. *Let (X, d) be a metric space. Then the map $d\colon X \times X \to \mathbb{R}^+$ is continuous. (Here, $X \times X$ is endowed with the product topology.)*

Proof. Let $(x_0, y_0) \in X \times X$ and let $\varepsilon > 0$. Set for short $\delta := d(x_0, y_0)$. We have to check that $d^{-1}(]\delta - \varepsilon, \delta + \varepsilon[)$, the inverse image by d of the interval $]\delta - \varepsilon, \delta + \varepsilon[$, is a neighborhood of (x_0, y_0). Let us show that it contains the set $B(x_0, \frac{\varepsilon}{2}) \times B(y_0, \frac{\varepsilon}{2})$. Indeed,

$$\begin{aligned} &d(x, y) \leq d(x, x_0) + \delta + d(y_0, y), \\ &\delta \leq d(x_0, x) + d(x, y) + d(y, y_0). \end{aligned}$$

Hence, for $x \in B(x_0, \frac{\varepsilon}{2})$ and $y \in B(y_0, \frac{\varepsilon}{2})$, we have

$$\begin{aligned} &d(x, y) < \delta + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \\ &\delta < d(x, y) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \end{aligned}$$

Therefore $|d(x, y) - d(x_0, y_0)| < \varepsilon$. q.e.d.

Distances of sets and diameters

Definition 2.1.8. (i) Let $A \subset X$. The diameter of A, denoted $\delta(A)$, is given by $\delta(A) = \sup_{x,y \in A} d(x, y)$.

(ii) Let $A, B \subset X$. The distance from A to B, denoted $d(A, B)$, is given by $d(A, B) = \inf_{x \in A, y \in B} d(x, y)$.

Proposition 2.1.9. (i) *Let $A \subset X$. Then $\delta(A) = \delta(\overline{A})$.*

(ii) *Let $A, B \subset X$. Then $d(A, B) = d(\overline{A}, \overline{B})$.*

Proof. (i) Denote by $\operatorname{Im} A$ the image in $\mathbb{R}$ of the set $A \times A$ by the map d. Since this map is continuous, we have $\operatorname{Im} \overline{A} \subset \overline{\operatorname{Im} A}$ by the result of Exercise 1.7. Therefore,

$$\delta(A) = \sup(\operatorname{Im} A) = \sup(\overline{\operatorname{Im} A}) \geq \sup(\operatorname{Im} \overline{A}) = \delta(\overline{A}).$$

On the other hand, $\delta(\overline{A}) \geq \delta(A)$.
(ii) Denote by $\operatorname{Im}(A \times B)$ the image in $\mathbb{R}$ of the set $A \times B$ by the map d. Since d is continuous, we have $\operatorname{Im}(\overline{A} \times \overline{B}) \subset \overline{\operatorname{Im}(A \times B)}$ by the result of Exercise 1.7. Therefore

$$d(A, B) = \inf(\operatorname{Im}(A, B)) = \inf(\overline{\operatorname{Im}(A, B)}) \leq \inf(\operatorname{Im}(\overline{A}, \overline{B})) = d(\overline{A}, \overline{B}).$$

On the other hand, $d(\overline{A}, \overline{B}) \leq d(A, B)$.

q.e.d.

One shall be aware that $d(A, B) = 0$ does not imply that $A = B$ and even, does not imply that $A \cap B$ is non empty. For example, take $X = \mathbb{R}^2$ endowed with its Euclidian distance and consider the sets $A = \mathbb{R} \times \{0\}$, $B = \{(x, y); y = \exp(-x)\}$. Then $d(A, B) = 0$.

Uniformly continuous maps

Let (X, d_X) and (Y, d_Y) be two metric spaces and let $f\colon X \to Y$ be a map.

Definition 2.1.10. One says that f is uniformly continuous if

> for any $\varepsilon > 0$, there exist $\eta > 0$ such that for any $x, x' \in X$, $d_X(x, x') < \eta$ implies $d_Y(f(x), f(x')) < \varepsilon$.

The difference with the notion of being continuous is that in general, if f is only continuous at each $x \in X$, then the number η above depends on the point x. The notion of uniform continuity depends on the distances and has no meaning on general topological spaces.

Example 2.1.11. The map $f\colon \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$ is continuous but is not uniformly continuous for the usual distance on $\mathbb{R}$.

Definition 2.1.12. One says that $f\colon X \to Y$ is Lipschitzian (or simply, f is Lipschitz) if there exists a real number $\lambda \geq 0$ such that

$$d_Y(f(x), f(x')) \leq \lambda d_X(x, x') \text{ for all } x, x' \in X. \tag{2.2}$$

If moreover $\lambda < 1$, one says that f is contracting.

We shall study contracting maps in § 2.4.

Clearly, if f is Lipschitz, then it is uniformly continuous.

Uniform convergence

Recall that a generalized distance d is a map with values in $[0, +\infty]$ satisfying the axioms D1-D2-D3 of (1.4). The notion of a Cauchy sequence and of a complete space still makes sense for such a distance.

Let X be a set and let (Y, d) be a metric space. Denote by Y^X the set of maps from X to Y. One defines a generalized distance on Y^X by setting, for $f, g \in Y^X$:

$$\delta(f, g) = \sup_{x \in X} d(f(x), g(x)). \tag{2.3}$$

The generalized distance given by (2.3) is called the distance of uniform convergence.

Hence, we have now two different notions of convergence for functions from X to Y. Consider a sequence $(f_n)_n$ in Y^X an let $f \in Y^X$. Then

$$(2.4)\qquad \begin{cases} \text{(i) } (f_n)_n \text{ converges simply to } f \text{ if for any } x \in X,\\ d(f(x), f_n(x)) \xrightarrow{n} 0,\\ \text{(ii) } (f_n)_n \text{ converges uniformly to } f \text{ if } \delta(f, f_n) \xrightarrow{n} 0. \end{cases}$$

One translates (i) as follows:
for any $x \in X$ and any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $d(f(x), f_n(x)) \leq \varepsilon$.

One translates (ii) as follows:
for any $\varepsilon > 0$ ther exists $N \in \mathbb{N}$ such that for any $n \geq N$ and any $x \in X$, $d(f(x), f_n(x)) \leq \varepsilon$.

Of course, saying that $d(f(x), f_n(x)) \leq \varepsilon$ for any $x \in X$ is equivalent to saying that $\sup_{x\in X} d(f(x), f_n(x)) \leq \varepsilon$.

Proposition 2.1.13. *Let X be a topological space, $x_0 \in X$, and let (Y, d) be a metric space. Let $(f_n)_n$ be a sequence of functions from X to Y. Assume that the sequence converges uniformly to a function $f : X \to Y$ and each f_n is continuous at x_0. Then f is continuous at x_0.*

Proof. Let $\varepsilon > 0$. There exists $N > 0$ such that

$$(2.5)\qquad d(f(x), f_n(x)) \leq \frac{\varepsilon}{3} \text{ for all } n \geq N \text{ and all } x \in X.$$

Let us choose such an $n \geq N$ and let us choose a neighborhood V of x_0 such that

$$d(f_n(x), f_n(x_0)) \leq \frac{\varepsilon}{3} \text{ for all } x \in V.$$

Then

$$\begin{aligned} d(f(x), f(x_0)) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(x_0)) + d(f_n(x_0), f(x_0)) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore, $f(x) \to f(x_0)$ when $x \to x_0$. q.e.d.

2.2 Complete metric space

Let (X, d) be a metric space.

Definition 2.2.1. (i) A sequence $(x_n)_{n\in\mathbb{N}}$ in X is a Cauchy sequence if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n, m \geq N$ one has $d(x_n, x_m) \leq \varepsilon$.

(ii) A metric space is complete if any Cauchy sequence is convergent.

(iii) A Banach space over the field $\mathbb{R}$ or $\mathbb{C}$ is a vector space endowed with a norm and complete for the distance defined by this norm.

Example 2.2.2. (i) The space $\mathbb{R}$ endowed with its usual distance, is complete.
(ii) The space $\mathbb{R}^n$ endowed with one of the norms of Example 1.2.1 is a Banach space.
(iii) If two distances d_1 and d_2 are equivalent, then (X, d_1) is complete if and only if (X, d_2) is complete.
(iv) If $f\colon (X, d_X) \to (Y, d_Y)$ is a surjective isometry, then X is complete if and only if Y is complete.
(v) The space $]-\frac{\pi}{2}, +\frac{\pi}{2}[$ endowed with the distance induced by the one of $\mathbb{R}$, that is, $d(x, y) = |x - y|$ is not complete. However, the same space endowed with the distance $d_2(x, y) = |\tan(x) - \tan(y)|$ is complete since it is isomorphic to $\mathbb{R}$ as a metric space.

Proposition 2.2.3. (i) *A convergent sequence is a Cauchy sequence.*

(ii) *If a Cauchy sequence admits an extracted sequence which converges, then the Cauchy sequence converges.*

The proof is left as an exercise.

Proposition 2.2.4. (i) *Let X be a metric space and let Y be a subspace. Assume that Y is complete. Then Y is closed in X.*

(ii) *Let X be a complete metric space and let Y be a closed subspace. Then Y is complete.*

Proof. (i) Let us prove that $Y = \overline{Y}$. Let $x \in \overline{Y}$. For any n there exists $x_n \in B(x, \frac{1}{n}) \cap Y$. The sequence $(x_n)_n$ is clearly a Cauchy sequence. Hence it converges in Y, which implies $x \in Y$.
(ii) A Cauchy sequence in Y converges in X since X is complete, and the limit belongs to Y since Y is closed in X. Hence, it converges in Y. q.e.d.

Product

Let (X_i, d_i) $(i = 1, 2)$ be metric spaces. One may endow $X_1 \times X_2$ with various distances. For example, setting $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we set:

$$\delta_1(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2), \tag{2.6}$$

$$\delta_2(x, y) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}, \tag{2.7}$$

$$\delta_\infty(x, y) = \max(d_1(x_1, y_1), d_2(x_2, y_2)). \tag{2.8}$$

These distances are clearly equivalent and we shall simply denote by d one of them.

Proposition 2.2.5. *Assume that (X_1, d_1) and (X_2, d_2) are complete. Then $(X_1 \times X_2, d)$ is complete.*

Proof. Let us denote by $x = (x', x'')$ a point of $X_1 \times X_2$. If $(x_n)_n$ is a Cauchy sequence in $X_1 \times X_2$, then $(x'_n)_n$ is a Cauchy sequence in X_1 and $(x''_n)_n$ is a Cauchy sequence in X_2. Therefore there exists $x = (x', x'')$ such that $x'_n \xrightarrow{n} x'$ and $x''_n \to x''$. This implies that $x_n \xrightarrow{n} x$. q.e.d.

Prolongation of continuous maps

Proposition 2.2.6. *Let X and Y be two metric spaces. Let $A \subset X$ and let $f\colon A \to Y$ be a map. We assume that A is dense in X, f is uniformly continuous and Y is complete. Then there exists a unique continuous map $\widetilde{f}\colon X \to Y$ such that $\widetilde{f}|_A = f$. Moreover f is uniformly continuous.*

Proof. (i) The unicity follows from Corollary 1.6.6.
(ii) Let $x \in X$. We define $\widetilde{f}(x)$ as follows. Let $(x_n)_n$ be a sequence in A with $x_n \xrightarrow{n} x$. Then $(x_n)_n$ is a Cauchy sequence. Since f is uniformly continuous on A, the sequence $(f(x_n))_n$ is a Cauchy sequence in Y. Indeed, given $\varepsilon > 0$ there exists $\eta > 0$ such that

$$d_X(x', x'') \le \eta, x', x'' \in A \Rightarrow d(f(x'), f(x'')) \le \varepsilon. \tag{2.9}$$

Moreover, there exists $N \in \mathbb{N}$ such that

$$d_X(x_n, x_m) \le \eta \text{ for } n, m \ge N.$$

This implies

$$d_Y(f(x_n), f(x_m)) \le \varepsilon \text{ for } n, m \ge N.$$

Since Y is complete, there exists $y \in Y$ such that $f(x_n) \xrightarrow{n} y$.

(iii) Let us show that y depends only on x, not on the choice of the sequence $(x_n)_n$. Consider two sequences $(x'_n)_n$ and $(x''_n)_n$ which converge to x and let $f(x'_n) \xrightarrow{n} y'$, $f(x''_n) \xrightarrow{n} y''$. Define the sequence $(x_n)_n$ by setting

$$x_{2n} = x'_n, \quad x_{2n+1} = x''_n.$$

Then $x_n \xrightarrow{n} x$ and $f(x_n)$ has a limit y. Since $\{f(x'_n)\}_n$ and $(f(x''_n))_n$ are extracted sequences from $(f(x_n))_n$, they have the same limit. Hence, $y' = y = y''$.
(iv) Let us define $\widetilde{f}$ on X as follows. For $x \in X$, choose a sequence $(x_n)_n$ in A with $x_n \xrightarrow{n} x$ and define $f(x)$ as the limit of the sequence $(f(x_n))_n$. Then $\widetilde{f}|_A = f$. Indeed, if $x \in A$ we may choose $x_n = x$ for all n.
(v) It remains to show that $\widetilde{f}$ is uniformly continuous. Given $\varepsilon > 0$, let us choose $\eta > 0$ as in (2.9). Let $x', x'' \in X$ with $d_X(x', x'') \leq \frac{\eta}{2}$. Choose sequences $(x'_n)_n$ and $(x''_n)_n$ in A with $x'_n \xrightarrow{n} x'$ and $x''_n \xrightarrow{n} x''$. There exists N such that $n \geq N$ implies $d_X(x'_n, x') \leq \frac{\eta}{4}$ and $d_X(x''_n, x'') \leq \frac{\eta}{4}$. Therefore, $d_X(x'_n, x''_n) \leq \eta$ and this implies $d_Y(f(x'_n), f(x''_n)) \leq \varepsilon$. The sequence $((x'_n, x''_n))_n$ converges to (x', x'') in $X \times X$ and the function $d_Y \circ (f, f)$ is continuous on $X \times X$. Therefore $d_Y(f(x'), f(x'')) \leq \varepsilon$. q.e.d.

Remark 2.2.7. Let (X, d) be a metric space. Then one can prove that there exists $(\iota, \widehat{X}, \widehat{d})$ such that: $(\widehat{X}, \widehat{d})$ is a complete metric space and $\iota \colon (X, d) \to (\widehat{X}, \widehat{d})$ is an isometry with dense image. Moreover $(\widehat{X}, \widehat{d})$ is unique up to a bijective isometry.

For example, taking $X = \mathbb{Q}$ with the usual distance (that is, $d(x, y) = |x - y|$), one obtains $\widehat{X} = \mathbb{R}$ endowed with its usual distance.

