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Hyperfunctions on hypo-analytic manifolds, by Paulo D. Cordaro and Francois Trèves, Ann. of Mathematics Studies, vol. 136, Princeton Univ. Press, Princeton, NJ, 1994, xx + 377 pp., $29.95, ISBN 0-691-92992-X

Hyperfunctions were introduced by Mikio Sato [8] in the late fifties as cohomological objects built from holomorphic functions. More precisely, if M denotes an n-dimensional real analytic manifold and X a complexification of M, the sheaf $\mathcal{B}_M$ of hyperfunctions is defined as:

$$\mathcal{B}_M := H^n_M(\mathcal{O}_X) \otimes \mathrm{or}_M,$$

where $\mathcal{O}_X$ is the sheaf of holomorphic functions on X and or_M the orientation sheaf on M. Using Čech cohomology, it is then possible to represent hyperfunctions as "boundary values" of holomorphic functions defined on tuboids having M as an edge.

For example, if M is an open interval of the real line $\mathbb{R}$ and X is an open subset of $\mathbb{C}$, with $X \cap \mathbb{R} = M$, then

$$\mathcal{B}(M) \simeq \mathcal{O}(X \setminus M)/\mathcal{O}(X)$$

and the natural map $b : \mathcal{O}(X \setminus M) \to \mathcal{B}(M)$ is called the boundary value morphism. Set $X^+ = X \cap \{z \in \mathbb{C}; \mathrm{Im}\, z > 0\}$. If $\phi \in \mathcal{O}(X^+)$, one also writes $\phi(x + i0)$ instead of $b(\phi)$. This is because if $b(\phi)$ is a distribution, then it is the limit (for the topology of the space of distributions) of $\phi(x + i\epsilon)$ when $\epsilon \to 0, \epsilon > 0$, but in general $\phi(x + i\epsilon)$ has no limit in any reasonable topology and in fact $\mathcal{B}(M)$ has no natural separated topology (since $\mathcal{O}(X)$ is dense in $\mathcal{O}(X \setminus M)$). Hence Sato's definition forces any holomorphic function on $X \setminus M$ to have a boundary value, just as Schwartz's definition of distributions forces continuous functions to have derivatives.

Once one has the concept of hyperfunctions in hand, a natural problem is to recognize where the boundary values come from. It was known after Martineau's work [6] that on $\mathbb{R}^n$, any hyperfunction $u \in \mathcal{B}(\mathbb{R}^n)$ may be written as a finite sum

$$u = \sum_\alpha b(\phi_\alpha),$$

where each ϕ_α is a holomorphic function in a tube $\mathbb{R}^n + i\Gamma_\alpha$, Γ_α denoting a convex proper open cone in $\mathbb{R}^n$, as soon as the family of the interiors of the polar cones of the Γ_α's is a covering of $\mathbb{R}^n \setminus \{0\}$. Hence, to define a hyperfunction, one needs in general at least $n + 1$ holomorphic functions. Moreover the famous "edge of the wedge theorem" ([7]) asserts that the above sum is zero if and only if there exist convex proper open cones $\Gamma_{\alpha\beta}$ such that $\Gamma^\circ_{\alpha\beta}$ (the polar cone of $\Gamma_{\alpha\beta}$) is a neighborhood of $\Gamma^\circ_\alpha \cap \Gamma^\circ_\beta$ (hence, $\Gamma_{\alpha\beta}$ is almost the convex hull of $\Gamma_\alpha \cup \Gamma_\beta$) and holomorphic functions $\phi_{\alpha\beta}$ defined in the tubes $\mathbb{R}^n + i\Gamma_{\alpha\beta}$, each ϕ_α being decomposed as a sum of $\phi_{\alpha\beta}$.

In 1969, M. Sato introduced a fundamentally new idea, which is the cornerstone of the so-called "Microlocal Analysis" developed in the 70's, both by Sato's school and Hörmander's school. Roughly speaking, Sato's idea is as follows. Let M be a real submanifold of a *real* manifold X. Then for some mathematical objects (e.g. a cohomology class of some sheaf defined on X), to be supported by M is a global property which can be read on $T^*_M X$, the conormal bundle to M in X. For that purpose one considers the family of closed tuboids in X with edge M: these tuboids define closed tubes in the normal bundle $T_M X$ to which one associates their polar sets in $T^*_M X$. For example, a hyperfunction on $\mathbb{R}$ is a cohomology class of $\mathcal{O}_{\mathbb{C}}$ supported by $\mathbb{R}$, and this class may be non-zero although considered as a cohomology class supported by the closed tube $\{z; \mathrm{Im}\, z \leq 0\}$ it is zero. This idea led Sato to introduce $\mu_M(F)$, the microlocalization along M of a sheaf F on X, a sheaf (in fact, an object of the derived category of sheaves) on $T^*_M X$ whose direct image by the projection $\pi : T^*_M X \to X$ will be $R\Gamma_M(F)$, the cohomology of F with support in M. When M is a real analytic manifold, X is a complexification of M and $F = \mathcal{O}_X$, one gets the sheaf $\mathcal{C}_M$ of Sato's microfunctions. A hyperfunction u on M is a globally defined microfunction $sp(u)$, and the support of $sp(u)$ in $T^*_M X$ is the analytic wave front set of u (see [9], and see [3] for a systematic study of sheaves in the microlocal framework).