Intervertion of limits

[1] Let X and Z be topological spaces and let (Y, d) be a metric space. Let $A \subset X$ and $B \subset Z$. Consider a map

$$f \colon A \times B \to Y.$$

Proposition 2.2.8. *Let $x_0 \in \overline{A}$ and $z_0 \in \overline{B}$. Assume*

(a) *Y is complete,*

(b) *for each $z \in B$, $f(x, z)$ has a limit denoted $f(x_0, z)$ when $x \to x_0$, $x \in A$ (that is, for any $z \in B$, any $\varepsilon > 0$, there exists a neighborhood U of x_0 in X such that $d(f(x, z), f(x_0, z)) \leq \varepsilon$ for any $x \in U \cap A$),*

[1] The subsection “Intervertion of limits” may be skipped by the students.

(c) *for each* $x \in A$, $f(x,z)$ *has a limit denoted* $f(x,z_0)$ *when* $z \to z_0$, $z \in B$ *and the limit* $f(x,z) \to f(x,z_0)$ *is uniform with respect to* $x \in A$ *(that is, for any* $\varepsilon > 0$*, there exists a neighborhood* V *of* z_0 *in* Z *such that* $d(f(x,z), f(x,z_0)) \le \varepsilon$ *for any* $x \in A$, $z \in V \cap B$*).*

Then $f(x,z_0)$ *has a limit* $f(x_0,z_0) \in Y$ *when* $x \to x_0$ *and* $f(x_0,z)$ *converges to* $f(x_0,z_0)$ *when* $z \to z_0$.

Proof. Let $\varepsilon > 0$. Using hypothesis (c), we find a neighborhood V of z_0 in Z such that

$$d(f(x,z), f(x,z_0)) \le \frac{\varepsilon}{3} \text{ for any } x \in A,\ z \in V \cap B. \tag{2.10}$$

Let us choose $z_1 \in V \cap B$. Using hypothesis (b), there exists a neighborhood U of x_0 in X such that

$$d(f(x,z_1), f(x',z_1)) \le \frac{\varepsilon}{3} \text{ for any } x, x' \in U \cap A. \tag{2.11}$$

It follows that for $x, x' \in U \cap A$

$$\begin{aligned} d(f(x,z_0), f(x',z_0)) &\le d(f(x,z_0), f(x,z_1)) + d(f(x,z_1), f(x',z_1)) \\ &\qquad + d(f(x',z_1), f(x',z_0)) \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Choosing $\varepsilon = \frac{1}{n}$, we choose $x_{2n} = x$ and $x_{2n+1} = x'$ in $U \cap A$. Then the sequence $(f(x_n,z_0))_n$ is a Cauchy sequence in Y, hence converges to some limit $l \in Y$. This limit does not depend of the choice of the sequence $(x_n)_n$ (we skip this point) and we denote it by $f(x_0,z_0)$. We have thus proved that $f(x,z_0)$ has a limit $f(x_0,z_0) \in Y$ when $x \to x_0$. Using (2.10) and passing to the limit when $x \to x_0$, we get

$$d(f(x_0,z), f(x_0,z_0)) \le \frac{\varepsilon}{3} \text{ for } z \in V \cap B.$$

Therefore, $f(x_0,z)$ converges to $f(x_0,z_0)$ when $z \to z_0$. q.e.d.

One should be aware that, given three topological spaces X, Y, Z, a function $f\colon X \times Z \to Y$ may be separately continuous without being continuous. Separately continuous means that for any $x \in X$, $z \mapsto f(x,z)$ is continuous and for any $z \in Z$, $x \mapsto f(x,z)$ is continuous.

Example 2.2.9. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1,x_2) = \dfrac{x_1x_2}{x_1^2 + x_2^2}$ for $(x_1,x_2) \neq (0,0)$ and $f(0,0) = 0$. Then for any x_1 fixed, $f(x_1,\cdot)$ is continuous and for any x_2 fixed, $f(\cdot,x_2)$ is continuous. However, $f(\cdot,\cdot)$ is not continuous since $f(x,x) = \frac{1}{2}$ for $x \neq 0$ and $f(0,0) = 0$.

In Exercise 2.5, we give a sufficient condition in order that a separately continuous function be continuous.

2.3 The Baire Theorem

Proposition 2.3.1. *Let (X, d) be a complete metric space. Let $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$ be a decreasing sequence of non empty closed subspaces. Assume that the diameter $\delta(F_n)$ converges to 0. Then there exists $x \in X$ such that $\bigcap_n F_n = \{x\}$.*

Proof. Set $F = \bigcap_n F_n$. Then $\delta(F) = 0$ and it is enough to check that $F \neq \emptyset$. For each n, choose $a_n \in F_n$. For $p \geq n$, $a_p \in F_n$. Hence, $d(a_p, a_n) \leq \delta(F_n)$. Since $\delta(F_n) \xrightarrow{n} 0$, the sequence $(a_n)_n$ is a Cauchy sequence. Denote by a its limit. Since the sequence $(a_p)_{p \geq n}$ is contained in F_n, its limit a belongs to F_n. Hence $a \in F$. q.e.d.

Remark that the condition $\delta(F_n) \xrightarrow{n} 0$ is necessary. For example, the sequence of closed subsets $([n, +\infty[)_n$ in $\mathbb{R}$ has an empty intersection.

Theorem 2.3.2. (The Baire theorem.) *Let X be a complete metric space and let $(U_n)_n$ be a sequence of subsets of X. Assume that all U_n are open and dense in X. Then the intersection $\bigcap_n U_n$ is dense in X.*

Proof. Set $A = \bigcap_n U_n$ and let $B(a, \varepsilon)$ be a non empty open ball. One has to show that $A \cap B(a, \varepsilon) \neq \emptyset$. Since $B(a, \varepsilon) \cap U_1$ is open and non empty, it contains a closed ball $\overline{B}(a_1, \varepsilon_1)$ for some $a_1 \in U_1$ and $\varepsilon_1 > 0$. Since $B(a_1, \varepsilon_1) \cap U_2$ is open and non empty, it contains a closed ball $\overline{B}(a_2, \varepsilon_2)$ for some $a_2 \in U_2$ and $\varepsilon_2 > 0$. By induction, we find a sequence $(a_n)_n$ with $a_n \in U_n$ and a sequence $(\varepsilon_n)_n$ of positive numbers such that

$$B(a_n, \varepsilon_n) \cap U_{n+1} \supset \overline{B}(a_{n+1}, \varepsilon_{n+1}).$$

We may choose the sequence $(\varepsilon_n)_n$ decreasing to 0. Then consider the sequence of subsets $(\overline{B}(a_n, \varepsilon_n))_n$: this is a decreasing sequence of closed subsets whose diameters tends to 0. Applying Proposition 2.3.1, we find $x \in \bigcap_n \overline{B}(a_n, \varepsilon_n) \subset B(a, \varepsilon) \cap A$. q.e.d.

There is an equivalent formulation of the Baire's theorem using closed sets instead of open sets:

Theorem 2.3.3. *Let X be a complete metric space and let $(Z_n)_n$ be a sequence of subsets of X. Assume that all Z_n's are closed and without interior points (that is, $\mathrm{Int(Z_n)} = \emptyset$). Then $\bigcup_n Z_n$ has no interior points.*

Proof. Apply Theorem 2.3.2 with $U_n = X \setminus Z_n$. q.e.d.

Corollary 2.3.4. *Let E be a Banach space and let $(E_n)_{n\in\mathbb{N}}$ be an increasing sequence of closed vector subspaces. Assume that $E = \bigcup_n E_n$. Then, there exists some $n \in \mathbb{N}$ such that $E = E_n$. In other words, the sequence is stationnary.*

Proof. Let F be a vector subspace of E and assume that F admits an interior point. This means that there exists a non empty open subset U of E with $U \subset F$. Hence there exist some $\varepsilon > 0$ and $a \in F$ with $B(a;\varepsilon)$ contained in F. Since F is invariant by translation, $B(0;\varepsilon)$ is contained in F. Since F is invariant by scalar multiplication, this implies $F = E$.

Now assume that the sequence $(E_n)_{n-in\mathbb{N}}$ is not stationnary. This implies that the E_n's have no interior points. Applying Theorem 2.3.3, we get that $\bigcup_n E_n$ has no interior points and hence is not egual to E. q.e.d.

2.4 Contracting maps

Let us repeat Definition 2.1.12 in a particular case.

Definition 2.4.1. Let X be a metric space and let $f\colon X \to X$ be a map. One says that f is contracting if there exists $0 \leq \lambda < 1$ such that $d(f(x), f(y)) \leq \lambda d(x,y)$ for all $x, y \in X$.

Theorem 2.4.2. *Let X be a complete metric space and let $f\colon X \to X$ be a contracting map. Then there exists $a \in X$ such that $f(a) = a$ and such an element a is unique. Moreover, for any $y_0 \in X$, the sequence defined by induction by setting $x_0 = y_0$ and $x_n = f(x_{n-1})$ converges to a.*

Proof. (i) Unicity. Assume that $f(a) = a$ and $f(b) = b$. Then $d(a,b) = d(f(a), f(b)) \leq \lambda d(a,b)$, which implies $d(a,b) = 0$ since $\lambda < 1$.
(ii) Consider a sequence $(x_n)_n$ as in the statement, that is, $x_n = f(x_{n-1})$. We have

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq \lambda d(x_{n-1}, x_n)$$

Therefore,

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \leq \lambda^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq \lambda^n d(x_0, x_1).$$

Hence,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n+p-1}) d(x_1, x_0) \\ &\leq \frac{\lambda^n}{1-\lambda} d(x_1, x_0), \end{aligned}$$

and this quantity converges to 0 when $n \to \infty$. We have thus proved that the above sequence $(x_n)_n$ is a Cauchy sequence. Denote by a its limit. Since $x_n \xrightarrow{n} a$ and f is continuous, $f(x_n) \xrightarrow{n} f(a)$. The sequence $(f(x_n))_n$ is nothing but the sequence $(x_{n+1})_n$. Since the sequences $(x_n)_n$ and $(x_{n+1})_n$ have the same limit, we have $f(a) = a$. q.e.d.

Let us give an important application to differential equations. In the sequel, $||\cdot||$ is one of the norms $||\cdot||_p$ ($p = 1, 2, \infty$) on $\mathbb{R}^n$.

Theorem 2.4.3. (A particular case of the Cauchy-Lipschitz theorem.) *Let $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and assume that there exists $C \geq 0$ such that*

$$||f(t,x) - f(t,y)|| \leq C||x-y|| \text{ for all } (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n. \tag{2.12}$$

Then there exists $\alpha > 0$ such that, for all $x_0 \in \mathbb{R}^n$, there exists a unique continuously derivable function $x\colon [-\alpha, \alpha] \to \mathbb{R}^n$ which is a solution of the system:

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(0) = x_0. \end{cases} \tag{2.13}$$

In the classical Cauchy-Lipschitz theorem, one only assumes that f is defined in an open subset of $\mathbb{R} \times \mathbb{R}^n$ and it makes the proof more delicate.

Proof. Let I be a closed bounded interval of $\mathbb{R}$. It follows from Proposition 2.1.13 that the space $C^0(I;\mathbb{R}^n)$ endowed with the distance of uniform convergence

$$d(x,y) = \sup_{t \in I} ||(x(t) - y(t)||$$

is complete. We shall prove this result in a more general setting in Proposition 3.2.4.

Set for short $X_\alpha := C^0([-\alpha,\alpha];\mathbb{R}^n)$. Proving that (2.13) has a unique solution is equivalent to proving that the equation

$$x(t) = x_0 + \int_0^t f(s, x(s))ds \tag{2.14}$$

has a unique solution $x \in X_\alpha$.

Consider the map

$$F\colon X_\alpha \to X_\alpha, \quad x \mapsto x_0 + \int_0^t f(s,x(s))ds.$$

For $x, y \in X_\alpha$ and $t \in [-\alpha, \alpha]$, we have

$$\begin{aligned}
||F(x)(t) - F(y)(t)|| &\leq ||\int_0^t f(s, x(s))ds - \int_0^t f(s, y(s))ds|| \\
&\leq C\int_0^t ||x(s) - y(s)||ds \\
&\leq C\alpha d(x, y).
\end{aligned}$$

Hence,

$$d(F(x), F(y)) \leq C\alpha d(x, y).$$

Choosing α such that $C\alpha < 1$, we may apply Theorem 2.4.2 to the map F and get the result. q.e.d.

Exercises to Chapter 2

Exercise 2.1. Let (X, d) be a metric space and let Z_0 and Z_1 be two closed subsets with $Z_0 \cap Z_1 = \emptyset$. Using the distance d, construct a continuous function $f\colon X \to \mathbb{R}$ such that $f|_{Z_0} \equiv 0$ and $f|_{Z_1} \equiv 1$.

Exercise 2.2. Let (X, d) be a metric space, let $F \subset X$ be a closed subspace and let $a \in X$. Denote by δ the distance from $\{a\}$ to F. Show that $\delta = 0$ if and only if $a \in F$.

Exercise 2.3. Let (X, d) be a metric space and assume d is bounded (that is, X has a finite diameter). Denote by $\mathcal{F}$ the family of closed subsets of X. For $A, B \in \mathcal{F}$, set

$$\rho(A, B) = \sup_{x\in A} d(x, B), \quad \lambda(A, B) = \sup(\rho(A, B), \rho(B, A)).$$

(i) Prove that λ is a distance on $\mathcal{F}$.
(ii) Assume that X is complete and let $(F_n)_n$ be a Cauchy sequence in $\mathcal{F}$. For $n \in \mathbb{N}$, set $Y_n = \bigcup_{p\in\mathbb{N}} F_{n+p}$ and $Y = \bigcap_n \overline{Y}_n$. Prove that $F_n \xrightarrow{n} Y$ when $n \to \infty$ and deduce that $\mathcal{F}$ is complete.

Exercise 2.4. Let (X, d) be a complete metric space.
(i) For (x, t) and (x', t') in $X \times \mathbb{R}$, one sets $d_1((x, t), (x', t')) = d(x, x') + |t - t'|$. Prove that d_1 is a distance on $X \times \mathbb{R}$ and that $X \times \mathbb{R}$ is complete for d_1.
(ii) Let U be an open subset of X and let $Z = X \setminus U$. Consider the set A and the map h

$$\begin{aligned}
&A = \{(x, t) \in X \times \mathbb{R}; 1 - t \cdot d(x, Z) = 0\}, \\
&h\colon U \to A, \quad x \mapsto (x, \frac{1}{d(x, Z)}).
\end{aligned}$$

Prove that the map $X \times \mathbb{R} \to \mathbb{R}$, $(x,t) \mapsto 1 - t \cdot d(x,Z)$ is continuous and deduce that A is closed in $X \times \mathbb{R}$.
(iii) Prove that $h\colon U \to A$ is well defined and is a topological isomorphism.
(iii) One endows U with the distance $d_2(x,x') = d_1(h(x),h(x'))$. Prove that (U,d_2) is complete.

Exercise 2.5. Let X and Z be topological spaces and let (Y,d) be a complete metric space. Let $A \subset X$, $B \subset Z$, $x_0 \in \overline{A}$ and $z_0 \in \overline{B}$. Consider a map $f\colon A \times B \to Y$ and assume that
(i) for each $z \in B$, $f(x,z)$ has a limit denoted $f(x_0,z)$ when $x \to x_0$, $x \in A$ and the limit $f(x,z) \to f(x_0,z)$ is uniform with respect to $z \in B$
(ii) for each $x \in A$, $f(x,z)$ has a limit denoted $f(x,z_0)$ when $z \to z_0$, $z \in B$ and the limit $f(x,z) \to f(x,z_0)$ is uniform with respect to $x \in A$.

Prove that $f(x,z)$ has a limit when $(x,z) \to (x_0,z_0)$ with $(x,z) \in A \times B$.

Chapter 3

Compact spaces

3.1 Basic properties of compact spaces

Let X be a set. Recall that a family $\{A_i\}_{i\in I}$ of subsets of X is a covering of X if $X = \bigcup_{i\in I} A_i$. Let now X be a topological space. Consider the two conditions below.

$$(3.1)\quad \begin{cases} \text{K1: For any open covering } X = \bigcup_{i\in I} U_i, \text{ there exists a finite subset} \\ J \subset I \text{ such that } X = \bigcup_{j\in J} U_j. \\ \text{K2: For any family } \{F_i\}_{i\in I} \text{ of closed subsets with } \bigcap_{i\in I} F_i = \emptyset, \\ \text{there exists a finite subset } J \subset I \text{ such that } \bigcap_{j\in J} F_j = \emptyset. \end{cases}$$

By choosing $F_i = X \setminus U_i$, one sees that the conditions K1 and K2 are equivalent.

Definition 3.1.1. A topological space X is compact if it is Hausdorff and satisfies one of the equivalent conditions K1 or K2 above.

If Y is a subset of a topological space X, one says that Y is compact if it is so for the induced topology.

Proposition 3.1.2. *Let X be a topological space and let $Y \subset X$. Assume Y is Hausdorff (for the induced topology). Then Y is compact if and only if, for any family $\{U_i\}_{i\in I}$ of open subsets of X such that $Y \subset \bigcup_{i\in I} U_i$, there exists a finite set $J \subset I$ with $Y \subset \bigcup_{j\in J} U_j$.*

The easy proof is left to the reader.

Proposition 3.1.3. *Let X be a Hausdorff space and let $Y \subset X$.*

(i) *If X is compact and Y is closed, then Y is compact.*

(ii) *If Y is compact, then Y is closed in X.*

Proof. (i) Let $\{F_i\}_{i\in I}$ be a family of closed subsets of Y with empty intersection. Since the F_i's are closed in X, there exists $J \subset I$ with J finite such that the family $\{F_j\}_{j\in J}$ has an empty intersection.
(ii) Let us show that $X \setminus Y$ is open. Let us choose $x \in X \setminus Y$. For each $y \in Y$, there exists an open neighborhood V_y of y in X and an open neighborhood U_x^y of x in X such that $V_y \cap U_x^y = \emptyset$. The family $\{V_y\}_{y\in Y}$ is an open covering of Y. Hence, we may extract a finite covering:

$$Y \subset \bigcup_{j=1}^{n} V_{y_j}.$$

Set $U_x = \bigcap_{i=1}^{n} U_x^{y_i}$. Then $U_x \cap Y = \emptyset$ and U_x is an open neighborhood of x. q.e.d.

Proposition 3.1.4. *Let X be a Hausdorff space and let A, B two compact subsets of X with $A \cap B = \emptyset$. Then there exist two open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

Proof. As in the proof of Proposition 3.1.3 (ii), for each $x \in A$, we construct an open neighborhood U_x of x and an open neighborhood V_x of B such that $U_x \cap V_x = \emptyset$. The family $\{U_x\}_{x\in A}$ is an open covering of A from which we extract a finite covering $\{U_{x_j}\}_{j=1,\dots,m}$. It remains to set $U = \bigcup_{j=1}^{m} U_{x_j}$ and $V = \bigcap_{j=1}^{m} V_{x_j}$. q.e.d.

Corollary 3.1.5. *Let X be a compact space and let $x \in X$. Then x admits a fundamental system of closed neighborhoods.*

Proof. Let V be an open neighborhood of x. Then $X \setminus V$ is compact. Hence, there exists open neighborhoods U of x and W of $X \setminus V$ with $U \cap W = \emptyset$. Therefore, $\overline{U} \cap W = \emptyset$, that is, $\overline{U} \subset V$. The family $\{\overline{U}\}$ so constructed is a fundamental system of closed neighborhoods of x. q.e.d.

Proposition 3.1.6. *Let X and Y be two compact spaces. Then $X \times Y$ is compact.*

Proof. We have already proved in Proposition 1.6.4 that $X \times Y$ is Hausdorff. Let $\{W_i\}_{i\in I}$ be an open covering of $X \times Y$. For each $z = (x, y) \in X \times Y$, let us choose an elementary open set $U_z \times V_z$ contained in one of the W_i's. Let $x_0 \in X$. The space $\{x_0\} \times Y$ is topologically isomorphic to Y, thus is compact. Consider the open covering $\{U_{x_0,y} \times V_{x_0,y}\}_{y\in Y}$ of $\{x_0\} \times Y$. We may extract a finite covering $\{U_{x_0,y_j} \times V_{x_0,y_j}\}_{j\in J(x_0)}$. Set $U_{x_0} = \bigcap_{j\in J(x_0)} U_{x_0,y_j}$. Then

$$\{x_0\} \times Y \subset \bigcup_{j\in J(x_0)} U_{x_0} \times V_{x_0,y_j}.$$

The family $\{U_{x_0}\}_{x_0\in X}$ is an open covering of X from which we extract a finite covering $\{U_{x_k}\}_{k\in K}$. Then

$$X \times Y = \bigcup_{k\in K} \bigcup_{j\in J(x_k)} (U_{x_k} \times V_{x_k,y_j}).$$

q.e.d.

Corollary 3.1.7. *A finite product of compact spaces is compact.*

Proposition 3.1.8. *Let X and Y be topological spaces, with X compact and Y Hausdorff. Let $f\colon X \to Y$ be a continuous map. Then $f(X) \subset Y$ is compact.*

Proof. Let $\{V_i\}_{i\in I}$ be an open covering of $f(X)$. Since $\{f^{-1}(V_i)\}_{i\in I}$ is an open covering of X, there exists a finite subset $J \subset I$ such that $\{f^{-1}(V_j)\}_{j\in J}$ is an open covering. Then $\{V_j\}_{j\in J}$ be an open covering of $f(X)$. q.e.d.

Proposition 3.1.9. *Let $f\colon X \to Y$ be a continuous map. Assume that X is compact, Y is Hausdorff and f is bijective. Then f^{-1} is continuous, that is, f is a topological isomorphism.*

Proof. It is enough to prove that f is closed. Let $A \subset X$ be a closed subset. Then A is compact. Therefore, $f(A)$ is compact by Proposition 3.1.8, hence closed. q.e.d.

Example 3.1.10. Recall (see Example 1.7.2) that $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Denote by $f\colon [0, 2\pi] \to \mathbb{T}$ the quotient map. Since $\mathbb{T}$ is Hausdorff and $\mathbb{T} = f([0, 2\pi])$, this space is compact.

On the other hand, we have seen that the topological circle is isomorphic to the Euclidian circle. Hence, it is compact. It follows that the map

$$f\colon \mathbb{T} \to \mathbb{S}^1, \quad \theta \mapsto \exp(i\theta)$$

is a topological isomorphism.

Example 3.1.11. Denote by $\mathbb{S}^n$ the Euclidian n-sphere (see Example 1.7.3), that is, the unit sphere of the space $\mathbb{R}^{n+1}$ endowed with its Euclidian norm. Being closed and bounded in $\mathbb{R}^{n+1}$, it is a compact space. Being topologically isomorphic to the topological sphere $(\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^+$, this last space is compact.

Remark 3.1.12. A topological space which is both compact and discrete is finite. Indeed, the points are open, hence the family of points is an open covering from which one can extract a finite covering.

Remark 3.1.13. The space $\mathbb{R}$ is not compact. However, the space $\overline{\mathbb{R}}$ introduced in Exercise 1.4 is compact since it is isomorphic to $[-\frac{\pi}{2}, +\frac{\pi}{2}]$.

3.2 Compact spaces and real numbers

Theorem 3.2.1. (The Borel-Lebesgue Theorem.) *Let $a \leq b$ be two real numbers. Then the closed interval $[a, b]$ is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open covering of $[a, b]$. Set

$$A = \{x \in [a, b];\ [a, x] \text{ is contained in a finite subcovering}\}.$$

Then $A \neq \emptyset$ since $a \in A$. Set $m = \sup A$, $a \leq m \leq b$. There exists $i_0 \in I$ such that $m \in U_{i_0}$. If $m \notin A$, then $]m - \varepsilon, m] \cap A \neq \emptyset$ for all $\varepsilon > 0$. Let $\varepsilon > 0$ such that $]m - \varepsilon, m] \subset U_{i_0}$ and let $x \in A\cap]m - \varepsilon, m]$. There exists a finite subset $J \subset I$ such that $[a, x] \subset \bigcup_{j\in J} U_j$. Therefore,

$$[a, m] \subset \bigcup_{j\in J} U_j \cup U_{i_0}.$$

This shows that $m \in A$. If $m < b$, there exists $\varepsilon > 0$ such that $[m, m + \varepsilon] \subset U_{i_0}$, which contradicts $m = \sup A$. Hence, $m = b$. q.e.d.

Proposition 3.2.2. *Let $A \subset \mathbb{R}^n$. Then A is compact if and only if A is closed and bounded.*

Proof. (i) Assume A is compact. We already know that it implies that A is closed. If A were not bounded, A would not be contained in some open ball $B(a, R)$ $(R \in \mathbb{R})$. Hence it would not be possible to extract a finite covering from the open covering $A = \bigcup_{R>0}(B(0, R) \cap A)$.
(ii) Assume that A is closed and bounded. Then A is closed in some set $[-R, +R]^n$ and this last set is compact by the Borel-Lebesgue theorem and Corollary 3.1.7. Therefore A is compact by Proposition 3.1.3 (i). q.e.d.

Proposition 3.2.3. *Let X be a compact space and let $f\colon X \to \mathbb{R}$ be a continuous map. Then f is bounded and there exist x_0 and x_1 in X such that* $\inf f = f(x_0)$ *and* $\sup f = f(x_1)$.

Proof. This follows from Proposition 3.1.8. q.e.d.

The space $C^0(X;Y)$ for X compact

For two topological spaces X and Y, let us denote by $C^0(X;Y)$ the subspace of Y^X consisting of continuous maps. Now assume that (Y,d) is a metric space and let us endow $C^0(X;Y)$ with the quasi-distance of uniform convergence defined in (2.3).

$$\delta(f,g) = \sup_{x\in X} d(f(x),g(x)).$$

If X is compact, this quasi-distance is a true distance. Indeed, consider two continuous functions $f,g\colon X\to Y$. The function d_Y on $Y\times Y$ being continuous, the function

$$\Phi\colon X\to\mathbb{R},\quad x\mapsto d_Y(f(x),g(x))$$

is continuous. Hence, $\Phi(X)\subset\mathbb{R}$ is compact and in particular bounded. by Proposition 3.1.8.

Proposition 3.2.4. *Let X be a compact topological space and let (Y,d_Y) be a complete metric space. Then the space $C^0(X;Y)$ endowed with the distance of uniform convergence is complete.*

Proof. Consider a Cauchy sequence $\{f_n\}_n$ in $C^0(X;Y)$. For any $\varepsilon>0$, there exists $N>0$ such that for any $n,p\geq N$, one has

$$\sup_{x\in X} d_Y(f_n(x),f_p(x))\leq\varepsilon. \tag{3.2}$$

For each $x\in X$, the sequence $(f_n(x))_n$ is thus a Cauchy sequence in Y, hence converges. Denote by $f(x)$ this limit. This defines a function $f\colon X\to Y$. Making p goes to ∞ in (3.2), we get

$$\sup_{x\in X} d_Y(f_n(x),f(x))\leq\varepsilon.$$

Hence, the sequence $(f_n)_n$ converges uniformly to f. Then f is continuous by Proposition 2.1.13. q.e.d.

Now let $(E,||\cdot||)$ be a normed space. We endow the space $C^0(X;E)$ of the norm of uniform convergence:

$$||f|| = \sup_{x\in X}||f(x)||. \tag{3.3}$$

Corollary 3.2.5. *Let X be a compact topological space and let $(E,||\cdot||)$ be a Banach space. Then the space $C^0(X;E)$ endowed with the norm of uniform convergence is a Banach space.*

This result will be particularly important when considering the case $E=\mathbb{R}$ or $E=\mathbb{C}$. In this case, $C^0(X;E)$ is a Banach algebra (see Definition 4.2.3).

3.3 Compact metric spaces

In this section, (X, d) is a metric space.

Proposition 3.3.1. *The conditions below are equivalent:*

(i) *the space X is compact,*

(ii) *any sequence $(a_n)_n$ in X admits at least one limit point,*

(iii) *for any sequence $(a_n)_n$ in X, there exists an extracted sequence which converges.*

Proof. (i) $\Rightarrow$ (ii). Let $(a_n)_n$ be a sequence in X. Set

$$F_p = \overline{(a_n)_{n \geq p}}.$$

The set of limits points of the sequence $(a_n)_n$ is the set $\bigcap_p F_p$ and this set is non empty by the hypothesis that X is compact.
(ii) $\Rightarrow$ (i). Consider an open covering $\{U_i\}_{i \in I}$ of X. First, we shall show that

(3.4) There exists $\alpha > 0$ such that for any $x \in X$ there exists $i \in I$ with $B(x, \alpha) \subset U_i$.