At the opposite of Sato's, the point of view chosen by Cordaro and Trèves is highly non-cohomological. Their attempt is to give a presentation of the theory of hyperfunctions and their analytic wave front set "with bare hands", as sums of boundary values of holomorphic functions, with the only tool of Sjöstrand's so-called FBI transform. At the same time, they present a variant of the theory by considering C^∞-manifolds locally embedded in $\mathbb{C}^n$ and by replacing the sheaf $\mathcal{O}_X$ by the sheaf of holomorphic solutions of some holomorphic vector fields.

Let us describe with some details the contents of the book.

Let M be a C^∞-manifold. A hypo-analytic structure on M is the data of an open covering $M = \bigcup_\alpha M_\alpha$ and C^∞-maps $\phi_\alpha : M_\alpha \to \mathbb{C}^m$, such that if $\phi_\alpha = (\phi^1_\alpha, ..., \phi^m_\alpha)$, then $d\phi^1_\alpha \wedge ... \wedge d\phi^m_\alpha \neq 0$, and transition functions $\phi_{\alpha\beta}$, holomorphic in a neighborhood of $\phi_\alpha(M_\alpha) \cap \phi_\beta(M_\beta)$. Let us express this in our own language with our own notation and let us consider a complex manifold X of dimension m and a C^∞-map $\phi : M \to X$ such that $d\phi$, considered as a linear map $(TM)^{\mathbb{C}} \to TX$, has rank m.

Chapters I and II are devoted to the case where M is "totally real", namely, when $m = \dim M$. Then the authors develop the theory of hyperfunctions and microfunctions. As for hyperfunctions, they follow Martineau's approach that consists in gluing together analytic functionals thanks to the Runge property of M in X. Next, they define the boundary value of a holomorphic function f in a tube with edge M by assuming first that f extends holomorphically outside a compact subset of M, then by gluing together these boundary values modulo the relation given by the edge of the wedge theorem. This technical procedure has been made possible thanks to the deep analytical tool introduced by J. Sjöstrand [10, 11] under the name of FBI-transform (FBI for Fourier-Bros-Iagolnitzer).

At this stage, let us allow a digression. The FBI transform is a Fourier-integral transformation with complex phase, which corresponds in the language of Sato's school to a complex quantized contact transformation (CQCT for short). For a suitable choice of the phase, such a CQCT interchanges (locally on T^*X), $T^*_M X$ and $T^*_N X$ where now N is the boundary of a strictly pseudo-convex open subset $\Omega \subset X$, and the sheaf $\mathcal{C}_M$ of microfunctions on M is interchanged with the sheaf $j_*\mathcal{O}_\Omega/\mathcal{O}_X$ (where $j : \Omega \hookrightarrow X$), that is, with the sheaf of boundary values on N of holomorphic functions on Ω. This last sheaf being extremely easy to manipulate, one sees why CQCT or else the FBI transform are powerful tools, and that is the method used by Kashiwara for proving that the sheaf $\mathcal{C}_M$ is flabby. The idea of

CQCT emerged around 1977 (Boutet de Monvel, Kashiwara) and has been used by several people (Lebeau [5], Hörmander [2], Komatsu [4]) for giving an elementary approach to Sato's theory.

Let us come back to the book under review, keeping as above our own notation (in particular, avoiding the heavy use of local coordinates). We assume now that $\phi : M \to X$ factors as:

$$M \xrightarrow{i} Y \xrightarrow{f} X$$

where i turns M into a totally real submanifold of Y and $f : Y \to X$ is a smooth morphism of complex manifolds. Let $m = \dim_{\mathbb{C}} X, m + n = \dim_{\mathbb{C}} Y = \dim_{\mathbb{R}} M$.