If (3.4) were false, there would exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that the balls $B(x_n, \frac{1}{n})$ are not contained in any of the U_i's. Let x be a limit point of the sequence $(x_n)_n$. Then $x \in U_{i_0}$ for some $i_0 \in I$, hence there exists some integer N such that $B(x, \frac{1}{N}) \subset U_{i_0}$. On the other hand, there exists $n \geq 2N$ such that $x_n \in B(x, \frac{1}{2N})$. Then

$$B(x_n, \frac{1}{n}) \subset B(x_n, \frac{1}{2N}) \subset B(x, \frac{1}{N}) \subset U_{i_0}.$$

This is a contradiction and this proves (3.4).

Consider now the covering $X = \bigcup_{x \in X} B(x, \alpha)$. Since any $B(x, \alpha)$ is contained in some U_i, it is enough to prove that one can extract a finite covering of this last covering. Let us argue by contradiction and assume there is no extracted finite covering. Then one constructs by induction a sequence $(x_n)_n$ such that

$$x_n \notin B(x_1, \alpha) \cup B(x_2, \alpha) \cup \cdots \cup B(x_{n-1}, \alpha).$$

Clearly, such a sequence has no limit points.
(ii) $\Leftrightarrow$ (iii) by Corollary 2.1.4. q.e.d.

Proposition 3.3.2. *The conditions below are equivalent.*

(i) *The space X is compact,*

(ii) *the space X is complete and moreover, for any $\varepsilon > 0$ there exists a finite covering by open balls of radius ε.*

Proof. (i) $\Rightarrow$ (ii). Let $(x_n)_n$ be a Cauchy sequence. There exists an extracted sequence which converges, and this implies that the sequence itself converges.
(ii) $\Rightarrow$ (i). Let $(x_n)_n$ be a sequence in X. By Proposition 3.3.1, it is enough to show that the sequence admits at least one limit point. Consider a finite covering of X by open balls of radius $\frac{1}{2}$. One of these balls contains infinite many x_n. Hence, we find an extracted sequence in which the distance of two elements is ≤ 1. We may apply the same argument to this sequence, after replacing the open balls of radius $\frac{1}{2}$ by open balls of radius $\frac{1}{4}$, $\frac{1}{6}$, etc.

Therefore, we find sequences, each one being extracted from the previous one:

$$
\begin{aligned}
&(x_1^1, x_2^1, \dots, x_n^1, \dots) \text{ with } d(x_n^1, x_m^1) \leq 1 \text{ for all } n, m,\\
&(x_1^2, x_2^2, \dots, x_n^2, \dots) \text{ with } d(x_n^2, x_m^2) \leq \frac{1}{2} \text{ for all } n, m,\\
&\cdots\cdots\\
&(x_1^p, x_2^p, \dots, x_n^p, \dots) \text{ with } d(x_n^p, x_m^p) \leq \frac{1}{p} \text{ for all } n, m,\\
&\cdots\cdots .
\end{aligned}
$$

Consider the diagonal sequence

$$(x_1^1, x_2^2, \dots, x_n^n, \dots).$$

Since $d(x_n^n, x_m^m) \leq \frac{1}{p}$ for any $n, m \geq p$, this diagonal sequence is a Cauchy sequence. Its limit x will be a limit point of the initial sequence $(x_n)_n$. q.e.d.

Proposition 3.3.3. *Let X and Y be two metric spaces, let $f\colon X \to Y$ be a continuous map and assume that X is compact. Then f is uniformly continuous.*

Proof. Since f is continuous at each $x \in X$, for any $\varepsilon > 0$ and any $x \in X$, there exists $\eta(x)$ such that $d(x', x) \leq \eta(x)$ implies $d(f(x'), f(x)) \leq \frac{\varepsilon}{2}$.

Consider the covering $X = \bigcup_{x \in X} B(x, \frac{\eta(x)}{2})$. One can extract a finite covering:

$$X = B(x_1, \frac{\eta(x_1)}{2}) \cup \cdots \cup B(x_n, \frac{\eta(x_n)}{2}).$$

We set $\eta = \inf_i \frac{\eta(x_i)}{2}$. Let $x', x'' \in X$ with $d(x', x'') \leq \eta$. There exists $i \in \{1, \dots, n\}$ such that $x' \in B(x_i, \frac{\eta(x_i)}{2})$. Since $\eta \leq \frac{\eta(x_i)}{2}$, $x'' \in B(x_i, \frac{\eta(x_i)}{2})$. Therefore,

$$d(f(x_i), f(x')) \leq \frac{\varepsilon}{2}, \quad d(f(x_i), f(x'')) \leq \frac{\varepsilon}{2},$$

which implies $d(f(x'), f(x'')) \leq \varepsilon$. q.e.d.

One shall be aware not to confuse the notion of uniform continuity (for a function) and the notion of uniform convergence (for a sequence of functions).

3.4 Locally compact spaces

Definition 3.4.1. A topological space is locally compact if it is Hausdorff and any point admits a compact neighborhood.

- In a locally compact space, any point admits a fundamental system of compact neighborhoods.
- A compact space is clearly locally compact.
- The space $\mathbb{R}^n$ is locally compact. Indeed, closed balls with finite radius are compact.
- An open subset U of a locally compact space X is locally compact. Indeed, let $x \in U$. Since x admits a fundamental system of compact neighborhoods, there exists a compact neighborhood K of x with $K \subset U$ and K is compact in U.
- A closed subset Z of a locally compact space X is locally compact. Indeed, let $x \in Z$ and let K be a compact neighborhood of x in X. Then $K \cap Z$ is a compact neighborhood of x in Z.

If X is compact and $x \in X$, then $X \setminus \{x\}$ is locally compact and one can recover the topology of X from that of $X \setminus \{x\}$. Indeed, U is open in X if and only if $U = X \setminus K$ for a compact subset K of X. Hence, the open subsets of X are either the open sets of $X \setminus \{x\}$ or the sets $(X \setminus \{x\} \setminus K) \cup \{x\}$, with K compact in $X \setminus \{x\}$. We shall show that any locally compact space is isomorphic to a space $Y \setminus \{x\}$, with Y compact.

Let X be a locally compact space and set $Y = X \sqcup \{\omega\}$ where $\{\omega\}$ is a set with one point. Let us call "open" the subsets of Y defined as follows:

$$U \text{ is open} \Leftrightarrow \text{(i) if } \omega \notin U, \text{ then } U \text{ is open in } X, \text{ (ii) if } \omega \in U, \text{ then } U \setminus \{\omega\} = X \setminus K,\ K \text{ compact in } X. \tag{3.5}$$

Proposition 3.4.2. (i) *The family of open sets given in* (3.5) *defines a topology on* Y*,*

(ii) *this topology on* Y *induces its previous topology on* X*,*

(iii) Y *is compact.*

The space Y so constructed is called the Alexandroff compactification of X.

Proof. (i) and (ii) are easily checked and left to the reader.
(iii)-(a) First, let us show that Y is Hausdorff. Let $x \neq y$ in Y. If x and y belong to X, there exist open neighborhoods U and V in X of x and y respectively with $U \cap V = \emptyset$. We get the result in this case since U and V are open in Y. Now assume for example that $y = \omega$. Let K be a compact neighborhood of x in X. Set $U = (X \setminus K) \sqcup \{\omega\}$. Then U is an open neighborhood of ω and $U \cap K = \emptyset$.
(iii)-(b) Let $\{U_i\}_{i\in I}$ be an open covering of Y. There exists $i_0 \in I$ with $\omega \in U_{i_0}$. Then $U_{i_0} = (X \setminus K) \sqcup \{\omega\}$. Let us choose $J \subset I$, J finite such that $K \subset \bigcup_{j\in J} U_j$. Then $Y = \bigcup_{j\in J} U_j \cup U_{i_0}$. q.e.d.

Example 3.4.3. (i) The space $]-\frac{\pi}{2}, +\frac{\pi}{2}[$ is locally compact. One can embed it in a compact space by choosing the embedding

$$]-\frac{\pi}{2}, +\frac{\pi}{2}[\hookrightarrow [-\frac{\pi}{2}, +\frac{\pi}{2}]$$

but its Alexandroff compactification is obtained by identifying $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, a space isomorphic to the circle $\mathbb{S}^1$. Similarly, one can compactify $\mathbb{R}$ by adding two points, $-\infty$ and $+\infty$, but one can also add a single point by identifying $-\infty$ and $+\infty$.
(ii) The n-sphere $\mathbb{S}^n$ is isomorphic to the Alexandroff compactification of $\mathbb{R}^n$. Indeed, denote by $(x_0, x_1, \dots, x_n)$ the coordinates on $\mathbb{R}^{n+1}$ and recall that the n-sphere is the set

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1}; \sum_{i=0}^{n} x_i^2 = 1\}.$$

Denote by N the "north pole" of $\mathbb{S}^n$, $N = (1, 0, \dots, 0)$. The stereographic projection is the map

$$\begin{aligned} \rho\colon \mathbb{S}^n \setminus \{N\} &\to \mathbb{R}^n, \\ (x_0, x_1, \dots, x_n) &\mapsto y_i = \frac{x_i}{1 - x_0}. \end{aligned}$$

Clearly, this maps is bijective and is a topological isomorphism. The result follows since $\mathbb{S}^n$ is a Alexandroff compactification of $\mathbb{S}^n \setminus \{N\}$.
(iii) Recall Example 1.3.22: we have set $\overline{\mathbb{N}}:=\mathbb{N}\cup\{+\infty\}$ and we have endowed $\overline{\mathbb{N}}$ with the topology for which the open sets are the union of the subsets of $\mathbb{N}$ and the sets $[n, +\infty]$ ($n \in \mathbb{N}$). Then $\overline{\mathbb{N}}$ is the Alexandroff compactification of $\mathbb{N}$.

Exercises to Chapter 3

Exercise 3.1. Let X be a compact space and let $Y = X \sqcup \{\omega\}$ be its Alexandroff compactification. Prove that ω is isolated in Y.

Exercise 3.2. Let X be a Hausdorff space and let $(x_n)_n$ be a convergent sequence in X. Denote by x_∞ its limit.
(i) Prove that the set $A = \bigcup_n \{x_n\} \cup \{x_\infty\}$ is compact.
(ii) Prove that the space $B = \{\frac{1}{n}\}_{n\in\mathbb{N}_{>0}} \subset \mathbb{R}$ is discrete.

Exercise 3.3. Let X be a compact space and let $Y = X \sqcup \{\omega\}$ be its Alexandroff compactification. Prove that ω is isolated in Y.

Exercise 3.4. Let $X = B(0, R)$ be the open ball of $\mathbb{R}^n$ with radius $R > 0$. Define the space Y as the quotient space $\overline{B}(0, R)/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation which identifies x and y if and only if $x, y \in \partial B(0, R)$. Prove that Y is topologically isomorphic to the Alexandroff compactification of X.

Exercise 3.5. Let X and Y be two locally compact spaces and let $f\colon X \to Y$ be a continuous map. One says that f is *proper* if the inverse image by f of any compact set of Y is compact in X.
(i) Prove that a proper map is closed.
(ii) Let E and F be two real vector spaces of finite dimension. Prove that a linear map $u\colon E \to F$ is proper if and only if it is injective.
(iii) Let $(x_1, \dots, x_n)$ denote the coordinates on $\mathbb{R}^n$. Let $\varepsilon_i = \pm 1$. Give a necessary and sufficient condition on the ε_i's in order that the map $f\colon \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \sum_{i=1}^n \varepsilon_i x_i^2$ be proper.

Exercise 3.6. Let X be a compact space and let $(f_n)_{n\in\mathbb{N}}$ be an deacreasing sequence of continuous real valued functions defined on X, meaning that $f_n \in C^0(X;\mathbb{R})$ and $f_{n+1}(x) \le f_n(x)$ for all $n \in \mathbb{N}$ and all $x \in X$. Assume that the sequence $(f_n)_n$ converges simply to a continuous function f, that is, $f_n(x) \xrightarrow{n} f(x)$ for all $x \in X$. Prove that the convergence is uniform.

(Hint: Replacing f_n with $f_n - f$, we may assume from the beginning that $f \equiv 0$. Let $\varepsilon > 0$ be given. Set $E_n = \{x \in X; f_n(x) < \varepsilon\}$. Since the sequence $\{f_n\}_n$ is deacreasing, $E_{n+1} \subset E_n$ for all n. On the other hand, the hypothesis that f_n converges to 0 implies that $X = \bigcup_n E_n$. Conclude by using the compactness hypothesis.)
This result is known as the Dini theorem.

Chapter 4

Banach spaces

4.1 Normed spaces

Let $\mathbf{k}$ denote either the field $\mathbb{R}$ or the field $\mathbb{C}$ and let E be a $\mathbf{k}$-vector space. Recall that we have already defined in Section 1.2 the notions of a quasi-norm and a norm on E. A normed space is a metric space for the distance $d(x,y) = ||x-y||$, hence it is a topological space.

Example 4.1.1. Let E be a real finite dimensional vector space endowed with a basis $(e_1,\dots,e_n)$. We may endow E with the norm $||\cdot||_1$ defined as follows. For $x = \sum_{i=1}^n x_i e_i$, we set $||x||_1 = \sum_{i=1}^n |x_i|$.

Consider the linear isomorphism $u\colon \mathbb{R}^n \xrightarrow{\sim} E$,

$$x = (x_1,\dots,x_n) \mapsto \sum_{i=1}^n x_i e_i.$$

The norm $||\cdot||_1$ on E is nothing but the image of the norm $||\cdot||_1$ on $\mathbb{R}^n$ by this isomorphism.

One defines similarly the norms $||\cdot||_2$ and $||\cdot||_\infty$ on E.

Example 4.1.2. Recall (see Example 1.2.1) that $\mathbb{C}^{\mathbb{N}}$ denote the space of all sequences $x = (a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{C}$. We have already introduced the quasi-norms $||\cdot||_p$ on $\mathbb{C}^{\mathbb{N}}$ when $p = 1, 2\infty$. For $p \in [1,\infty[$, one sets

$$||x||_p = \big(\sum_n |a_n|^p\big)^{\frac{1}{p}},$$

We shall admit that these are quasi-norms for all p. One defines the spaces $l^p(\mathbb{C})$ as

$$l^p(\mathbb{C}) = \{x \in \mathbb{C}^{\mathbb{N}}; ||x||_p < \infty.$$

Hence, $l^p(\mathbb{C})$ is a normed space.

One denotes by $\mathbf{k}^{(\mathbb{N})}$ the subspace of $\mathbf{k}^{\mathbb{N}}$ consisting of sequences $x = (a_n)_n$ such that all a_n but a finite number are 0. For $p \neq q \in [1, \infty]$, the norms $||x||_p$ and $||x||_q$ on $\mathbf{k}^{(\mathbb{N})}$ are not equivalent (see Example 1.2.1).

Example 4.1.3. Let $(E, ||\cdot||)$ be a normed space and let X be a compact space. We have already endowed the space $C^0(X; E)$ of the norm of uniform convergence, also called the "the sup norm" (see (3.3)):

$$||f|| = \sup_{x \in X} ||f(x)||.$$

This applies in particular when $E = \mathbb{R}$ or $E = \mathbb{C}$.

Proposition 4.1.4. *Let E be a $\mathbf{k}$-vector space endowed with a norm $||\cdot||$. Then*

(a) $x \mapsto ||x||$ *is a continuous function from E to $\mathbb{R}$,*

(b) $(\lambda, x) \mapsto \lambda x$ *is a continuous function from $\mathbf{k} \times E$ to E,*

(c) $(x, y) \mapsto x + y$ *is a continuous function from $E \times E$ to E.*

Proof. (i) Let $x_0 \in E$. Then $x \mapsto ||x||$ is continuous at x_0 since

$$|||x_0|| - ||x||| \leq ||x - x_0||.$$

(ii) Let $(\lambda_0, a) \in \mathbf{k} \times E$. The map $(\lambda, x) \mapsto \lambda x$ is a continuous at (λ_0, a) since

$$||\lambda x - \lambda_0 x_0|| \leq |\lambda - \lambda_0|||x|| + |\lambda_0|||x - x_0||.$$

(iii) The map $(x, y) \mapsto x + y$ is continuous at (x_0, y_0) since

$$||(x + y) - (x_0 + y_0)|| \leq ||x - x_0|| + ||y - y_0||.$$

q.e.d.

Let E and F be two normed spaces over $\mathbf{k}$. For short, we denote by the same symbol $||\cdot||$ the norm on E and the norm on F. Let $u \colon E \to F$ be a linear map. One sets

$$||u|| = \sup_{||x|| \leq 1} ||u(x)||. \tag{4.1}$$

This function is clearly a quasi-norm on the space of all linear maps from E to F.

Proposition 4.1.5. *The conditions below are equivalent.*

(i) *u is continuous at $0 \in E$,*

(ii) *u is continuous,*

(iii) *u is uniformly continuous,*

(iv) $||u|| < \infty$.

Proof. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is obvious.
(i) $\Rightarrow$ (iv). For any $\varepsilon > 0$, there exists $\eta > 0$ such that $||x|| \leq \eta$ implies $||u(x)|| \leq \varepsilon$. Hence, $||x|| \leq 1$ implies $||u(x)|| \leq \frac{\varepsilon}{\eta}$.
(iv) $\Rightarrow$ (iii). Set $A = ||u||$ and let $\varepsilon > 0$. Then $||x - y|| \leq \frac{\varepsilon}{A}$ implies $||u(x) - u(y)|| \leq \varepsilon$. q.e.d.

Notation 4.1.6. Let E and F be two normed spaces over $\mathbf{k}$. One denotes by $L(E, F)$ the $\mathbf{k}$-vector space of continuous linear maps from E to F. One endows this space with the norm given in (4.1).

Definition 4.1.7. Let E be a normed space. A subset $A \subset E$ is bounded if $\sup_{x \in A} ||x|| < \infty$.

- A subset A is bounded if and only if it is contained in some ball $B(0, R)$ of radius $R < \infty$.
- A compact subset of E is bounded since it is contained in some ball $B(0, R)$ of radius $R < \infty$.
- A linear map $u \colon E \to F$ is continuous if and only if the image by u of any bounded set in E is bounded in F.

Example 4.1.8. Denote by E the $\mathbb{R}$-linear space of functions $f \colon [0, 1] \to \mathbb{R}$ such that f is continuously derivable. Let us endow E with the norm induced by that of $C^0([0, 1], \mathbb{R})$ (see Example 4.1.3). Consider the linear map

$$u \colon E \to \mathbb{R}, \quad f \mapsto f'(0).$$

Then u is not continuous. In fact, consider the subset $A = \{f_c\}_{c \geq 0}$ of E where $f_c(t) = \dfrac{ct}{1 + ct}$. Then A is bounded since $||f_c|| \leq 1$. On the other hand the set $u(A)$ is not bounded since $u(f_c) = c$.

Lemma 4.1.9. *Let E be a finite dimensional real vector space endowed with a basis $(e_1, \ldots, e_n)$ and with the norm $||\cdot||_1$ of Example 4.1.1. Let $(F, ||\cdot||_F)$ be a real normed space and let $u \colon E \to F$ be a linear map. Then u is continuous. More precisely, $||u|| \leq \sup_{i=1}^{n} ||u(e_i)||_F$.*

Proof. One has

$$\begin{aligned}||u(x)||_F &= ||u(\sum_{i=1}^{n} x_i e_i)||_F \\ &\leq \sum_{i=1}^{n} |x_i| ||u(e_i)||_F.\end{aligned}$$

q.e.d.

Theorem 4.1.10. *On a real or complex vector space of finite dimension all the norms are equivalent.*

Proof. (i) Since a complex vector space of finite dimension is in particular a real vector space of finite dimension and a norm on a complex vector space induces a norm on the real associated vector space, we may assume from the beginning that E is a real vector space.
(ii) We choose a basis $(e_1, \ldots, e_n)$ on E and we endow E with the norm $|| \cdot ||_1$ of Example 4.1.1. Let $|| \cdot ||$ be another norm on E. We shall prove that $|| \cdot ||_1$ and $|| \cdot ||$ are equivalent.

By Lemma 4.1.9 applied with $F = E$, setting $A = \sup_i ||e_i||$, we have

$$||x|| \leq \sum_{i=1}^{n} |x_i| A = A||x||_1.$$

(iii) Set $S = \{x \in E; ||x||_1 = 1\}$. Since E endowed with the norm $|| \cdot ||_1$ is topologically isomorphic to $\mathbb{R}^n$ endowed with its usual topology, the set S is compact in $(E, || \cdot ||_1)$. By (ii), the identity map $(E, || \cdot ||_1) \to (E, || \cdot ||)$ is continuous. Therefore, the set S is compact in $(E, || \cdot ||)$. The function $|| \cdot ||$ being continuous on this space by Proposition 4.1.4 (i), it takes its minimum on S. Hence, there exists $B > 0$ such that

$$B \leq ||x|| \text{ for any } x \in S.$$

Since $||x||_1 = 1$ for $x \in S$, we get

$$B||x||_1 \leq ||x|| \text{ for any } x \in S.$$

Replacing x with λx for any scalar λ, we get:

$$B||\lambda x||_1 \leq ||\lambda x|| \text{ for any } x \in S.$$

Now let $y \in E$ with $y \neq 0$. Set $\lambda = ||y||_1$. Then $y = \lambda x$ with $x \in S$. Therefore

$$||y|| \geq B||y||_1.$$

q.e.d.

The above theorem shows that on a finite dimensional vector space, there exists only one topology of normed space. Hence, when working on a finite dimensional vector space, unless otherwise specified, we shall always assume that it is endowed with this topology.

Corollary 4.1.11. *A finite dimensional normed vector space E is complete for the distance associated with the norm.*

Proof. Since all the norm are equivalent, it is enough to prove the result for the norm $||\cdot||_1$ associated with a basis. But in this case, $(E, ||\cdot||_1)$ is isomorphic, as a metric space, to $(\mathbb{R}^n, ||\cdot||_1)$ and this last space is complete. q.e.d.

Corollary 4.1.12. *Let $u\colon E \to F$ be a linear map and assume that E is finite dimensional. Then u is continuous.*

Proof. By Lemma 4.1.9 the result is true if E is endowed with the norm $||\cdot||_1$ associated with a basis. By Theorem 4.1.10 it is true for any norm on E. q.e.d.

Corollary 4.1.13. *Let F be a normed space and let L be a finite dimensional subspace of F. Then L is closed in F.*

Proof. The space L is complete by Corollary 4.1.11. q.e.d.

Theorem 4.1.14. (The Riesz Theorem.) *Let E be a normed space. Then E is finite dimensional if and only if its unit closed ball $\overline{B}(0,1)$ is compact.*

Proof. (i) If E is finite dimensional, it is isomorphic to $\mathbb{R}^n$ and we have already proved that the unit ball of $\mathbb{R}^n$ (for any norm) is compact.
(ii) Assume that the closed unit ball $\overline{B}(0,1)$ of E is compact. Note that this immediately implies that any closed ball is compact. Let $\varepsilon > 0$ and consider the open covering of $\overline{B}(0,1)$ by the open balls $\{B(a,\varepsilon)\}_{a\in\overline{B}(0,1)}$. We may extract a finite covering

$$\overline{B}(0,1) \subset \bigcup_{i=1}^{N} B(a_i,\varepsilon). \tag{4.2}$$

Denote by L the vector subspace of E generated by $\{a_1,\dots,a_N\}$. Let us show that

(4.3) for any $x \in E$ there exists $y \in L$ such that $||x-y|| \le \varepsilon||x||$.

Indeed, if $x = 0$ this is clear (choose $y = 0$) and otherwise there exists $i \in \{1, \dots, N\}$ such that $||\frac{x}{||x||} - a_i|| \leq \varepsilon$ by (4.2). Therefore,

$$||x - ||x|| \cdot a_i|| \leq \varepsilon ||x||$$

which proves (4.3).

Now assume that $L \neq E$ and choose $u \in E \setminus L$. Set $\delta = d(u, L)$. Since L is closed (by Corollary 4.1.13) and $u \notin L$, $\delta = d(u, L)$ is stricly positive (see Exercise 2.2). Hence there exists $z \in L$ with

$$||u - z|| \leq 2\delta. \tag{4.4}$$

By (4.3) applied to $x = u - z$, there exists $y \in L$ such that

$$||u - z - y|| \leq \varepsilon ||u - z||. \tag{4.5}$$

By (4.4) and (4.5), we get

$$||u - (z + y)|| \leq 2\varepsilon\delta.$$

Choosing $\varepsilon = \frac{1}{4}$, we get a contradiction since $z + y \in L$. q.e.d.

4.2 Banach spaces

As above, $\mathbf{k}$ is the field $\mathbb{R}$ or $\mathbb{C}$.

Recall that a Banach space over the field $\mathbf{k}$ is a $\mathbf{k}$-vector space endowed with a norm and complete for the distance defined by this norm.

Example 4.2.1. (i) Any finite dimensional normed space is a Banach space.
(ii) The spaces l^p ($1 \leq p \leq +\infty$) are complete. (The proof is not given here.)
(iii) Let X be a compact topological space and let E be a Banach space. Then the space $C^0(X; E)$ endowed with the norm of uniform convergence given in (4.1) is a Banach space by Corollary 3.2.5.
(iv) Let E be a normed space and let F be a Banach space. The space $L(E, F)$ of continuous linear maps endows with the norm (4.1) is a Banach space. For the proof, see Exercise 4.5.

Proposition 4.2.2. *Let E be a Banach space and let $(x_n)_n$ be a sequence in E. Assume that $\sum_{n=0}^{\infty} ||x_n|| < \infty$. Then the sequence $\{\sum_{n=0}^{p} x_n\}_p$ has a limit when $p \to \infty$. In other words, the series $\sum_{n=0}^{\infty} x_n$ is convergent in E.*

Proof. Set $y_p = \sum_{n=0}^{p} x_n$. Then

$$\begin{aligned} ||y_{p+m} - y_p|| &= ||\sum_{i=p+1}^{p+m} x_i|| \\ &\leq \sum_{i=p+1}^{p+m} ||x_i||. \end{aligned}$$

The sequence $(y_p)_p$ being a Cauchy sequence, it converges. q.e.d.

Banach algebras

Definition 4.2.3. A Banach algebra A is a Banach space which is also a unital algebra, and such that, denoting by $x \cdot y$ the product in A, this product satisfies $||x \cdot y|| \leq ||x|| \cdot ||y||$.

We shall often denote by $\mathbf{1}$ the unit in A.

Example 4.2.4. (i) Let E be a Banach space. Then $L(E, E)$ is a Banach algebra. In particular, consider $E = \mathbb{R}^n$ endowed with the norm $|| \cdot ||_1$. We may identify the algebra $L(\mathbb{R}^n, \mathbb{R}^n)$ with the algebra $M_{n,n}(\mathbb{R})$, of $n \times n$ real matrices. As a vector space, $M_{n,n}(\mathbb{R})$ is isomorphic to $\mathbb{R}^{n^2}$ and we may endow it with the norm $|| \cdot ||_1$ of $\mathbb{R}^{n^2}$. One shall be aware that this norm does not coincide (if $n > 1$) with the norm (4.1) on $L(\mathbb{R}^n, \mathbb{R}^n)$. Also note that for $n > 1$, the algebra $M_{n,n}(\mathbb{R})$ is not commutative.
(ii) Let X be a compact space. Then $C^0(X; \mathbf{k})$ ($\mathbf{k} = \mathbb{C}$ of $\mathbb{R}$) is a Banach algebra. Note that this algebra is commutative.
(iii) Let X be a compact space and let A be a Banach algebra. Then $C^0(X; A)$ is a Banach algebra.

Proposition 4.2.5. *Let A be a Banach algebra and let $a \in A$ with $||a|| < 1$. Then $\mathbf{1} - a$ is invertible in A.*

Proof. The series $\sum_{n=0}^{\infty} a^n$ is convergent by Proposition 4.2.2. Since

$$\begin{aligned} (\mathbf{1} - a)(\mathbf{1} + a + \cdots + a^n) &= \mathbf{1} - a^{n+1}, \\ (\mathbf{1} + a + \cdots + a^n)(\mathbf{1} - a) &= \mathbf{1} - a^{n+1}, \end{aligned}$$

this series converges to $(\mathbf{1} - a)^{-1}$. q.e.d.

Corollary 4.2.6. *Let A be a Banach algebra and denote by Ω the subset of A consisting of invertible elements. Then Ω is open in A and denoting by $J \colon \Omega \to \Omega$ the map $a \mapsto a^{-1}$, then J is continuous.*

Note that Ω is multiplicative, that is, if a and b belong to Ω then $a\dot{b}$ also belongs to Ω.

Proof. (i) Ω is open. Let $a \in \Omega$. Then $a + h = a(\mathbf{1} + a^{-1}h)$ and $\mathbf{1} + a^{-1}h$ is invertible as soon as $||h|| \cdot ||a^{-1}|| < 1$, hence as soon as $||h|| < (||a^{-1}||)^{-1}$. Therefore, the ball $B(a, \varepsilon)$ is contained in Ω as soon as $\varepsilon < (||a^{-1}||)^{-1}$.
(ii) J is continuous. This is equivalent to saying that $(a+h)^{-1} - a^{-1}$ goes to 0 when h goes to 0. When multiplying $(a+h)^{-1} - a^{-1}$ by a, this is equivalent to saying that $(\mathbf{1} - u)^{-1}$ goes to $\mathbf{1}$ when u goes to 0. Since

$$|| \mathbf{1} - (\mathbf{1} - u)^{-1} || = || \sum_{n \geq 1} u^n || \leq \sum_{n \geq 1} ||u||^n,$$

this follows from the fact that $\sum_{n \geq 1} \varepsilon^n$ goes to 0 when ε goes to 0. q.e.d.