Let us denote by Θ_Y the sheaf of holomorphic vector fields on Y, and by Θ_f the subsheaf of vector fields tangent to the leaves of f. Let us denote by $\mathcal{D}_Y$ the ring of holomorphic differential operators on Y and by $\mathcal{I}_f$ the left ideal generated by Θ_f. Chapter III is essentially devoted to the proof of the isomorphism, for U open in M:

$$Ext^q_{\mathcal{D}_Y}(U; \mathcal{D}_Y/\mathcal{I}_f, \mathcal{B}_M) \simeq H^{m+n+q} R\Gamma_U(Y_{\mathbb{R}}; R\mathcal{H}om_{\mathcal{D}_{Y\times\bar{Y}}}(\mathcal{D}_Y/\mathcal{I}_f \underline{\boxtimes} \mathcal{O}_{\bar{Y}}, \mathcal{B}_{Y_{\mathbb{R}}})$$

where $\bar{Y}$ denotes the complex manifold Y endowed with the conjugate complex structure, $Y_{\mathbb{R}}$ the real underlying manifold of Y, and $\mathcal{D}_Y/\mathcal{I}_f \underline{\boxtimes} \mathcal{O}_{\bar{Y}}$ is the system associated both to the vector fields of Θ_f and to the Cauchy-Riemann system on Y. Nevertheless the symbols $R\mathcal{H}om, Ext, R\Gamma_U$ never appear in this book and the theorem is formulated differently, using cohomology of explicit Koszul complexes. Notice that the result is purely algebraic and quite easy when formulated as above, in the language of derived categories.

In Chapter IV the notion of hyperfunction on $M = M' \times \mathbb{R}^p$ with C^∞ parameters w.r.t. $\mathbb{R}^p$ is introduced, but, as pointed out by the authors, this notion is not intrinsically defined. Let us call this sheaf $\mathcal{BC}^\infty$. Then it is shown that if the sheaf Θ_f mentioned above is "elliptic" w.r.t. $\mathbb{R}^p$, then there are isomorphisms

$$\mathcal{E}xt^q_{\mathcal{D}_Y}(\mathcal{D}_Y/\mathcal{I}_f, \mathcal{BC}^\infty_M) \simeq \mathcal{E}xt^q_{\mathcal{D}_Y}(\mathcal{D}_Y/\mathcal{I}_f, \mathcal{B}_M),$$

and in particular the left hand side is intrinsically defined. Here again this isomorphism is written using explicit Koszul complexes.

Finally, in the last pages, the authors discuss various applications and in particular the (non-) solvability in the sheaf of hyperfunctions of systems of differential operators generated by complex vector fields. Indeed, keeping in mind the above notation, the problem of finding geometrical conditions on f in order that $\mathcal{E}xt^q_{\mathcal{D}_Y}(\mathcal{D}_Y/\mathcal{I}_f, \mathcal{B}_M)$ vanishes for a given q is an important and open problem. The case where $n = 1$ is now classical, and the other extreme case where $\dim X = 1$ has been solved only very recently by the authors [1], and is not treated in full generality in this book.

The reader will have understood that I am not totally convinced by the point of view adopted here by the authors. In fact, hyperfunctions are cohomology classes, so why represent them by functions? Most analysts feel secure working with good concrete holomorphic functions rather than with abstract cohomology classes, but there are plenty of situations where this is simply not possible (see [12]). Even when this is possible, we cannot go very far with such methods: for example, take a hyperfunction which satisfies some partial differential equation $Pu = 0$. Then, in general, u will not be a sum of boundary values of holomorphic functions satisfying this equation. Moreover, there is a high price to pay for the lack of functoriality:

it forces one to repeat the proofs in each particular situation and hides the deep geometrical meaning of the results. A few tools of homological algebra would avoid writing down each time explicit complexes (use the functor $\mathcal{E}xt$), a little more will allow treating simultaneously various derived functors (such as $R\mathcal{H}om$, and $R\Gamma$) and many complicated formulas will be highly simplified. Contrary to a general opinion, derived categories are not so difficult tools to manipulate (definitely easier than, say, Lebesgue integral). For example, in this language, a system of partial differential equations becomes a left $\mathcal{D}_Y$-module $\mathcal{M}$, its complex of holomorphic solutions is the object $F = R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}, \mathcal{O}_Y)$; and if one is interested in replacing holomorphic solutions by other kind of solutions derived from the sheaf $\mathcal{O}_Y$, such as cohomology classes supported by a submanifold M, then one studies $R\Gamma_M(F)$. One important piece of information is the characteristic variety of $\mathcal{M}$, and this can be recovered from the knowledge of F (this is the micro-support of F). At this stage, one can even forget that we are working with linear PDE, and sometimes one can also forget the complex structure of Y, many problems being in fact purely topological. This is the point of view developed in [3], and this is another story.

However, even if cohomological methods are in many situations a really efficient tool, one should keep in mind that they have their own limits. They rapidly become very hard to manipulate when dealing with precise growth conditions, and so far, don't apply at all to non-linear problems, contrary to traditional methods which often are suitable with slight modifications. This is a good reason not to reject approaches based on explicit representations of cohomology, and this book is an attempt in this direction.

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