4.3 Study of the space $C^0(K; \mathbb{R})$

Let K be a compact topological space. We have already proved that the space $C^0(K; \mathbb{R})$ endowed with the sup norm (see (4.1))

$$||f|| = \sup_{x \in K} |f(x)|.$$

is a Banach space. Hence, it is a Banach algebra.

Theorem 4.3.1. (The Stone-Weierstrass theorem.) *Let $\mathcal{H}$ be a unital subalgebra of $C^0(K; \mathbb{R})$ with the property that for any $x \neq y$ in K, there exists $f \in \mathcal{H}$ with $f(x) \neq f(y)$. Then $\mathcal{H}$ is dense in $C^0(K; \mathbb{R})$.*

In order to prove this result, we need a few lemmas.

Lemma 4.3.2. *Let $\mathcal{L}$ be a vector subspace of $C^0(K; \mathbb{R})$ satisfying:*
(i) $f, g \in \mathcal{L}$implies $\sup(f, g) \in \mathcal{L}$ and $\inf(f, g) \in \mathcal{L}$,
(ii) for any $x, y \in K$ and any $\alpha, \beta \in \mathbb{R}$ satisfying $\alpha = \beta$ if $x = y$, there exists $f \in \mathcal{L}$ such that $f(x) = \alpha$ and $f(y) = \beta$.
Then $\mathcal{L}$ is dense in $C^0(K; \mathbb{R})$.

Proof. Let $f \in C^0(K; \mathbb{R})$ and let $\varepsilon > 0$.
(i) Let $y \in K$. For any $x \in K$, there exists $g_x \in \mathcal{L}$ such that $g_x(y) = f(y)$ and $g_x(x) = f(x)$. Set

$$U_x = \{z \in K; g_x(z) > f(z) - \varepsilon\}.$$

[1]Section 4.3 will not be treated during the course 2010/2011

Then U_x is an open neighborhood of x. Since $K = \bigcup_{x\in K} U_x$, there exist $x_1, \dots, x_n$ such that $K = \bigcup_{i=1}^n U_{x_i}$. Set

$$h_y = \sup_{i=1,\dots,n} g_{x_i}.$$

Then $h_y \in \mathcal{L}$, $h_y(y) = f(y)$ and $h_y \geq f - \varepsilon$.
(ii) Set

$$V_y = \{z \in K; h_y(z) < f(z) + \varepsilon\}.$$

Then V_y is an open neighborhood of y. Since $K = \bigcup_{y\in K} V_x$, there exist $y_1, \dots, y_m$ such that $K = \bigcup_{j=1}^m V_{y_j}$. Set

$$k = \inf_{j=1,\dots,m} h_{y_j}.$$

Then $k \in \mathcal{L}$ and $k \leq f + \varepsilon$.
(iii) Since $h_{y_j} \geq f - \varepsilon$, we have $k \geq f - \varepsilon$. Therefore $||k - f|| \leq \varepsilon$. q.e.d.

Lemma 4.3.3. *On the interval* $[0, 1]$ *endowed with the coordinate* t*, the function* $\sqrt{t}$ *is the uniform limit of a sequence of real polynomials.*

Proof. Define the sequence of polynomials $p_n(t)$ by induction, by setting:

$$p_0(t) = 0, \quad p_{n+1}(t) = p_n(t) + \frac{(t - p_n^2(t))}{2}. \tag{4.6}$$

Let us prove by induction that

$$t \geq p_n^2(t). \tag{4.7}$$

This is true for $n = 0$. Assuming this is true for n, we have

$$p_{n+1}(t) - \sqrt{t} = (p_n(t) - \sqrt{t})\big(1 - \frac{1}{2}(\sqrt{t} + p_n(t))\big). \tag{4.8}$$

Since $p_n(t) \leq \sqrt{t}$, $(\sqrt{t} + p_n(t)) \leq 2$. Therefore, the left hand side in (4.8) is ≤ 0 and the induction proceeds.

From (4.6) and (4.7) we deduce that the sequence of functions $(p_n)_n$ is increasing. Hence, for each t the sequence $(p_n(t))_n$ has a limit $p(t)$ which satisfies

$$p(t) = p(t) + \frac{1}{2}(t - p^2(t)).$$

Therefore, $p(t) = \sqrt{(t)}$. In other words, for each $t \in [0, 1]$, $\sqrt{t} - p_n(t)$ converges to 0. The fact that the convergence is uniform will follow from Lemma 4.3.4 below. q.e.d.

Lemma 4.3.4. (Dini's lemma.) *Let I be a compact space and let $g_n\colon I \to \mathbb{R}$ ($n \in \mathbb{N}$) be a sequence of continuous functions satisfying*
(i) $g_{n+1} \leq g_n$,
(ii) *for any $t \in I$, $g_n(t) \xrightarrow{n} 0$.*
Then $g_n \xrightarrow{n} 0$ uniformly.

Proof. Let $\varepsilon > 0$. Set

$$I_n = \{t \in I; g_n(t) \geq \varepsilon\}.$$

The I_n's are closed, $I_{n+1} \subset I_n$ and $\bigcap_n I_n = \emptyset$. Therefore, I being compact, there exists N such that $I_n = \emptyset$ for any $n \geq N$. This is equivalent to saying that there exists N such that $|g_n(t)| \leq \varepsilon$ for any $t \in I$ and any $n \geq N$. q.e.d.

Proof of Theorem 4.3.1. Denote by $\overline{\mathcal{H}}$ the closure of $\mathcal{H}$ in $C^0(K;\mathbb{R})$. Then $\overline{\mathcal{H}}$ is an algebra. Let $x, y \in K$ with $x \neq y$ and let $\alpha, \beta \in \mathbb{R}$. Let $g \in \mathcal{H}$ with $g(x) \neq g(y)$. Set

$$f(\cdot) = \frac{\alpha - \beta}{g(x) - g(y)}\big(g(\cdot) - g(y)\big) + \beta.$$

Then $f \in \mathcal{H}$, $f(x) = \alpha$ and $f(y) = \beta$. In order to apply Lemma 4.3.2 to $\overline{\mathcal{H}}$, it remains to check that $\overline{\mathcal{H}}$ is stable by sup and inf of two functions. Since

$$\sup(f, g) = \frac{1}{2}(f + g + |f - g|,$$
$$\inf(f, g) = \frac{1}{2}(f + g - |f - g|,$$

it is enough to check that $f \in \overline{\mathcal{H}}$ implies $|f| \in \overline{\mathcal{H}}$.

One has $|f| = \sqrt{f^2}$. Moreover, after multiplying f by a scalar, we may assume that $|f| \leq 1$. Let $\varepsilon > 0$. By Lemma 4.3.3 there exists a polynomial $p_n(t)$ such that $\sup_t(\sqrt{t} - p_n(t)) \leq \varepsilon$. Therefore,

$$||\sqrt{f^2} - p_n(f^2)|| \leq \varepsilon.$$

Since $p_n(f^2)$ belongs to $\mathcal{H}$, this proves that $\sqrt{f^2}$ belongs to $\overline{\mathcal{H}}$. q.e.d.

Applications

Corollary 4.3.5. *Let $\mathcal{H}$ be a unital $\mathbb{C}$-sub-algebra of $C^0(K;\mathbb{C})$ with the property that*
(i) *for any $x \neq y$ in K, there exists $f \in \mathcal{H}$ with $f(x) \neq f(y)$,*
(ii) *if $f \in \mathcal{H}$ then $\overline{f} \in \mathcal{H}$ (recall that $\overline{f}$ denotes the complex conjugate of f).*
Then $\mathcal{H}$ is dense in $C^0(K;\mathbb{R})$.

Proof. Denote by $\mathcal{H}^{\mathbb{R}}$ the $\mathbb{R}$-sub-algebra of $\mathcal{H}$ consisting of $\mathbb{R}$-valued functions. Let $x \neq y$. There exists $f \in \mathcal{H}$ such that $f(x) \neq f(y)$. Then either $\Re f(x) \neq \Im f(y)$ or $\Im f(x) \neq \Im f(y)$ and both belong to $\mathcal{H}^{\mathbb{R}}$ by the second hypothesis. Therefore, $\mathcal{H}^{\mathbb{R}}$ satisfies the hypothesis of the Stone-Weierstrass theorem, hence, is dense in $C^0(K;\mathbb{R})$. Now let $f \in C^0(K;\mathbb{C})$. There exists sequences $(f_n^1)_n$ and $(f_n^2)_n$ in $\mathcal{H}^{\mathbb{R}}$ such that $f_n^1 \xrightarrow{n} \Re f$ and $f_n^2 \xrightarrow{n} \Im f$. Then $f_n^1 + f_n^2 \xrightarrow{n} f$. q.e.d.

Corollary 4.3.6. *Let K be a compact subset of $\mathbb{R}^n$. Then $\mathbb{R}[x_1,\dots,x_n]$ is dense in $C^0(K;\mathbb{R})$ and $\mathbb{C}[x_1,\dots,x_n]$ is dense in $C^0(K;\mathbb{C})$.*

Proof. It is enough to check that the polynomials separate the points, but the linear functions already separate points. Indeed, if $x \neq y$, then there exists $i \in \{1,\dots,n)$ such that $x_i \neq y_i$. q.e.d.

Now denote by $\mathbb{C}[\exp i\,\theta]$ the $\mathbb{C}$-algebra of trigonometric polynomials in one variable $\theta \in \mathbb{R}$. An element f of this algebra is a finite sum

$$f = \sum_n a_n \exp(in\theta),\ a_n \in \mathbb{C} \text{ and the sum is finite.}$$

Recall (see Example 1.7.2) that we have denoted by $\mathbb{T}$ the space $\mathbb{R}/2\pi\mathbb{Z}$, that is, the quotient $\mathbb{R}/\sim$ where $\sim$ is the equivalence relation which identifies $x, y \in \mathbb{R}$ if $x - y = 2\pi n$, $n \in \mathbb{Z}$. We may identify the space $C^0(\mathbb{T};\mathbb{C})$ with the subspace of $C^0(\mathbb{R};\mathbb{C})$ consisting of periodic functions of period 2π, that is, the space of functions f satisfying $f(x) = f(x+2\pi n)$. Clearly, $\mathbb{C}[\exp i\theta]$ is a $\mathbb{C}$-sub-algebra of $C^0(\mathbb{T};\mathbb{C})$.

Corollary 4.3.7. *The space $\mathbb{C}[\exp i\theta]$ is dense in $C^0(\mathbb{T};\mathbb{C})$.*

Proof. Recall that, denoting by $\mathbb{S}^1$ the unit circle in the Euclidian space $\mathbb{R}^2$, the map

$$\varphi\colon \mathbb{T} \to \mathbb{S}^1, \quad \theta \mapsto \exp(i\theta)$$

is an isomorphism (see Example 1.7.4). The map φ defines by composition, an isomorphism of Banach algebras

$$\varphi^*\colon C^0(\mathbb{S}^1;\mathbb{C}) \xrightarrow{\sim} C^0(\mathbb{T};\mathbb{C}).$$

Denote by (x, y) the coordinates on $\mathbb{R}^2$. Since $\varphi^*(\mathbb{C}[x, y]) = \mathbb{C}[\exp i\theta]$, it is enough to check that $\mathbb{C}[x, y]$ is dense in $C^0(\mathbb{S}^1;\mathbb{C})$, which follows from Corollary 4.3.5. q.e.d.

Exercises to Chapter 4

Exercise 4.1. Let us endow $\mathbb{R}^n$ with the norm $||\cdot||_1$. After identifying $L(\mathbb{R}^n,\mathbb{R}^n)$ with $M_{n,n}(\mathbb{R})$, describe the associated norm on $L(\mathbb{R}^n,\mathbb{R}^n)$ given in (4.1).

Exercise 4.2. Let E be a normed space and F a Banach space.
(i) Prove that the space $L(E,F)$ is a Banach space for the norm (4.1).
(ii) Let $\widehat{E}$ be a normed space and assume that E is a dense subspace of $\widehat{E}$ and the norm of E is induced by that of $\widehat{E}$. Prove that the natural map $L(\widehat{E},F)\to L(E,F)$ (which, to a linear map $u\colon \widehat{E}\to F$, associates its restriction to E) is an isomorphism of Banach spaces.
(Hint: for (ii), use Proposition 2.2.6.)

Exercise 4.3. Denote by E the Banach space $C^0([0,1];\mathbb{R})$ endowed with the sup-norm. Set

$$X_n=\{f\in E;\exists t\in[0,1] \text{ such that } \forall s\in[0,1], |f(t)-f(s)|\leq n|t-s|\}.$$

(i) Prove that X_n is closed in E.
(ii) Prove that X_n has no interior points, that is, $\mathrm{Int(X_n)}=\emptyset$, and deduce by Corollary 2.3.4 that $\bigcup_n X_n$ has no interior points.
(iii) let $f\in E$ and assume that $f'(t_0)$ exists for some $t_0\in[0,1]$. Prove that $f\in X_n$ for some $n\in\mathbb{N}$.
(iv) Deduce that there exists a continuous function f on $[0,1]$ which is nowhere derivable.

Exercise 4.4. Denote by E the Banach space $C^0([0,1];\mathbb{R})$ endowed with the sup-norm. Denote by $\mathcal{P}$ the vector subspace consisting of polynomial functions and, for $n\in\mathbb{N}$, denote by $\mathcal{P}_n$ vector subspace consisting of polynomial of order $\leq n$. One endows $\mathcal{P}$ and $\mathcal{P}_n$ of the norms induced by that of E.
(i) Show that $\mathcal{P}_n$ is a Banach space and is closed in E.
(ii) Show that the series $\sum_{j=0}^{\infty}(\frac{x}{2})^j$ converges uniformly on $[0,1]$ to the function $\dfrac{2}{2-x}$ and deduce that $\mathcal{P}$ is not a Banach space.
(iii) Let λ be the linear form on $\mathcal{P}$ given by $P\mapsto\sum_{j=0}^{\infty}P^{(j)}(0)$, where $P^{(j)}$ denotes the j-th derivative of the polynomial P. Show that the restriction of λ to each $\mathcal{P}_n$ is continuous but λ is not continuous on $\mathcal{P}$.

Exercise 4.5. Let X be a topological space and let $(F,||\cdot||)$ be a Banach space. Let us denote by $C^{0,\mathrm{b}}(X,F)$ the subspace of $C^0(X,F)$ consisting of bounded functions, that is, functions $f\colon X\to F$ such that $f(X)$ is bounded in F.

(i) Prove that the quasi-norm of uniform convergence is a norm on $C^{0,\mathrm{b}}(X, F)$ and that $C^{0,\mathrm{b}}(X, F)$ is complete for the associated distance, hence is a Banach space.
(ii) Let $(E, ||\cdot||)$ be a normed space and denote by X its closed unit ball. Prove that the natural map $L(E, F) \to C^{0,\mathrm{b}}(X, F)$ is injective and that the image of $L(E, F)$ in $C^{0,\mathrm{b}}(X, F)$ by this map is closed. Deduce that $L(E, F)$ is a Banach space.

Chapter 5

Connectness and homotopy

5.1 Connectness

Let X be a topological space.

Proposition 5.1.1. *The following conditions are equivalent:*

(i) *there does not exist* $A \subset X$, $A \neq \emptyset$, $A \neq X$, A *is both open and closed,*

(ii) *there does not exist* $A \subset X$, $B \subset X$ *such that* A *and* B *are both open and non empty,* $A \cap B = \emptyset$, $A \cup B = X$,

(iii) *same as in* (ii) *when replacing the hypothesis that* A *and* B *are open by the hypothesis that* A *and* B *are closed.*

The proof is obvious.

Definition 5.1.2. Let X be a topological space.

(a) One says that X is connected if one of the equivalent conditions in Proposition 5.1.1 is satisfied.

(b) One says that X is locally connected if any $x \in X$ admits a fundamental system of connected neighborhoods.

One shall be aware that there exist spaces which are connected without being locally connected (see Example 5.1.14).

Theorem 5.1.3. *The space* $\mathbb{R}$ *is connected.*

Proof. Let us argue by contradiction and assume $\mathbb{R}$ is not connected. Let $A \subset \mathbb{R}$ such that $\emptyset \neq A$, $\mathbb{R} \neq A$, A is closed and open in $\mathbb{R}$. Let $x \in \mathbb{R} \setminus A$ and assume for example that $A \cap [x, +\infty[\neq \emptyset$. Denote by B the set $A \cap [x, +\infty[$. Then B is non empty, closed and bounded from below. Hence, it admits a smallest element, say b. On the other hand, since $x \notin A$, $A \cap [x, +\infty[= A \cap]x, +\infty[$. Therefore, B is open. Hence there exists $\varepsilon > 0$ such that $]b - \varepsilon, b + \varepsilon] \subset B$. This contradicts the fact that b is the smallest element in B. q.e.d.

- An open interval of $\mathbb{R}$ is connected. Indeed, if it is not empty, then it is isomorphic to $\mathbb{R}$.

- $\mathbb{R} \setminus \{0\}$ is not connected, but (see below) $\mathbb{R}^n \setminus \{0\}$ is connected for $n > 1$.

One says that a subset A of a topological space X is connected if it is connected for the induced topology.

Proposition 5.1.4. *Let $A \subset B \subset \overline{A} \subset X$ and assume that A is connected. Then B is connected.*

Proof. Assume $B = U_1 \cup U_2$ with U_i $(i = 1, 2)$ open in B and $U_1 \cap U_2 = \emptyset$. We shall show that either U_1 or U_2 is empty.

There exist U_i' $(i = 1, 2)$ open in X such that $U_i' \cap B = U_i$. On the other hand, $A \cap U_i$ is open in A, $A = (U_1 \cap A) \cup (U_2 \cap A)$ and $(U_1 \cap A) \cap (U_2 \cap A) = \emptyset$. Hence $A \cap U_1$ or $A \cap U_2$ is empty. For example, $A \cap U_1 = \emptyset$. Since $A \cap U_1 = A \cap U_1' = \emptyset$, A is contained in $X \setminus U_1'$ which is closed and this implies $\overline{A} \subset X \setminus U_1'$. Hence, $B \subset X \setminus U_1'$ and $B \cap U_1' = B \cap U_1 = \emptyset$. q.e.d.

Corollary 5.1.5. *A subset of $\mathbb{R}$ is connected if and only if it is an interval.*

Proof. (i) We have already seen that an open interval is connected. Hence, any interval is connected by Proposition 5.1.4.
(ii) Conversely, let A be a connected subset of $\mathbb{R}$. By using the map $t \mapsto \tan(t)$, we may assume that A is contained in the interval $]-1, +1[$. Set $a = \inf A$, $b = \sup A$. Then $A \subset [a, b]$ and it is enough to prove that $]a, b[\subset A$. Let us argue by contradiction and assume there exists $x \in]a, b[$ with $x \notin A$. Then $A = (]-\infty, x[\cup]x, +\infty[$. Hence, A would be the union of two non empty open subsets with empty intersection. This contradicts the hypothesis that A is connected. q.e.d.

Remark 5.1.6. The closure $\overline{A}$ of a set A may be connected although A is not connected. For example $\mathbb{R} \setminus \{0\}$ is not connected and $\mathbb{R}$ is connected.

Proposition 5.1.7. *Let $f\colon X \to Y$ be a continuous map and assume that X is connected. Then $f(X)$ is connected.*

Proof. Let V_1 and V_2 be two non empty open subsets of Y such that $V_1 \cap V_2 = \emptyset$ and $f(X) \subset V_1 \cup V_2$. Then $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. Therefore, either $f^{-1}(V_1)$ of $f^{-1}(V_2)$ is empty. Then either $f(X) \subset V_1$ or $f(X) \subset V_2$. q.e.d.

Corollary 5.1.8. *Let $f\colon X \to \mathbb{R}$ be a continuous map and assume that X is connected. Then for any $x, x' \in X$, f takes all values between $f(x)$ and $f(x')$.*

Connected components

Proposition 5.1.9. *Let $\{A_i\}_{i \in I}$ be a family of connected subsets of A, let $A = \bigcup_i A_i$ and assume that for any $i, j \in I$, $A_i \cap A_j \neq \emptyset$. Then A is connected.*

Proof. Let us argue by contradiction. Assume U_1 and U_2 are two non empty open subsets such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = A$. Since the A_i's are connected and $A_i = (A_i \cap U_1) \cup (A_i \cap U_2)$, any A_i is contained either in U_1 or in U_2. Let $I = I_1 \sqcup I_2$, with $i \in I_j \Leftrightarrow A_i \subset U_j$ $(j = 1, 2)$. Then $A_l \cap A_k = \emptyset$ if $l \in I_1$ and $k \in I_2$. this contradicts the hypothesis. q.e.d.

Definition 5.1.10. Let $x \in X$. Denote by C_x the union of all connected subsets of X which contain x. Then C_x is called the connected component of x in X and is also called “a connected component of X”.

- A connected component is connected since the union of all connected subsets of X which contain a point x is connected by Proposition 5.1.9.

- Any connected component of X is closed in X by Proposition 5.1.4. One shall be aware that a connected component of X is not necessarily open in X as seen in Example 5.1.11.

- Two connected components are equal or disjoint (again by Proposition 5.1.9) and the relation $x \sim y$ if and only if x and y belong to the same connected component, is an equivalence relation. Hence X is the disjoint union of its connected components.

Example 5.1.11. (i) Let X be a topological space and let $a \in X$ be an isolated point, that is, the set $\{a\}$ is open and closed in X. Assume $X \neq \{a\}$. Then $X = (X \setminus \{a\}) \sqcup \{a\}$ is the disjoint union of two non empty open sets, hence is not connected.

(ii) Let X be the set $\{0\} \sqcup \{\frac{1}{n}; n \in \mathbb{N}_{>0}\}$ endowed with the topology induced by $\mathbb{R}$. Let A be the connected component of 0. If some point $\frac{1}{n}$ belongs to A, then A is not connected by (i). Hence, $\{0\}$ is the connected component of the point $0 \in X$, although this set is not open in X.

Proposition 5.1.12. *The space X is locally connected if and only if, for each open set U of X, the connected components of the space U are open.*

Proof. (i) Assume X is locally connected, let U be an open subset of X and let $C \subset U$ be a a connected component of U. Let $x \in C$. By the hypothesis, there exists a connected neighborhood V of x contained in U. Since $C \cup V$ is connected, V is contained in C. Hence, C, being a neighborhood of each of its points, is open.
(ii) Let us prove the converse. Let $x \in X$ and let U be an open neighborhood of x. Denote by V the connected component of x in U. By the hypothesis, V is open. Hence, V is an connected neighborhood of x contained in U. q.e.d.

Corollary 5.1.13. *Assume X is compact and locally connected. Then X has only finite many connected components.*

Proof. Consider the covering of X by its connected components. By the hypothesis and Proposition 5.1.12, this is an open covering. The space X being compact, we may extract a finite covering. q.e.d.

Example 5.1.14. Let

$$A = \{(x, y) \in \mathbb{R}^2; x > 0, y = \sin(\frac{1}{t})\},$$
$$B = \{(x, y) \in \mathbb{R}^2; x = 0, |y| \leq 1\}.$$

(i) The set A is connected. Indeed, this follows from Proposition 5.1.7 since A is the image of $\mathbb{R}_{>0}$ by the continuous map $f\colon \mathbb{R}_{>0} \to \mathbb{R}^2$, $t \mapsto (t, \sin(\frac{1}{x}))$.

Then $X = A \sqcup B$ is connected. Indeed, $X = \overline{A}$ and we may apply Proposition 5.1.4.
(ii) X is not locally connected. In fact consider the open set $U = X \cap \{(x, y) \in \mathbb{R}^2; |y| < \frac{1}{2}\}$. The set $\{(x, y) \in \mathbb{R}^2; x = 0, |y| < \frac{1}{2}\}$ is a connected component of U and is not open.

5.2 Homotopy

In the sequel, we denote by I the closed interval $I = [0, 1]$.

Definition 5.2.1. Let X and Y be two topological spaces.

(i) Let f_0 and f_1 be two continuous maps from X to Y. One says that f_0 and f_1 are homotopic if there exists a continuous map $h\colon I \times X \to Y$ such that $h(0, \cdot) = f_0$ and $h(1, \cdot) = f_1$.

(ii) Let $f\colon X \to Y$ be a continuous map. One says that f is a homotopy equivalence if there exists $g\colon Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. In such a case one says that X and Y are homotopically equivalent, or simply, are homotopic.

(iii) One says that a topological space X is contractible if X is homotopic to a point $\{x_0\}$.

Lemma 5.2.2. *The relation "f_0 is homotopic to f_1" is an equivalence relation.*

Proof. (i) Let $f\colon X \to Y$ be a continuous map. Then f is homotopic to f. Indeed, define $h\colon I \times X \to Y$ by $h(t, x) = f(x)$.
(ii) Let f_0 and f_1 be continuous maps from X to Y. Assume that f_0 and f_1 are homotopic by a map $h\colon I \times X \to Y$. Then f_1 and f_0 are homotopic by the map $\widetilde{h}$ given by $\widetilde{h}(t, x) = h(1 - t, x)$.
(iii) If f_0 and f_1 are homotopic by a map $h_1\colon I \times X \to Y$ and f_1 and f_2 are homotopic by a map $h_2\colon I \times X \to Y$, then f_0 and f_2 are homotopic by the map $h\colon I \times X \to Y$ given by $h(t, x) = h_1(2t, x)$ for $0 \le t \le \frac{1}{2}$ and $h(t, x) = h_2(2t - 1, x)$ for $\frac{1}{2} \le t \le 1$. q.e.d.

Of course, the relation of being homotopic is much weaker than the relation of being topologically isomorphic. For example, $\mathbb{R}^n$ is homotopic to $\{0\}$ (see below) but certainly not topologically isomorphic.

A topological space is contractible if and only if there exist $g\colon \{x_0\} \to X$ and $f\colon X \to \{x_0\}$ such that $f \circ g$ is homotopic to id_X. Replacing x_0 with $g(x_0)$, this means that there exists $h\colon I \times X \to X$ such that $h(1, x) = \mathrm{id}_X$ and $h(0, x)$ is the map $x \mapsto x_0$. Note that contractible implies non empty.

Examples 5.2.3. (i) Let V be a real vector space. Recall that a set A is convex if for any $a, b \in A$ we have $[a, b] \subset A$. Recall that the segment $[a, b]$ denotes the set

$$[a, b] = \{ta + (1 - t)b; 0 \le t \le 1\}.$$

Also recall that a set A is star-shaped if there exists $a \in A$ such that for any $b \in A$, the segment $[a, b]$ is contained in A.

A non empty convex set is star-shaped and a star-shaped is contractible. Indeed, choose $a \in A$. Then $h(t,x) = ta + (1-t)x$ is a homotopy.

In particular, let $\gamma \subset V$ be a closed non empty cone. Then γ is contractible. Indeed, γ is star-shaped at 0.

(ii) Let $X = \mathbb{S}^n$ be the unit sphere of the Euclidian space $\mathbb{R}^{n+1}$ and let $Y = \mathbb{R}^{n+1} \setminus \{0\}$. The embedding $f\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ is a homotopy equivalence. Indeed, denote by $g\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n$ the map $x \mapsto x/||x||$. Then $g \circ f = \mathrm{id}_X$ and $f \circ g$ is homotopic to id_Y. The homotopy is given by the map $h(x,t) = (t/||x|| + 1 - t)x$.

(iii) Consider the truncated closed cone

$$A = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 = x_3^2, 0 \leq x_3 \leq 1\}.$$

Clearly, it is homotopic to the origin, the homotopy being given by $x \mapsto tx$. However, the circle $\{x; x_1^2 + x_2^2 = 1\}$ is not homotopic to a point. This last fact is not obvious and follows for example from the result in Example 5.3.7 below.

Arcwise connected spaces

Definition 5.2.4. Let X be a topological space.

(a) A path in X is a continuous map $\gamma\colon [0,1] \to X$. One calls $\gamma(0)$ and $\gamma(1)$ the ends of the path. One also calls $\gamma(0)$ the origin of the path and $\gamma(1)$ its end.

(b) One says that X is arcwise connected if for any x and y in X, there exists a path in X with ends x and y.

(c) One says that X is locally arcwise connected if each $x \in X$ admits a fundamental system of arcwise connected neighborhoods.

One often identifies a path with its image $\gamma([0,1])$ in X.

A path is connected by Proposition 5.1.7.

Lemma 5.2.5. *The relation on X given by $x_0 \sim x_1$ if there is a path which starts at x_0 and ends at x_1 is an equivalence relation.*

Proof. Denote by pt a set with a single point. A point $x \in X$ may be regarded as a continuous map $f\colon \mathrm{pt} \to X$. By this identification, a path from x_0 to x_1 may be considered as an homotopy from the constant map $f_0\colon \mathrm{pt} \to X$, $f_0(\mathrm{pt}) = x_0$, to the constant map $f_1\colon \mathrm{pt} \to X$, $f_1(\mathrm{pt}) = x_t$. Hence, the result follows from Lemma 5.2.2. q.e.d.

Proposition 5.2.6. *If X is arcwise connected, then X is connected.*

Proof. We may assume that X is not empty. Let $x_0 \in X$. By the hypothesis, for any $x \in X$, there exists a path γ_x with ends x_0 and x. Then $X = \bigcup_{x \in X} \gamma_x$ is connected by Proposition 5.1.9. q.e.d.

Proposition 5.2.7. *If X is locally arcwise connected and connected, then X is arcwise connected.*

Proof. We may assume that X is not empty. Let $x_0 \in X$ and denote by A the set of points x such that there exists a path with ends x_0 and x.
(i) By the hypothesis, for any $x \in X$ there exists an open neighborhood U_x of x such that any point $y \in U_x$ may be joined to x by a path. Hence, if $x \in A$ and $y \in U_x$, y may be joined to x_0 by a path, which shows that A is open.
(ii) It remains to show that A is closed. Let $z \in \overline{A}$ and let U_z be an arcwise connected neighborhood of z. Let $y \in V_z \cap A$. There exists a path with ends z and y and there exists a path with ends x_0 and y. Hence, there exists a path with ends x_0 and z. q.e.d.

Consider the hypothesis

(5.1) X is locally arcwise connected.

Definition 5.2.8. Assume (5.1). The set of connected components of X is denoted by $\pi_0(X)$.

Let X and Y be two topological spaces satisfying (5.1) and let $f \colon X \to Y$ be a continuous map. Then f defines a map

$$\pi_0(f) \colon \pi_0(X) \to \pi_0(Y). \tag{5.2}$$

Indeed, if x_1 and x_2 in X are connected by a path $\gamma \colon I \to X$, then $f(x_1)$ and $f(x_2)$ in Y are connected by the path $f \circ \gamma$. Moreover,

- if f_0 and f_1 are homotopic, they define the same map:

  $$\pi_0(f_0) = \pi_0(f_1) \colon \pi_0(X, x_0) \to \pi_0(Y, y_0), \tag{5.3}$$

  Indeed, consider an homotopy $\{f_t\}_{t \in [0,1]}$ from f_0 to f_1. Let $x \in X$. Then $f_0(x)$ and $f_1(x)$ belong to the same connected component of Y since they are connected by the arc γ where $\gamma(t) = f_t(x)$.

- if $g\colon Y \to Z$ is a continuous map and Z satisfies (5.1), then

$$\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f). \tag{5.4}$$

  This means that for $x \in X$, the connected component of $(g \circ f)(x)$ is the image by g of the connected component of $f(x)$, which is clear.

Using (5.3) and (5.4), we get that the group $\pi_0(\cdot)$ is an homotopy invariant. More precisely:

Proposition 5.2.9. *Let X and Y be two topological spaces satisfying* (5.1)*. Assume that X and Y are homotopic. Then the sets $\pi_0(X)$ and $\pi_0(Y)$ are isomorphic.*

In other words, the cardinals of the set of connected components of X and Y are the same.

5.3 Fundamental group

Recall that I is the closed interval $I = [0, 1]$.

For the reader's convenience, we partly recall Definition 5.2.4.

Definition 5.3.1. (i) A path from x_0 to x_1 in X is a continuous map $\sigma\colon I \to X$, with $\sigma(0) = x_0$ and $\sigma(1) = x_1$. The two points x_0 and x_1 are called the ends of the path.

(ii) Two paths σ_0 and σ_1 are called homotopic if there exists a continuous function $\varphi\colon I \times I \to X$ such that $\varphi(i, t) = \sigma_i(t)$ for $i = 0, 1$. (See Definition 5.2.1.)

(iii) If the two paths have the same ends, x_0 and x_1, one says they are homotopic with fixed ends if moreover $\varphi(s, 0) = x_0, \varphi(s, 1) = x_1$ for all s. This is equivalent to saying that there exists a continuous function $\psi : D \to X$ such that $\psi(i, t) = \sigma_i(t)$ for $i = 0, 1$.

(iv) A loop in X is continuous map $\gamma : \mathbb{S}^1 \to X$. One can also consider a loop as a path γ such that $\gamma(0) = \gamma(1)$. A trivial loop is a constant map $\gamma : \mathbb{S}^1 \to \{x_0\}$. Two loops are homotopic if they are homotopic as paths.

[1]Section 5.3 is out of the scope of the course 2010/2011

By Lemma 5.2.5, being homotopic is an equivalence relation.

If σ is a path from x_0 to x_1 and τ a path from x_1 to x_2 one can define a new path $\tau\sigma$ (in this order) from x_0 to x_2 by setting $\tau\sigma(t) = \sigma(2t)$ for $0 \leq t \leq 1/2$ and $\tau\sigma(t) = \tau(2t-1)$ for $1/2 \leq t \leq 1$.

If σ is a path from x_0 to x_1, one can define the path σ^{-1} from x_1 to x_0 by setting $\sigma^{-1}(t) = \sigma(1-t)$.

Let us denote by $[\sigma]$ the homotopy class of a path σ. It is easily checked that the homotopy class of $\tau\sigma$ depends only on the homotopy classes of σ and τ. Hence, we can define $[\tau][\sigma]$ as $[\tau\sigma]$. The next result is left as an exercise.

Lemma 5.3.2. *The product $[\sigma][\tau]$ is associative, and $[\sigma\sigma^{-1}]$ is the homotopy class of the trivial loop at x_0.*

By this lemma, the set of homotopy classes of loops at x_0 is a group.

Definition 5.3.3. The set of homotopy classes of loops at x_0 endowed with the above product is called the fundamental group of X at x_0 and denoted $\pi_1(X; x_0)$.

Let X and Y be two topological spaces satisfying (5.1) and let $f\colon X \to Y$ be a continuous map. Let $x_0 \in X$ and set $y_0 = f(x_0)$. Then f defines a map

$$\pi_1(f)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0). \tag{5.5}$$

Indeed, if γ is a loop at x_0, then $f \circ \gamma$ is a loop at y_0, and if two loops are homotopic, their images by f will remain homotopic. Moreover,

- if f_0 and f_1 are homotopic, they define the same map:

$$\pi_1(f_0) = \pi_1(f_1)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \tag{5.6}$$

- if $g\colon Y \to Z$ is a continuous map and Z satisfies (5.1), then

$$\pi_1(g \circ f) = \pi_1(g) \circ \pi_1(f). \tag{5.7}$$

Assume (5.1). If σ is a path from x_0 to x_1 in X, then the map $\gamma \mapsto \sigma^{-1}\gamma\sigma$ defines an isomorphism

$$\pi_1(X; x_0) \simeq \pi_1(X; x_1).$$

Hence, if X is connected, all groups $\pi_1(X; x)$ are isomorphic for $x \in X$. In this case, one sometimes denote simply by $\pi_1(X)$ one of these groups and calls it the fundamental group of X.

Using (5.6) and (5.7), we get that the group $\pi_1(\cdot)$ is an homotopy invariant. More precisely:

Proposition 5.3.4. *Let X and Y be two spaces satisfying* (5.1)*. Assume that X and Y are homotopic. Then the two groups $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic.*

Definition 5.3.5. Let X be a topological space satisfying (5.1). One says that X is simply connected if any loop in X is homotopic to a trivial loop.

If X is non empty, connected and simply connected, then $\pi_1(X) \simeq \mathrm{pt}$.

Example 5.3.6. Let $n > 1$. Then $\pi_1(\mathbb{S}^n) \simeq \mathrm{pt}$. In other words, $\mathbb{S}^n$ is simply connected for $n > 1$.

Example 5.3.7. One has $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$.

Although this result is rather intuitive, its proof is not so easy, and will not be given here.

5.4 C^0-manifolds

Definition 5.4.1. Let $n \in \mathbb{N}$. A C^0-manifold, or topological manifold, of dimension n is a Hausdorff topological space X countable at infinity (which means that X is a countable union of compact subsets) and locally topologically isomorphic to an open subset of $\mathbb{R}^n$.

To be locally topologically isomorphic to an open subset of $\mathbb{R}^n$ means that there exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X, each U_i being topologically isomorphic to an open subset of $\mathbb{R}^n$. The U_i's are called local charts of X.

Example 5.4.2. (i) A non empty open subset of $\mathbb{R}^n$ is a C^0-manifold of dimension n.
(ii) The Euclidian sphere $\mathbb{S}^n$ is a C^0-manifold of dimension n.
(iii) A finite or countable discrete set is a C^0-manifold of dimension 0.
(iv) If X is a C^0-manifold of dimension n and Y is C^0-manifold of dimension p then $X \times Y$ is C^0-manifold of dimension $n + p$. In particular, the torus $\mathbb{T}$ is a C^0-manifold of dimension 2.
(v) The subset

$$Z = \{(x, y) \in \mathbb{R}^2; y = 0, x \leq 0 \cup x = 0, y \geq 0\}$$

is a C^0-manifold of dimension 1.

[1]Section 5.4 is out of the scope of the course 2010/2011

(vi) Consider the closed cone in $\mathbb{R}^{n+1}$ ($n \geq 1$):

$$\gamma = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}; \sum_{i=1}^{n} x_i^2 = x_0^2\}.$$

One checks easily that $\gamma \setminus \{0\}$ is a C^0-manifold of dimension n. Let us show that γ is not a C^0-manifold. Otherwise, there exists a connected open neighborhood U of 0 topologically isomorphic to an open set $V \subset \mathbb{R}^n$. Then $U \setminus \{0\}$ would be topologically isomorphic to $V \setminus \{x\}$ for some $x \in V$. This is not possible since such an open set $V \setminus \{x\}$ would be connected, and $U \setminus \{0\}$ is not.

Since a C^0-manifold is locally isomorphic to an open subset of $\mathbb{R}^n$, a C^0-manifold is locally arcwise connected.

In particular, if X is a compact C^0-manifold, its has a finite number of connected components.

The study and the classification of compact C^0-manifolds is an important problem, extremely difficult as soon as the dimension is ≥ 3. If X is a compact connected C^0-manifold of dimension 0, then X is a point, $X = \{x\}$.

Theorem 5.4.3. *Let X be a non empty compact connected C^0-manifold of dimension* 1*. Then X is isomorphic to the circle $\mathbb{S}^1$.*

Sketch of proof. (i) First, consider two open subsets U and V of X, each of them being isomorphic to a non empty open interval of $\mathbb{R}$ with a non empty intersection. Then the intersection $U \cap V$ is either connected and in this case $U \cup V$ is isomorphic to a non empty open interval of $\mathbb{R}$, or $U \cap V$ has two connected components and in this case, $U \cup V$ is isomorphic to $\mathbb{S}^1$. We shall admit this fact.
(ii) Now, consider an open covering $X = \bigcup_{i \in I} U_i$ where the U_i's are isomorphic to a non empty open interval of $\mathbb{R}$. We may extract a finite covering $X = \bigcup_{i=1}^{N} U_i$. We have $N \geq 2$ otherwise, X would be isomorphic to an open interval and such an interval is not compact. Let us prove the result by induction on N.
(iii) Assume $N = 2$. Hence, $X = U_1 \cup U_2$. Then $U_1 \cap U_2$ has two connected components and $U_1 \cup U_2 \simeq \mathbb{S}^1$.
(iv) Consider U_1. There exists $2 \leq i \leq N$ with $U_1 \cap U_i \neq \emptyset$. By reordering the set $\{2, \ldots, N\}$, we may assume $i = 2$. If $U_1 \cap U_2$ has a single connected component, $U_1 \cup U_2$ is isomorphic to an interval and replacing U_1 with $U_1 \cup U_2$, the induction proceeds. Otherwise, $U_1 \cup U_2$ is isomorphic to $\mathbb{S}^1$. Since $U_1 \cup U_2$ is open in X and $\mathbb{S}^1$ is compact, $\mathbb{S}^1$ is a connected component of X. Since X is connected, $X = U_1 \cup U_2$. q.e.d.

Remark 5.4.4. A compact connected and simply connected C^0-manifold of dimension 2 is topologically isomorphic to the 2-sphere. Although this theorem is not considered as very difficult, its proof will not be given here. A similar result holds in dimension 3: a compact connected and simply connected C^0-manifold of dimension 3 is topologically isomorphic to the 3-sphere, but this problem, known as the Poincaré conjecture, has only been solved very recently by Perelmann. In order that a compact connected and simply connected C^0-manifold of dimension $n > 3$ be isomorphic to the n-sphere, other conditions, whose formulation is out of the scope of this course, are necessary.

Exercises to Chapter 5

Exercise 5.1. Prove the assertions in Example 5.1.14.

Exercise 5.2. Let X be a connected space and let $f \colon X \to \mathbb{Z}$ be a continuous function. Here, $\mathbb{Z}$ is endowed with the discrete topology. Prove that f is constant.

Exercise 5.3. Prove that the connected components of $\mathbb{Q}$ are the sets $\{x\}$, $x \in \mathbb{Q}$.

Exercise 5.4. Let $X = \bigcup_{n\in\mathbb{N}} A_n$. Assume that all A_n's are connected and $A_n \cup A_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Prove that X is connected.

Exercise 5.5. Let X be a Hausdorff topological space and let $(K_n)_n$ be a decreasing sequence of compact subsets of X. Assume all K_n's are connected. Prove that $K := \bigcap_n K_n$ is connected.

Exercise 5.6. Let X be a topological space satisfying (5.1). Recall that a map $f \colon X \to \mathbb{R}$ is locally constant if any $x \in X$ admits a neighborhood on which f is constant. Denote by $LC(X, \mathbb{R})$ the real vector space of $\mathbb{R}$-valued locally constant functions.
(i) Prove that $LC(X, \mathbb{R}) \simeq C^0(X; \mathbb{R}_{dis})$ where $\mathbb{R}_{dis}$ is the set $\mathbb{R}$ endowed with the discrete topology.
(ii) Prove that there is an isomorphism $LC(X, \mathbb{R}) \simeq \mathbb{R}^{\pi_0(X)}$.

Exercise 5.7. Let $n \geq 1$ and let $a_1, a_2 \in \mathbb{R}^n$ with $a_1 \neq a_2$. Prove that $\mathbb{R}^n \setminus \{a_1, a_2\}$ is homotopic to the union $A \cup B \cup I$ where A and B are two disjoint $n-1$-spheres and I is a closed interval $[a, b]$ with $I \cap A = \{a\}$ and $I \cap B = \{b\}$.